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Some remarks on smooth renormings of Banach spaces

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ABSTRACT

We prove that in every separable Banach space X with a Schauder basis and a C^k-smooth norm it is possible to approximate, uniformly on bounded sets, every equivalent norm with a C^k-smooth one in a way that the approximation is improving as fast as we wish on the elements depending only on the tail of the Schauder basis. Our result solves a problem from the recent monograph of Guirao, Montesinos and Zizler.

1. Introduction

The problem of smooth approximation of continuous mappings is one of the classical themes in analysis. An important special case of this problem is the existence of C^k-smooth approximations of norms on an infinite-dimensional real Banach space. More precisely, assume that the real Banach space X admits a C^k-smooth norm. Let $\|\cdot\|$ be an equivalent norm on X, $\varepsilon > 0$. Does there exist a C^k-smooth renorming $|||\cdot|||$ of X such that $1 \le \frac{|||x|||}{\|x\|} \le 1 + \varepsilon$ holds for all $0 \neq x \in X$?

In its full generality, this problem is still open, even in the case $k = 1$ (no counterexample is known). For $k = 1$, the problem can be solved easily by using Smulyan's criterion, once a dual LUR norm is present on X^*. This covers a wide range of Banach spaces, in particular all WCG (hence all separable, and all reflexive) spaces [6]. In the absence of a dual LUR renorming, the problem appears to be completely open.

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For $k \geq 2$ the problem seems to be more difficult, and no dual approach is available. To begin with, Deville [3] proved that the existence of C^2-smooth norm has profound structural consequence for the space. In some sense, such spaces are either superreflexive, or close to c_0. To get an idea of the difficulty of constructing smooth norms, we refer to e.g. [15,13,14,9,1].

Broadly speaking, the construction of the smooth norm is carried out by techniques locally using only finitely many ingredients. Of course, this idea is present already in the concept of partitions of unity, but in the setting of norms it is harder to implement as we need to preserve the convexity of the involved functions. Probably the first explicit use of this technique in order to construct smooth norms is found in the work of Pechanec, Whitfield and Zizler [16]. The authors construct a particular LUR and C^1-smooth norm on $c_0(\Gamma)$ which admits C^∞-approximations. This result has later been generalized to arbitrary WCG spaces [11]. Recently, Bible and Smith [2] have succeeded in solving the smooth approximation problem for norms on $c_0(\Gamma)$, $k = \infty$. This is essentially the only known nonseparable space where the problem has been solved.

In the separable setting, the problem has been completely solved for every separable Banach space and every k, in a series of papers [8,4,5], and the final solution in [12].

We refer to the monographs [6] and [10] for a more complete discussion and references, too numerous to be included in our note.

The main result of the present note delves deeper into the fine behavior of C^k-smooth approximations of norms in the separable setting. It is in some sense analogous to the condition (ii) in Theorem VIII.3.2 in [6], which claims that in the Banach space with C^k-smooth partitions of unity, the C^k-smooth approximations to continuous functions exist with a prescribed precision around each point. Our result solves Problem 170 (stated somewhat imprecisely) in [7]. We also hope that the result may be of some use in the context of metric fixed point theory, where several notions are present of properties which asymptotically improve with growing codimension. For example, let us mention the notion of *asymptotically non-expansive function* or the ones of *asymptotically isometric copy* of ℓ_1 or c_0.

Let us now state our main result.

Theorem 1.1. *Let $(X, \|\cdot\|)$ be a separable real Banach space with a Schauder basis $\{e_i\}_{i\geq 1}$ that admits a C^k-smooth renorming. Then for every sequence $\{\varepsilon_N\}_{N\geq 0}$ of positive numbers, there is a C^k-smooth renorming $|||\cdot|||$ of X such that for every $N \geq 0$*

$$\Big| |||x||| - \|x\| \Big| \leq \varepsilon_N \|x\| \qquad \text{for } x \in X^N,$$

where $X^N := \overline{\operatorname{span}}\,\{e_i\}_{i\geq N+1}$.

In other words, we can approximate the original norm with a C^k-smooth one in a way that on the "tail vectors" the approximation is improving as fast as we wish.

The proof of Theorem 1.1 will be presented in the next section. The rough idea is the following. By the result in [12], for every N one can find a C^k-smooth norm $\|\cdot\|_N$ such that $\Big| \|\cdot\|_N - \|\cdot\| \Big| \leq \varepsilon_N \|\cdot\|$. One is tempted to use the standard gluing together in a C^k-smooth way and hope that the resulting norm will be as desired. Unfortunately, in this way there is no possibility to assure that on X^N only the $\|\cdot\|_n$ norms with $n \geq N$ will enter into the gluing procedure. To achieve this feature it is necessary that the norms $\|\cdot\|_N$ be quantitatively different on X^N and $X_N = \operatorname{span}\{e_i\}_{i=1}^N$. The first part of the argument, consisting of the geometric Lemma 2.1 and some easy deductions, is exactly aimed at finding new norms which are quantitatively different on tail vectors and "head vectors". The second step consists in iterating this renorming for every n and rescaling the norms. Finally, we suitably approximate these norms with C^k-smooth ones and we glue everything together using the standard technique.

2. Proof of the main result

In this section we shall prove Theorem 1.1.

Let X be a separable (real) Banach space with norm $\|\cdot\|$ and a Schauder basis $\{e_i\}_{i\geq 1}$. We denote by $K := \text{b.c.}\{e_i\}_{i\geq 1}$ the basis constant of the Schauder basis (of course K depends on the particular norm we are using). We also let P_k be the usual projection $P_k(\sum_{j\geq 1}\alpha^j e_j) = \sum_{j=1}^k \alpha^j e_j$ and $P^k := I_X - P_k$, i.e. $P^k(\sum_{j\geq 1}\alpha^j e_j) = \sum_{j\geq k+1}\alpha^j e_j$. It is clear that $\|P_k\| \leq K$ and $\|P^k\| \leq K+1$. Finally, we denote $X_k := \operatorname{span}\{e_i\}_{i=1}^k$ and $X^k = \overline{\operatorname{span}}\{e_i\}_{i\geq k+1}$, i.e. the ranges of the two projections respectively.

We will make extensive use of convex sets: let us recall that a convex set C in a Banach space X is a *convex body* if it has nonempty interior. Obviously a symmetric convex body is in particular a neighborhood of the origin and the unit ball B_X of X is a bounded, symmetric convex body (we shorthand this fact by saying that it is a BCSB). Any other BCSB B in X induces an equivalent norm on X via its Minkowski functional

$$\mu_B(x) := \inf\{t > 0 : x \in tB\}.$$

We will also denote by $\|\cdot\|_B$ the norm induced by B, i.e. $\|x\|_B := \mu_B(x)$; obviously $\|\cdot\|_{B_X}$ is the original norm of the space. Moreover we clearly have

$$B \subseteq C \Longrightarrow \mu_B \geq \mu_C,$$
$$\mu_{\lambda B} = \frac{1}{\lambda}\mu_B$$

and passing to the induced norms we see that

$$B \subseteq C \subseteq (1+\delta)B \Longrightarrow \frac{1}{1+\delta}\|\cdot\|_B \leq \|\cdot\|_C \leq \|\cdot\|_B.$$

We now start with the first part of the argument.

Lemma 2.1. *Let $(X, \|\cdot\|)$ be a Banach space with a Schauder basis $\{e_i\}_{i\geq 1}$ with basis constant K. Denote the unit ball of X by B, fix $k \in \mathbb{N}$, two parameters $\lambda > 0$ and $0 < R < 1$, and consider the sets*

$$D := \left\{x \in X : \left\|P^k x\right\| \leq R\right\} \cap (1+\lambda)\cdot B,$$
$$C := \overline{\operatorname{conv}}\{D, B\}.$$

Then C is a BCSB and

$$C \cap X^k \subseteq \left(1 + \lambda\frac{K}{K+1-R}\right)\cdot B.$$

Heuristically, if we modify the unit ball in the direction of X_k, this modification results in a perturbation of the ball also in the remaining directions, but this modification is significantly smaller.

Proof. The fact that C is a BCSB is obvious. Let $x \in C \cap X^k$ and notice that $0 \in \operatorname{Int}C$ (as $B \subseteq C$); by the cone argument we deduce that $tx \in \operatorname{Int}C$ for $t \in [0, 1)$. Moreover $\operatorname{conv}\{D, B\}$ has non-empty interior, so it is easily seen that its interior equals the interior of its closure, hence $tx \in \operatorname{Int}C = \operatorname{Int}(\operatorname{conv}\{D, B\}) \subseteq \operatorname{conv}\{D, B\}$. If we can show that $tx \in \left(1 + \lambda\frac{K}{K+1-R}\right)\cdot B$ we then let $t \to 1$ and conclude the proof. In other words we can assume without loss of generality that $x \in X^k \cap \operatorname{conv}\{D, B\}$. Hence we can write

$x = ty + (1-t)z$ with $t \in [0,1]$, $y \in D$ and $z \in B$, in particular $\left\|P^k y\right\| \leq R$ and $\|z\| \leq 1$. Moreover $x \in X^k$ implies

$$\|x\| = \left\|P^k x\right\| \leq t \left\|P^k y\right\| + (1-t) \left\|P^k z\right\| \leq tR + (1-t)(K+1);$$

if $\|x\| \leq 1$ the conclusion of the lemma is clearly true, so we can assume $\|x\| \geq 1$. Thus we have $1 \leq K + 1 - t(K+1-R)$ and this yields $t \leq \frac{K}{K+1-R}$.

Next, we move the points y, z slightly, in such a way that x is still a convex combination of them: fix two parameters $\tau, \eta > 0$ to be chosen later and consider $u := (1-\tau)y$ and $v := (1+\eta)z$. Obviously $x = \frac{t}{1-\tau}u + \frac{1-t}{1+\eta}v$ and we require this to be a convex combination:

$$1 = \frac{t}{1-\tau} + \frac{1-t}{1+\eta} \qquad \Longrightarrow \qquad \tau = \frac{(1-t)\eta}{t+\eta} \leq 1$$

(of course this choice implies $1 - \tau \geq 0$). Since $y \in D$, we have $\|v\| \leq 1 + \eta$ and $\|u\| \leq (1-\tau)\|y\| \leq (1-\tau)(1+\lambda)$; we want these norms to be both small, so we require (here we use the previous choice of τ)

$$1 + \eta = (1-\tau)(1+\lambda) \qquad \Longrightarrow \qquad \eta = \lambda t.$$

With this choice of τ and η we have $\|u\|, \|v\| \leq 1 + \eta = 1 + \lambda t \leq 1 + \lambda \cdot \frac{K}{K+1-R}$; by convexity the same holds true for x and the proof is complete. □

We now modify again the obtained BCSB in such a way that on X^k the body is an exact multiple of the original ball; this modification does not destroy the properties achieved before. It will be useful to denote by $S := \left\{x \in X : \left\|P^k x\right\| \leq R\right\}$; with this notation we have $D := S \cap (1+\lambda) \cdot B$.

Corollary 2.1. *In the above setting, let* $\gamma := \frac{K}{K+1-R}$ *and*

$$\tilde{B} := \overline{\operatorname{conv}} \left\{C, X^k \cap (1+\lambda\gamma) \cdot B\right\}.$$

Then $\tilde{B}$ *is a BCSB and*

$$B \subseteq \tilde{B} \subseteq (1+\lambda) \cdot B,$$
$$S \cap \tilde{B} = S \cap (1+\lambda) \cdot B,$$
$$X^k \cap \tilde{B} = X^k \cap (1+\lambda\gamma) \cdot B.$$

Proof. It is obvious that $\tilde{B}$ is a BCSB. Of course $B \subseteq C$, so $B \subseteq \tilde{B}$ too; also $D \subseteq (1+\lambda) \cdot B$ implies $C \subseteq (1+\lambda) \cdot B$. Since $\gamma \leq 1$ we deduce that $\tilde{B} \subseteq (1+\lambda) \cdot B$.

The $\subseteq$ in the second assertion follows from what we have just proved; for the converse inclusion, just observe that $S \cap (1+\lambda) \cdot B = D \subseteq \tilde{B}$.

For the last equality, obviously $X^k \cap (1+\lambda\gamma) \cdot B \subseteq \tilde{B}$, so the $\supseteq$ inclusion follows. For the converse inclusion, let $p \in X^k \cap \tilde{B}$; exactly the same argument as in the first part of the previous proof (with C replaced by $\tilde{B}$) shows that we can assume $p \in \operatorname{conv}\left\{C, X^k \cap (1+\lambda\gamma) \cdot B\right\} \cap X^k$. So we can write $p = ty + (1-t)z$ with $y \in C$ and $z \in X^k \cap (1+\lambda\gamma) \cdot B$. If $t = 0$, $p = z \in X^k \cap (1+\lambda\gamma) \cdot B$ and we are done. On the other hand if $t > 0$, from $p \in X^k$ we deduce that $y \in X^k$ too; hence in fact $y \in C \cap X^k \subseteq (1+\lambda\gamma) \cdot B$, by the previous lemma. By convexity $p \in (1+\lambda\gamma) \cdot B$ and the proof is complete. □

The next proposition is essentially a restatement of the above corollary in terms of norms rather than convex bodies; we write it explicitly since in what follows we will use the approach using norms. The general setting is the one above: we have a separable Banach space X with a Schauder basis $\{e_i\}_{i\geq 1}$ and we denote by $X_k := \operatorname{span}\{e_i\}_{i=1}^k$ and $X^k = \overline{\operatorname{span}}\{e_i\}_{i\geq k+1}$.

Proposition 2.1. *Let B be a BCSB in X and let $\|\cdot\|_B$ be the induced norm; also let K be the basis constant of $\{e_i\}_{i\geq 1}$ relative to $\|\cdot\|_B$. Fix $k \in \mathbb{N}$ and two parameters $\lambda > 0$ and $0 < R < 1$. Then there is a BCSB $\tilde{B}$ in X such that the induced norm $\|\cdot\|_{\tilde{B}}$ satisfies the following properties:*

(a)

$$\|\cdot\|_{\tilde{B}} \leq \|\cdot\|_B \leq (1+\lambda)\|\cdot\|_{\tilde{B}},$$

(b)

$$\|\cdot\|_B = (1+\lambda\gamma)\|\cdot\|_{\tilde{B}} \qquad \text{on } X^k,$$

(c)

$$\|x\|_B = (1+\lambda)\|x\|_{\tilde{B}} \qquad \text{whenever } \left\|P^k x\right\| \leq \frac{R}{1+\lambda}\|x\|,$$

where $\gamma := \frac{K}{K+1-R}$.

Proof. We let $\tilde{B}$ be the convex body defined in the corollary. Then (a) follows immediately from the corollary and (b) is immediate too: for $x \in X^k$

$$\|x\|_{\tilde{B}} = \inf\left\{t > 0 : x \in t \cdot \tilde{B}\right\} = \inf\left\{t > 0 : x \in t \cdot \left(\tilde{B} \cap X^k\right)\right\} =$$
$$\inf\left\{t > 0 : x \in t \cdot \left(X^k \cap (1+\lambda\gamma)\cdot B\right)\right\} = \inf\left\{t > 0 : x \in t\,(1+\lambda\gamma)\cdot B\right\}$$
$$= \frac{1}{1+\lambda\gamma}\inf\{t > 0 : x \in t \cdot B\} = \frac{1}{1+\lambda\gamma}\|x\|_B.$$

The last equality is not completely trivial since S is not a cone, so we first modify it and we define

$$S_1 := \left\{x \in X : \left\|P^k x\right\| \leq \frac{R}{1+\lambda}\|x\|\right\}.$$

We observe that replacing S with S_1 does not modify the construction: if we set $D_1 := S_1 \cap (1+\lambda)\cdot B$, then we have $C_1 := \overline{\operatorname{conv}}\{D_1, B\} = C$. In fact $S_1 \cap (1+\lambda)\cdot B \subseteq S$ implies $C_1 \subseteq C$ and the converse inclusion follows from $D \subseteq \operatorname{conv}\{D_1, B\}$. In order to prove this, fix $x \in D$; then $\left\|P^k x\right\| \leq R < 1$ and in particular $P^k x \in B$. Now set $x_t := P^k x + t(x - P^k x)$ and choose $t \geq 1$ such that $\|x_t\| = 1+\lambda$; with this choice of t we get $\left\|P^k x_t\right\| = \left\|P^k x\right\| \leq R = \frac{R}{1+\lambda}\|x_t\|$, so $x_t \in D_1$. Since x is a convex combination of x_t and $P^k x$ we deduce $D \subseteq \operatorname{conv}\{D_1, B\}$.

Next, we claim that

$$S_1 \cap \tilde{B} = S_1 \cap (1+\lambda)\cdot B.$$

In fact $\supseteq$ follows from the analogous relation with S, proved in the corollary, and $S_1 \cap (1+\lambda)\cdot B \subseteq S$. The converse inclusion follows from the usual $\tilde{B} \subseteq (1+\lambda)\cdot B$.

Finally we prove (c): pick $x \in S_1$ and notice that

$$\left\{t>0: x\in t\tilde{B}\right\} = \left\{t>0: x\in t\tilde{B}\cap S_1\right\} = \left\{t>0: x\in t\left(\tilde{B}\cap S_1\right)\right\}$$
$$= \{t>0: x\in t\,(S_1\cap(1+\lambda)\cdot B)\} = \{t>0: x\in t\,(1+\lambda)\cdot B\};$$

hence

$$\inf\left\{t>0: x\in t\tilde{B}\right\} = \frac{1}{1+\lambda}\inf\{t>0: x\in t\cdot B\},$$

which is exactly (c). □

We now iterate the above renorming procedure: we start with the Banach space X with unit ball B and corresponding norm $\|\cdot\| := \|\cdot\|_B$ and we apply the proposition with $k=1$, a certain $\lambda_1>0$ and $R=1/2$. We let $B_1 := \widetilde{B}$ be the obtained body and $\|\cdot\|_1 := \|\cdot\|_{B_1}$ be the corresponding norm. Then we have

$$\|\cdot\|_1 \le \|\cdot\| \le (1+\lambda_1)\|\cdot\|_1,$$
$$\|\cdot\| = (1+\lambda_1\gamma_1)\|\cdot\|_1 \qquad \text{on } X^1,$$
$$\|x\| = (1+\lambda_1)\|x\|_1 \qquad \text{whenever } \left\|P^1x\right\| \le \frac{1/2}{1+\lambda_1}\|x\|,$$

where $\gamma_1 := \frac{K}{K+1/2}$.

We proceed inductively in the obvious way: fix a sequence $\{\lambda_n\}_{n\ge1}\subseteq(0,\infty)$ such that $\prod_{i=1}^{\infty}(1+\lambda_i)<\infty$ and, in order to have a more concise notation, denote by $\|\cdot\|_0 := \|\cdot\|$ the original norm of X and by $K_0 := K$. Apply inductively the previous proposition: at the step n we use the proposition with $\lambda=\lambda_n$, $R=1/2$, $k=n$ and $B=B_{n-1}$ and we set $B_n := \widetilde{B_{n-1}}$ and $\|\cdot\|_n := \|\cdot\|_{B_n}$. This gives a sequence of norms $\{\|\cdot\|_n\}_{n\ge0}$ on X such that for every $n\ge1$ we have:

$$\|\cdot\|_n \le \|\cdot\|_{n-1} \le (1+\lambda_n)\|\cdot\|_n, \tag{1}$$
$$\|\cdot\|_{n-1} = (1+\lambda_n\gamma_n)\|\cdot\|_n \qquad \text{on } X^n, \tag{2}$$
$$\|x\|_{n-1} = (1+\lambda_n)\|x\|_n \qquad \text{whenever } \|P^nx\|_{n-1} \le \frac{1/2}{1+\lambda_n}\|x\|_{n-1}, \tag{3}$$

where K_n denotes the basis constant of $\{e_i\}_{i\ge1}$ relative to $\|\cdot\|_n$ and $\gamma_n := \frac{K_{n-1}}{K_{n-1}+1/2}\in(0,1)$.

Remark 2.1. The condition $\|P^nx\|_{n-1}\le\frac{1/2}{1+\lambda_n}\|x\|_{n-1}$ appearing in (3) is somewhat unpleasing since the involved norms change with n; we thus replace it with the following more uniform, but weaker, condition.

$$\|x\|_{n-1} = (1+\lambda_n)\|x\|_n \qquad \text{whenever } \|P^nx\|_0 \le \frac{1}{2}\prod_{i=1}^{\infty}(1+\lambda_i)^{-1}\cdot\|x\|_0. \tag{4}$$

The validity of (4) is immediately deduced from the validity of (1) and (3): in fact if x satisfies $\|P^nx\|_0\le\frac{1}{2}\prod_{i=1}^{\infty}(1+\lambda_i)^{-1}\cdot\|x\|_0$, then by (1)

$$\|P^nx\|_{n-1} \le \|P^nx\|_0 \le \frac{1}{2}\prod_{i=1}^{\infty}(1+\lambda_i)^{-1}\cdot\|x\|_0 \le \frac{1}{2}\prod_{i=1}^{\infty}(1+\lambda_i)^{-1}\cdot\prod_{i=1}^{n-1}(1+\lambda_i)\cdot\|x\|_{n-1}$$
$$= \frac{1/2}{1+\lambda_n}\prod_{i=n+1}^{\infty}(1+\lambda_i)^{-1}\cdot\|x\|_{n-1} \le \frac{1/2}{1+\lambda_n}\|x\|_{n-1};$$

hence (3) implies that $\|x\|_{n-1} = (1+\lambda_n)\|x\|_n$.

In order to motivate the next step, let us notice that for a fixed $x \in X$ the sequence $\{\|x\|_n\}_{n\geq 0}$ has the same qualitative behavior since it is a decreasing sequence; on the other hand the quantitative rate of decrease changes with n. In fact it is clear that for a fixed $x \in X$, the condition $\|P^n x\|_0 \leq \frac{1}{2}\prod_{i=1}^{\infty}(1+\lambda_i)^{-1} \cdot \|x\|_0$ is eventually satisfied, so the sequence $\{\|x\|_n\}_{n\geq 0}$ eventually decreases with rate $(1+\lambda_n)^{-1}$. On the other hand if $x \in X^N$, then for the terms $n = 1, \dots, N$ the rate of decrease is $(1+\lambda_n\gamma_n)^{-1}$. This makes it possible to rescale the norms $\{\|\cdot\|_n\}_{n\geq 0}$, obtaining norms $\{|||\cdot|||_n\}_{n\geq 0}$, in a way to have a qualitatively different behavior, increasing for $n = 1, \dots, N$ and eventually decreasing. This property is crucial since it allows us to assure that, for $x \in X^N$, the norms $|||x|||_n$ for $n = 0, \dots, N-1$ are quantitatively smaller than $|||x|||_N$ and thus do not enter in the gluing procedure. As we have hinted at the end of the previous section and as it will be apparent in the proof of Lemma 2.2, this is exactly what we need in order the approximation on X^N to improve with N.

Definition 2.1. Let

$$C := \prod_{i=1}^{\infty} \frac{1+\lambda_i\gamma_i}{1+\lambda_i\frac{1+\gamma_i}{2}},$$

$$|||\cdot|||_n := C \cdot \prod_{i=1}^{n}\left(1+\lambda_i\frac{1+\gamma_i}{2}\right) \cdot \|\cdot\|_n .$$

For later convenience, let us also set

$$|||\cdot|||_\infty = \sup_{n\geq 0} |||\cdot|||_n .$$

The qualitative behavior of $\{|||x|||_n\}_{n\geq 0}$ is expressed in the following obvious, though crucial, properties of the norms $|||\cdot|||_n$. In particular, (a) will be used to show that the gluing together locally takes into account only finitely many terms; this will allow us to preserve the smoothness in Lemma 2.3. (b) expresses the fact that on X^N the norms $\{|||\cdot|||_n\}_{n=0}^{N-1}$ are smaller than $|||\cdot|||_N$ and will be used in Lemma 2.2 to obtain the improvement of the approximation.

Fact 2.1. (a) *For every $x \in X$ there is $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$*

$$|||x|||_n = \frac{1+\lambda_n\frac{1+\gamma_n}{2}}{1+\lambda_n} |||x|||_{n-1} .$$

In particular, it suffices to take any n_0 such that $\|P^n x\|_0 \leq \frac{1}{2}\prod_{i=1}^{\infty}(1+\lambda_i)^{-1} \cdot \|x\|_0$ for every $n \geq n_0$.

(b) *If $x \in X^N$, then for $n = 1, \dots, N$ we have*

$$|||x|||_n = \frac{1+\lambda_n\frac{1+\gamma_n}{2}}{1+\lambda_n\gamma_n} |||x|||_{n-1} .$$

Proof. (a) Since $P^n x \to 0$ as $n \to \infty$, condition (4) implies that there is n_0 such that for every $n \geq n_0$ we have $\|x\|_n = (1+\lambda_n)^{-1}\|x\|_{n-1}$. Then it suffices to translate this to the $|||\cdot|||_n$ norms:

$$|||x|||_n = \left(1+\lambda_n\frac{1+\gamma_n}{2}\right) \cdot C \cdot \prod_{i=1}^{n-1}\left(1+\lambda_i\frac{1+\gamma_i}{2}\right) \cdot \|x\|_n =$$

$$\frac{1+\lambda_n\frac{1+\gamma_n}{2}}{1+\lambda_n} \cdot C \cdot \prod_{i=1}^{n-1}\left(1+\lambda_i\frac{1+\gamma_i}{2}\right) \cdot \|x\|_{n-1} = \frac{1+\lambda_n\frac{1+\gamma_n}{2}}{1+\lambda_n} |||x|||_{n-1} .$$

(b) If $x \in X^N$ and $n = 1, \ldots, N$, then $x \in X^n$ too; thus by (2) we have $\|x\|_n = (1 + \lambda_n \gamma_n)^{-1} \|x\|_{n-1}$. Now exactly the same calculation as in the other case gives the result. □

We can now conclude the renorming procedure: first we smoothen up the norms $|||\cdot|||_n$ and then we glue together all these smooth norms. Fix a decreasing sequence $\delta_n \searrow 0$ such that for every $n \geq 0$

$$(\dagger) \qquad (1 + \delta_n) \frac{1 + \lambda_{n+1}\gamma_{n+1}}{1 + \lambda_{n+1}\frac{1+\gamma_{n+1}}{2}} \leq 1 - \delta_n$$

(of course this is possible since $\gamma_{n+1} < 1$). Then we apply the main result in [12] (Theorem 2.10 in their paper) to find C^k-smooth norms $\left\{|||\cdot|||_{(s),n}\right\}_{n\geq 0}$ such that for every n

$$|||\cdot|||_n \leq |||\cdot|||_{(s),n} \leq (1 + \delta_n)\, |||\cdot|||_n \,.$$

Next, let $\varphi_n : [0, \infty) \to [0, \infty)$ be C^∞-smooth, convex and such that $\varphi_n \equiv 0$ on $[0, 1 - \delta_n]$ and $\varphi_n(1) = 1$; note that of course the φ_n's are strictly monotonically increasing on $[1-\delta_n, \infty)$. Finally define $\Phi : X \to [0, \infty]$ by

$$\Phi(x) := \sum_{n\geq 0} \varphi_n \left(|||x|||_{(s),n}\right)$$

and let $|||\cdot|||$ be the Minkowski functional of the set $\{\Phi \leq 1\}$.

The fact that $|||\cdot|||$ is the desired norm is now an obvious consequence of the next two lemmas. In the first one we show that $|||\cdot|||$ is indeed a norm and that the approximation on X^N improves with N.

Lemma 2.2. $|||\cdot|||$ *is a norm, equivalent to the original norm* $\|\cdot\|$ *of* X.

Moreover for every $N \geq 0$ *we have*

$$\prod_{i=N+1}^{\infty} (1 + \lambda_i)^{-1} \cdot \|\cdot\| \leq |||\cdot||| \leq \frac{1 + \delta_N}{1 - \delta_N} \cdot \prod_{i=N+1}^{\infty} (1 + \lambda_i) \cdot \|\cdot\| \qquad \text{on } X^N.$$

Proof. We start by observing that for every $N \geq 0$

$$\left\{x \in X^N : |||x|||_\infty \leq \frac{1 - \delta_N}{1 + \delta_N}\right\} \subseteq \left\{x \in X^N : \Phi(x) \leq 1\right\} \subseteq \left\{x \in X^N : |||x|||_\infty \leq 1\right\}.$$

In fact, pick $x \in X^N$ such that $\Phi(x) \leq 1$, so in particular $\varphi_n \left(|||x|||_{(s),n}\right) \leq 1$ for every n. The inequality $|||\cdot|||_n \leq |||\cdot|||_{(s),n}$ and the properties of φ_n then imply $|||x|||_n \leq 1$ for every n. This proves the right inclusion. For the first inclusion, we actually show that if $x \in X^N$ satisfies $|||x|||_\infty \leq \frac{1-\delta_N}{1+\delta_N}$, then $\Phi(x) = 0$. To see this, fix any $n \geq N$; since the function $t \mapsto \frac{1-t}{1+t}$ is decreasing on $[0, 1]$ and the sequence δ_n is decreasing too, we deduce

$$|||x|||_n \leq |||x|||_\infty \leq \frac{1 - \delta_N}{1 + \delta_N} \leq \frac{1 - \delta_n}{1 + \delta_n}.$$

Hence $|||x|||_{(s),n} \leq 1-\delta_n$ and $\varphi_n \left(|||x|||_{(s),n}\right) = 0$ for every $n \geq N$. For the remaining values $n = 0, \ldots, N-1$ we use (b) in Fact 2.1 and condition $(\dagger)$:

$$
\begin{aligned}
|||x|||_{(s),n} \leq (1+\delta_n)\,|||x|||_n &= (1+\delta_n)\frac{1+\lambda_{n+1}\gamma_{n+1}}{1+\lambda_{n+1}\frac{1+\gamma_{n+1}}{2}} \cdot |||x|||_{n+1} \\
&\leq (1-\delta_n)\,|||x|||_{n+1} \leq 1-\delta_n;
\end{aligned}
$$

hence $\varphi_n\left(|||x|||_{(s),n}\right) = 0$ for $n = 0, \dots, N-1$ too. This implies $\Phi(x) = 0$ and proves the first inclusion.

Taking in particular $N = 0$, we see that $\{\Phi \leq 1\}$ is a bounded neighborhood of the origin in $(X, |||\cdot|||_\infty)$. Since it is clearly convex and symmetric, we deduce that $\{\Phi \leq 1\}$ is a BCSB relative to $|||\cdot|||_\infty$. Hence $|||\cdot|||$ is a norm on X, equivalent to $|||\cdot|||_\infty$. The fact that $|||\cdot|||$ is equivalent to the original norm $\|\cdot\|$ follows immediately from the case $N = 0$ in the second assertion, which we now prove.

Fix $N \geq 0$; in order to estimate the distortion between $|||\cdot|||$ and $\|\cdot\|$ on X^N, we show that, on X^N, $|||\cdot|||$ is close to $|||\cdot|||_\infty$, that $|||\cdot|||_\infty$ is close to $|||\cdot|||_N$ and finally that $|||\cdot|||_N$ is close to $\|\cdot\|$.

First, passing to the associated Minkowski functionals, the above inclusions yield

$$
(*) \qquad |||\cdot|||_\infty \leq |||\cdot||| \leq \frac{1+\delta_N}{1-\delta_N}\,|||\cdot|||_\infty \qquad \text{on } X^N.
$$

Next, we compare $|||\cdot|||_\infty$ with $|||\cdot|||_N$. Of course $|||\cdot|||_N \leq |||\cdot|||_\infty$ and by property (b) in Fact 2.1 already used above we also have $|||\cdot|||_n \leq |||\cdot|||_N$ for $n \leq N$. We thus fix $n > N$ and observe

$$
\begin{gathered}
|||\cdot|||_n := C\prod_{i=1}^{n}\left(1+\lambda_i\frac{1+\gamma_i}{2}\right) \cdot \|\cdot\|_n \leq \\
\prod_{i=N+1}^{n}\left(1+\lambda_i\frac{1+\gamma_i}{2}\right) \cdot C \cdot \prod_{i=1}^{N}\left(1+\lambda_i\frac{1+\gamma_i}{2}\right) \cdot \|\cdot\|_N = \\
\prod_{i=N+1}^{n}\left(1+\lambda_i\frac{1+\gamma_i}{2}\right) \cdot |||\cdot|||_N \leq \prod_{i=N+1}^{\infty}(1+\lambda_i) \cdot |||\cdot|||_N\,.
\end{gathered}
$$

This yields

$$
(*) \qquad |||\cdot|||_N \leq |||\cdot|||_\infty \leq \prod_{i=N+1}^{\infty}(1+\lambda_i) \cdot |||\cdot|||_N \qquad \text{on } X^N.
$$

Finally, we compare $|||\cdot|||_N$ with $\|\cdot\|_0$. The subspaces X^N are decreasing, so (2) implies $\|\cdot\| = \prod_{i=1}^{N}(1+\lambda_i\gamma_i) \cdot \|\cdot\|_N$ on X^N; hence

$$
\begin{aligned}
\|\cdot\| &= \prod_{i=1}^{N}(1+\lambda_i\gamma_i) \cdot \prod_{i=1}^{\infty}\frac{1+\lambda_i\frac{1+\gamma_i}{2}}{1+\lambda_i\gamma_i} \cdot \prod_{i=1}^{N}\left(1+\lambda_i\frac{1+\gamma_i}{2}\right)^{-1} \cdot |||\cdot|||_N \\
&= \prod_{i=N+1}^{\infty}\frac{1+\lambda_i\frac{1+\gamma_i}{2}}{1+\lambda_i\gamma_i} \cdot |||\cdot|||_N\,.
\end{aligned}
$$

This implies in particular

$$
(*) \qquad |||\cdot|||_N \leq \|\cdot\| \leq \prod_{i=N+1}^{\infty}(1+\lambda_i) \cdot |||\cdot|||_N \qquad \text{on } X^N;
$$

combining the $(*)$ inequalities concludes the proof of the lemma. $\square$

Remark 2.2. The estimate of the distortion in the particular case $N = 0$ is in fact shorter than the general case given above. In fact, property (1) obviously implies $\|\cdot\|_n \le \|\cdot\| \le \prod_{i=1}^{n}(1+\lambda_i) \cdot \|\cdot\|_n$. It easily follows that for every n

$$\prod_{i=1}^{\infty}(1+\lambda_i)^{-1} \cdot \|\cdot\| \le |||\cdot|||_n \le \prod_{i=1}^{\infty}(1+\lambda_i) \cdot \|\cdot\|;$$

it is then sufficient to combine this with the first of the $(*)$ inequalities.

We finally check the regularity of $|||\cdot|||$.

Lemma 2.3. *The norm $|||\cdot|||$ is C^k-smooth.*

Proof. We first show that for every x in the set $\{\Phi < 2\}$ there is a neighborhood $\mathcal{U}$ of x (in X) where the function Φ is expressed by a finite sum. We have already seen in the proof of Lemma 2.2 that $\Phi = 0$ in a neighborhood of 0, so the assertion is true for $x = 0$; hence we can fix $x \neq 0$ such that $\Phi(x) < 2$. Observe that clearly the properties of φ_n imply $\varphi_n(1+\delta_n) \ge 2$; thus x satisfies $|||x|||_n \le |||x|||_{(s),n} \le 1+\delta_n$ for every n.

Denote by $c := \frac{1}{2}\prod_{i=1}^{\infty}(1+\lambda_i)^{-1}$ and choose n_0 such that $\|P^n x\| \le \frac{c}{2} \cdot \|x\|$ for every $n \ge n_0$ (this is possible since $P^n x \to 0$). Next, fix $\varepsilon > 0$ small so that $\frac{c}{2} + K\varepsilon \le (1-\varepsilon)c$ and $(1+\varepsilon)(1-\delta_{n_0}) \le 1$, and let $\mathcal{U}$ be the following neighborhood of x:

$$\mathcal{U} := \left\{ y \in X : \|y - x\| < \varepsilon \|x\| \text{ and } |||y|||_{n_0} < (1+\varepsilon)\, |||x|||_{n_0} \right\}.$$

Clearly for $y \in \mathcal{U}$ we have $\|x\| \le \frac{1}{1-\varepsilon}\|y\|$; thus for $y \in \mathcal{U}$ and $n \ge n_0$ we have

$$\|P^n y\| \le \|P^n y - P^n x\| + \|P^n x\| \le K\varepsilon \|x\| + \frac{c}{2} \cdot \|x\| \le (1-\varepsilon)c \|x\| \le c\|y\|.$$

Hence (a) of Fact 2.1 implies that $|||y|||_n = \frac{1+\lambda_n \frac{1+\gamma_n}{2}}{1+\lambda_n} |||y|||_{n-1}$ for every $n \ge n_0$ and $y \in \mathcal{U}$ (let us explicitly stress the crucial fact that n_0 does not depend on $y \in \mathcal{U}$).

We have $|||y|||_{n_0} < (1+\varepsilon)\, |||x|||_{n_0} \le (1+\varepsilon)(1+\delta_{n_0})$; using this bound and the previous choices of the parameters (in particular we use twice $(\dagger)$ and twice the fact that δ_n is decreasing), for every $n \ge n_0 + 2$ and $y \in \mathcal{U}$ we estimate

$$\begin{aligned}
|||y|||_{(s),n} &\le (1+\delta_n)\, |||y|||_n = (1+\delta_n) \prod_{i=n_0+1}^{n} \frac{1+\lambda_i \frac{1+\gamma_i}{2}}{1+\lambda_i} \cdot |||y|||_{n_0} \\
&\le (1+\delta_n) \prod_{i=n_0+1}^{n} \frac{1+\lambda_i \frac{1+\gamma_i}{2}}{1+\lambda_i} \cdot (1+\varepsilon)(1+\delta_{n_0}) \overset{(\dagger)}{\le} (1+\delta_n) \prod_{i=n_0+2}^{n} \frac{1+\lambda_i \frac{1+\gamma_i}{2}}{1+\lambda_i} \cdot (1+\varepsilon)(1-\delta_{n_0}) \\
&\le (1+\delta_n) \prod_{i=n_0+2}^{n} \frac{1+\lambda_i \frac{1+\gamma_i}{2}}{1+\lambda_i} \le (1+\delta_{n-1}) \frac{1+\lambda_n \frac{1+\gamma_n}{2}}{1+\lambda_n} \cdot \prod_{i=n_0+2}^{n-1} \frac{1+\lambda_i \frac{1+\gamma_i}{2}}{1+\lambda_i} \\
&\le (1+\delta_{n-1}) \frac{1+\lambda_n \frac{1+\gamma_n}{2}}{1+\lambda_n} \overset{(\dagger)}{\le} 1 - \delta_{n-1} \le 1 - \delta_n.
\end{aligned}$$

It follows that $\varphi_n\left(|||y|||_{(s),n}\right) = 0$ for $n \ge n_0 + 2$ and $y \in \mathcal{U}$, hence

$$\Phi = \sum_{n=0}^{n_0+2} \varphi_n\left(|||\cdot|||_{(s),n}\right) \qquad \text{on } \mathcal{U}.$$

This obviously implies that Φ is C^k-smooth on the set $\{\Phi < 2\}$ and in particular $\{\Phi < 2\}$ is an open set. Concerning the regularity of Φ, we also observe here that Φ is lower semi-continuous on X (this follows immediately from the fact that Φ is the sum of a series of positive continuous functions).

The last step consists in applying the Implicit Function theorem (see e.g. [10], Theorem 1.87) and deduce the C^k-smoothness of $|||\cdot|||$ from the one of Φ; this argument is quite well known, but equally short, so we decided to present it. The set

$$V := \left\{(x, \rho) \in (X \setminus \{0\}) \times (0, \infty) : \rho^{-1} \cdot x \in \{\Phi < 2\}\right\}$$

is open in $X \times (0, \infty)$ and the function $\Psi : V \to \mathbb{R}$ defined by $\Psi(x, \rho) := \Phi(\rho^{-1} \cdot x)$ is C^k-smooth on V.

We notice that for every $h \in X \setminus \{0\}$ there is a unique $\rho > 0$ such that $(h, \rho) \in V$ and $\Psi(h, \rho) = 1$; moreover, $\rho = |||h|||$. In fact the functions φ_n are strictly increasing on the set where they are positive, so $t \mapsto \Phi(th)$ is strictly increasing where it is positive; hence there is at most one ρ as above. Also, $|||h||| = \inf\left\{t > 0 : \Phi(t^{-1}h) \le 1\right\}$, so for every $\varepsilon > 0$ we have $\Phi\left(\frac{1}{|||h|||+\varepsilon} h\right) \le 1$; as Φ is lower semi-continuous, we deduce $\Phi\left(|||h|||^{-1} h\right) \le 1$. If it were that $\Phi\left(|||h|||^{-1} h\right) < 1$, then from the continuity of Φ on $\{\Phi < 2\}$ we would deduce $\Phi\left(\frac{1}{|||h|||-\varepsilon} h\right) \le 1$ for $\varepsilon > 0$ small; however this contradicts $|||h|||$ being the infimum. Hence $\Phi\left(|||h|||^{-1} h\right) = 1$ and in particular the unique ρ as above is $\rho = |||h|||$.

In other words, the equation $\Psi = 1$ on V globally defines a unique implicit function on $X \setminus \{0\}$, which is given by $\rho(h) = |||h|||$. Since

$$D_2\Psi(h, \rho) = \frac{-1}{\rho^2} \Phi'(\rho^{-1}h)h = \frac{-1}{\rho^2} \sum_{n \ge 0} \varphi_n' \left(\left|\left|\left|\rho^{-1}h\right|\right|\right|_{(s),n}\right) |||h|||_{(s),n}$$

(where $D_2\Psi$ denotes the partial derivative of Ψ in its second variable), we have

$$D_2\Psi(h, |||h|||) = \frac{-1}{|||h|||^2} \sum_{n \ge 0} \varphi_n' \left(\frac{1}{|||h|||} |||h|||_{(s),n}\right) |||h|||_{(s),n} .$$

The condition $\Phi\left(|||h|||^{-1} h\right) = 1$ implies $\varphi_n\left(\frac{1}{|||h|||} |||h|||_{(s),n}\right) > 0$ for some n, hence $\varphi_n'\left(\frac{1}{|||h|||} |||h|||_{(s),n}\right) > 0$ too and $D_2\Psi(h, |||h|||) \neq 0$ on $X \setminus \{0\}$. Thus the Implicit Function theorem yields that the implicitly defined function shares the same regularity as Ψ, i.e. $|||\cdot|||$ is C^k-smooth on $X \setminus \{0\}$. $\square$

Proof of Theorem 1.1. Fix a separable Banach space as in the statement and a sequence $\{\varepsilon_N\}_{N \ge 0}$ of positive numbers. We find a sequence $\{\lambda_i\}_{i \ge 1} \subseteq (0, \infty)$ such that

$$\prod_{i=N+1}^{\infty} (1 + \lambda_i) < 1 + \varepsilon_N$$

for every $N \ge 0$; next, we find a decreasing sequence $\{\delta_N\}_{N \ge 0}$, $\delta_N \searrow 0$, that satisfies (†) and such that

$$\frac{1 + \delta_N}{1 - \delta_N} \cdot \prod_{i=N+1}^{\infty} (1 + \lambda_i) \le 1 + \varepsilon_N$$

for every $N \ge 0$. We then apply the renorming procedure described in this section with these parameters $\{\lambda_i\}_{i \ge 1}$ and $\{\delta_N\}_{N \ge 0}$ and we obtain a C^k-smooth norm $|||\cdot|||$ on X that satisfies

$$(1-\varepsilon_N)\cdot\|\cdot\| \le \prod_{i=N+1}^{\infty}(1+\lambda_i)^{-1}\cdot\|\cdot\| \le |||\cdot||| \le \frac{1+\delta_N}{1-\delta_N}\cdot\prod_{i=N+1}^{\infty}(1+\lambda_i)\cdot\|\cdot\| \le (1+\varepsilon_N)\cdot\|\cdot\|$$

on X^N for every $N \ge 0$; since these inequalities are obviously equivalent to

$$\Big| |||x||| - \|x\| \Big| \le \varepsilon_N \|x\| \qquad \text{for } x \in X^N,$$

the proof is complete. □

3. Final remarks

In this short section we present some improvements of our main result in the particular case of polyhedral Banach spaces. Recall that a finite-dimensional Banach space X is said to be *polyhedral* if its unit ball is a polyhedron, i.e. finite intersection of closed half-spaces; an infinite-dimensional Banach space X is *polyhedral* if its finite-dimensional subspaces are polyhedral. It is proved in [5] that if X is a separable polyhedral Banach space, then every equivalent norm on X can be approximated (uniformly on bounded sets) by a polyhedral norm (see Theorem 1.1 in [5], where the approximation is stated in terms of closed, convex and bounded bodies).

In analogy with our main result, it is natural to ask if this result can be improved in the sense that the approximation can be chosen to be improving on the tail vectors. It is not difficult to see that if we replace the C^k-smooth norms $|||\cdot|||_{(s),n}$ with polyhedral norms $|||\cdot|||_{(p),n}$ (thus using Theorem 1.1 in [5]) and we replace the C^∞-smooth functions φ_n with piecewise linear ones, the resulting norm $|||\cdot|||$ is still polyhedral. We thus have:

Proposition 3.1. *Let X be a polyhedral Banach space with a Schauder basis $\{e_i\}_{i\ge1}$ and let $\|\cdot\|$ be any renorming of X. Then for every sequence $\{\varepsilon_N\}_{N\ge0}$ of positive numbers, there is a polyhedral renorming $|||\cdot|||$ of X such that for every $N \ge 0$*

$$\Big| |||x||| - \|x\| \Big| \le \varepsilon_N \|x\| \qquad \textit{for } x \in X^N.$$

We say that $\|\cdot\|$ *depends locally on finitely many coordinates* if for each $x \in S_X$ there exists an open neighborhood O of x, a finite set $\{x_1^*, \dots, x_k^*\} \subset X^*$ and a function $f : \mathbb{R}^k \to \mathbb{R}$ such that $\|y\| = f(x_1^*(y), \dots, x_k^*(y))$ for $y \in O$. It was also shown in [5] that if X is a separable polyhedral space, then every equivalent norm on X can be approximated by a C^∞-smooth norm that depends locally on finitely many coordinates. By inspection of our argument it follows that if we use such approximations in our proof, the resulting C^∞-smooth norm $|||\cdot|||$ will also depend locally on finitely many coordinates. Explicitly, we obtain:

Proposition 3.2. *Let X be a polyhedral Banach space with a Schauder basis $\{e_i\}_{i\ge1}$ and let $\|\cdot\|$ be any renorming of X. Then for every sequence $\{\varepsilon_N\}_{N\ge0}$ of positive numbers, there is a C^∞-smooth renorming $|||\cdot|||$ of X that locally depends on finitely many coordinates and such that for every $N \ge 0$*

$$\Big| |||x||| - \|x\| \Big| \le \varepsilon_N \|x\| \qquad \textit{for } x \in X^N.$$

In conclusion of our note, we mention that we do not know whether our main result can be generalized replacing Schauder basis with Markushevich basis. The argument presented here is not directly applicable, since, for example, we have made use of the canonical projections on the basis and their uniform boundedness.

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References

[1] V. Bible, Using boundaries to find smooth norms, Studia Math. 224 (2014) 169–181.
[2] V. Bible, R.J. Smith, Smooth and polyhedral approximations in Banach spaces, J. Math. Anal. Appl. 435 (2016) 1262–1272.
[3] R. Deville, Geometrical implications of the existence of very smooth bump functions in Banach spaces, Israel J. Math. 6 (1989) 1–22.
[4] R. Deville, V.P. Fonf, P. Hájek, Analytic and C^k approximations of norms in separable Banach spaces, Studia Math. 120 (1996) 61–74.
[5] R. Deville, V.P. Fonf, P. Hájek, Analytic and polyhedral approximation of convex bodies in separable polyhedral Banach spaces, Israel J. Math. 105 (1998) 139–154.
[6] R. Deville, G. Godefroy, V. Zizler, Smoothness and Renormings in Banach Spaces, Pitman Monographs and Surveys in Pure and Applied Mathematics, vol. 64, 1993.
[7] A.J. Guirao, V. Montesinos, V. Zizler, Open Problems in the Geometry and Analysis of Banach Spaces, Springer, 2016.
[8] P. Hájek, Smooth norms that depend locally on finitely many coordinates, Proc. Amer. Math. Soc. 123 (1995) 3817–3821.
[9] P. Hájek, R. Haydon, Smooth norms and approximation in Banach spaces of the type $C(K)$, Q. J. Math. 58 (2007) 221–228.
[10] P. Hájek, M. Johanis, Smooth Analysis in Banach Spaces, De Gruyter, Berlin, 2014.
[11] P. Hájek, A. Procházka, C^k-smooth approximations of LUR norms, Trans. Amer. Math. Soc. 366 (2014) 1973–1992.
[12] P. Hájek, J. Talponen, Smooth approximations of norms in separable Banach spaces, Q. J. Math. 65 (2014) 957–969.
[13] R. Haydon, Smooth functions and partitions of unity on certain Banach spaces, Q. J. Math. 47 (1996) 455–468.
[14] R. Haydon, Trees in renorming theory, Proc. Lond. Math. Soc. 78 (1999) 541–584.
[15] R.P. Maleev, S. Troyanski, Smooth norms in Orlicz spaces, Canad. J. Math. 34 (1991) 74–82.
[16] J. Pechanec, J.H.M. Whitfield, V. Zizler, Norms locally dependent on finitely many coordinates, An. Acad. Brasil. Ciênc. 53 (1981) 415–417.