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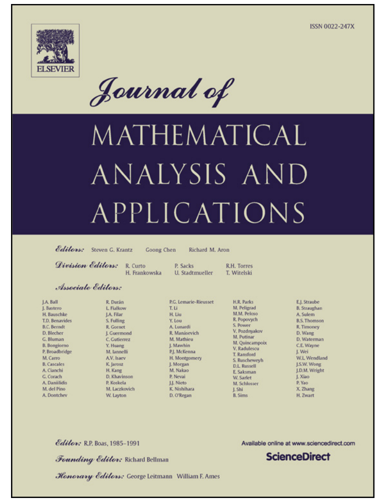

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A variational approach to symmetry, monotonicity, and comparison for doubly-nonlinear equations*

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Abstract

We advance a variational method to prove qualitative properties such as symmetries, monotonicity, upper and lower bounds, sign properties, and comparison principles for a large class of doubly-nonlinear evolutionary problems including gradient flows, some nonlocal problems, and systems of nonlinear parabolic equations.

Our method is based on the so-called Weighted-Energy-Dissipation (WED) variational approach. This consists in defining a global parameter-dependent functional over entire trajectories and proving that its minimizers converge to solutions to the target problem as the parameter goes to zero. Qualitative properties and comparison principles can be easily proved for minimizers of the WED functional and, by passing to the limit, for the limiting problem.

Several applications of the abstract results to systems of nonlinear PDEs and to fractional/nonlocal problems are presented. Eventually, we present some extensions of this approach in order to deal with rate-independent systems and hyperbolic problems.

Key words: Qualitative properties, comparison principles, variational approach, WED functionals.

AMS (MOS) subject classification: 35B06, 35B51, 49J27.

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1 Introduction

In this paper we illustrate a general procedure to prove qualitative properties and comparison principles for the abstract doubly-nonlinear system given by

$$\mathrm{d}_V\psi(u') + \eta^1 - \eta^2 - f(u) = 0 \text{ a.e. in } (0,T), \tag{1.1}$$

$$\eta^1 \in \partial\varphi^1(u),\ \eta^2 \in \partial\varphi^2(u), \tag{1.2}$$

$$u(0) = u_0. \tag{1.3}$$

Here u' denotes the time derivative of the unknown trajectory $t \in (0,T) \longmapsto u(t) \in V$, $\varphi^1, \varphi^2, \psi$ are proper, lower semicontinuous, and convex functionals on a Banach space V, $\partial\varphi^1$ and $\partial\varphi^2$ denote the subdifferentials of φ^1 and φ^2 respectively, $\mathrm{d}_V\psi$ is the Fréchet differential of ψ, and $f : V \to V^*$ is a continuous map. Note that we do not assume a differential structure on f, thus f is *nonpotential*.

The abstract system (1.1)-(1.3) describes a variety of dissipative problems, e.g., (degenerate) parabolic equations, doubly-nonlinear equations, fractional and nonlocal problems, some ODEs, and systems of reaction-diffusion equations [25]. Such a nonpotential perturbation of doubly-nonlinear problems have been studied by many authors, see, e.g., [31, 32] (see also [10, 11] for the potential case: $f \equiv 0$).

Recently, a variational approach to the doubly-nonlinear system (1.1)-(1.3) has been proposed in [1]. This approach relies on the so-called *Weighted-Energy-Dissipation* (WED) procedure for doubly nonlinear systems [1, 2, 3, 4, 28]. Given a target evolutionary problem, the WED approach consists in defining a global parameter-dependent functional I_ε over entire trajectories and proving that its minimizers converge, up to subsequences, to solutions to the target problem, as the parameter ε goes to 0.

The WED formalism has been used by Ilmanen [19] in the context of mean-curvature flows, and later reconsidered by Mielke and Ortiz [26] for rate-independent systems. The gradient flow case with λ-convex potentials has been studied by Mielke and Stefanelli [28]. Akagi and Stefanelli have extended the theory to the genuinely nonconvex case for gradient flows [2] and to convex doubly-nonlinear systems [3, 4], namely to problem (1.1)-(1.3) with $\varphi^2 = 0$ and $f = 0$. Finally, an analogous approach has been applied to some hyperbolic problems, e.g., the semilinear wave equation [23, 35, 39], and to Lagrangian Mechanics equations [24].

In the case of $f \neq 0$, the lack of potential for f opens on the one hand the possibility of considering systems instead of equations. On the other hand, it determines an obstruction to the application of the WED procedure described above to problem (1.1)-(1.3), for the latter has in general no variational nature. In particular, it is not possible to build a WED functional for problem (1.1)-(1.3). This difficulty may be tamed by combining the WED technique with a fixed-point argument [1, 25]. Since our argument relies on the WED procedure for problem (1.1)-(1.3), we now briefly sketch the results in [1], for the reader's convenience. Under the assumption of Fréchet differentiability of φ^2 and of p-growth for the dissipation potential ψ, for all $v \in L^p(0,T;V)$ the WED-type functional

$I_{\varepsilon,w} : L^p(0,T;V) \to (-\infty,+\infty]$ is defined by

$$I_{\varepsilon,w}(u) = \begin{cases} \displaystyle\int_0^T \mathrm{e}^{-t/\varepsilon} \left(\varepsilon\psi(u') + \varphi^1(u) - \langle w, u\rangle_V\right) \mathrm{d}t & \text{if } u \in K(u_0) : \varphi^1(u) \in L^1(0,T), \\ +\infty & \text{else,} \end{cases} \tag{1.4}$$

$$w = F(v) := \mathrm{d}_V\varphi^2(v) + f(v), \tag{1.5}$$

where $K(u_0) = \{u \in W^{1,p}(0,T;V) : u(0) = u_0\}$. For all $v \in L^p(0,T;V)$, an approximation of the Direct Method [12] ensures that the functional $I_{\varepsilon,w}$ admits a unique minimizer $u_{\varepsilon,v}$ over $K(u_0)$. Moreover, the map

$$S : v \in L^p(0,T;V) \longmapsto u_{\varepsilon,v} \in L^p(0,T;V) \tag{1.6}$$

can be proved to have a fixed-point u_ε fulfilling

$$\begin{aligned} u_\varepsilon &= \operatorname*{argmin}_{\tilde u} I_{\varepsilon,F(u_\varepsilon)}(\tilde u) \\ &= \operatorname*{argmin}_{\tilde u} \int_0^T \mathrm{e}^{-t/\varepsilon} \left(\varepsilon\psi(\tilde u') + \varphi^1(\tilde u) - \left\langle \mathrm{d}_V\varphi^2(u_\varepsilon) + f(u_\varepsilon), \tilde u\right\rangle_V\right) \mathrm{d}t \end{aligned} \tag{1.7}$$

and solving an elliptic-in-time regularization of (1.1)-(1.3) given by

$$-\varepsilon\left(\mathrm{d}_V\psi(u_\varepsilon')\right)' + \mathrm{d}_V\psi(u_\varepsilon') + \eta^1 - \eta^2 - f(u_\varepsilon) = 0 \text{ a.e. in } (0,T), \tag{1.8}$$

$$\eta^1 \in \partial\varphi^1(u_\varepsilon),\ \eta^2 = \mathrm{d}_V\varphi^2(u_\varepsilon), \tag{1.9}$$

$$\mathrm{d}_V\psi(u_\varepsilon')(T) = 0, \tag{1.10}$$

$$u_\varepsilon(0) = u_0. \tag{1.11}$$

Finally, u_ε converges, up to subsequences, to solutions to (1.1)-(1.3).

In the first part of this work we prove qualitative properties such as symmetries, monotonicity, upper and lower bounds, sign properties, for solutions to system (1.1)-(1.3), provided some compatibility conditions (e.g. symmetry of the domain or compatibility of the initial data). A standard approach suggests to describe qualitative properties (e.g., the axial symmetry of a function) as invariance under the action of a map [9] (e.g. the reflection with respect to a given axis). Following this idea we aim to prove existence of solutions u to system (1.1)-(1.3) which are invariant under the action of a map R, namely such that $u = Ru$. This will follow by i) proving that the functional $I_{\varepsilon,w}$ is nonincreasing under the action of the map R and ii) checking that the invariance property is preserved by taking the limit $\varepsilon \to 0$.

Let us now briefly comment on some peculiarities and advantages of our method and compare it with other techniques used to prove qualitative properties of solutions to PDEs. We start by observing that our result is extremely versatile. Indeed it applies to a large number of qualitative properties (symmetries, upper and lower bounds, monotonicity, sign properties, and combinations of them, see Corollary 3), and a variety of evolution equations, e.g., dissipative systems of the form (1.1)-(1.3) (see Section 3), but also rate-independent systems and hyperbolic problems (see Section 4).

Let us also note that our technique applies to maps R which are not necessary invertible (such as rearrangements or truncations). In particular, R does not generate a group of transformations. This implies that the theory of invariance under the action of Lie groups (see, e.g., [9]) may not be directly used in our setting.

As a byproduct of our results, we get also existence of R-invariant solutions to the elliptic-in-time regularization (1.8)-(1.11) of (1.1)-(1.3).

It is worth noting that our technique does not require regularity of solutions to the target problem. This is not the case for others methods used for proving qualitative properties of solutions to PDEs. Moving planes and sliding methods [6, 8] for instance require classical regularity, as they rely on classical comparison principles and on the Hopf Lemma.

Furthermore, we can treat the case of problems with nonunique solutions. Indeed, the uniqueness of solution to (1.1)-(1.3) may genuinely fail (e.g., the sublinear heat equation $u_t - \Delta u = u^q$, $0 < q < 1$ has positive solutions even for zero initial data). In this case it might be trivial to prove that R maps solutions into solutions (namely the problem is invariant under the action of R). However, due to the lack of uniqueness one cannot conclude the existence of invariant solutions. Our method is hence particularly useful in the case of nonuniqueness of solutions.

In the second part of this work we use the WED approach to prove a comparison principle for system (1.1)-(1.3) in the case of f being independent of u. Our strategy consists in combining the WED minimization with an abstract comparison principle, see Lemma 9 below. More precisely, we i) prove a comparison principle for minimizers of the WED functional and ii) pass to the limit as $\varepsilon \to 0$. It is noteworthy that the comparison principle established in the present paper is not standard: given two initial data u_0, v_0 such that $u_0 \leq v_0$ in a suitable sense, we show the existence of at least two solutions u, v such that $u(0) = u_0$, $v(0) = v_0$, and $u \leq v$. We emphasize that we cannot expect the relation $u \leq v$ to hold for all u, v solutions to (1.1)-(1.3) such that $u(0) = u_0$, $v(0) = v_0$, as problem (1.1)-(1.3) has in general nonunique solutions.

Section 4 addresses by similar techniques different types of evolution equations which does not fit in the abstract formulation (1.1)-(1.3). In particular, we prove a comparison principle for rate-independent systems of the form

$$\partial\psi(u') + \partial\phi(u) - \Delta u \ni 0 \text{ in } \Omega \times (0, T),$$

where ψ and ϕ are proper, lower semicontinuous, convex functionals, ψ is 1-homogeneous, and Ω is a bounded subset of $\mathbb{R}^d$.

Moreover, we check symmetries of solutions to the semilinear wave equation

$$\rho u_{tt} + \nu u_t - \Delta u + F'(u) = 0 \text{ in } \Omega \times (0, T),$$

where $\Omega \subset \mathbb{R}^d$ is open, $\rho > 0$, $v \geq 0$ are constants, and $F \in C^1(\mathbb{R})$ has polynomial growth.

Finally, we tackle the lagrangian system

$$M u_{tt} + \nu u_t + \nabla U(u) = 0 \text{ in } (0, T),$$

where $u : (0, T) \to \mathbb{R}^d$, M is a positive definite $d \times d$ matrix, $U \in C^1(\mathbb{R}^d)$ is bounded from below and convex, and $\nu \geq 0$.

The paper is organized as follows. We fix the notation, enlist assumptions, and we state and prove our abstract results in Section 2. We present several examples of application to PDEs and integrodifferential problems in Section 3. Finally, Section 4 is devoted to rate-independent systems and hyperbolic problems.

2 Notation, assumptions, and main results

Given any real Banach space E, we denote by E^* its dual, by $|\cdot|_E$ its norm, and by $\langle\cdot,\cdot\rangle_E$ the duality pairing between E^* and E. Let $\phi : E \to (-\infty,+\infty]$ be a convex functional, we denote its subdifferential by $\partial_E\phi$ and its Fréchet differential by $\mathrm{d}_E\phi$, whenever it exists.

For all $h > 1$, $\Theta_h(E)$ denotes the set of all lower semicontinuous convex functionals $\phi : E \to [0,+\infty)$ such that there exists a strictly positive constant C such that

$$|u|_E^h \le C\,(\phi(u)+1) \ \text{ for all } u \in E,$$
$$|\xi|_{E^*}^{h'} \le C(|u|_E^h+1), h' = \frac{h}{h-1} \ \text{ for all } \xi \in \partial_E\phi(u).$$

Given a set A and a map $R : A \to A$, we denote the set of fixed points of R by A_R, namely $A_R = \{a \in A : Ra = a\}$ is the set of R-*invariant* elements of A.

The symbols γ^+ and γ^- stand for the positive and the negative part in $\mathbb{R}$, namely $\gamma^+ = \max\{\gamma,0\}$ and $\gamma^- = -\min\{\gamma,0\}$, while the symbols $\vee$ and $\wedge$ denote the maximum and the minimum respectively: $a\vee b = \max\{a,b\}$, $a\wedge b = \min\{a,b\}$.

Let V be a uniformly convex Banach space and X be a reflexive Banach space such that

$$X \hookrightarrow V \text{ and } V^* \hookrightarrow X^*$$

with densely-defined compact canonical injections. Let $\psi,\varphi^1,\varphi^2 : V \to [0,\infty)$ be proper, lower semicontinuous (l.s.c.), and convex functionals. Furthermore, we assume ψ to be Fréchet differentiable and φ^1 to be strictly convex. Let $p,m \in (1,\infty)$ be fixed. Assume that $\psi \in \Theta_p(V)$, $\varphi^1 \in \Theta_m(X)$. Moreover, we ask for constants $k \in [0,1)$, $C_1 > 0$, and a nondecreasing function ℓ on $[0,+\infty)$ such that

$$\varphi^2(u) \le k\varphi^1(u) + C_1 \tag{2.1}$$

for all $u \in D(\varphi^1)$ and

$$|\eta^2|_{V^*}^{p'} \le \ell(|u|_V)(\varphi^1(u)+1) \tag{2.2}$$

for all $u \in D(\varphi^2)$ and $\eta^2 \in \partial_V\varphi^2(u)$. Let $f : V \to V^*$ be such that

$$|f(u)|_{V^*}^{p'} \le C_2(|u|_V^p+1) \tag{2.3}$$

for some constant $C_2 \ge 0$ and $f : L^p(0,T;V) \to L^{p'}(0,T;V^*)$ be continuous. Finally, we assume $u_0 \in D(\varphi^1)$. We remark that Fréchet differentiability of ψ is assumed since we need the term $\mathrm{d}_V\psi$ to be single valued in order to differentiate it in time (cf. equation (1.8)). Furthermore, this condition is typically satisfied in applications where ψ is often chosen as a power of a norm (cf. Section 3).

Before stating our main results, let us now introduce the definition of strong solution to system (1.1)-(1.3).

Definition 1 (Strong solution) *A function $u : [0,T] \to V$ is a* strong solution *of system* (1.1)-(1.3) *if*

1. $u \in L^m(0,T;X) \cap W^{1,p}(0,T;V)$, $u(t) \in D(\partial_V \varphi^1(u))$ *for a.e.* $t \in (0,T)$, $\mathrm{d}_V\psi(u') \in L^{p'}(0,T;V^*)$,
2. *there exist* $\eta^1 \in L^{m'}(0,T;X^*) \cap L^{p'}(0,T;V^*)$ *and* $\eta^2 \in L^{p'}(0,T;V^*)$ *such that* $\eta^1 \in \partial_V\varphi^1(u)$ *and* $\eta^2 \in \partial_V\varphi^2(u)$,
3. $\mathrm{d}_V\psi(u') + \eta^1 - \eta^2 - f(u) = 0$ *in* V^* *a.e. in* $(0,T)$, *and* $u(0) = u_0$.

2.1 Main result 1: qualitative properties

In order to define a single-valued map S as in (1.6), in this subsection we additionally assume that $\varphi^2 : V \to (-\infty, +\infty)$ is Fréchet differentiable.

We now introduce assumptions on the abstract maps $R : V \to V$ which describe qualitative properties.

(R1) V_R is nonempty, convex, and closed in V. Assume that $Ru \in W^{1,p}(0,T;V)$ for every $u \in W^{1,p}(0,T;V)$.

(R2) Define $F(v) := (\mathrm{d}_V\varphi^2 + f)(v)$, and assume either $\delta V_R \subset V_R$ for every $\delta \in (0,1)$ and

$$I_{\varepsilon,w}(Ru) \le I_{\varepsilon,w}(u) \text{ for all } u \in K(u_0) \text{ and } w \in \{F(v) : v \in L^p(0,T;V_R)\} \tag{2.4}$$

or

$$I_{\varepsilon,w}(Ru) \le I_{\varepsilon,w}(u) \text{ for all } u \in K(u_0) \text{ and } w = F(u). \tag{2.5}$$

where

$$I_{\varepsilon,w}(u) = \begin{cases} \displaystyle\int_0^T \mathrm{e}^{-t/\varepsilon}\left(\varepsilon\psi(u') + \varphi^1(u) - \langle w, u\rangle_V\right)\mathrm{d}t & \text{if } u \in K(u_0) : \varphi^1(u) \in L^1(0,T), \\ +\infty & \text{else,} \end{cases}$$

$$K(u_0) = \{u \in W^{1,p}(0,T;V) : u(0) = u_0\}.$$

Before commenting our assumptions let us state our main results.

Theorem 2 (Existence of invariant solutions) *Let the above assumptions be satisfied and $Ru_0 = u_0$. Then, system* (1.1)-(1.3) *admits a strong solution u which is invariant under the action of R. Namely, $u = Ru$.*

The latter result can be extended to composition of maps. More precisely, we prove the following.

Corollary 3 (Composition of maps) *Let the assumptions of Theorem* 2 *be satisfied, let R_1 satisfy* (R1) *and* (2.4), *and R_2 satisfy* (R1)-(R2). *Moreover, assume $R_1u_0 = R_2u_0 = u_0$. Then, there exists a strong solution u to system* (1.1)-(1.3) *invariant under the action of both $R_1 \circ R_2$ and $R_2 \circ R_1$.*

We now comment briefly our abstract assumptions. Loosely speaking condition (R1) ensures the compatibility of the map R with the WED approach. More precisely (R1), together with $Ru_0 = u_0$ is sufficient to guarantee the R-invariance of the domain of the WED functional, i.e., $RK(u_0) \subset K(u_0)$. Assumption (R2) is the crucial assumption; it allows us to prove that the map S defined by (1.6) has a R-invariant fixed point u_ε, i.e., $Ru_\varepsilon = u_\varepsilon$. Having this, by using again (R1), we can easily pass to the limit $\varepsilon \to 0$ and prove $Ru = u = \lim_{\varepsilon\to 0} u_\varepsilon$. Let us note that in concrete applications (see Section 3) we check (R2) by proving the following.

(R2.1) $\varphi^1(Ru) \le \varphi^1(u)$ for all $u \in V$.

(R2.2) $\int_0^T \mathrm{e}^{-t/\varepsilon}\psi\left(\frac{\mathrm{d}}{\mathrm{d}t}(Ru)\right)\mathrm{d}t \le \int_0^T \mathrm{e}^{-t/\varepsilon}\psi\left(\frac{\mathrm{d}}{\mathrm{d}t}u\right)\mathrm{d}t$ for all $u \in W^{1,p}(0,T;V)$.

(R2.3) Either

$$\langle w, Ru\rangle_V \ge \langle w, u\rangle_V \text{ for all } w = F(v),\ v \in V_R,\ u \in V, \tag{2.6}$$

or

$$\langle F(u), Ru\rangle_V \ge \langle F(u), u\rangle_V \text{ for all } u \in V. \tag{2.7}$$

Lemma 4 ((R2.1)-(R2.3) imply (R2)) *Let* (R1) *be satisfied. Then,*

i) (R2.1)*,* (R2.2)*, and* (2.6) *imply* (2.4)*,*

ii) (R2.1)*,* (R2.2)*, and* (2.7) *imply* (2.5)*.*

In particular, (R2.1)-(R2.3) *imply* (R2)*.*

Proof. For all $u \in L^p(0,T;V)$ and $w \in L^{p'}(0,T;V^*)$, decompose $I_{\varepsilon,w}(u) = I^1_\varepsilon(u) + I^2_{\varepsilon,w}(u)$, where

$$I^1_\varepsilon(u) = \begin{cases} \displaystyle\int_0^T \mathrm{e}^{-t/\varepsilon}\left(\varepsilon\psi(u') + \varphi^1(u)\right)\mathrm{d}t & \text{if } u \in K(u_0) \cap L^m(0,T;X), \\ +\infty & \text{else,} \end{cases}$$

$$I^2_{\varepsilon,w}(u) = -\int_0^T \mathrm{e}^{-t/\varepsilon}\langle w, u\rangle_V\,\mathrm{d}t.$$

Note that, as a consequence of $\varphi^1 \in \Theta_m(X)$ and of (R2.1), we have that $Ru \in X$ for all $u \in X$. This fact, (R2.1), and (R2.2) imply that $I^1_\varepsilon(Ru) \le I^1_\varepsilon(u)$ for all $u \in K(u_0)$. Moreover, inequality (2.6) ensures that $I^2_{\varepsilon,w}(Ru) \le I^2_{\varepsilon,w}(u)$ for all $u \in K(u_0)$ and $w \in \{F(v) : v \in L^p(0,T;V_R)\}$, which yields inequality (2.4), and inequality (2.7) implies $I^2_{\varepsilon,w}(Ru) \le I^2_{\varepsilon,w}(u)$ for all $u \in K(u_0)$ and $w = F(u)$, i.e. inequality (2.5). ■

2.1.1 Preliminary results for the proof of Theorem 2

In order to prove Theorem 2, we first collect some preliminary results. We record here a slightly modified version of the Schaefer fixed-point Theorem, which will be used in the proof of Theorem 2.

Theorem 5 (Modified Schaefer's fixed-point Theorem) *Let B be a reflexive Banach space and $L \subset B$ be nonempty, convex, and closed. Assume $\delta L \subset L$ for every $\delta \in (0,1)$. Let $S : B \to B$ be continuous, compact, and such that $S(L) \subset L$. Moreover, let the set $\{u \in B : \alpha S(u) = u$ for some $\alpha \in [0,1]\}$ be bounded. Then, S has a fixed point in L.*

Proof. Our proof is a minor modification of the proof of the Schaefer fixed-point Theorem presented in [17, Thm. 4, Ch. 9]. Choose M so large that $|u|_B < M$ for every $u \in \{u \in B : \alpha S(u) = u$ for some $\alpha \in [0,1]\}$. Then, define

$$T(u) = \begin{cases} S(u) & \text{if } |S(u)|_B \leq M, \\ \frac{S(u)M}{|S(u)|_B} & \text{if } |S(u)|_B > M. \end{cases}$$

Observe that $T : B_M(0) \cap L \to B_M(0) \cap L$, where $B_M(0) = \{b \in B : |b|_B \leq M\}$. Define $\tilde{K}$ to be the convex hull of $T(B_M(0) \cap L)$ and K the closure of $\tilde{K}$. Note that, as L is closed and convex, we have that $K \subseteq B_M(0) \cap L$. Moreover, $T : K \to K$ is continuous and $T(K)$ is relatively compact in K. Hence, we can apply the Schauder fixed-point Theorem and prove the existence of $u \in K \subset L$ such that $T(u) = u$. We now show that u is a fixed point for S. Suppose by contradiction that $S(u) \neq u$. Then, $|S(u)|_B > M$ and $u = \alpha S(u)$ for $\alpha = \frac{M}{|S(u)|_B} < 1$. Hence, $|u|_B < M$. As u is a fixed point for T, we conclude that $|T(u)|_B = |u|_B < M = \left|\frac{MS(u)}{|S(u)|_B}\right|_B = |T(u)|_B$, a contradiction. ■

We shall now summarize the WED approach to system (1.1)-(1.3) studied in [1].

Proposition 6 (WED approach I) *Let the assumptions of Theorem 2 be satisfied. Then, the functional $I_{\varepsilon,w}$ defined by*

$$I_{\varepsilon,w}(u) = \begin{cases} \displaystyle\int_0^T \mathrm{e}^{-t/\varepsilon} \left(\varepsilon\psi(u') + \varphi^1(u) - \langle w, u\rangle_V\right) \mathrm{d}t & \text{if } u \in K(u_0) \cap L^m(0,T;X), \\ +\infty & \text{else}, \end{cases} \tag{2.8}$$

admits an unique minimizer $u_{\varepsilon,w}$ over the set $K(u_0) = \{u \in W^{1,p}(0,T;V) : u(0) = u_0\}$ for every $w \in L^{p'}(0,T;V^*)$ and $\varepsilon > 0$ small enough.

Define the map $S : L^p(0,T;V) \to L^p(0,T;V)$ by

$$S : v \longmapsto w = F(v) := f(v) + \mathrm{d}_V\varphi^2(v) \longmapsto u_{\varepsilon,v}, \tag{2.9}$$

where

$$u_{\varepsilon,v} = \operatorname*{argmin}_{\tilde{u} \in K(u_0)} I_{\varepsilon,w}(\tilde{u}).$$

Then, S is continuous and compact, the set $\{v \in L^p(0,T;V) : \alpha S(v) = v$ for $\alpha \in [0,1]\}$ is bounded in $L^p(0,T;V)$, and S has a fixed point u_ε for every $\varepsilon > 0$ small enough. Moreover, $u_\varepsilon \in C([0,T];V)$ and fulfills

$$u_\varepsilon = \operatorname*{argmin}_{\tilde{u} \in K(u_0)} I_{\varepsilon,F(u_\varepsilon)}(\tilde{u}) = \operatorname*{argmin}_{\tilde{u} \in K(u_0)} \int_0^T \mathrm{e}^{-t/\varepsilon}(\varepsilon\psi(\tilde{u}') + \varphi^1(\tilde{u}) - \langle F(u_\varepsilon), \tilde{u}\rangle_V)\mathrm{d}t. \tag{2.10}$$

Finally, there exists a sequence $\varepsilon_n \to 0$ such that

$$u_{\varepsilon_n} \to u \text{ strongly in } C([0,T];V),$$

where u is a strong solution of (1.1)-(1.3).

2.1.2 Proof of Theorem 2 and Corollary 3

Let us first prove Theorem 2. We start by recalling that the map $S : L^p(0,T;V) \to L^p(0,T;V)$ defined as in (2.9) has at least a fixed point (see Proposition 6). We now check that one of this fixed points $u_\varepsilon \in C([0,T];V)$ is such that

$$Ru_\varepsilon = u_\varepsilon \text{ in } [0,T]. \tag{2.11}$$

In case condition (2.5) is satisfied we proceed as follows. Every fixed point u_ε of S fulfills

$$u_\varepsilon = \arg \min_{\tilde{u} \in K(u_0)} I_{\varepsilon, F(u_\varepsilon)}(\tilde{u}),$$

i.e., u_ε is the unique minimizer of $I_{\varepsilon,F(u_\varepsilon)}$ over $K(u_0)$. As a consequence of (R1), we have that $Ru_\varepsilon \in K(u_0)$, and, thanks to assumption (2.5),

$$I_{\varepsilon,F(u_\varepsilon)}(Ru_\varepsilon) \leq I_{\varepsilon,F(u_\varepsilon)}(u_\varepsilon).$$

By uniqueness of the minimizer, $Ru_\varepsilon = u_\varepsilon$.

In case condition (2.4) holds true, we show that S maps the set $L^p(0,T;V_R)$ into itself and we than infer the existence of a fixed point $u_\varepsilon \in L^p(0,T;V_R)$ of S. Let $v \in L^p(0,T;V_R)$ and $w = F(v)$. Using assumptions (R1) and (2.4) and arguing as above, we deduce that the unique minimizer $u_{\varepsilon,v}$ of $I_{\varepsilon,w}$ satisfies $Ru_{\varepsilon,v} = u_{\varepsilon,v}$. This yields $S : L^p(0,T;V_R) \to L^p(0,T;V_R)$. By virtue of Proposition 6, the map $S : L^p(0,T;V_R) \to L^p(0,T;V_R)$ is continuous, compact, and such that the set $\{v \in L^p(0,T;V_R) : \alpha S(v) = v \text{ for } \alpha \in [0,1]\}$ is bounded. Moreover, we have that $\delta L^p(0,T;V_R) \subset L^p(0,T;V_R)$ for all $\delta \in (0,1)$. Therefore, by applying the Schaefer fixed-point Theorem 5, we conclude that S has a fixed point $u_\varepsilon \in L^p(0,T;V_R)$. Moreover, $u_\varepsilon \in C([0,T];V_R)$. In particular, it fulfills (2.11).

Thanks to Proposition 6, we have (after extraction of a not relabeled subsequence) that $u_\varepsilon \to u$ strongly in $C([0,T];V)$, where u solves (1.1)-(1.3). As V_R is closed in V, from (2.11) we deduce that

$$Ru(t) = u(t) \text{ for all } t \in [0,T].$$

This proves Theorem 2.

We now move to Corollary 3. Assume that R_1 and R_2 satisfy (2.4). Then, by restricting the map S to $L^p(0,T;V_{R_1} \cap V_{R_2})$ and arguing as above, we can easily deduce that $S : L^p(0,T;V_{R_1} \cap V_{R_2}) \to L^p(0,T;V_{R_1} \cap V_{R_2})$ and that it has a fixed point $u_\varepsilon \in L^p(0,T;V_{R_1} \cap V_{R_2})$.

In case R_1 satisfies condition (2.4) and R_2 satisfies condition (2.5), we still have that $S : L^p(0,T,V_{R_1}) \to L^p(0,T,V_{R_1})$ and it has a fixed point $u_\varepsilon \in L^p(0,T,V_{R_1})$. Moreover, as a consequence of assumption (2.5) for R_2

$$I_{\varepsilon,F(u_\varepsilon)}(R_2 u_\varepsilon) \leq I_{\varepsilon,F(u_\varepsilon)}(u_\varepsilon).$$

Thus, $R_2 u_\varepsilon = u_\varepsilon$ in $[0,T]$, which yields $u_\varepsilon \in L^p(0,T;V_{R_1} \cap V_{R_2})$.

Then, in both cases, by applying Proposition 6, we can pass to the limit as $\varepsilon \to 0$ and obtain that there exists u solution of system (1.1)-(1.3) such that

$$u \in C([0,T]; V_{R_1} \cap V_{R_2}).$$

In particular,

$$R_2 R_1 u = R_1 R_2 u = u \text{ in } [0,T].$$

This concludes the proof of Corollary 3.

Note that, as a byproduct of Theorem 9 and Corollary 3, we also have that the elliptic-in-time regularization (1.8)-(1.11) of (1.1)-(1.3) admits R-invariant solutions, i.e., there exists u_ε solution to (1.8)-(1.11), invariant under the action of R.

2.2 Main result 2: comparison principles

We now state a comparison principle for doubly-nonlinear systems. Here, we assume F to have a potential structure, namely $F(u) = g + \partial_V \varphi^2(u)$: indeed comparison principles cannot be expected for genuinely nonpotential terms. Counterexamples can be found already in ODE systems. On the other hand, we allow for possibly noncontinuous/nondifferentiable functionals φ^2 as our argument does not require uniqueness of the minimizer of the WED functional.

Let ψ and φ^1 satisfy assumptions of Theorem 2. Let $\varphi^2 : V \to [0,\infty)$ be proper, convex, l.s.c., and satisfy conditions (2.1) and (2.2). Assume additionally that the reaction term f does not depend on u, namely

$$f(u) = g \in L^{p'}(0,T;V^*) \text{ for all } u \in L^p(0,T;V).$$

In order to avoid unnecessary complications deriving by the definition of an abstract concept of order in Banach spaces, we restrict now our attention to the case of problems (1.1)-(1.3) whose solutions can be represented as real-valued functions. More precisely, we assume that X and V are Banach spaces composed by real-valued functions satisfying assumptions of Theorem 2 and such that

$$\{(a,b) \in V \times V : a \leq b\} \text{ is closed in } V \times V \tag{2.12}$$

and

$$u \wedge v, u \vee v \in W^{1,p}(0,T;V) \text{ for all } u, v \in W^{1,p}(0,T;V). \tag{2.13}$$

Let us remark that the above assumptions are satisfied by the Lebesgue spaces $L^q(\Omega)$, Sobolev spaces $W^{1,q}(\Omega)$, and fractional Sobolev spaces $W^{s,r}(\Omega)$, for all $q \in [1,\infty]$, $s \in (0,1)$, $r \in [1,\infty)$ and $\Omega = \mathbb{R}^d$ or measurable, bounded, and with Lipschitz boundary. We define the WED functional

$$I_\varepsilon(\tilde{u}) = \begin{cases} \displaystyle\int_0^T \mathrm{e}^{-t/\varepsilon} \left(\varepsilon\psi(\tilde{u}') + \varphi^1(\tilde{u}) - \varphi^2(\tilde{u}) - \langle g, \tilde{u}\rangle_V\right) \mathrm{d}t & \text{if } \tilde{u} \in K(u_0) \\ & \cap L^m(0,T;X), \\ +\infty & \text{else,} \end{cases} \tag{2.14}$$

where $K(u_0) = \{\tilde{u} \in W^{1,p}(0,T;V) : \tilde{u}(0) = u_0\}$, and assume that for all $u_0, v_0 \in D(\varphi^1)$, such that $u_0 \leq v_0$

$$I_\varepsilon(u \wedge v) + I_\varepsilon(u \vee v) \leq I_\varepsilon(u) + I_\varepsilon(v) \text{ for all } u \in K(u_0),\, v \in K(v_0). \tag{2.15}$$

Before stating the main result of this section let us remark that assumption (2.15) is crucial as it allows us to prove a comparison principle for minimizers of the WED functional by applying Lemma 9 below.

The main result of this section states a comparison principle for problem (1.1)-(1.3).

Theorem 7 (Comparison principle) *Let $u_0, v_0 \in X$ be such that $u_0 \leq v_0$. Assume conditions* (2.12)-(2.15)*. Then, there exist two strong solutions u, v to equation* (1.1)-(1.3) *corresponding to the initial data u_0 and v_0, respectively, such that $u \leq v$ for all $t \in [0,T]$.*

Note that solutions to (1.1)-(1.3) are, in general, nonunique. Thus, we can not expect the statement of the theorem to hold for every couple u, v of solutions corresponding to the initial data u_0 and v_0 (take $u_0 = v_0$).

Several applications of Theorem 7 to local and nonlocal PDE problems will be presented in Section 3.

2.2.1 Preliminary results for the proof of Theorem 7

In this section we collect some preliminary results, which will be used in the proof of Theorem 7. Under the assumptions of Theorem 7, namely f independent of u, the WED procedure simplifies as a fixed-point argument is no longer necessary. More precisely, the following proposition has been proved in [1] (see also [2, 3, 4, 28]).

Proposition 8 (WED approach 2) *Let the assumptions of Theorem 7 be satisfied. Then, for each $g \in L^{p'}(0,T;V^*)$ and $u_0 \in X$ the* WED *functional I_ε, defined by* (2.14)*, admits at least one global minimizer u_ε over the set $K(u_0) = \{u \in W^{1,p}(0,T;V) : u(0) = u_0\}$. Moreover, for every sequence $\varepsilon_n \to 0$ there exists a (not relabeled) subsequence such that*

$$u_{\varepsilon_n} \to u \text{ strongly in } C([0,T];V) \tag{2.16}$$

and u is a strong solution of system (1.1)-(1.3)*.*

In order to prove Theorem 7, we take advantage of the following abstract comparison principle for minimizers of functionals.

Lemma 9 (Abstract comparison principle) *Let A, B be sets. Let $\alpha, \beta : A \times A \to A$ be two maps. Let $M_0 : A \to B$ be a function. Let $I : A \to \mathbb{R} \cup \{+\infty\}$ be such that for every $\bar{u} \in B$ there exists at least a minimizer of I over the set $K(\bar{u}) = \{w \in A : M_0(w) = \bar{u}\}$. Assume*

$$I(\alpha(u,v)) + I(\beta(u,v)) \leq I(u) + I(v), \tag{2.17}$$

for all $u \in K(u_0)$*,* $v \in K(v_0)$*,* $M_0\left(\alpha(u,v)\right) = v_0$*, and* $M_0\left(\beta(u,v)\right) = u_0$*. Fix*

$$u \in \arg\min_{w \in K(u_0)} I(w),$$
$$v \in \arg\min_{w \in K(v_0)} I(w).$$

Then, $\alpha(u,v)$ *and* $\beta(u,v)$ *are minimizers of* I *over* $K(v_0)$ *and* $K(u_0)$ *respectively. Furthermore, if additionally, the functional* I *has a unique minimizer over* $K(\bar{u})$ *for all* $\bar{u} \in B$*, then* $\alpha(u,v) = v$ *and* $\beta(u,v) = u$*.*

Proof. Let u and v be minimizers of I over $K(u_0)$ and $K(v_0)$ respectively and let $M_0\left(\beta(u,v)\right) = u_0$ and $M_0\left(\alpha(u,v)\right) = v_0$. Then, we have

$$I\left(u\right) \le I\left(\beta(u,v)\right),$$
$$I\left(v\right) \le I\left(\alpha(u,v)\right).$$

By using the property (2.17), we get

$$I\left(u\right) \le I\left(\beta(u,v)\right) \le I(u) + I(v) - I(\alpha(u,v)),$$
$$I\left(v\right) \le I\left(\alpha(u,v)\right) \le I(u) + I(v) - I(\beta(u,v)),$$

Thus,

$$I(\beta(u,v)) \le I(u),$$
$$I(\alpha(u,v)) \le I(v).$$

Therefore, $\alpha(u,v)$ minimizes I over $K(v_0)$ and $\beta(u,v)$ minimizes I over $K(u_0)$. If additionally the minimizers are unique, then $v = \alpha(u,v)$ and $\beta(u,v) = u$. ∎

2.2.2 Proof of Theorem 7

With this preparation we are now in the position of proving Theorem 7. Let u_ε and v_ε be minimizers of I_ε over $K(u_0)$ and $K(v_0)$ respectively. Recalling that $K(\bar{u}) = \{\tilde{u} \in W^{1,p}(0,T;V) : \tilde{u}(0) = \bar{u}\}$, and using assumptions (2.13), we have that $u_\varepsilon^1 := u_\varepsilon \wedge v_\varepsilon \in K(u_0)$ and $v_\varepsilon^1 := u_\varepsilon \vee v_\varepsilon \in K(v_0)$. By applying Lemma 9 with $A = W^{1,p}(0,T;V)$, $B = V$, $\alpha(u,v) = u \vee v$, $\beta(u,v) = u \wedge v$, $M_0(w) = w(0)$, we get that u_ε^1 and v_ε^1 minimize I_ε over $K(u_0)$ and $K(v_0)$ respectively and $u_\varepsilon^1 \le v_\varepsilon^1$ a.e. in $[0,T]$. Thanks to Proposition 8, there exists a sequence $\varepsilon_n \to 0$ such that $u_{\varepsilon_n}^1 \to u$ a.e. in $[0,T]$ and u is a strong solution to the doubly-nonlinear problem (1.1)-(1.3). Moreover, there exists a (not relabeled) subsequence such that $v_{\varepsilon_n}^1 \to v$ a.e. in $[0,T]$ and v solves system (1.1)-(1.3). Thus, thanks to the closedness condition (2.12), we get $u \le v$.

Let us remark that the uniqueness of minimizers of the WED functional was not used here.

3 Applications

In this section we present several applications of Theorem 2 and Theorem 7 to some PDE systems of local and nonlocal type. We remark that, although some of the results presented in this section are known and can be obtained by standard techniques, e.g., comparison principles or sliding methods [6], some applications are, to the best of the authors knowledge, new. In particular, results concerning the composition of maps for problems with nonunique solutions.

3.1 Doubly-nonlinear parabolic equations

Consider the family of *doubly-nonlinear equations of m-Laplace type* given by

$$\alpha(u_t) - \operatorname{div}(B(x)|\nabla u|^{m-2}\nabla u) + C(x)|u|^{m-2}u \\ - D(x)|u|^{q-2}u - h(x,t) = 0 \text{ in } \Omega \times (0,T), \tag{3.1}$$

$$u + b\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega \times (0,T), \tag{3.2}$$

$$u(0) = u_0 \text{ in } \Omega. \tag{3.3}$$

Here, we assume that $\Omega \subset \mathbb{R}^d$ is bounded with Lipschitz boundary $\partial\Omega$ and $\alpha : \mathbb{R} \to \mathbb{R}$ is maximal monotone. Moreover, let exist a constant $\tilde{C}$ such that

$$\frac{1}{\tilde{C}}|s|^p - \tilde{C} \le A(s) := \int_0^s \alpha(r)\mathrm{d}r \text{ and } |\alpha(s)|^{p'} \le \tilde{C}\left(|s|^p + 1\right) \text{ for all } s \in \mathbb{R}. \tag{3.4}$$

We assume m, q, p to satisfy the following relations: $m \ge 2$, $1 < p < m^* := dm/(d-m)^+$, $1 < q \le p$. We consider b constant and strictly positive. We remark that this choice is made for the sake of simplicity and other types of boundary conditions, e.g, Neumann or Dirichlet boundary conditions can be treated similarly and with no additional difficulties. Let $h \in L^{p'}(0,T;L^{p'}(\Omega))$. We assume the coefficients $B, C, D \in L^\infty(\Omega)$ to be positive a.e. in Ω. Moreover, $0 < b_1 \le B(x)$ for a.e. $x \in \Omega$ and some $b_1 \in \mathbb{R}^+$.

With the aim of applying the abstract theory of Section 2.1, we recast system (3.1)-(3.3) into the abstract form (1.1)-(1.3). To this end, we set $V = L^p(\Omega)$, $X = W^{1,m}(\Omega)$, and

$$\psi(u) = \int_\Omega A(u), \tag{3.5}$$

$$\varphi^1(u) = \begin{cases} \displaystyle\int_\Omega \left(\frac{1}{m}B|\nabla u|^m + \frac{1}{m}C|u|^m\right)\mathrm{d}x + \int_{\partial\Omega}\frac{1}{2b}|u|^2, & \text{if } u \in W^{1,m}(\Omega) \\ +\infty & \text{otherwise,} \end{cases} \tag{3.6}$$

$$f(u) = 0, \qquad \varphi^2(u) = D\frac{1}{q}|u|^q + \langle h, u\rangle_V\,. \tag{3.7}$$

System (3.1)-(3.3) is a doubly-nonlinear version of the Allen-Cahn equation coupled with Robin boundary conditions. The existence of a strong solution u to (3.1)-(3.3) in the sense of Definition 1 follows by a direct application of Proposition 6 (for checking that assumptions of Proposition 6 are satisfied we refer the reader to [2, Sec. 6.1],[3, Sec. 7]). We recall that, if u solves (3.1)-(3.3) in the sense of Definition 1, then, $u \in W^{1,p}(0,T;L^p(\Omega)) \cap L^m(0,T;W^{1,m}(\Omega))$, $\operatorname{div}(B|\nabla u(t)|^{m-2}\nabla u(t)) \in L^{p'}(\Omega)$ for a.e. $t \in (0,T)$, and u solves (3.1) pointwise a.e. in $\Omega \times (0,T)$. It is worth mentioning that the uniqueness of solution may essentially fail, e.g., in the case of the sublinear heat equation $u_t - \Delta u = |u|^{q-2}u$, $1 < q < 2$. Indeed, the latter admits positive solutions even for zero initial data. [1]

We aim at proving existence of solutions to (3.1)-(3.3) which satisfy qualitative properties such as symmetries, monotonicity, and upper and lower bounds. To this end, we introduce some maps $R : L^p(\Omega) \to L^p(\Omega)$ to describe the mentioned properties, together with compatibility assumptions on the data.

1. *Linear rigid transformation of the space:* $Ru(x) = u(rx)$ for some $r \in SL(d,\mathbb{R}) = \{r \in M(\mathbb{R}^{d\times d}) : |\det r| = 1\}$, $r\Omega = \Omega$, and B, C, D, h are R-invariant;

2. *Symmetric decreasing rearrangement* or *Schwartz symmetrization* [20]*:* $Ru = (u^+)^*$, Ω is radially symmetric, B, C, and D are constant a.e. in Ω, and $h = (h^+)^*$ a.e. in $\Omega \times (0,T)$;

3. *Symmetric decreasing rearrangement with respect to the hyperplane* $H \subset \mathbb{R}^d$ *(*or *Steiner symmetrization* in case $\dim H = 1$) [20]: $Ru = (u^+)^{*,H}$, Ω is invariant under the action of any rotation and reflection which maps H into H, B, C, and D are constant a.e. in Ω, $h = (h^+)^{*,H}$ a.e. in $\Omega \times (0,T)$;

4. *Monotone decreasing rearrangement with respect to the direction* $y \in \mathbb{R}^d$ [20]: $Ru = (u^+)^{*,y}$, $\Omega = \Omega^{*,y}$, B, C, and D are constant in the direction of y, a.e. in Ω, and $h = (h^+)^{*,y}$ a.e. in $\Omega \times (0,T)$;

5. *Lower truncation*: $R(u) = (u - M)^+ + M$, where $M \leq 0$ is constant, $h \geq 0$ a.e. in $\Omega \times (0,T)$, and either $M = 0$ or $D = 0$;

6. *Upper truncation:* $R(u) = M - (M - u)^+$, where $M \geq 0$ is constant, $h \leq 0$ a.e. in $\Omega \times (0,T)$, and either $M = 0$ or $D = 0$.

The definitions and some basic properties of the rearrangement maps are collected in the appendix for the reader's convenience (see also [20] for a survey).

Assume $Ru_0 = u_0$. By applying Theorem 2, we conclude that there exists a solution u to the Cauchy problem (3.1)-(3.3) such that $Ru = u$ a.e. in $\Omega \times [0,T]$. Furthermore, by applying Theorem 7, we can also deduce a comparison principle for solutions to (3.1)-(3.3). More precisely, we have the following theorem.

Theorem 10 (Doubly-nonlinear parabolic equation (3.1)-(3.3)) *Let the above assumptions be satisfied and let R_i, $i = 1, ..., k$, be any collection of maps as defined above. Assume $R_i u_0 = u_0$ for $i = 1, ..., k$. Then, there exists a strong solution u (in the sense of Definition 1) to the Cauchy problem* (3.1)-(3.3) *such that $R_1 \circ ... \circ R_k u = u$.*

Furthermore, let $u_0 \leq v_0$ a.e. in Ω. Then, there exist u, v strong solutions to (3.1)-(3.2) *such that $u(0) = u_0$, $v(0) = v_0$, and $u \leq v$ a.e. in $\Omega \times (0,T)$.*

[1] Indeed the constant-in-space functions $u = 0$ and $u = t^2/4$ are solutions.

Note that any map R_i is associated with some compatibility conditions on the data. Let us note that, in case $k \geq 2$, these conditions have to be satisfied simultaneously and, hence, they have to be compatible. This fact is implicitly guaranteed by the assumption of the existence of some u_0 satisfying $u_0 = R_i u_0$ for all $i = 1, ..., k$.

Proof. In order to apply Theorem 2, it suffices to check conditions (R1)-(R2). Note that R satisfies

$$\int_\Omega J(Ru - Rv) \leq \int_\Omega J(u - v) \tag{3.8}$$

for every $u, v \in V$ and $J : \mathbb{R} \to \mathbb{R}$ nonnegative, convex, and such that $J(0) = 0$ (see the appendix or [20] for the case of rearrangements). Thus,

$$\lim_{s\to t} \int_\Omega \left| \frac{Ru(s) - Ru(t)}{s - t} \right|^p \mathrm{d}x \leq \lim_{s\to t} \int_\Omega \left| \frac{u(s) - u(t)}{s - t} \right|^p \mathrm{d}x \text{ for a.a. } t \in (0, T).$$

This fact, together with the Dominated Convergence Theorem, proves that

$$Ru \in W^{1,p}(0, T; V) \text{ for all } u \in W^{1,p}(0, T; V).$$

This easily yields (R1). It is standard matter to check (R2.1) (see the appendix or [20] for more details in the case of rearrangement maps). By definition of ψ, for a.a. $t \in (0, T)$, we have

$$\psi\left(\frac{\mathrm{d}}{\mathrm{d}t}(Ru(t))\right) = \int_\Omega A\left(\lim_{s\to t} \frac{Ru(s) - Ru(t)}{s - t}\right) \mathrm{d}x.$$

By using the continuity of A,

$$\begin{aligned} \psi\left(\frac{\mathrm{d}}{\mathrm{d}t}(Ru(t))\right) &= \int_\Omega \lim_{s\to t} A\left(\frac{Ru(s) - Ru(t)}{s - t}\right) \mathrm{d}x \\ &= \lim_{s\to t} \int_\Omega A\left(\frac{Ru(s) - Ru(t)}{s - t}\right) \mathrm{d}x. \end{aligned}$$

Thanks to inequality (3.8) (applied to $w \longmapsto J(w) = A(\frac{w}{s-t})$),

$$\begin{aligned} \lim_{s\to t} \int_\Omega A\left(\frac{Ru(s) - Ru(t)}{s - t}\right) \mathrm{d}x &\leq \lim_{s\to t} \int_\Omega A\left(\frac{u(s) - u(t)}{s - t}\right) \mathrm{d}x \\ &= \int_\Omega A\left(\lim_{s\to t} \frac{u(s) - u(t)}{s - t}\right) \mathrm{d}x = \psi\left(\frac{\mathrm{d}}{\mathrm{d}t} u(t)\right). \end{aligned}$$

The above computations hold true for a.a. $t \in (0, T)$. Thanks to the upper bound in (3.4), we have $|A(s)| \leq |\alpha(s)s| \leq C(|s|^p + 1)$ and hence,

$$\int_\Omega A\left(\frac{Ru(s) - Ru(t)}{s - t}\right) \mathrm{d}x \leq C\left(\int_\Omega \left|\frac{u(s) - u(t)}{s - t}\right|^p \mathrm{d}x + 1\right).$$

As a consequence of (R2), we have that the right hand side is uniformly bounded for a.a. t and s. Moreover, for a.a. $t \in (0, T)$, we note that

$$\frac{Ru(s) - Ru(t)}{s - t} \to \frac{\mathrm{d}}{\mathrm{d}t}(Ru(t))$$

for $s \to 0$. By applying the Dominated Convergence Theorem, we get

$$\int_0^T \mathrm{e}^{-t/\varepsilon}\psi\left(\frac{\mathrm{d}}{\mathrm{d}t}(Ru(t))\right)\mathrm{d}t \leq \int_0^T \mathrm{e}^{-t/\varepsilon}\psi\left(\frac{\mathrm{d}}{\mathrm{d}t}u(t)\right)\mathrm{d}t \text{ for all } T>0.$$

This yields (R2.2). We readily check that (2.6) is satisfied. In particular, in the case of rearrangements maps R, we have $R\mathrm{d}_V\varphi^2(v) = \mathrm{d}_V\varphi^2(v)$ for all $v \in V_R$. Thus, condition (2.6) follows from the well known rearrangement inequality (see the appendix or [20]):

$$\int_\Omega Ra \cdot Rb \geq \int_\Omega a \cdot b \text{ for all } a \in L^p(\Omega),\ b \in L^p(\Omega),\ a, b \geq 0.$$

A direct application of Theorem 2 and Corollary 3 yields the first part of Theorem 10.

To prove the second part, we aim at applying Theorem 7. To this end, we now verify condition (2.15). Note that, for all $u, v \in W^{1,p}(0,T;L^p(\Omega)) \cap L^m(0,T;X)$, one has $u \vee v, u \wedge v \in W^{1,p}(0,T;L^p(\Omega)) \cap L^m(0,T;X)$. Furthermore, the following relations hold true a.e. in $\Omega \times (0,T)$

$$(u \vee v)' = \begin{cases} u' & \text{if } u \geq v, \\ v' & \text{if } u < v, \end{cases}$$
$$(u \wedge v)' = \begin{cases} v' & \text{if } u \geq v, \\ u' & \text{if } u < v, \end{cases}$$
$$\nabla(u \vee v) = \begin{cases} \nabla u & \text{if } u \geq v, \\ \nabla v & \text{if } u < v, \end{cases}$$
$$\nabla(u \wedge v) = \begin{cases} \nabla v & \text{if } u \geq v, \\ \nabla u & \text{if } u < v, \end{cases}$$
$$(u \vee v)_{|\partial\Omega} = \begin{cases} u_{|\partial\Omega} & \text{if } u \geq v, \\ v_{|\partial\Omega} & \text{if } u < v, \end{cases}$$
$$(u \wedge v)_{|\partial\Omega} = \begin{cases} v_{|\partial\Omega} & \text{if } u \geq v, \\ u_{|\partial\Omega} & \text{if } u < v, \end{cases}$$

where $w_{|\partial\Omega}$ denotes the trace of w on $\partial\Omega$. Moreover,

$$\begin{aligned}
&I_\varepsilon(u \vee v) + I_\varepsilon(u \wedge v) \\
&= \int\int_{\Omega\times(0,T)\cap\{u\geq v\}} \mathrm{e}^{-t/\varepsilon}G(u) + \int\int_{\Omega\times(0,T)\cap\{u<v\}} \mathrm{e}^{-t/\varepsilon}G(v) \\
&+ \int\int_{\Omega\times(0,T)\cap\{u\geq v\}} \mathrm{e}^{-t/\varepsilon}G(v) + \int\int_{\Omega\times(0,T)\cap\{u<v\}} \mathrm{e}^{-t/\varepsilon}G(u) \\
&+ \int\int_{\partial\Omega\times(0,T)\cap\{u\geq v\}} \mathrm{e}^{-t/\varepsilon}\frac{1}{2b}|u|^2 + \int\int_{\partial\Omega\times(0,T)\cap\{u<v\}} \mathrm{e}^{-t/\varepsilon}\frac{1}{2b}|v|^2 \\
&+ \int\int_{\partial\Omega\times(0,T)\cap\{u\geq v\}} \mathrm{e}^{-t/\varepsilon}\frac{1}{2b}|u|^2 + \int\int_{\partial\Omega\times(0,T)\cap\{u<v\}} \mathrm{e}^{-t/\varepsilon}\frac{1}{2b}|u|^2 \\
&= I_\varepsilon^1(u) + I_\varepsilon^1(v),
\end{aligned}$$

where $G(u) = \varepsilon\alpha(u') + \frac{1}{m}B|\nabla u|^m + \frac{1}{m}C|u|^m - D\frac{1}{q}|u|^q - hu$. By applying Theorem 2, we conclude the proof of Theorem 10. ■

3.2 Fractional heat equation

We consider the *fractional heat equation*

$$u_t + (-\Delta)^s u + \gamma u = g \text{ in } \Omega \times (0,T), \tag{3.9}$$

$$u = 0 \text{ in } \left(\mathbb{R}^d \setminus \Omega\right) \times (0,T), \tag{3.10}$$

$$u(0) = u_0 \text{ a.e. in } \mathbb{R}^d, \tag{3.11}$$

where $\Omega \subset \mathbb{R}^d$ bounded with Lipschitz boundary, $\gamma > 0$, $s \in (0,1)$, $u_0 \in H_0^s(\Omega)$, and $g \in L^2(\Omega)$. Here, $(-\Delta)^s$ denotes the *s-fractional Laplace operator* [16].

Before stating the main result of this section, let us first recall some definitions and known results. For every $s \in (0,1)$ and $d \in \mathbb{N}$, we denote by $H^s(\mathbb{R}^d) = W^{s,2}(\mathbb{R}^d)$ the usual s-fractional Sobolev space equipped with the norm

$$\begin{aligned}\|u\|^2_{H^s(\mathbb{R}^d)} &= \|u\|^2_{L^2(\mathbb{R}^d)} + [u]^2_{\mathbb{R}^d,s} \\ &= \|u\|^2_{L^2(\mathbb{R}^d)} + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

We use the notation

$$H_0^s(\Omega) = \{u \in H^s(\mathbb{R}^d) : u = 0 \text{ a.e. in } \mathbb{R}^d \setminus \Omega\}.$$

Let $Q = \mathbb{R}^{2d} \setminus \left(\mathbb{R}^d \setminus \Omega\right) \times \left(\mathbb{R}^d \setminus \Omega\right)$. Then, the space $H_0^s(\Omega)$, equipped with the norm

$$\begin{aligned}\|u\|^2_{H_0^s(\Omega)} &= \|u\|^2_{L^2(\Omega)} + \iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\mathrm{d}y \\ &= \|u\|^2_{L^2(\Omega)} + [u]^2_{\mathbb{R}^d,s}\end{aligned}$$

and with the scalar product

$$(u,v)_{H_0^s(\Omega)} = (u,v)_{L^2(\Omega)} + \iint_Q \frac{(u(x)-u(y))\,(v(x)-v(y))}{|x-y|^{d+2s}}\mathrm{d}x\mathrm{d}y,$$

is a Hilbert space [37, Lemma 7]. Note that the functional $\varphi : H_0^s(\Omega) \to (H_0^s(\Omega))^*$ defined by

$$u \mapsto \varphi(u) = \frac{1}{2}\iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\mathrm{d}y$$

is Fréchet differentiable over $H_0^s(\Omega)$ and

$$\left\langle \mathrm{d}_{H_0^s(\Omega)}\varphi(u), v\right\rangle_{H_0^s(\Omega)} = \iint_Q \frac{(u(x)-u\,(y))\,(v(x)-v\,(y))}{|x-y|^{d+2s}}\mathrm{d}x\mathrm{d}y$$

for every $v \in H_0^s(\Omega)$ [37]. We define the fractional Laplacian operator as [16]

$$-(-\Delta)^s u(x) = \int_{\mathbb{R}^d} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{d+2s}}\mathrm{d}y,\ x \in \mathbb{R}^d$$

and we note that $\partial_{L^2(\Omega)}\varphi(u) = (-\Delta)^s u$ for all $u \in D(\partial_{L^2(\Omega)}\varphi)$.

Thanks to the above preparation, we can rewrite equation (3.9)-(3.11) in the gradient flow form

$$u_t + \partial_V \varphi^1(u) - \mathrm{d}_V \varphi^2(u) = g \text{ in } V^*, \text{ a.e. in } (0,T), \tag{3.12}$$
$$u(0) = u_0, \tag{3.13}$$

where

$$X = H_0^s(\Omega), \qquad V = L^2(\Omega),$$
$$\varphi^1(u) = \frac{\gamma}{2}\|u\|_{L^2(\Omega)}^2 + \frac{1}{2}\iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{d+2s}}\mathrm{d}x\mathrm{d}y,$$
$$\varphi^2(u) = 0, \qquad f(u) = g.$$

The existence of solutions to problem (3.12)-(3.13) follows, e.g., from Proposition 6. The continuous dependence of solutions from initial data (and hence the uniqueness of solutions) can be directly proved by testing the equation with the solution, taking advantage of the linearity of the equation and performing standard estimates. We observe that the (unique) solution to the problem (3.12)-(3.13) in the sense of Definition 1 is a function $u \in L^2(0,T;H_0^s(\Omega)) \cap H^1(0,T;L^2(\Omega))$ such that $(-\Delta)^s u(t) \in L^2(\Omega)$ for a.e. $t \in (0,T)$ that solves equation (3.9) a.e. in $\Omega \times (0,T)$.

Aiming at applying Theorem 2 to prove qualitative properties of the solution of (3.9)-(3.11), we now introduce some maps $R : L^2(\Omega) \to L^2(\Omega)$ which describe qualitative properties and we fix some compatibility conditions for the data.

1. *Linear rigid transformation of the space:* $Ru(x) = u(rx)$ for some $r \in SL(d,\mathbb{R})$, $r\Omega = \Omega$, and g is R-invariant;
2. *Symmetric decreasing rearrangement* or *Schwartz symmetrization* [20]: $Ru = (u^+)^*$, Ω is radially symmetric, $g = (g^+)^*$ a.e. in Ω;
3. *Positive part:* $R(u) = u^+$ and $g \geq 0$ a.e. in Ω;
4. *Negative part:* $R(u) = -u^-$ and $g \leq 0$ a.e. in Ω.

By applying Theorem 2, Corollary 3, and Theorem 7, we get the following.

Theorem 11 (Fractional heat equation) *Let R_i, $i = 1, ..., k$, be any collection of the maps defined above. Then, for every $u_0 \in H_0^s(\Omega)$ such that $R_i u_0 = u_0$ for $i = 1, ..., k$, the strong solution u to* (3.9)-(3.11) *fulfills $R_1 \circ ... \circ R_k u = u$ a.e. in $\Omega \times (0,T)$. Moreover, let u and v be the two strong solutions to* (3.9)-(3.10) *corresponding to initial conditions $u(0) = u_0$ and $v(0) = v_0$, with $u_0 \leq v_0$ a.e. in Ω. Then, $u \leq v$ a.e. in $\Omega \times [0,T]$.*

As already mentioned in the previous section, in the case $k \geq 2$, assumption $R_i u_0 = u_0$ for all $i = 1, ..., k$ implies that the compatibility conditions associated with any of the maps R_i are satisfied simultaneously.

Proof. Taking advantage of the above preparation, we readily check conditions (R1), (2.4) (and hence (R2)) (see the appendix or [33] for the case of the symmetric decreasing rearrangement). Thus, the first part of Theorem 11 follows directly from Theorem 2 and Corollary 3.

In order to prove the second part of Theorem 11, we now check that assumptions of Theorem 7 are satisfied. For all $u, v \in H^s(\mathbb{R}^d)$, one has

$$\begin{aligned}
[u \vee v]^2_{\mathbb{R}^d,s} + [u \wedge v]^2_{\mathbb{R}^d,s} &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{|(u \vee v)(x) - (u \vee v)(y)|^2}{|x-y|^{d+2s}} \mathrm{d}y \right) \mathrm{d}x \\
&+ \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{|(u \wedge v)(x) - (u \wedge v)(y)|^2}{|x-y|^{d+2s}} \mathrm{d}y \right) \mathrm{d}x \\
&= A_1 + A_2 + A_3 + A_4
\end{aligned}$$

where,

$$\begin{aligned}
A_1 &= \int_{u \geq v} \left(\int_{u \geq v} \left(\frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} + \frac{|v(x) - v(y)|^2}{|x-y|^{d+2s}} \right) \mathrm{d}y \right) \mathrm{d}x, \\
A_2 &= \int_{u < v} \left(\int_{u < v} \left(\frac{|v(x) - v(y)|^2}{|x-y|^{d+2s}} + \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} \right) \mathrm{d}y \right) \mathrm{d}x, \\
A_3 &= \int_{u \geq v} \left(\int_{u < v} \left(\frac{|u(x) - v(y)|^2}{|x-y|^{d+2s}} + \frac{|v(x) - u(y)|^2}{|x-y|^{d+2s}} \right) \mathrm{d}y \right) \mathrm{d}x, \\
A_4 &= \int_{u < v} \left(\int_{u \geq v} \left(\frac{|v(x) - u(y)|^2}{|x-y|^{d+2s}} + \frac{|u(x) - v(y)|^2}{|x-y|^{d+2s}} \right) \mathrm{d}y \right) \mathrm{d}x.
\end{aligned}$$

We now prove that

$$A_3 \leq \int_{u \geq v} \left(\int_{u < v} \left(\frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} + \frac{|v(x) - v(y)|^2}{|x-y|^{d+2s}} \right) \mathrm{d}y \right) \mathrm{d}x. \tag{3.14}$$

To this aim, let us denote $a = u(x) - v(x)$, $b = v(x) - v(y)$, $c = v(y) - u(y)$. Note that, as $u(x) \geq v(x)$ and $u(y) < v(y)$ a.e. over the integration domain, $ac \geq 0$. Thus, (3.14) follows by a direct application of inequality $(a+b)^2 + (b+c)^2 \leq b^2 + (a+b+c)^2$. Similarly, we can prove

$$A_4 \leq \int_{u < v} \left(\int_{u \geq v} \left(\frac{|v(x) - v(y)|^2}{|x-y|^{d+2s}} + \frac{|u(x) - u(y)|^2}{|x-y|^{d+2s}} \right) \mathrm{d}y \right) \mathrm{d}x.$$

Combining these estimates, we get

$$A_1 + A_2 + A_3 + A_4 \leq [u]^2_{\mathbb{R}^d,s} + [v]^2_{\mathbb{R}^d,s}.$$

In particular, $u \wedge v, u \vee v \in H^s(\mathbb{R}^d)$ for every $u, v \in H^s(\mathbb{R}^d)$ and conditions (2.13) and (2.15) are fulfilled.

Finally, note that the spaces $L^2(\Omega)$ and $H^s_0(\Omega)$, $s \in (0,1)$ satisfy condition (2.12). Thus, the second assertion in Theorem 11 follows directly from Theorem 7. ■

3.3 Systems of reaction-diffusion equations

We consider the *diffusive Lotka-Volterra prey-predator system* given by

$$u_t - D_1\Delta u = Au\left(1 - \frac{u}{K}\right) - \frac{Buv}{1+Ev} - F_1 u \text{ in } \Omega \times (0,T), \tag{3.15}$$
$$v_t - D_2\Delta v = \frac{Cuv}{1+Ev} - F_2 v \text{ in } \Omega \times (0,T), \tag{3.16}$$
$$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega \times (0,T), \tag{3.17}$$
$$v(0) = v_0,\ u(0) = u_0 \text{ in } \Omega, \tag{3.18}$$

where $A, K, D_1, D_2, F_1, F_2 > 0$ and $B, C, E \geq 0$ are constants and $\Omega \subset \mathbb{R}^d$ is bounded with Lipschitz boundary $\partial\Omega$. The model describes the evolution of two interacting populations, cf., e.g., [29, 14, 15]. Here u and v denote the concentrations of a prey species and a predator species respectively, D_1, D_2, and F_1, F_2 are the diffusion rates and the spontaneous-death rates of preys and predators respectively. The parameters C, B describe the interaction rates of the two species while E measures the so-called predator satiation [29, 14, 15]. Finally, A represents the preys' birth rate (at predators low density) and K the so-called carrying capacity of the environment.

Note that negative values of u and v or values of u larger than K are meaningless from the biological viewpoint. By applying Theorem 2 together with the choice $R(u,v) = \left((\min\{u,K\})^+, v^+\right)$, we can prove the existence of solutions to system (3.15)-(3.18) starting from initial data $(u_0, v_0) \in [0,K] \times [0,\infty)$ a.e. in $\Omega \times \Omega$, satisfy the same bounds at any time. To this end, we first reformulate system (3.15)-(3.18) in the abstract form (1.1)-(1.3) by defining

$$\begin{aligned}
V &= L^2(\Omega) \times L^2(\Omega),\\
X &= H^1(\Omega) \times H^1(\Omega),\\
\varphi^1(u,v) &= \frac{1}{2}\int_\Omega D_1|\nabla u|^2 + D_2|\nabla v|^2 + F_1|u|^2 + F_2|v|^2,\\
\psi(u,v) &= \frac{1}{2}\int_\Omega |u|^2 + |v|^2,\\
\varphi^2(u) &= 0,\\
f(u,v) &= \left(AU\left(1 - \frac{U}{K}\right) - \frac{BUV}{1+EV}, \frac{CUV}{1+EV}\right),
\end{aligned}$$

where $U := (\min\{u,K\})^+$ and $V = v^+$. Note that in case (u,v) solves (1.1)-(1.3) in the sense of Definition 1 and $0 \leq u \leq K$, $v \geq 0$, then,

$$\begin{aligned}
&u, v \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),\\
&\Delta u(t), \Delta v(t) \in L^2(\Omega) \text{ for a.e. } t \in (0,T),
\end{aligned}$$

and (u,v) fulfills identities (3.15)-(3.18) pointwise a.e. in $\Omega \times (0,T)$. Indeed, $f(u,v) = \left(Au\left(1 - \frac{u}{K}\right) - \frac{Buv}{1+Ev}, \frac{Cuv}{1+Ev}\right)$ if $0 \leq u \leq K$ and $v \geq 0$.

Theorem 12 (System of reaction-diffusion equations) *For all $u_0, v_0 \in H^1(\Omega)$ such that $0 \leq u_0 \leq K$ and $v_0 \geq 0$ a.e. in Ω, system (3.15)-(3.18) admits a strong solution (u,v) such that $0 \leq u \leq K$ and $v \geq 0$ a.e. in $\Omega \times (0,T)$.*

Proof. Define $R(u,v) = (U,V) = \left((\min\{u,K\})^{+}, v^{+}\right)$. It is standard matter to check that assumptions of Proposition 6 are satisfied [25]. Moreover, (R1) can be easily proved. We now verify condition (2.5) (and, thus, (R2)). Note that

$$\begin{aligned}
&\int_0^T \mathrm{e}^{-t/\varepsilon}\left(\varepsilon\psi\left(\frac{\mathrm{d}}{\mathrm{d}t}R(u,v)\right) + \varphi^1\left(R(u,v)\right)\right)\mathrm{d}t \\
&\leq \int_0^T \mathrm{e}^{-t/\varepsilon}\left(\varepsilon\psi\left(\frac{\mathrm{d}}{\mathrm{d}t}(u,v)\right) + \varphi^1(u,v)\right)\mathrm{d}t.
\end{aligned}$$

Let us prove that, for all $(u,v) \in L^2(\Omega)\times L^2(\Omega)$,

$$\langle f(u,v), R(u,v)\rangle_{L^2(\Omega)\times L^2(\Omega)} \geq \langle f(u,v), (u,v)\rangle_{L^2(\Omega)\times L^2(\Omega)}, \tag{3.19}$$

i.e.,

$$\begin{aligned}
&\int_\Omega AU\left(1-\frac{U}{K}\right)U - \frac{BUV}{1+EV}U + \frac{CUV}{1+EV}V \\
&\geq \int_\Omega AU\left(1-\frac{U}{K}\right)u - \frac{BUV}{1+EV}u + \frac{CUV}{1+EV}v.
\end{aligned}$$

Note that $AU\left(1-\frac{U}{K}\right)U = AU\left(1-\frac{U}{K}\right)u$ a.e. in Ω. Moreover,

$$U^2 = \left\{\begin{array}{cc} 0 & \text{if } u<0 \\ u^2 & \text{if } u\in[0,K] \\ K^2 & \text{if } u>K \end{array}\right\} \leq \left\{\begin{array}{cc} 0 & \text{if } u<0 \\ u^2 & \text{if } u\in[0,K] \\ Ku & \text{if } u>K \end{array}\right\} = Uu \text{ a.e. in } \Omega.$$

Thus, as $V \geq 0$,

$$-\frac{BUV}{1+EV}U \geq -\frac{BUV}{1+EV}u \text{ a.e. in } \Omega.$$

Finally, $V^2 = Vv$ a.e in Ω, which implies

$$\frac{CUV}{1+EV}V = \frac{CUV}{1+EV}v \text{ a.e. in } \Omega.$$

Combining these estimates, we get (3.19). Thus, $I_{\varepsilon,f(u,v)}(R(u,v)) \leq I_{\varepsilon,f(u,v)}(u,v)$, which yields (2.5). Hence, Theorem 12 follows from a direct application of Theorem 2. ∎

Remark 13 Analogous systems with nonquadratic dissipation and energy functionals of the form

$$\begin{aligned}
\varphi^1(u,v) &= \frac{1}{m}\int_\Omega D_1|\nabla u|^m + D_2|\nabla v|^m + F_1|u|^m + F_2|v|^m, \\
\psi(u,v) &= \frac{1}{p}\int_\Omega |u|^p + |v|^p, \qquad m\in(1,\infty),\ p\in(2,\infty)
\end{aligned}$$

can be treated in a similar way (see [1]). The argument may be easily generalized also to systems with nonconstant spatially-dependent coefficients.

4 More examples

The WED variational procedure and its analogous for hyperbolic problems, the *weighted-inertia-energy-dissipation* (WIDE) procedure, have been applied to a larger class of problems including rate-independent systems [26] and some hyperbolic problems [23, 35, 39, 24]. In this section we use these variational approaches to prove a comparison principle for a rate-independent system and to check qualitative properties of solutions of a nonlinear wave equation, and of a lagrangian-mechanics system. It is worth mentioning that the results presented in this section can be widely generalized. In particular, an abstract theory for rate-independent systems in Banach spaces may be developed in the spirit of Section 2.1 and Section 2.2. This, however, is beyond our scope. Moreover, the (relatively simple) examples we present here suffice to provide the main ideas and a guide line for the developing of abstract results in the spirit of what we have done above for doubly-nonlinear problems. In each subsection, we introduce introduce a notion of solution adequate to the type of problem considered and we briefly review the WED or WIDE variational approach in the specific case for the reader's convenience.

4.1 Rate-independent systems

In this section we prove a comparison principle for *energetic solutions* to the following rate-independent inclusion

$$\operatorname{sign}(u') + \tilde{\phi}'(u) - a\Delta u \ni h(t) \text{ in } \Omega \times (0,T), \tag{4.1}$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{4.2}$$

$$u(0) = u_0, \tag{4.3}$$

where $\Omega \subset \mathbb{R}^d$ is bounded with Lipschitz boundary $\partial\Omega$ with outward normal unit vector n, $u_0 \in H^1(\Omega) \cap L^p(\Omega)$ for some $p \geq 2$, $a \geq 0$, $\tilde{\phi} \in C^1(\mathbb{R})$ is assumed to be convex and satisfying

$$\phi(u) \leq C(|u|^p + 1) \text{ for some positive constant } C \text{ and all } u \in \mathbb{R},$$

and $h \in L^\infty(0,T;L^2(\Omega))$. Aiming at applying the WED theory for rate-independent problems developed in [26], we start by rewriting inclusion (4.1)-(4.3) in the form

$$0 \in \partial_{L^2(\Omega)}\psi(u') + \partial_{L^2(\Omega)}\phi(u), \tag{4.4}$$

where

$$\phi(u) = \begin{cases} \displaystyle\int_\Omega \tilde{\phi}(u) + \frac{a}{2}|\nabla u|^2 - hu & \text{if } u \in H^1(\Omega),\ \tilde{\phi}(u) \in L^1(\Omega), \\ +\infty & \text{else,} \end{cases}$$

$\psi(v) = \int_\Omega |v|$, and

$$\partial_{L^2(\Omega)}\psi(v)(x) = \partial_{\mathbb{R}}|v|(x) = \operatorname{sign}(v(x)) = \begin{cases} \{-1\} & \text{for } v(x) \in [-\infty, 0), \\ [-1,1] & \text{for } v(x) = 0, \\ \{1\} & \text{for } v(x) \in (0, +\infty], \end{cases}$$

for a.e. $x \in \Omega$. The abstract inclusion (4.4) arise ubiquitously in applications, from mechanics and electromagnetism to economics (see, e.g, [27]). An elliptic operator as in (4.1) appears frequently in models concerning micromagnetics and plasticity.

The notion of *energetic solutions* to rate-independent systems is given by the following definition.

Definition 14 (Energetic solution) *We define* $u \in \mathrm{BV}([0,T]; L^2(\Omega))$ energetic solution *to the rate independent problem* (4.1)-(4.3) *if it satisfies*

$$\phi(u(t)) \le \phi(w) + \psi(w - u(t)) \text{ for all } w \in L^2(\Omega) \text{ and a.e. } t \in [0,T],$$
$$\phi(u(t)) + \int_0^t \psi(\mathrm{d}u) = \phi(u_0) \text{ for a.e. } t \in [0,T],$$

where $\int_0^t \psi(\mathrm{d}u)$ *is defined by*

$$\int_0^T \psi(\mathrm{d}u) = \sup\left\{ \sum_{j=1}^N \psi(u(s_j) - u(s_{j-1})) : N \in \mathbb{N},\ 0 \le s_1 < ... < s_N \le T \right\}.$$

Existence of energetic solutions to (4.1)-(4.3) is classical and a proof can be found, e.g., in [27]. We remark that solutions are in general not unique.

Our technique is based on the WED approach to rate-independent problems studied in [26]. Thus, before stating our comparison principle, we sketch the results in [26] for the reader's convenience. For every $\varepsilon > 0$ small enough, the functional I_ε defined by

$$I_\varepsilon(u) = \begin{cases} \mathrm{e}^{-T/\varepsilon}\phi(u(T)) + \int_0^T \mathrm{e}^{-t/\varepsilon}\varepsilon\psi(\mathrm{d}u) & \\ \quad + \int_0^T \mathrm{e}^{-t/\varepsilon}\phi(u(t))\mathrm{d}t & \text{if } u \in K(u_0), \\ \quad +\infty & \text{else}, \end{cases}$$

admits a minimizer u_ε over

$$K(u_0) = \{u \in \mathrm{BV}([0,T]; L^2(\Omega)) : u(0) = u_0\}. \tag{4.5}$$

Moreover, for every sequence $\varepsilon_n \to 0$, there exists a (not relabeled) subsequence $\varepsilon_n \to 0$ such that

$$u_{\varepsilon_n} \to u \text{ a.e. in } [0,T] \tag{4.6}$$

and u is an energetic solution to inclusion (4.1)-(4.3).

Taking advantage of the WED approach and arguing as in Theorem 7, we can prove the following comparison principle.

Theorem 15 (Comparison principle for rate-independent systems) *Let* $v_0, u_0 \in D(\phi)$ *be such that* $u_0 \le v_0$ *a.e. in* Ω. *Then, there exist two energetic solutions* u, v *to inclusion* (4.1) *corresponding to initial conditions* $u(0) = u_0$ *and* $v(0) = v_0$ *such that* $u \le v$ *for a.e. in* $\Omega \times (0,T)$.

Proof. For all $\varepsilon > 0$ sufficiently small let u_ε and v_ε be minimizers of I_ε over $K(u_0)$ and $K(v_0)$ respectively. Recalling that $w_1 \vee w_2, w_1 \wedge w_2 \in \mathrm{BV}([0,T]; L^2(\Omega))$ for all $w_1, w_2 \in \mathrm{BV}([0,T]; L^2(\Omega))$, we have that $u_\varepsilon \wedge v_\varepsilon \in K(u_0)$ and $u_\varepsilon \vee v_\varepsilon \in K(v_0)$. Moreover, it is easy to prove that

$$I_\varepsilon(u_\varepsilon \vee v_\varepsilon) + I_\varepsilon(u_\varepsilon \wedge v_\varepsilon) \le I_\varepsilon(u_\varepsilon) + I_\varepsilon(v_\varepsilon).$$

Thus, by applying the abstract comparison principle given by Lemma 9, we deduce that $\tilde{u_\varepsilon} := u_\varepsilon \wedge v_\varepsilon$ and $\tilde{v_\varepsilon} := u_\varepsilon \vee v_\varepsilon$ minimize I_ε over $K(u_0)$ and $K(v_0)$ respectively. Trivially, $\tilde{u_\varepsilon} \le \tilde{v_\varepsilon}$. By using convergence (4.6), we have (up to some not relabeled subsequences) that

$$\begin{aligned}\tilde{u}_\varepsilon &\to u \text{ a.e. in } [0,T],\\ \tilde{v}_\varepsilon &\to u \text{ a.e. in } [0,T],\end{aligned}$$

and u and v are energetic solutions to inclusion (4.1) corresponding to the initial conditions $u(0) = u_0$ and $v(0) = v_0$ respectively. Moreover, we have that $u \le v$ a.e. in $[0,T]$. ■

Here, we have chosen to deal with a simple problem for sake of brevity and simplicity and in order to avoid technicalities. We remark that the results presented in this section can be generalized. In particular, under suitably assumptions on the energy functional ϕ and on the dissipation potential ψ, rate-independent problems on abstract Banach spaces can be treated similarly in the spirit of Section 2 (see [26] for the WED approach).

4.2 Nonlinear wave equation

In this section we deal with the hyperbolic problem given by

$$\rho u_{tt} + \nu u_t - \Delta u + F'(u) = 0 \text{ in } \Omega \times (0,T), \tag{4.7}$$

$$u(0) = u_0,\ u_t(0) = v_0, \tag{4.8}$$

where $\rho > 0$, $\nu \ge 0$ are constants, formulated in a bounded or unbounded domain Ω and coupled with different types of boundary conditions. We consider initial data $u_0, v_0 \in H^1(\Omega) \cap L^p(\Omega)$ for some $p \ge 2$. We restrict ourself to the case of $0 < T < +\infty$ for simplicity, although the case of unbounded time intervals (i.e., $T = +\infty$) can be treated analogously (see [35] for the WIDE procedure in this case). We assume i) $\Omega \subset \mathbb{R}^d$ to be nonempty, open, and Lipschitz and that the problem is coupled with Dirichlet or Neumann boundary conditions, or ii) $\Omega = \mathbb{T}^d$, where $\mathbb{T}^d = [0, 2\pi)^d$ is the d-dimensional flat torous, together with periodic boundary conditions, or iii) $\Omega = \mathbb{R}^d$. Moreover, let $F \in C^1(\mathbb{R})$ be λ-convex for some $\lambda \in \mathbb{R}$ and let exist $C > 0$ such that

$$\frac{1}{C}|s|^p - C \le F(s),\ |F'(s)|^{p'} \le C(1 + |s|^p).$$

Furthermore, if $\Omega = \mathbb{R}^d$, we ask $F(s) = |s|^p$ and $\nu = 0$. We remark here that our assumptions on F are chosen in accordance with [39] and [23] where the WIDE approach for system (4.7)-(4.8) is discussed. We note however that slightly more general conditions on F, can be considered without additional difficulties. Examples may be space dependent potentials F or some compact nonlocal operators, e.g., $F'(u)$ replaced by $F'(u) + J * u$ where J is a sufficiently regular positive kernel and * denotes the convolution product.

We are interested in weak solutions to the Cauchy problem (4.7)-(4.8) with regularity $u \in Q$, where

$$Q = H^2(0,T;L^2(\Omega)) \cap L^2(0,T;X) \cap L^p(0,T;L^p(\Omega)).$$

Here $X = H^1_0(\Omega)$ in the case of bounded domain Ω and Dirichlet boundary conditions, $X = H^1(\mathbb{T}^d)$ in the case of periodic boundary conditions, and $X = H^1(\Omega)$ in the case of Neumann boundary conditions or $\Omega = \mathbb{R}^d$. In order to prove qualitative properties for solutions to equation (4.7)-(4.8), we follow the same idea presented in the above sections. To this aim, we now illustrate the WIDE approach to problem (4.7)-(4.8) for the reader's convenience. The following result was first conjectured by De Giorgi (in the case $\nu = 0$) [13] and than proved in [23, 39] (see also [35, 36]). The WIDE functional $I_\varepsilon : Q \to \mathbb{R} \cup \{\infty\}$, defined by

$$I_\varepsilon(u) = \int_0^T \int_\Omega e^{-t/\varepsilon} \left(\frac{\varepsilon^2 \rho}{2} |u''|^2 + \frac{\varepsilon \nu}{2} |u'|^2 + \frac{1}{2} |\nabla u|^2 + F(u) \right) \mathrm{d}x\mathrm{d}t$$

admits a unique minimizer u_ε over the set

$$K(u_0, v_0) = \{u \in Q : u(0) = u_0, \partial_t u(0) = v_0\}$$

for every $\varepsilon > 0$ sufficiently small. Furthermore, up to (not relabeled) subsequences

$$u_\varepsilon(t) \to u(t) \text{ pointwise a.e. in } \Omega \text{ for all } t \in [0,T] \tag{4.9}$$

and the limit u is a weak solution to equation (4.7)-(4.8). We recall that uniqueness for p large is an open problem.

Taking advantage of this variational procedure, we can prove some symmetries for solutions to equation (4.7)-(4.8), in the spirit of Section 3. To this aim, we introduce maps R which describe symmetries and we fix some compatibility conditions on the problem's data.

1. *Linear rigid transformation*: $Ru(x) = u(rx)$, where $r \in SL(d, \mathbb{R})$ and $r\Omega = \Omega$.
2. *Translation*: $Ru(x) = u(x + \tilde{x})$ for some $\tilde{x} \in \mathbb{R}^d$ and $\Omega = \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$.
3. *Averaging in the direction* $y \in \mathbb{R}^d$: given $y \in \mathbb{R}^d$ such that $|y| = 1$, decompose every $x \in \mathbb{R}^d$ as $x = \alpha y + x_2$ where $\alpha \in \mathbb{R}$ and x_2 is orthogonal to y. Assume $\Omega = \{\alpha y + x_2 \in \mathbb{R}^d : \alpha \in (0, L),\ x_2 \in \Omega'\}$, where $L > 0$ and $\Omega^{'}$ is a subset of $\mathbb{R}^{d-1}$ (note that $\Omega = \mathbb{T}^d$ satisfies this assumption with $L = 2\pi$ and $\Omega' = \mathbb{T}^{d-1}$). Moreover, let F be convex. Define $Ru(x) = \frac{1}{L}\int_0^L u(sy + x_2)\mathrm{d}s$.

Remark 16 We observe that invariance under the action of a map R as in 2) implies periodicity in the direction of $\frac{\tilde{x}}{|\tilde{x}|}$ with period $|\tilde{x}|$. Functions u which are invariant under the action of R as in 3) are instead constant in the direction y.

We now prove existence of invariant solutions to equation (4.7)-(4.8).

Theorem 17 (Semilinear wave equations) *Let $u_0, v_0 \in X$. Let $\nu \geq 0$ and R_i, $i \in \{1, ..., k\}$, be any collection of the above maps. Assume $R_i u_0 = u_0$ and $R_i v_0 = v_0$ for all $i \in \{1, ..., k\}$. Then, there exists a weak solution u to equation (4.7)-(4.8) such that $u = R_1 \circ ... \circ R_k u$.*

Proof. As a direct consequence of the assumptions on the initial data, for any $i \in \{1, ..., k\}$, we have that $R_i u \in K(u_0, v_0)$ for all $u \in K(u_0, v_0)$. Moreover, one can easily prove that $I_\varepsilon(R_i u) \leq I_\varepsilon(u)$ for all $u \in K(u_{0,} v_0)$. Let $u_\varepsilon \in K(u_0, v_0)$ be the unique minimizer of I_ε over $K(u_0, v_0)$. By uniqueness, we deduce invariance $R_i u_\varepsilon = u_\varepsilon$ for all $i \in \{1, ..., k\}$. In particular, u_ε is $R_1 \circ ... \circ R_k$-invariant. Using convergence (4.9), we extract a subsequence $\varepsilon_n \to 0$ such that $u_{\varepsilon_n} \to u$ pointwise a.e. in $\Omega \times (0, T)$. Then, for a.a. $(x, t) \in \Omega \times (0, T)$, $u(x, t) = \lim_{\varepsilon_n \to 0} u_\varepsilon(x, t) = \lim_{\varepsilon_n \to 0} R_1 \circ ... \circ R_k u_\varepsilon(x, t) = R_1 \circ ... \circ R_k u(x, t)$. ■

Remark 18 Note that invariance under rearrangement transformations cannot be expected here. Indeed, monotonicity properties do not hold true for solutions to the wave equation. Similarly, we cannot apply the same idea to truncation maps R of the form $R(u) = \pm(u - M)^+ + M$ as comparison principles (with constant functions) are in general false for the wave equation. Moreover, $K(u_0, v_0)$ is a subset of $H^2(0, T; L^2(\Omega))$. Thus, $Ru \in K(u_0, v_0)$ can not be expected for every $u \in K(u_0, v_0)$ and R a rearrangement or truncation operator.

4.3 Lagrangian mechanics

Consider now the Lagrangian system

$$Mu_{tt} + \nu u_t + \nabla U(u) = 0 \text{ in } (0, T), \tag{4.10}$$
$$u(0) = u_0,\ u_t(0) = v_0, \tag{4.11}$$

where $u : (0, T) \to \mathbb{R}^d$, M is a positive definite $d \times d$ matrix, $\nu \geq 0$, and $U \in C^1(\mathbb{R}^d, \mathbb{R})$ is convex and bounded from below. We summarize here the WIDE approach to system (4.10)-(4.11) studied in [24]. For every $\varepsilon > 0$, the functional

$$I_\varepsilon(u) = \int_0^T e^{-t/\varepsilon} \left(\frac{\varepsilon^2}{2} u'' \cdot Mu'' + \frac{\varepsilon\nu}{2} |u'|^2 + U(u) \right) \mathrm{d}t$$

admits a unique minimizer over the set $K(u_0, v_0) = \{u \in H^2([0, T]; \mathbb{R}^d) : u(0) = u_0,\ \partial_t u(0) = v_0\}$. Moreover, there exists a (not relabeled) subsequence $\varepsilon \to 0$ such that $u_\varepsilon \to u$ pointwise a.e. in $(0, T) \times \mathbb{R}^d$ and u is a strong solution to system (4.10)-(4.11). We recall that solutions to (4.10)-(4.11) are, in general, not unique. Take, e.g., $d = 1$ and $U(s) = (s^+)^{3/2}$.

Taking advantage of these results, by following the idea presented in the previous sections, we can prove existence of solutions to system (4.10)-(4.11) invariant under the action of maps R defined as follows. Let $r \in M(\mathbb{R}^{d \times d})$ be such that $r^T r = 1$, $v \in \mathbb{R}^d$, and assume $U(ru + v) \leq U(u)$ (e.g., $U(u) = V(|u|)$ for some $V \in C^1(\mathbb{R}^+)$ if $v = 0$). Define $Ru = ru + v$.

Arguing as in Theorem 17, one can prove the following.

Theorem 19 (Lagrangian mechanics) *Let u_0, v_0, and R be such that $Ru_0 = u_0$ and $Rv_0 = v_0$. Then, there exists a solution u to* (4.10)-(4.11) *such that $u = Ru$.*

5 Appendix, rearrangement maps

We recall here the definitions and some basic properties of rearrangement maps, for the reader's convenience. For a fuller treatment, we refer the reader to [20].

Rearrangement maps transform a given function u into a new function u^* that has some desired property, e.g., symmetry. This is done by a rearrangement of the level sets of the function. Thus, in order to define rearrangement maps, we start by introducing some rearrangements of measurable sets $\Omega \subset \mathbb{R}^d$ of finite Lebesgue measure.

1. The *symmetric rearrangement* $\Omega^* = \{x \in \mathbb{R}^d : |x| < r\}$, where r is such that $|\Omega| = |\Omega^*|$.

2. The *symmetric rearrangement with respect to a hyperplane*. Let $H \subset \mathbb{R}^d$ be a m-dimensional hyperplane. Then, we decompose every $x \in \mathbb{R}^d$ as $x = x_1 + x_2$, $x_1 \in H$, $x_2 \in H^\perp$ and define $\Omega^{*,H} = \{x_1 + x_2 : x_1 \in (\Omega - x_2 \cap H)^* \text{ and } x_2 \in P^\perp(\Omega)\}$, where $\Omega - x_2 = \{x - x_2 : x \in \Omega\}$, $(\Omega - x_2 \cap H)^*$ denotes the m-dimensional symmetric rearrangement of $(\Omega - x_2 \cap H)$, and $P^\perp : \mathbb{R}^d \to H^\perp$ denotes the usual orthogonal-projection map.

3. The *monotone rearrangement with respect to the direction $y \in \mathbb{R}^d$*. Given $y \in \mathbb{R}^d$ such that $|y| = 1$, we decompose every $x \in \mathbb{R}^d$ as $x = \alpha y + x_2$ where $\alpha \in \mathbb{R}$ and x_2 is orthogonal to y. We define $\Omega^{*,y} = \{\alpha y + x_2 : 0 \leq \alpha < \mathcal{H}^1(\Omega \cap L(x_2)) \text{ and } x_2 \in P^\perp(\Omega)\}$, where $L(x_2) = \{\beta y + x_2 : \beta \in \mathbb{R}\}$, $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure and $P^\perp : \mathbb{R}^d \to \{y\}^\perp$ denotes the orthogonal projection on the subspace orthogonal to y.

Note that the rearrangements defined above are area preserving, i.e., $|\Omega| = |\Omega^*| = |\Omega^{*,H}| = |\Omega^{*,y}|$.

We now consider functions $u : \Omega \subseteq \mathbb{R}^d \to [0, \infty)$ such that $|\{x \in \Omega : u(x) > t\}|$ is finite for all $t > 0$, and we define rearrangement maps as follows.

1. The *symmetric decreasing rearrangement* $u^* : \Omega^* \to [0, \infty)$, $u^*(x) := \sup\{c \in \mathbb{R} : x \in \{u > c\}^*\}$ or equivalently $u^*(x) = \int_0^\infty \chi_{\{u>t\}^*}(x)\mathrm{d}t$.

2. The *symmetric decreasing rearrangement with respect to the hyperplane* $H \subset \mathbb{R}^d$ $u^{*,H} : \Omega^{*,H} \to [0, \infty)$, $u^*(x) = \int_0^\infty \chi_{\{u>t\}^{*,H}}(x)\mathrm{d}t$.

3. The *monotone decreasing rearrangement with respect to the direction* $y \in \mathbb{R}^d$ $u^{*,y} : \Omega^{*,y} \to [0, \infty)$, $u^*(x) = \int_0^\infty \chi_{\{u>t\}^{*,y}}(x)\mathrm{d}t$.

We note that, by a direct consequence of the definition, rearranged functions are measurable and lower semicontinuous. Moreover, their level sets are rearrangements of the level sets of u. We now recall some known properties of rearrangement maps.

Lemma 20 (Rearrangement inequalities) *Let $R\Omega$ be one of the rearrangement of the set Ω defined above and Ru be the corresponding rearrangement of the function $u : \Omega \to [0, \infty)$. Then, the following inequalities hold true.*

1. Conservation of L^p-norms*:* $\|u\|_{L^p(\Omega)} = \|Ru\|_{L^p(R\Omega)}$ *for all* $p \in [1, \infty]$*,* $u \in L^p(\Omega)$*.*
2. $\int_\Omega uv \le \int_{R\Omega} (Ru)\,(Rv)$ *for all* $u \in L^p(\Omega)$*,* $v \in L^{p/(p-1)}(\Omega)$*,* $p \in [1, \infty]$*.*
3. Nonexpansivity of rearrangements*:* $\int_{R\Omega} J(Ru - Rv) \le \int_\Omega J(u - v)$ *for all* $J : \mathbb{R} \to \mathbb{R}$ *nonnegative, convex, and such that* $J(0) = 0$*.*
4. Pólya-Szegö inequality*:* $\|\nabla u\|_{L^p(\Omega)} \ge \|\nabla(Ru)\|_{L^p(R\Omega)}$ *for all* $p \in [1, \infty]$*,* $u \in W^{1,p}(\Omega)$*. In particular,* $Ru \in W^{1,p}(R\Omega)$ *for all* $u \in W^{1,p}(\Omega)$*.*
5. Fractional Pólya-Szegö inequality [33]*:* $[u]_{s,\mathbb{R}^d} \ge [u^*]_{s,\mathbb{R}^d}$ *for all* $s \in (0, 1)$*,* $u \in H^s(\mathbb{R}^d)$*. Here,* $[u]_{s,\mathbb{R}^d}$ *denotes the usual s-Gagliardo seminorm and* $H^s(\mathbb{R}^d) = W^{s,2}(\mathbb{R}^d)$ *the usual fractional Sobolev space* [16]*. In particular,* $u^* \in H^s(\mathbb{R}^d)$ *for all* $u \in H^s(\mathbb{R}^d)$*.*

The above inequality are well known. We refer to [20] for a proof of 1-4 and to [33] for a proof of 5.

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