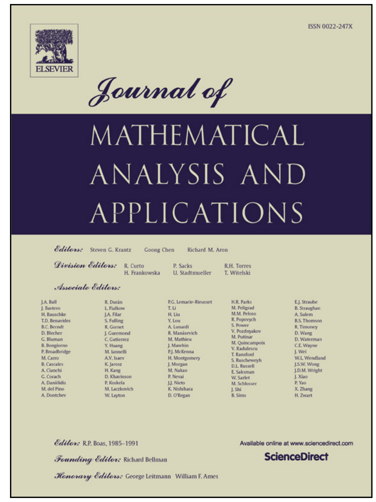

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SPECTRALITY OF CERTAIN MORAN MEASURES WITH THREE-ELEMENT DIGIT SETS

ZHI-YONG WANG, XIN-HAN DONG*, AND ZONG-SHENG LIU

ABSTRACT. Let $\mathcal{D}_n = \{0, a_n, b_n\} = \{0, 1, 2\}(mod\ 3)$, $p_n \in 3\mathbb{Z}^+$, $n \geq 1$, satisfy $\sup_{n\geq 1} \frac{\max\{|a_n|,|b_n|\}}{p_n} < \infty$. It is well-known that there exists a unique Borel probability measure $\mu_{\{p_n\},\{\mathcal{D}_n\}}$ generated by the following infinite convolution product

$$\mu_{\{p_n\},\{\mathcal{D}_n\}} = \delta_{p_1^{-1}\mathcal{D}_1} * \delta_{(p_1p_2)^{-1}\mathcal{D}_2} * \cdots$$

in the weak convergence. In this paper, we give some conditions to ensure that there exists a discrete set Λ such that the exponential function system $\{e^{2\pi i\lambda x}\}_{\lambda\in\Lambda}$ forms an orthonormal basis for $L^2(\mu_{\{p_n\},\{\mathcal{D}_n\}})$.

CONTENTS

1. Introduction

Let μ be a probability measure with compact support on $\mathbb{R}^n$. We call it a spectral measure if there exists a discrete set $\Lambda \subset \mathbb{R}^n$ such that $E_\Lambda := \{e^{2\pi i\langle\xi,\lambda\rangle} : \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$. The set Λ is then called a spectrum for μ. The existence of spectrum for μ was initiated by Fuglede in his seminal paper [11]. The first example of a singular, non-atomic, spectral measure was constructed by Jorgensen and Pedersen in [14]. This surprising discovery received a lot of attention. The spectral property of fractal measures becomes an active research area, and more spectral fractal measures were found in [15, 19] and [1–3, 5, 9, 12, 13, 17]. The spectral property, Fourier transform [16, 21, 22] and Cauchy transform [6–8, 18] of fractal measure form the main topics in the analysis on fractals.

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In [19] and [1,4,13], the authors constructed some classes of Moran spectral measures. Motivated by their work, we focus on certain Moran measures with three-element digit sets. Before the statement of the main results, we first give some definitions and notations. Let

$$\mathcal{D}_n = \{0, a_n, b_n\} = \{0, 1, 2\}(mod\ 3) \quad \text{and} \quad p_n \in 3\mathbb{Z}^+ \tag{1.1}$$

be a digit set in $\mathbb{Z}$ with $|a_n| < |b_n|$ and an integer for all $n \geq 1$, respectively. In this paper, we always assume that

$$c := \limsup_{n\to\infty} \frac{|a_n|}{p_n} < \infty, \qquad d := \limsup_{n\to\infty} \frac{|b_n|}{p_n} < \infty. \tag{1.2}$$

Write $P_n = p_1 p_2 \cdots p_n$. Under the assumptions of (1.1) and (1.2), then associated to the sequence $\{p_n, \mathcal{D}_n\}$, there exists a Borel probability measure $\mu_{\{p_n\},\{\mathcal{D}_n\}}$, which is defined by the following infinite convolutions of finite measures:

$$\mu_{\{p_n\},\{\mathcal{D}_n\}} = \delta_{P_1^{-1}\mathcal{D}_1} * \delta_{P_2^{-1}\mathcal{D}_2} * \cdots \tag{1.3}$$

in the weak convergence, where $*$ is the convolution sign, $\delta_E = \frac{1}{\#E}\sum_{e\in E}\delta_e$, $\#E$ is the cardinality of a set E and δ_e is the Dirac measure at the point $e \in \mathbb{R}$. It is known that the support of $\mu_{\{p_n\},\{\mathcal{D}_n\}}$ is the Moran set:

$$T(\{p_n\}, \{\mathcal{D}_n\}) = \sum_{n=1}^{\infty} P_n^{-1}\mathcal{D}_n.$$

Strichartz [19] first studied the spectrality of $\mu_{\{R_n\},\{B_j\}}$ where expanding matrices $\{R_j\}_{j=1}^{\infty}$ and digit sets $\{B_j\}_{j=1}^{\infty}$ satisfy $\#\{R_1, R_2, \cdots\}$, $\#\{B_1, B_2, \cdots\} < \infty$. In [5], Ding considered the case p_n and $\mathcal{D}_n$ defined by (1.1) such that $\mathcal{D}_n \subset \mathbb{N}$, $\#\{a_n, b_n : n \geq 1\} < \infty$ and $\gcd(a_n, b_n) = 1$ for $n \geq 1$. In this paper, we will study the spectrality of $\mu_{\{p_n\},\{\mathcal{D}_n\}}$ defined by (1.3) without the boundedness assumptions : $\#\{a_n, b_n : n \geq 1\} < \infty$.

Theorem 1.1. *Let $\mu_{\{p_n\},\{\mathcal{D}_n\}}$ be defined by* (1.3) *where p_n and $\mathcal{D}_n$ satisfy* (1.1)-(1.2). *Suppose that one of the following conditions holds:*

(a) $\liminf_{n\to\infty} \frac{|a_n|}{p_n} < \min\{\frac{2}{3}, \frac{2}{3c}\}$;

(b) $1 \leq a_n < b_n$ *for all* $n \geq 1$, *and* $\liminf_{n\to\infty} \frac{b_n}{p_n} < \min\{1, \frac{1}{d}\}$.

Then $\mu_{\{p_n\},\{\mathcal{D}_n\}}$ is a spectral measure with spectrum $\Lambda = \bigcup_{n=1}^{\infty}\left(\sum_{i=1}^{n} P_i\{0, u_i/3, v_i/3\}\right)$ for some choice $\{u_i\}, \{v_i\}$ satisfying $u_i, v_i \in \mathbb{Z}\setminus 3\mathbb{Z}$ and $u_i \neq v_i(mod\ 3)$ for all i, where $P_i = \prod_{k=1}^{i} p_k$.

From Theorem 1.1, we have the following corollary.

Corollary 1.2. *Let $\mu_{\{p_n\},\{\mathcal{D}_n\}}$ be defined by* (1.3) *where p_n and $\mathcal{D}_n$ satisfy* (1.1)-(1.2). *If $c < 2/3$ (or $d < 1$ and $1 \leq a_n < b_n$), then $\mu_{\{p_n\},\{\mathcal{D}_n\}}$ is a spectral measure.*

Remark 1.3. *Ding* [5] *proved that* $\mu_{\{p_n\},\{\mathcal{D}_n\}}$ *is a spectral measure if* p_n, $\mathcal{D}_n$ *are defined by* (1.1) *with* $a_n, b_n \geq 1$, $\gcd(a_n, b_n) = 1$ *for* $n \geq 1$, *and* $\sup_{n\geq 1} \max\{a_n, b_n\} < \infty$. *Its proof is divided into two cases: (i)* $\#\{p_n : n \geq 1\} < \infty$*; (ii)* $\#\{p_n : n \geq 1\} = \infty$. *In case (i), it follows from* [19, Theorem 2.8] *that* $\mu_{\{p_n\},\{\mathcal{D}_n\}}$ *is a spectral measure; for case (ii), we have* $\liminf\limits_{n\to\infty} \frac{a_n}{p_n} = 0$ *as* $\{a_n\}$ *is bounded, which is a special case of* Theorem 1.1(a)*, and hence* $\mu_{\{p_n\},\{\mathcal{D}_n\}}$ *is a spectral measure. Obviously, our theorem allows* $\sup_{n\geq 1} \max\{a_n, b_n\} = \infty$ *and* $\gcd(a_n, b_n) \neq 1$ *for* $n \geq 1$.

In the following, we construct an example to illustrate that the conditions (a) and (b) in Theorem 1.1 are not redundant.

Remark 1.4. *Let* $\mathcal{D}_1 = \{0, 1, 2\}$, $\mathcal{D}_m = \{0, 2, 4\}$ *for* $m \geq 2$, *and let* $p_n = 3$ *for* $n \geq 1$. *Then* $c = \lim\limits_{n\to\infty} \frac{a_n}{p_n} = \frac{2}{3}$ *and* $d = \lim\limits_{n\to\infty} \frac{b_n}{p_n} = \frac{4}{3}$. *Hence* $\{p_n\}$ *and* $\{\mathcal{D}_n\}$ *satisfy* (1.1)-(1.2)*, but do not satisfy the condition (a) or (b) in* Theorem 1.1. *However,* $\mu_{\{p_n\},\{\mathcal{D}_n\}}$ *is not a spectral measure. This follows from* [10, Corollary1.4] *and*

$$\mu_{\{p_n\},\{\mathcal{D}_n\}} = \delta_{3^{-1}\mathcal{D}_1} * \frac{3}{2}\mathcal{L}|_{[0,2/3]} = \frac{1}{2}\mathcal{L}|_{[0,1/3]\cup[1,4/3]} + \mathcal{L}|_{[1/3,1]}$$

as $\delta_{3^{-2}\mathcal{D}_2} * \delta_{3^{-3}\mathcal{D}_3} * \cdots = \frac{3}{2}\mathcal{L}|_{[0,2/3]}$.

The organization of the paper is as follows. In the next section, we summarize some of the definitions and preliminary results on Fourier transform of $\mu_{\{p_n\},\{\mathcal{D}_n\}}$. In Section 3, we prove Theorem 1.1.

2. **Preliminaries**

Let μ be a probability measure with compact support on $\mathbb{R}$. The Fourier transform of μ is defined by $\hat{\mu}(\xi) = \int e^{-2\pi i\xi x} d\mu(x)$. Then for our case, we have

$$\hat{\mu}_{\{p_n\},\{\mathcal{D}_n\}}(\xi) = \prod_{n=1}^{\infty} M_{a_n,b_n}(P_n^{-1}\xi), \tag{2.1}$$

where

$$M_{a_n,b_n}(\xi) = \frac{1}{3}(1 + e^{-2\pi i a_n\xi} + e^{-2\pi i b_n\xi}) \tag{2.2}$$

is known as the mask polynomial of $\mathcal{D}_n$. We use $\mathcal{Z}_h := \{\xi : h(\xi) = 0\}$ to denote the zero set of the function h. It is easy to see from $\{a_n, b_n\} = \{1, 2\}(mod\ 3)$ and (2.1)-(2.2) that

$$\mathcal{Z}_{M_{a_i,\,b_i}} = \frac{1}{3\gcd(a_i, b_i)}(\mathbb{Z} \setminus 3\mathbb{Z}) \quad \text{and} \quad \mathcal{Z}_{\hat{\mu}_{\{p_n\},\,\{\mathcal{D}_n\}}} = \bigcup_{i=1}^{\infty} \frac{P_i}{3\gcd(a_i, b_i)}(\mathbb{Z} \setminus 3\mathbb{Z}). \tag{2.3}$$

We say that Λ is an *orthogonal set* for μ if E_Λ is an orthonormal family for $L^2(\mu)$. It is easy to show that Λ is an orthogonal set for μ if and only if $\hat{\mu}(\lambda_i - \lambda_j) = 0$ for any $\lambda_i \neq \lambda_j \in \Lambda$, which is equivalent to

$$(\Lambda - \Lambda) \setminus \{0\} \subseteq \mathcal{Z}_{\hat{\mu}}. \tag{2.4}$$

Let $Q_\Lambda(\xi) = \sum_{\lambda\in\Lambda} |\hat{\mu}(\xi + \lambda)|^2$. Using the Parseval identity, we have the following basic criterion for orthogonality of E_Λ in $L^2(\mu)$.

Proposition 2.1. [14] *Let μ be a probability measure in $\mathbb{R}^n$ with compact support, and let $\Lambda \subset \mathbb{R}^n$ be a countable subset. Then*
(i) Λ is an orthonormal set for μ if and only if $Q_\Lambda(\xi) \leq 1$ for $\xi \in \mathbb{R}^n$; and
(ii) Λ is a spectrum for μ if and only if $Q_\Lambda(\xi) \equiv 1$ for $\xi \in \mathbb{R}^n$.

In this paper, we write

$$\mu_n = \delta_{P_1^{-1}\mathcal{D}_1} * \cdots * \delta_{P_n^{-1}\mathcal{D}_n} \quad \text{and} \quad \mu_{>n} = \delta_{P_{n+1}^{-1}\mathcal{D}_{n+1}} * \delta_{P_{n+2}^{-1}\mathcal{D}_{n+2}} * \cdots . \tag{2.5}$$

Then $\mu_{\{p_n\},\{\mathcal{D}_n\}} = \mu_n * \mu_{>n}$. We say $u \in (\mathbb{Z} \setminus 3\mathbb{Z})^{\mathbb{N}}$ if $u = \{u_i\}_{i=1}^\infty$ with all $u_i \in \mathbb{Z} \setminus 3\mathbb{Z}$. For any $u, v \in (\mathbb{Z} \setminus 3\mathbb{Z})^{\mathbb{N}}$ with all $u_i \neq v_i (mod\ 3)$, we write

$$\Lambda_n^{u,v} := \sum_{i=1}^{n} P_i\{0, u_i/3, v_i/3\} \quad \text{and} \quad \Lambda^{u,v} := \bigcup_{n=1}^{\infty} \Lambda_n^{u,v}. \tag{2.6}$$

The following lemma is used to illustrate that $\Lambda^{u,v}$ is an orthogonal set for $\mu_{\{p_n\},\{\mathcal{D}_n\}}$.

Lemma 2.2. *Let p_n and $\mathcal{D}_n$ be defined by* (1.1). *Then the set $\Lambda_k^{u,v}$ is a spectrum for μ_k, and $\Lambda^{u,v}$ is an orthogonal set for $\mu_{\{p_n\},\{\mathcal{D}_n\}}$ for any $u, v \in (\mathbb{Z} \setminus 3\mathbb{Z})^{\mathbb{N}}$ with $u_i \neq v_i (mod\ 3)$ for all i.*

Proof. From (2.3) and (2.4), it is easy to check that $\Lambda_k^{u,v}$ and $\Lambda^{u,v}$ are orthogonal sets for μ_k and $\mu_{\{p_n\},\{\mathcal{D}_n\}}$, respectively. Moreover, $\Lambda_k^{u,v}$ is a spectrum for μ_k because the dimension of the space $L^2(\mu_k)$ is 3^k, which is the cardinality of the set $\Lambda_k^{u,v}$. □

3. **Proof of Theorem 1.1**

We first state a proposition, which is weaker than Theorem 1.1. Indeed, we only use " sup " instead of " lim sup " in the hypotheses of Theorem 1.1.

Proposition 3.1. *Let $\mu_{\{p_n\},\{\mathcal{D}_n\}}$ be defined by* (1.3) *where p_n and $\mathcal{D}_n$ satisfy* (1.1) *and $\tilde{c} := \sup_{n\geq 1} \frac{|a_n|}{p_n} \leq \tilde{d} := \sup_{n\geq 1} \frac{|b_n|}{p_n} < \infty$. In addition, suppose that one of the following conditions holds: (i) $\liminf_{n\to\infty} \frac{|a_n|}{p_n} < \min\{\frac{2}{3}, \frac{2}{3\tilde{c}}\}$; (ii) $0 < a_n < b_n$ for all $n \geq 1$ and $\liminf_{n\to\infty} \frac{b_n}{p_n} < \min\{1, \frac{1}{\tilde{d}}\}$. Then $\mu_{\{p_n\},\{\mathcal{D}_n\}}$ is a spectral measure.*

To prove Proposition 3.1, we need the following lemmas. Noting that $p_n \in 3\mathbb{Z}^+$, we have $P_n = p_1 p_2 \cdots p_n \geq 3^n$, which will be used many times in the rest of this paper. We use the notation $\lceil x \rceil$ to denote the minimum integer greater than or equal to $x \in \mathbb{R}$, i.e., $\lceil x \rceil := \min\{n \in \mathbb{Z} : n \geq x\}$, and define a positive integer

$$N_0 := \lceil 3 + \log(\max\{1, \tilde{d}\})/\log 3 \rceil. \tag{3.1}$$

Lemma 3.2. *With the hypotheses of Proposition 3.1(i), let $\mu_{>n}$ be defined by* (2.5) *for $n \geq 1$ and $u = -v = \{1\}_{i=1}^{\infty}$. Then there exist an increasing positive integer sequence $\{n_k\}_{k=1}^{\infty}$ with $n_k - n_{k-1} \geq N_0$, and a constant $\alpha > 0$ such that*

(i) $|\hat{\mu}_{>n_k}(\xi + \lambda)| \geq \alpha$ for all $\lambda \in \Lambda_{n_k}^{u,v}$, $k \geq 1$ and $|\xi| \leq 1$;

(ii) $|\hat{\mu}_{>n_k}(\xi + \lambda)| \geq \exp\left(-9^{N_0+1-n_k+n_{k-1}}\right)$ for all $\lambda \in \Lambda_{n_{k-1}}^{u,v}$, $k > 1$ and $|\xi| \leq 1$.

Proof. For fixed $r \in (\liminf_{n\to\infty} \frac{|a_n|}{p_n},\ \min\{\frac{2}{3}, \frac{2}{3\tilde{c}}\})$, we choose a positive integer l such that

$$\frac{3^l + 2}{2 \cdot 3^l} < \min\{\frac{1}{3r}, \frac{1}{3\tilde{c}r}\}. \tag{3.2}$$

We can find an increasing sequence $\{n_k\}_{k=1}^{\infty}$ such that

$$n_k \geq l, \quad n_{k+1} - n_k \geq N_0, \quad \frac{|a_{n_k+1}|}{p_{n_k+1}} \leq r, \qquad \forall\ k \geq 1. \tag{3.3}$$

For convenience, we write

$$J_j(\xi + \lambda) := |M_{a_j,b_j}(P_j^{-1}(\xi + \lambda))| = \frac{1}{3}\left|1 + e^{-2\pi i P_j^{-1} a_j(\xi+\lambda)} + e^{-2\pi i P_j^{-1} b_j(\xi+\lambda)}\right|$$

and

$$\eta_{j,1} = \eta_{j,1}(\xi + \lambda) = -\frac{a_j(\xi + \lambda)}{P_j}, \qquad \eta_{j,2} = \eta_{j,2}(\xi + \lambda) = -\frac{b_j(\xi + \lambda)}{P_j}. \tag{3.4}$$

It follows from (2.5) that $|\hat{\mu}_{>n_k}(\xi + \lambda)| = \prod_{j=n_k+1}^{\infty} J_j(\xi + \lambda)$. Now, we estimate $J_j(\xi + \lambda)$ for $j \geq n_k + 1$. For

$$\lambda \in \Lambda_n^{u,v} = \sum_{i=1}^{n} P_i\{0, 1/3, -1/3\} \ \ (\text{see (2.6) with } u = -v = \{1\}_{i=1}^{\infty}), \tag{3.5}$$

we can find $\{c_i\}_{i=1}^{n} \in \{0, 1, -1\}^n$ such that $\lambda = 3^{-1}\sum_{i=1}^{n} c_i P_i$, and hence $|\lambda| \leq 3^{-1}\sum_{i=1}^{n} P_i$. Noting that $p_n \in 3\mathbb{Z}^+$ and $\frac{P_i}{P_j} = (\prod_{\ell=i+1}^{j} p_\ell)^{-1} \leq 3^{-j+i}$ for $i < j$, we have

$$\frac{|\xi + \lambda|}{P_{n_j}} \leq \frac{|\xi|}{3^{n_j}} + \frac{\sum_{i=1}^{n_j} P_i}{3P_{n_j}} \leq \frac{1}{3^{n_j}} + \sum_{i=1}^{n_j} \frac{1}{3^i} \leq \frac{1}{3^l} + \frac{1}{2}, \quad \lambda \in \Lambda_{n_j}^{u,v},\ |\xi| \leq 1. \tag{3.6}$$

Combining (3.3) and (3.6),

$$\begin{cases} |\eta_{n_k+1,1}(\xi+\lambda)| = \frac{|a_{n_k+1}|}{p_{n_k+1}}\frac{|\xi+\lambda|}{P_{n_k}} \le r\left(\frac{1}{3^l}+\frac{1}{2}\right), \\ |\eta_{j,1}(\xi+\lambda)| = \frac{|a_j|}{p_j}\,\frac{|a_{n_k+1}(\xi+\lambda)|}{P_{n_k+1}}\,\frac{1}{|a_{n_k+1}|}\,\frac{P_{n_k+1}}{P_{j-1}} \le \tilde{c}r\left(\frac{1}{3^l}+\frac{1}{2}\right)\frac{1}{3^{j-n_k-2}}, \quad j\ge n_k+2, \end{cases} \tag{3.7}$$

where $\sup_{n\ge 1}\frac{|a_n|}{p_n}=\tilde{c}$. Put $\varepsilon=\frac{1}{3}-(\frac{1}{3^l}+\frac{1}{2})\max\{\tilde{c}r, r\}$, (3.2) implies $\varepsilon\in(0,\frac{1}{3})$. By (3.7),

$$(\eta_{j,1}(\xi+\lambda),\eta_{j,2}(\xi+\lambda))\in[-\frac{1}{3}+\epsilon,\frac{1}{3}-\epsilon]\times\mathbb{R}=\cup_{k\in\mathbb{Z}}(\Omega+(0,k)) \tag{3.8}$$

for $|\xi|\le 1,\ \lambda\in\Lambda_{n_k}^{u,v}$ and $j\ge n_k+1$, where $\Omega:=[-\frac{1}{3}+\varepsilon,\ \frac{1}{3}-\varepsilon]\times[0,1]$. We consider the continuous function

$$f(x,y):=\frac{1}{3}|1+e^{2\pi ix}+e^{2\pi iy}|,\quad (x,y)^t\in\mathbb{R}^2.$$

It is easy to see that $f(\eta_{j,1},\eta_{j,2})=J_j(\xi+\lambda)$ and $\mathcal{Z}_f=\{(m,2m+k)/3: m\in\mathbb{Z}\setminus 3\mathbb{Z}, k\in 3\mathbb{Z}\}$, where $\mathcal{Z}_f$ is the zero set of f. Since $\mathrm{dist}(\Omega,\mathcal{Z}_f)>0$ (see Figure 1), it follows from (3.8) and the integer-periodicity of f that

$$\min_{|\xi|\le 1,\ \lambda\in\Lambda_{n_k}^{u,v}} J_j(\xi+\lambda)\ge\min_{(x,\,y)\in\Omega} f(x,y):=\beta>0,\qquad j\ge n_k+1.$$

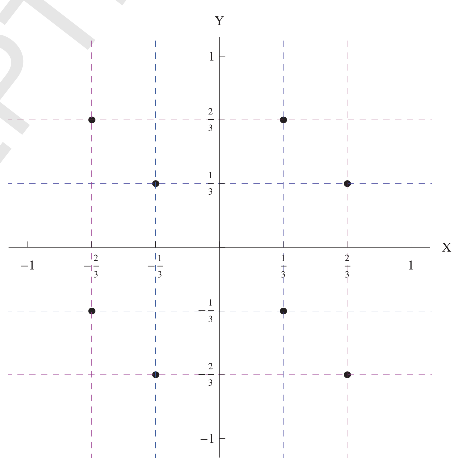

Figure 1. The zero set $\mathcal{Z}_f$ of f.

Thus for integer $p\ge 1$,

$$\prod_{j=n_k+1}^{n_k+p} J_j(\xi+\lambda)\ge\beta^p>0,\qquad |\xi|\le 1,\quad \lambda\in\Lambda_{n_k}^{u,v}. \tag{3.9}$$

In the following, we estimate the lower bound of $\prod_{j=n_k+N_0+1}^{\infty} J_j(\xi+\lambda)$ where N_0 is given in (3.1). It is easy to check that $3^{2-N_0}\tilde{d} \le 1$ where $\tilde{d} = \sup_{n\ge1} \frac{|b_n|}{p_n}$. Similar to (3.7), we have

$$|\eta_{j,2}(\xi+\lambda)| = \frac{|b_j|}{p_j}\frac{|\xi+\lambda|}{P_{n_k}}\frac{P_{n_k}}{P_{j-1}} \le \frac{3^l+2}{2\cdot 3^l}\frac{\tilde{d}}{3^{N_0-2}}\frac{1}{3^{j+1-n_k-N_0}} \le \frac{3^l+2}{2\cdot 3^l}\frac{1}{3^{j+1-n_k-N_0}}$$

for $j \ge n_k+N_0+1$, $|\xi| \le 1$ and $\lambda \in \Lambda_{n_k}^{u,v}$. Together with (3.2) and (3.7), it yields

$$2\pi|\eta_{j,1}(\xi+\lambda)|,\ 2\pi|\eta_{j,2}(\xi+\lambda)| \le 3^{-j+n_k+N_0+1},\ \ j \ge n_k+N_0+1,\ |\xi|\le 1,\ \lambda\in\Lambda_{n_k}^{u,v}. \quad (3.10)$$

Hence $J_j(\xi+\lambda) \ge \frac{1}{3}|1+\cos\eta_{j,1}+\cos\eta_{j,2}| \ge \frac{1}{3}(1+2\cos 3^{-j+n_k+N_0+1})$ for $j \ge n_k+N_0+1$. Applying $\cos x \ge 1-\frac{1}{2}x^2$, we obtain that

$$\prod_{j=n_k+N_0+1}^{\infty} J_j(\xi+\lambda) \ge \prod_{j=n_k+N_0+1}^{\infty}\frac{1}{3}(1+2\cos\frac{1}{3^{j-n_k-N_0-1}}) \ge \prod_{j=0}^{\infty}(1-\frac{1}{3}\frac{1}{9^j}) > \frac{1}{2} \quad (3.11)$$

for $|\xi|\le 1$ and $\lambda \in \Lambda_{n_k}^{u,v}$. This together with (3.9) gives $|\hat{\mu}_{>n_k}(\xi+\lambda)| = \prod_{j=n_k+1}^{\infty} J_j(\xi+\lambda) > \frac{1}{2}\beta^{N_0} := \alpha$ for $|\xi|\le 1$, $\lambda\in\Lambda_{n_k}^{u,v}$ and $k\ge 1$, and (i) follows.

In the following, we prove (ii). Let $\lambda \in \Lambda_{n_{k-1}}^{u,v}$, $k>1$ and $|\xi|\le 1$. By (3.3), $j \ge n_k+1$ implies $j \ge n_{k-1}+N_0+1$; hence (3.10) becomes

$$2\pi|\eta_{j,1}(\xi+\lambda)|,\quad 2\pi|\eta_{j,\,2}(\xi+\lambda)| < 3^{-j+n_{k-1}+N_0+1},\quad j\ge n_k+1. \quad (3.12)$$

Similar to (3.11), we have

$$\begin{aligned}|\hat{\mu}_{>n_k}(\xi+\lambda)| &\ge \prod_{j=0}^{\infty}(1-\frac{1}{3}\frac{1}{9^{j+n_k-n_{k-1}-N_0}}) = \exp\Big(\sum_{j=0}^{\infty}\log(1-\frac{1}{3}\frac{1}{9^{j+n_k-n_{k-1}-N_0}})\Big)\\ &\ge \exp\Big(\sum_{j=0}^{\infty}-\frac{5}{3}\frac{1}{9^{j+n_k-n_{k-1}-N_0}}\Big) \ge \exp\Big(\frac{-1}{9^{n_k-n_{k-1}-N_0-1}}\Big),\end{aligned}$$

where the second inequality follows from $\log(1-x) \ge -5x$ $(0\le x\le 4/5)$. The proof of (ii) is complete. □

Lemma 3.3. *Under the conditions of Proposition 3.1(ii), then there exist an increasing positive integer sequence $\{n_k\}_{k=1}^{\infty}$ satisfying $n_k - n_{k-1} \ge N_0$ where N_0 is given in* (3.1)*, two sequences $u, v \in (\mathbb{Z}\setminus 3\mathbb{Z})^{\mathbb{N}}$, and a constant $\beta > 0$ such that*

(i) $|\hat{\mu}_{>n_k}(\xi+\lambda)| \ge \beta$ *for all* $\lambda\in\Lambda_{n_k}^{u,v}$, $k>1$ *and* $\xi\in[0,1]$*;*

(ii) $|\hat{\mu}_{>n_k}(\xi+\lambda)| \ge \exp\big(-9^{N_0+1-n_k+n_{k-1}}\big)$ *for all* $\lambda\in\Lambda_{n_{k-1}}^{u,v}$, $k>1$ *and* $\xi\in[0,1]$.

Proof. Note that $\liminf_{n\to\infty}\frac{b_n}{p_n} < \min\{1,\frac{1}{\tilde{d}}\}$ and $0<a_n<b_n$ $(n\ge1)$. For fixed $r \in (\liminf_{n\to\infty}\frac{b_n}{p_n},\ \min\{1,\frac{1}{\tilde{d}}\})$, we can find a integer $l \ge N_0$ such that

$$\frac{3^{l-1}+1}{3^l-1} < \min\{\frac{1}{3r},\frac{1}{3\tilde{d}r}\}. \quad (3.13)$$

It is obvious that there exists an increasing positive integer sequence $\{n_k\}_{k=1}^{\infty}$ with $n_{k+1} - n_k > l$ and $n_1 \geq l$ such that

$$\frac{a_{n_k+1}}{p_{n_k+1}} < \frac{b_{n_k+1}}{p_{n_k+1}} \leq r < \min\{1, \frac{1}{\tilde{d}}\} \leq \min\{1, \frac{1}{\tilde{c}}\}, \quad k \geq 1. \tag{3.14}$$

Define $\delta = \{\delta_i\}_{i=1}^{\infty} \in \{-1, 1\}^{\mathbb{N}}$ such that $\delta_i = 1$ for $i \in \{n_k : k \geq 1\}$ and $\delta_i = -1$ otherwise. Let $u = 2v = \{2\delta_i\}_{i=1}^{\infty}$. For

$$\lambda \in \Lambda_n^{u,v} = \sum_{i=1}^{n} \delta_i P_i \{0, 1/3, 2/3\} \quad (\text{see } (2.6)),$$

we can find $\{c_i\}_{i=1}^{n} \in \{0, 1, 2\}^n$ such that $\lambda = 3^{-1} \sum_{i=1}^{n} c_i \delta_i P_i$. Hence

$$\frac{|\lambda|}{P_{n_k}} \leq \frac{2 \sum_{i=1}^{n_k-1} P_i}{3P_{n_k}} \leq \frac{2}{3} \sum_{i=1}^{n_k-1} \frac{1}{3^i} < \frac{1}{3}, \quad \lambda \in \Lambda_{n_k-1}^{u,v}, \quad k \geq 1. \tag{3.15}$$

We further claim that

$$0 < \frac{\lambda}{P_{n_k}} \leq \frac{2 \cdot 3^{l-1}}{3^l - 1}, \quad \lambda \in \Lambda_{n_k}^{u,v} \setminus \Lambda_{n_k-1}^{u,v}, \quad k \geq 1. \tag{3.16}$$

In fact, $\lambda \in \Lambda_{n_k}^{u,v} \setminus \Lambda_{n_k-1}^{u,v}$ implies $c_{n_k} \in \{1, 2\}$ and $\delta_{n_k} = 1$, and hence

$$\frac{\lambda}{P_{n_k}} = \sum_{i=1}^{n_k} \frac{c_i \delta_i P_i}{3P_{n_k}} = \frac{c_{n_k}}{3} + \sum_{i=1}^{n_k-1} \frac{c_i \delta_i}{3} (\prod_{j=i+1}^{n_k} p_j)^{-1} \geq \frac{1}{3} - 2 \sum_{i=1}^{n_k-1} \frac{1}{3^{n_k-i+1}} > 0.$$

On the other hand,

$$\frac{\lambda}{P_{n_k}} = \sum_{j=1}^{k} \frac{c_{n_j} P_{n_j}}{3P_{n_k}} - \sum_{1 \leq i \leq n_k-1, i \neq n_j} \frac{c_i P_i}{3P_{n_k}} \leq \sum_{i=1}^{k} \frac{2}{3} \frac{P_{n_i}}{P_{n_k}} \leq \frac{2}{3} \sum_{i=1}^{k} \frac{1}{3^{n_k - n_i}} < \frac{2 \cdot 3^{l-1}}{3^l - 1},$$

where the last inequality follows from $n_k - n_i = \sum_{j=i}^{k-1} (n_{j+1} - n_j) \geq (k - i)l$. Hence, (3.16) holds.

We now prove Lemma 3.3 (i). For $\lambda \in \Lambda_{n_k}^{u,v} = \Lambda_{n_k-1}^{u,v} \bigcup (\Lambda_{n_k}^{u,v} \setminus \Lambda_{n_k-1}^{u,v})$, we divide it into two situations for discussion: $\lambda \in \Lambda_{n_k-1}^{u,v}$ and $\lambda \in \Lambda_{n_k}^{u,v} \setminus \Lambda_{n_k-1}^{u,v}$.

Case I: $\lambda \in \Lambda_{n_k-1}^{u,v}$ and $\xi \in [0, 1]$. Then (3.15) implies $\frac{|\xi+\lambda|}{P_{n_k}} < \frac{1}{3^l} + \frac{1}{3} < \frac{3^{l-1}+1}{3^l-1}$. Using this instead of (3.6), and applying a similar argument as in Lemma 3.2(i), we can prove that there exists $c_0 > 0$ such that $|\hat{\mu}_{>n_k}(\xi + \lambda)| \geq c_0$ for $\lambda \in \Lambda_{n_k-1}^{u,v}$, $\xi \in [0, 1]$.

Case II: $\lambda \in \Lambda_{n_k}^{u,v} \setminus \Lambda_{n_k-1}^{u,v}$ and $\xi \in [0, 1]$. By (3.14) and (3.16), we have

$$\begin{cases} 0 \leq \frac{\max\{a_{n_k+1}, b_{n_k+1}\}}{p_{n_k+1}} \frac{\xi+\lambda}{P_{n_k}} \leq r\Big(\frac{1}{3^{(k-1)l+n_1}} + \frac{2 \cdot 3^{l-1}}{3^l-1}\Big) < 2r \frac{3^{l-1}+1}{3^l-1}, \\ 0 \leq \frac{\max\{a_j, b_j\}}{p_j} \frac{\xi+\lambda}{P_{n_k+1}} \frac{P_{n_k+1}}{P_{j-1}} \leq 2\tilde{d}r \frac{3^{l-1}+1}{3^l-1} \frac{1}{3^{j-n_k-2}}, \quad j \geq n_k + 2, \quad k \geq 1. \end{cases} \tag{3.17}$$

Let $\epsilon := \frac{2}{3} - 2r \max\{1, \tilde{d}\} \frac{3^{l-1}+1}{3^l-1}$. Then $\epsilon \in (0, \frac{2}{3})$ by (3.13). This together with (3.17) gives $(\eta_{j,1}, \eta_{j,2}) \in [-\frac{2}{3} + \epsilon, 0]^2 := D$ for $j > n_k$, where $\eta_{j,1}, \eta_{j,2}$ are in (3.4). Similar to (3.9), we have

$$\prod_{j=n_k+1}^{n_k+3} J_j(\xi+\lambda) \geq (\min_{(x,y)\in D} f(x,y))^3 := \kappa^3 > 0\ , \quad \lambda \in \Lambda_{n_k}^{u,v} \setminus \Lambda_{n_k-1}^{u,v},\ \ \xi \in [0,1]. \tag{3.18}$$

It is easy to see, by (3.13) and (3.17), that $2\pi|\eta_{j,1}|$, $2\pi|\eta_{j,2}| \leq 4\pi\frac{3^{l-1}+1}{3^l-1}\frac{\tilde{d}r}{3^{j-n_k-2}} < \frac{1}{3^{j-n_k-4}}$ for $j \geq n_k+4$. Hence $\prod_{j=n_k+4}^{\infty} J_j(\xi+\lambda) > \frac{1}{2}$ (a similar argument as in (3.11)). This and (3.18) give $|\hat{\mu}_{>n_k}(\xi+\lambda)| > \frac{1}{2}\kappa^3$ for $\lambda \in \Lambda_{n_k}^{u,v} \setminus \Lambda_{n_k-1}^{u,v}, \xi \in [0,1]$.

The above two cases tell us that (i) holds for $\beta := \min\{c_0, \frac{1}{2}\kappa^3\}$.

The proof of (ii) is similar to that of Lemma 3.2(ii); we only need to use $2\pi|\eta_{j,1}|$, $2\pi|\eta_{j,2}| \leq 4\pi\frac{3^{l-1}+1}{3^l-1}\frac{\tilde{d}r}{3^{j-n_{k-1}-2}} < \frac{1}{3^{j-n_{k-1}-4}}$ for $j \geq n_k+1$ and $\lambda \in \Lambda_{n_{k-1}}^{u,v}$ instead of (3.12). Therefore,

$$|\hat{\mu}_{>n_k}(\xi+\lambda)| \geq \exp\left(-9^{4-n_k+n_{k-1}}\right) \geq \exp\left(-9^{N_0+1-n_k+n_{k-1}}\right),$$

where the last inequality follows from $N_0 \geq 3$ (see (3.1)). □

Proof of Proposition 3.1. (i) Let $\Lambda^{u,v} = \bigcup_{n=1}^{\infty} \Lambda_n^{u,v}$, where $\Lambda_n^{u,v}$ is given in (3.5). Define

$$Q_m(\xi) = \sum_{\lambda\in\Lambda_m^{u,v}} |\hat{\mu}_{\{p_n\},\{\mathcal{D}_n\}}(\xi+\lambda)|^2 \quad \text{and} \quad Q_{\Lambda^{u,v}}(\xi) = \sum_{\lambda\in\Lambda^{u,v}} |\hat{\mu}_{\{p_n\},\{\mathcal{D}_n\}}(\xi+\lambda)|^2.$$

Let $\{n_k\}_{k=1}^{\infty}$ be given in Lemma 3.2. It follows from $\hat{\mu}_{\{p_n\},\{\mathcal{D}_n\}} = \hat{\mu}_{n_k}\hat{\mu}_{>n_k}$ and Lemma 3.2 (i) that $|\hat{\mu}_{\{p_n\},\{\mathcal{D}_n\}}(\xi+\lambda)|^2 \geq \alpha^2|\hat{\mu}_{n_k}(\xi+\lambda)|^2$ for $\lambda \in \Lambda_{n_k}^{u,v}$ and $|\xi| \leq 1$. By $\sum_{\lambda\in\Lambda_{n_k}^{u,v}} |\hat{\mu}_{n_k}(\xi+\lambda)|^2 = 1$(see Lemma 2.2), we have

$$\begin{aligned} Q_{n_k}(\xi) &= Q_{n_{k-1}}(\xi) + \sum_{\lambda\in\Lambda_{n_k}^{u,v}\setminus\Lambda_{n_{k-1}}^{u,v}} |\hat{\mu}_{\{p_n\},\{\mathcal{D}_n\}}(\xi+\lambda)|^2 \\ &\geq Q_{n_{k-1}}(\xi) + \alpha^2\Big(1 - \sum_{\lambda\in\Lambda_{n_{k-1}}^{u,v}} |\hat{\mu}_{n_k}(\xi+\lambda)|^2\Big), \qquad |\xi| \leq 1. \end{aligned} \tag{3.19}$$

If $Q_{\Lambda^{u,v}}(\xi) \not\equiv 1 (\xi \in \mathbb{R})$, then by Proposition 2.1 and the uniqueness theorem of analytic function, we can find $|\xi_0| < 1$ such that $Q_{\Lambda^{u,v}}(\xi_0) < 1$. Let η_0 satisfy $\max\{Q_{\Lambda^{u,v}}(\xi_0), e^{-2}\} < \eta_0 < 1$. Further, by selecting subsequences, we can assume that $n_{k+1} - n_k \geq N_0 + 1 - \log_9 \ln \eta_0^{-1/2}$ for $k \geq 1$, where N_0 is given in (3.1). It follows from Lemma 3.2(ii) that

$$|\hat{\mu}_{>n_k}(\xi_0+\lambda)| \geq \exp(-9^{N_0+1-n_k+n_{k-1}}) \geq \sqrt{\eta_0} > 0, \qquad \lambda \in \Lambda_{n_{k-1}}^{u,v}.$$

Since $|\hat{\mu}_{\{p_n\},\{\mathcal{D}_n\}}(\xi_0+\lambda)|^2 = |\hat{\mu}_{n_k}(\xi_0+\lambda)|^2|\hat{\mu}_{>n_k}(\xi_0+\lambda)|^2 \geq \eta_0|\hat{\mu}_{n_k}(\xi_0+\lambda)|^2$, summing with respect to $\lambda \in \Lambda_{n_{k-1}}^{u,v}$, we have

$$\sum_{\lambda\in\Lambda_{n_{k-1}}^{u,v}} |\hat{\mu}_{n_k}(\xi_0+\lambda)|^2 \leq \frac{1}{\eta_0}\sum_{\lambda\in\Lambda_{n_{k-1}}^{u,v}} |\hat{\mu}_{\{p_n\},\{\mathcal{D}_n\}}(\xi_0+\lambda)|^2 \leq \frac{1}{\eta_0}Q_{\Lambda^{u,v}}(\xi_0) < 1.$$

This and (3.19) give $Q_{n_k}(\xi_0) \geq Q_{n_{k-1}}(\xi_0) + \alpha^2 \left(1 - \eta_0^{-1} Q_{\Lambda^{u,v}}(\xi_0)\right)$. By recursion,

$$1 \geq Q_{\Lambda^{u,v}}(\xi_0) \geq Q_{n_k}(\xi_0) \geq Q_{n_1} + (k-1)\alpha^2 \left(1 - \eta_0^{-1} Q_{\Lambda^{u,v}}(\xi_0)\right) \to +\infty \quad \text{as} \quad k \to +\infty.$$

This contradiction shows $Q_{\Lambda^{u,v}}(\xi) \equiv 1 (\xi \in \mathbb{R})$, and the proof of Proposition 3.1(i) is complete by Proposition 2.1.

The proof of Proposition 3.1(ii) is similar to that of Proposition 3.1(i), we only need to use Lemma 3.3 instead of Lemma 3.2. □

Proof of Theorem 1.1. For any positive integer N, we let $\tilde{c} = c_N = \sup_{n \geq N} \frac{|a_n|}{p_n}$ and $\tilde{d} = d_N = \sup_{n \geq N} \frac{|b_n|}{p_n}$. It is easy to see $\tilde{c} \geq c$, $\tilde{d} \geq d$ (see (1.2)). By the assumptions of Theorem 1.1(a) and (b), we can find a sufficiently large N such that (i) $\liminf_{n\to\infty} \frac{|a_n|}{p_n} < \min\{\frac{2}{3}, \frac{2}{3c_N}\} \leq \min\{\frac{2}{3}, \frac{2}{3c}\}$, (ii) $\liminf_{n\to\infty} \frac{b_n}{p_n} < \min\{1, \frac{1}{d_N}\} \leq \min\{1, \frac{1}{d}\}$ under the assumption of $1 \leq a_n \leq b_n$.

Let $\mu_N = \delta_{P_1^{-1}\mathcal{D}_1} * \cdots * \delta_{P_N^{-1}\mathcal{D}_N}$ and $\mu_{N+} = \delta_{P_N P_{N+1}^{-1}\mathcal{D}_{N+1}} * \delta_{P_N P_{N+2}^{-1}\mathcal{D}_{N+2}} * \cdots$. Then $\mu_{\{p_n\},\{\mathcal{D}_n\}} = \mu_N * (\mu_{N+} \circ P_N)$ and $\hat{\mu}_{\{p_n\},\{\mathcal{D}_n\}}(\xi) = \hat{\mu}_N(\xi)\hat{\mu}_{N+}(P_N^{-1}\xi)$. Let $\Lambda_N = \sum_{i=1}^N P_i\{0, 1/3, -1/3\}$. It follows from Lemma 2.2 that Λ_N is spectrum for μ_N. On the other hand, by the proof of Proposition 3.1, it is clear that there exists a spectrum $\Lambda_{N+} \subset \mathbb{Z}$ for μ_{N+}. Now we show that $\Lambda_N + P_N\Lambda_{N+}$ is a spectrum for $\mu_{\{p_n\},\{\mathcal{D}_n\}}$ by Proposition 2.1.

Note that $\hat{\mu}_N(\xi) = \prod_{i=1}^N M_{a_i,b_i}(P_i^{-1}\xi)$, where $M_{a_i,b_i}(\xi) = \frac{1}{3}(1 + e^{-2\pi i a_i \xi} + e^{-2\pi i b_i \xi})$. Since $P_N P_i^{-1}$ $(i \leq N)$ is an integer and $M_{a_i,b_i}(\xi)$ has integer-periodicity, we can conclude that $\hat{\mu}_N(\xi + P_N\eta) = \hat{\mu}_N(\xi)$ for any $\eta \in \mathbb{Z}$, so

$$\begin{aligned}\sum_{\lambda \in \Lambda_N + P_N\Lambda_{N+}} |\hat{\mu}_{\{p_n\},\{\mathcal{D}_n\}}(\xi + \lambda)|^2 &= \sum_{\lambda_1 \in \Lambda_N, \lambda_2 \in \Lambda_{N+}} |\hat{\mu}_N(\xi + \lambda_1 + P_N\lambda_2)|^2 \left|\hat{\mu}_{N+}\left(P_N^{-1}(\xi + \lambda_1 + P_N\lambda_2)\right)\right|^2 \\ &= \sum_{\lambda_1 \in \Lambda_N} |\hat{\mu}_N(\xi + \lambda_1)|^2 \sum_{\lambda_2 \in \Lambda_{N+}} \left|\hat{\mu}_{N+}\left(P_N^{-1}(\xi + \lambda_1) + \lambda_2\right)\right|^2 \equiv 1.\end{aligned}$$

Hence, we complete the proof. □

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