



# Asymptotic formulas for spt-crank of partitions

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## ABSTRACT

In this paper, using a variant of the Hardy–Ramanujan Circle Method originally due to E. Wright, we prove asymptotic formulas for spt-crank of ordinary partitions.

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## 1. Introduction

A partition of a positive integer  $n$  is a sequence of non-increasing positive integers whose sum equals  $n$ . Let  $p(n)$  denote the number of partitions of  $n$ . The following three famous congruences for  $p(n)$  were found and later proved by S. Ramanujan:

$$p(5n + 4) \equiv 0 \pmod{5},$$

$$p(7n + 5) \equiv 0 \pmod{7},$$

$$p(11n + 6) \equiv 0 \pmod{11}.$$

Ranks [7,16] and cranks [4,18] of partitions provide combinatorial interpretations for Ramanujan's congruences. Let  $N(m, n)$  (resp.  $M(m, n)$ ) be the number of partitions of  $n$  whose rank (resp. crank) is  $m$ . Recently, K. Bringmann and J. Dousse [10] prove the following formula which was first conjectured by F.J. Dyson [17].

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**Theorem 1.1** (Bringmann–Dousse). *If  $|m| \leq \frac{1}{\pi\sqrt{6}}\sqrt{n}\log n$ , we have, as  $n \rightarrow \infty$*

$$M(m, n) = \frac{\gamma}{4} \operatorname{sech}^2\left(\frac{\gamma m}{2}\right) p(n) \left(1 + O\left(|m|^{\frac{1}{3}} \gamma^{\frac{1}{2}}\right)\right), \quad (1.1)$$

where  $\gamma = \frac{\pi}{\sqrt{6n}}$ .

Later, Dousse and Mertens [15] obtain the same formula for  $N(m, n)$ . For more results on asymptotic formulas for ranks and cranks, see [3, 19–22], for example. In this paper, we prove an analogue of (1.1) for the spt-crank of partitions.

Let  $\operatorname{spt}(n)$  denote the total number of appearances of the smallest parts in all the partitions of  $n$ . G.E. Andrews [1] proved

$$\operatorname{spt}(5n + 4) \equiv 0 \pmod{5}, \quad (1.2)$$

$$\operatorname{spt}(7n + 5) \equiv 0 \pmod{7}, \quad (1.3)$$

$$\operatorname{spt}(13n + 6) \equiv 0 \pmod{13}.$$

In order to provide combinatorial explanations for the above congruences, Andrews, F.G. Garvan and J. Liang introduce the spt-crank of an  $S$ -partition. Let  $\mathcal{P}$  denote the set of partitions and  $\mathcal{D}$  denote the set of partitions into distinct parts. For  $\pi \in \mathcal{P}$ , define  $s(\pi)$  as the smallest part of  $\pi$  with  $s(\emptyset) = \infty$  for the empty partition. Let  $l(\pi)$  to be the number of parts of  $\pi$ . Define

$$S = \{(\pi_1, \pi_2, \pi_3) \in \mathcal{D} \times \mathcal{P} \times \mathcal{P} : \pi_1 \neq \emptyset \text{ and } s(\pi_1) \leq \min\{s(\pi_2), s(\pi_3)\}\}.$$

For  $\vec{\pi} = (\pi_1, \pi_2, \pi_3) \in S$ , we define the weight  $\omega_1(\vec{\pi}) = (-1)^{l(\pi_1)-1}$ , the spt-crank  $(\vec{\pi}) = l(\pi_2) - l(\pi_3)$ , and  $|\vec{\pi}| = |\pi_1| + |\pi_2| + |\pi_3|$  where  $|\pi_j|$  is the sum of the parts of  $\pi_j$ . The number of vector partitions of  $n$  in  $S$  with spt-crank  $m$  counted according to the weight  $\omega_1$  is denoted by  $N_S(m, n)$ , that is

$$N_S(m, n) := \sum_{\substack{\vec{\pi} \in S, |\vec{\pi}|=n \\ \operatorname{spt-crank}(\vec{\pi})=m}} \omega_1(\vec{\pi}).$$

Then [5, Corollary 2.2] states that

$$\sum_{m=-\infty}^{\infty} N_S(m, n) = \operatorname{spt}(n).$$

And Andrews, Garvan and Liang [5] also show that

$$N_S(k, 5, 5n + 4) = \frac{\operatorname{spt}(5n + 4)}{5}, \text{ for } 0 \leq k \leq 4,$$

and

$$N_S(k, 7, 7n + 5) = \frac{\operatorname{spt}(7n + 5)}{7}, \text{ for } 0 \leq k \leq 6,$$

where  $N_S(m, t, n) := \sum_{k \equiv m \pmod{t}} N_S(k, n)$ . Clearly, the above two equations give combinatorial interpretations of (1.2) and (1.3), respectively. Motivated by the works in [5], many interesting properties of spt-crank have been discovered. For example, Chen, Ji and Zang [13] prove that for  $m \geq 0$  and  $n \geq 0$ , we have

$$N_S(m, n) \geq N_S(m+1, n),$$

which is first conjectured in [3]. Andrews, F.J. Dyson and R.C. Rhoades establish asymptotic formulas for the spt-crank moments (see [3, Theorem 1.4]). For definitions of the spt-crank for ordinary partitions, see [14] and [2, Section 5]. In this paper, we establish asymptotic formulas for  $N_S(m, n)$ . The main result is given in the following theorem.

**Theorem 1.2.** Assume that  $\beta = \frac{\pi}{\sqrt{6n}}$  and  $|m| \leq n^{\frac{3}{8}}$ . We have, as  $n \rightarrow \infty$ ,

$$N_S(m, n) = \frac{\beta}{4} \operatorname{sech}^2\left(\frac{m\beta}{2}\right) \operatorname{spt}(n) \left(1 + O\left(m^{\frac{1}{3}}\beta^{\frac{1}{2}}\right)\right).$$

### Remarks.

1. Since  $N_S(m, n) = N_S(-m, n)$  for all  $m$  and  $n$ , we assume  $m \geq 0$  throughout the rest of the paper.
2. K. Bringmann [9] proved that as  $n \rightarrow \infty$

$$\operatorname{spt}(n) \sim \frac{\sqrt{6n}}{\pi} p(n) \sim \frac{1}{2\sqrt{2\pi}\sqrt{n}} e^{\pi\sqrt{\frac{2n}{3}}}.$$

Thus we have for fixed  $m$

$$N_S(m, n) \sim \frac{1}{4} p(n) \sim \frac{1}{16\sqrt{3}\pi n} e^{\pi\sqrt{\frac{2n}{3}}}$$

as  $n \rightarrow \infty$ . This generalizes [11, Theorem 1.4]: as  $n \rightarrow \infty$

$$\operatorname{ospt}(n) \sim \frac{1}{4} p(n),$$

where the definition of ospt-function is given in [2] and we have  $\operatorname{ospt}(n) = N_S(0, n)$ .

The proof of Theorem 1.2 is motivated by the works in [10,15] and depends on a variant of the Hardy–Ramanujan Circle Method due to E. Wright [24].

The paper is organized as follows. In Section 2, we rewrite the two variables generating function of  $N_S(m, n)$  in terms of  $\theta$ -functions, Dedekind- $\eta$  functions and Appell–Lerch sums. Modular properties of these functions are given in this section. And we also recall some basic facts on the Euler numbers and Euler polynomials in Section 2. Next we study the asymptotic behavior of  $S_m(q) := \sum_{n=0}^{\infty} N_S(m, n)q^n$  with  $q$  approaching the roots of unity in Section 3. In Section 4, we apply Wright’s method to prove Theorem 1.2.

We remark that the generating function of spt-crank is closely related to those of rank and crank (see Equation (2.1) below). Hence we are able to apply the idea of Bringmann and Dousse in [10] and this paper closely follows the works in [10,15].

## 2. Preliminaries

### 2.1. Appell–Lerch sums and generating function of spt-crank

Throughout, we assume  $\tau = \frac{is}{2\pi}$  with  $s = \beta(1 + ixm^{-\frac{1}{3}})$  where  $x \in \mathbb{R}$  satisfying  $|x| \leq \frac{\pi m^{\frac{1}{3}}}{\beta}$ ,  $z \in \mathbb{R}$ ,  $q := e^{2\pi i\tau}$  and  $\zeta := e^{2\pi iz}$ . Define the generating functions of rank, crank and spt-crank of partitions as follows:

$$R(z; \tau) := \sum_{n \geq 0} \sum_{m \in \mathbb{Z}} N(m, n) \zeta^m q^n$$

$$C(z; \tau) := \sum_{n \geq 0} \sum_{m \in \mathbb{Z}} M(m, n) \zeta^m q^n$$

and

$$S(z; \tau) := \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_S(m, n) \zeta^m q^n.$$

By [5, Corollary 2.5], we have the following identity on the above three generating functions:

$$S(z; \tau) = \frac{-1}{(1-\zeta)(1-\zeta^{-1})} (C(z; \tau) - R(z; \tau)). \quad (2.1)$$

We remark that although the asymptotic behaviors of  $C(z; \tau)$  and  $R(z; \tau)$  when  $q$  is near their singularities are obtained in [10] and [15], respectively, we fail to apply these results to determine the asymptotic behavior of  $S(z; \tau)$  due to the appearance of the factor  $\frac{-1}{(1-\zeta)(1-\zeta^{-1})}$  on the right side of (2.1). To analyze the asymptotic behavior of  $S(z; \tau)$  as  $q$  approaching the roots of unity, especially when  $z$  is close to 0, we rewrite the rank and crank generating functions in terms of  $\theta$ -functions, Dedekind- $\eta$  functions and Appell–Lerch sums. Recall that

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{k=1}^{\infty} (1 - q^k)$$

$$\theta(z; \tau) := i q^{\frac{1}{8}} \zeta^{\frac{1}{2}} \prod_{n=1}^{\infty} (1 - q^n)(1 - \zeta q^n)(1 - \zeta^{-1} q^{n-1})$$

and the Appell–Lerch sum

$$A_1(u, v; \tau) := e^{\pi i u} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(n^2+n)/2} e^{2\pi i n v}}{1 - e^{2\pi i u} q^n}. \quad (2.2)$$

**Lemma 2.1** (see [25, Theorem 7.1] or [15, Lemma 3.1]). *We have*

$$R(z; \tau) = \frac{q^{\frac{1}{24}}}{\eta(\tau)} \left[ i \frac{\left( \zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}} \right) \eta^3(3\tau)}{\theta(3z; 3\tau)} - \zeta^{-1} \left( \zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}} \right) A_1(3z, -\tau; 3\tau) \right. \\ \left. - \zeta \left( \zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}} \right) A_1(3z, \tau; 3\tau) \right]. \quad (2.3)$$

Substituting (2.3) into (2.1) and noting that  $C(z; \tau) = \frac{i(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}) q^{\frac{1}{24}} \eta^2(\tau)}{\theta(z; \tau)}$ , after simplifying, we obtain

$$S(z; \tau) = \frac{-\zeta^{\frac{1}{2}} q^{\frac{1}{24}}}{(1-\zeta)\eta(\tau)} \left( \frac{i\eta^3(\tau)}{\theta(z; \tau)} - \frac{i\eta^3(3\tau)}{\theta(3z; 3\tau)} + \zeta^{-1} A_1(3z, -\tau; 3\tau) + \zeta A_1(3z, \tau; 3\tau) \right). \quad (2.4)$$

We also need the following transformation laws (see e.g. [23] or [10, Lemma 2.1]).

**Lemma 2.2.** We have

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau) \quad (2.5)$$

$$\theta\left(\frac{z}{\tau}; -\frac{1}{\tau}\right) = -i\sqrt{-i\tau}e^{\frac{\pi iz^2}{\tau}}\theta(z; \tau) \quad (2.6)$$

and also define the Mordell integral as

$$h(z) = h(z; \tau) := \int_{-\infty}^{\infty} \frac{e^{\pi i \tau w^2 - 2\pi zw}}{\cosh(\pi w)} dw, \quad (2.7)$$

then there holds

$$-\frac{1}{\tau}e^{\frac{\pi i(u^2-2uv)}{\tau}}A_1\left(\frac{u}{\tau}, \frac{v}{\tau}; -\frac{1}{\tau}\right) + A_1(u, v; \tau) = \frac{1}{2i}h(u-v; \tau)\theta(v; \tau) \quad ([26, \text{Proposition 1.5}]). \quad (2.8)$$

In the next lemma, we determine the modular behaviors of the four terms in the brackets on the right hand side of (2.4).

**Lemma 2.3.** Let  $a = e^{-\frac{4\pi^2 z}{s}}$ ,  $q_1 = e^{-\frac{4\pi^2}{3s}}$ ,  $\rho = e^{\frac{2\pi i}{3}}$ . Then we have

$$\frac{i\eta^3(3\tau)}{\theta(3z; 3\tau)} = \frac{2\pi}{3is}e^{\frac{6\pi^2 z^2}{s}}a^{\frac{1}{2}}\left(\frac{1}{1-a} + \sum_{n=1}^{\infty} \frac{(-1)^n q_1^{(n^2+n)/2}}{1-q_1^n a} - \sum_{n=1}^{\infty} \frac{(-1)^n q_1^{(n^2+n)/2}/a}{1-q_1^n/a}\right), \quad (2.9)$$

$$\frac{i\eta^3(\tau)}{\theta(z; \tau)} = \frac{2\pi}{is}e^{\frac{2\pi^2 z^2}{s}}a^{\frac{1}{2}}\left(\frac{1}{1-a} + \sum_{n=1}^{\infty} \frac{(-1)^n q_1^{3(n^2+n)/2}}{1-q_1^{3n} a} - \sum_{n=1}^{\infty} \frac{(-1)^n q_1^{3(n^2+n)/2}/a}{1-q_1^{3n}/a}\right) \quad (2.10)$$

and

$$\begin{aligned} A_1(3z, \mp\tau; 3\tau) = & \pm \frac{1}{2}e^{\frac{s}{6}}h\left(3z \pm \frac{is}{2\pi}; \frac{3is}{2\pi}\right)\eta\left(\frac{is}{2\pi}\right) - \frac{2\pi i}{3s}e^{\frac{6\pi^2 z^2}{s}}\zeta^{\pm 1} \times \\ & \left(\frac{a^{\frac{1}{2}}}{a-1} + a^{-\frac{1}{2}}\sum_{n=1}^{\infty} \frac{(-1)^n q_1^{(n^2+n)/2}\rho^{\mp n}}{1-q_1^n/a} - a^{\frac{1}{2}}\sum_{n=1}^{\infty} \frac{(-1)^n q_1^{(n^2+n)/2}\rho^{\pm n}}{1-aq_1^n}\right). \end{aligned} \quad (2.11)$$

**Proof.** First, we prove (2.9). By the transformation formulas (2.5) and (2.6), we have

$$\frac{i\eta^3(3\tau)}{\theta(3z; 3\tau)} = \frac{2\pi e^{\frac{6\pi^2 z^2}{s}}}{3ise^{\frac{2\pi^2 z}{s}}} \prod_{k=1}^{\infty} \frac{\left(1 - e^{-\frac{4\pi^2 k}{3s}}\right)^2}{\left(1 - e^{-\frac{4\pi^2 k}{3s} + \frac{4\pi^2 z}{s}}\right)\left(1 - e^{-\frac{4\pi^2 (k-1)}{3s} - \frac{4\pi^2 z}{s}}\right)}.$$

After invoking [8, Theorem 2.1] with  $q = q_1$ ,  $x = a$ , we find that

$$\begin{aligned} \frac{i\eta^3(3\tau)}{\theta(3z; 3\tau)} &= \frac{2\pi}{3is}e^{\frac{6\pi^2 z^2}{s} - \frac{2\pi^2 z}{s}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q_1^{(n^2+n)/2}}{1-q_1^n a} \\ &= \frac{2\pi}{3is}e^{\frac{6\pi^2 z^2}{s}}a^{\frac{1}{2}}\left(\frac{1}{1-a} + \sum_{n=1}^{\infty} \frac{(-1)^n q_1^{(n^2+n)/2}}{1-q_1^n a} - \sum_{n=1}^{\infty} \frac{(-1)^n q_1^{(n^2+n)/2}/a}{1-q_1^n/a}\right). \end{aligned}$$

This gives (2.9).

Replacing  $s$  (resp.  $z$ ) by  $\frac{s}{3}$  (resp.  $\frac{z}{3}$ ) in (2.9), we obtain (2.10).

Next, we prove (2.11). Applying (2.8) with  $\tau, u, v$  replaced by  $3\tau, 3z, \mp\tau$ , respectively, we find that

$$A_1(3z, \mp\tau; 3\tau) = \pm \frac{1}{2} e^{\frac{\pi}{6}} h \left( 3z \pm \frac{is}{2\pi}; \frac{3is}{2\pi} \right) \eta \left( \frac{is}{2\pi} \right) - \frac{2\pi i}{3s} e^{\frac{6\pi^2 z^2}{s}} \zeta^{\pm 1} A_1 \left( \frac{2\pi z}{is}, \mp \frac{1}{3}; \frac{2\pi i}{3s} \right). \quad (2.12)$$

By (2.2), we have

$$\begin{aligned} A_1 \left( \frac{2\pi z}{is}, \mp \frac{1}{3}; \frac{2\pi i}{3s} \right) &= a^{-\frac{1}{2}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q_1^{(n^2+n)/2} \rho^{\mp n}}{1 - q_1^n/a} \\ &= \frac{a^{\frac{1}{2}}}{a-1} + a^{-\frac{1}{2}} \sum_{n=1}^{\infty} \frac{(-1)^n q_1^{(n^2+n)/2} \rho^{\mp n}}{1 - q_1^n/a} - a^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{(-1)^n q_1^{(n^2+n)/2} \rho^{\pm n}}{1 - a q_1^n}, \end{aligned}$$

which together with (2.12) implies (2.11).  $\square$

## 2.2. Euler polynomials

Define the Euler polynomials  $E_r(x)$  by:

$$\frac{2e^{xt}}{1+e^t} =: \sum_{r=0}^{\infty} E_r(x) \frac{t^r}{r!}, \quad |t| < \pi.$$

**Lemma 2.4.** *We have*

$$\sum_{r=0}^{\infty} E_{2r+1}(0) \frac{t^{2r}}{(2r)!} = -\frac{1}{2} \operatorname{sech}^2 \left( \frac{t}{2} \right) \quad (2.13)$$

and as  $t \rightarrow 0$ , there holds

$$\sum_{r=0}^{\infty} (2r+3)(2r+2) E_{2r+1}(0) \frac{(2t)^{2r}}{(2r)!} = O(\operatorname{sech}^2(t)), \quad (2.14)$$

$$\sum_{r=0}^{\infty} (2r+4)(2r+3)(2r+2) E_{2r+1}(0) \frac{(2t)^{2r}}{(2r)!} = O(\operatorname{sech}^2(t)). \quad (2.15)$$

**Proof.** Equation (2.13) is proved by Bringmann and Dousse in [10, Lemma 2.2].

To prove (2.14), we first recall the following equation in [19, Lemma 2.3]:

$$\sum_{r=0}^{\infty} (2r+2) E_{2r+1}(0) \frac{(2t)^{2r}}{(2r)!} = \operatorname{sech}^2(t) (t \tanh t - 1)$$

Multiplying both sides of the above equation by  $t^3$  and differentiating with respect to  $t$ , we obtain

$$\sum_{r=0}^{\infty} (2r+3)(2r+2) E_{2r+1}(0) \frac{2^{2r} t^{2r+2}}{(2r)!} = \frac{d(t^3 \operatorname{sech}^2(t) (t \tanh t - 1))}{dt}.$$

After simplifying, we have

$$\sum_{r=0}^{\infty} (2r+3)(2r+2) E_{2r+1}(0) \frac{(2t)^{2r}}{(2r)!} = \operatorname{sech}^2(t) (-2t^2 \tanh^2(t) + t^2 \operatorname{sech}^2(t) + 6t \tanh(t) - 3).$$

This implies (2.14) since  $\tanh(t) \rightarrow 0$  and  $\operatorname{sech}(t) \rightarrow 1$  as  $t \rightarrow 0$ .

Similarly, we can prove (2.15).  $\square$

We also recall the following integral representation of Euler polynomials ([10, Lemma 2.3]):

$$\mathcal{I}_{2j+1} = \frac{(-1)^{j+1} E_{2j+1}(0)}{2}, \quad (2.16)$$

where

$$\mathcal{I}_j := \int_0^\infty \frac{z^j}{\sinh(\pi z)} dz.$$

The integral  $\mathcal{I}_j$  can also be evaluated by the Riemann zeta functions.

**Lemma 2.5.** *We have*

$$\mathcal{I}_j = \frac{j! (2 - 2^{-j})}{\pi^{j+1}} \zeta(j+1). \quad (2.17)$$

**Proof.** Following the proof of [19, Lemma 2.2] closely, we rewrite the integrand in terms of the geometric series of exponential functions as follows:

$$\frac{z^j}{\sinh(\pi z)} = 2 \frac{z^j e^{-\pi z}}{1 - e^{-2\pi z}} = 2z^j \sum_{k=0}^{\infty} e^{-(2k+1)\pi z}.$$

This yields that

$$\mathcal{I}_j = 2 \sum_{k=0}^{\infty} \int_0^\infty z^j e^{-(2k+1)\pi z} dz.$$

Noting

$$\int_0^\infty z^j e^{-(2k+1)\pi z} dz = \frac{j!}{((2k+1)\pi)^{j+1}},$$

we find that

$$\mathcal{I}_j = 2 \frac{j!}{\pi^{j+1}} \sum_{k=0}^{\infty} (2k+1)^{-j-1} = 2 \frac{j! (1 - 2^{-j-1})}{\pi^{j+1}} \zeta(j+1). \quad \square$$

### 3. Asymptotic behavior of $S_m(q)$

By Cauchy's residue theorem, we have the following equation:

$$\begin{aligned} S_m(q) &= \sum_{n \geq 0} N_S(m, n) q^n = \int_{-\frac{1}{2}}^{\frac{1}{2}} S(z; \tau) e^{-2\pi i m z} dz \\ &= 2 \int_0^{\frac{1}{2}} S(z; \tau) \cos(2\pi m z) dz. \end{aligned} \quad (3.1)$$

### 3.1. Asymptotic estimation of $S(z; \tau)$

To study asymptotic behavior of  $S_m(q)$  as  $q \rightarrow 1$ , we need to estimate  $S(z; \tau)$  as  $n \rightarrow \infty$ . To do this, we first prove the transformation formula of  $S(z; \tau)$  under  $\tau \rightarrow \frac{-1}{\tau}$ .

Applying (2.9), (2.10) and (2.11) and rearranging, we find that

$$\begin{aligned} & \frac{i\eta^3(\tau)}{\theta(z; \tau)} - \frac{i\eta^3(3\tau)}{\theta(3z; 3\tau)} + \zeta^{-1}A_1(3z, -\tau; 3\tau) + \zeta A_1(3z, \tau; 3\tau) \\ &= \frac{2\pi}{is} \left( \frac{a^{\frac{1}{2}} e^{\frac{2\pi^2 z^2}{s}}}{1-a} - \frac{a^{\frac{1}{2}} e^{\frac{6\pi^2 z^2}{s}}}{3(1-a)} - \frac{2a^{\frac{1}{2}} e^{\frac{6\pi^2 z^2}{s}}}{3(1-a)} \right) \\ &+ \frac{2\pi a^{\frac{1}{2}} e^{\frac{2\pi^2 z^2}{s}}}{is} \left( \sum_{n=1}^{\infty} \frac{(-1)^n q_1^{3(n^2+n)/2}}{1-q_1^{3n}a} - \sum_{n=1}^{\infty} \frac{(-1)^n q_1^{3(n^2+n)/2}/a}{1-q_1^{3n}/a} \right) \\ &- \frac{2\pi a^{\frac{1}{2}} e^{\frac{6\pi^2 z^2}{s}}}{3is} \sum_{n=1}^{\infty} \frac{(-1)^n q_1^{(n^2+n)/2} (1+\rho^n + \rho^{-n})}{1-q_1^n a} \\ &+ \frac{2\pi a^{-\frac{1}{2}} e^{\frac{6\pi^2 z^2}{s}}}{3is} \sum_{n=1}^{\infty} \frac{(-1)^n q_1^{(n^2+n)/2} (1+\rho^n + \rho^{-n})}{1-q_1^n/a} \\ &+ \frac{1}{2} e^{\frac{s}{6}} \eta \left( \frac{is}{2\pi} \right) \left[ \zeta^{-1} h \left( 3z + \frac{is}{2\pi}; \frac{3is}{2\pi} \right) - \zeta h \left( 3z - \frac{is}{2\pi}; \frac{3is}{2\pi} \right) \right] \end{aligned}$$

After simplifying, we obtain

$$\begin{aligned} & \frac{i\eta^3(\tau)}{\theta(z; \tau)} - \frac{i\eta^3(3\tau)}{\theta(3z; 3\tau)} + \zeta^{-1}A_1(3z, -\tau; 3\tau) + \zeta A_1(3z, \tau; 3\tau) \\ &= \frac{2\pi a^{\frac{1}{2}} \left( e^{\frac{2\pi^2 z^2}{s}} - e^{\frac{6\pi^2 z^2}{s}} \right)}{is(1-a)} + \frac{2\pi a^{\frac{1}{2}} e^{\frac{2\pi^2 z^2}{s}}}{is} \sum_{n=1}^{\infty} \frac{(-1)^n q_1^{3(n^2+n)/2} (1-a^{-1})(1+q_1^{3n})}{(1-q_1^{3n}a)(1-q_1^{3n}/a)} \\ &- \frac{2\pi a^{\frac{1}{2}} e^{\frac{6\pi^2 z^2}{s}}}{is} \sum_{n=1}^{\infty} \frac{(-1)^n q_1^{(9n^2+3n)/2}}{1-q_1^{3n}a} + \frac{2\pi a^{-\frac{1}{2}} e^{\frac{6\pi^2 z^2}{s}}}{is} \sum_{n=1}^{\infty} \frac{(-1)^n q_1^{(9n^2+3n)/2}}{1-q_1^{3n}/a} \\ &+ \frac{1}{2} e^{\frac{s}{6}} \eta \left( \frac{is}{2\pi} \right) \left[ \zeta^{-1} h \left( 3z + \frac{is}{2\pi}; \frac{3is}{2\pi} \right) - \zeta h \left( 3z - \frac{is}{2\pi}; \frac{3is}{2\pi} \right) \right] \\ &= \frac{2\pi a^{\frac{1}{2}} \left( e^{\frac{2\pi^2 z^2}{s}} - e^{\frac{6\pi^2 z^2}{s}} \right)}{is(1-a)} + \frac{2\pi a^{\frac{1}{2}} e^{\frac{2\pi^2 z^2}{s}}}{is} \sum_{n=1}^{\infty} \frac{(-1)^n q_1^{3(n^2+n)/2} (1-a^{-1})(1+q_1^{3n})}{(1-q_1^{3n}a)(1-q_1^{3n}/a)} \\ &- \frac{2\pi a^{\frac{1}{2}} e^{\frac{6\pi^2 z^2}{s}}}{is} \sum_{n=1}^{\infty} \frac{(-1)^n q_1^{(9n^2+3n)/2} (1-a^{-1})(1+q_1^{3n})}{(1-q_1^{3n}a)(1-q_1^{3n}/a)} \\ &+ \frac{1}{2} e^{\frac{s}{6}} \eta \left( \frac{is}{2\pi} \right) \left[ \zeta^{-1} h \left( 3z + \frac{is}{2\pi}; \frac{3is}{2\pi} \right) - \zeta h \left( 3z - \frac{is}{2\pi}; \frac{3is}{2\pi} \right) \right] \\ &=: g_1(z; s) + g_2(z; s) + g_3(z; s) + g_4(z; s) \end{aligned} \tag{3.2}$$

Substituting (3.2) into (2.4), we have

$$S(z; \tau) = \frac{-\zeta^{\frac{1}{2}} q^{\frac{1}{24}}}{(1-\zeta)\eta(\tau)} (g_1(z; s) + g_2(z; s) + g_3(z; s) + g_4(z; s)). \tag{3.3}$$



**Lemma 3.1.** Assume that  $0 \leq z \leq \frac{1}{2}$ . Then for  $|x| \leq 1$ , we have as  $n \rightarrow \infty$

$$\frac{\zeta^{\frac{1}{2}} g_2(z; s)}{1 - \zeta} = O\left(|s|^{-1} e^{-\frac{5}{2}\pi^2 \operatorname{Re}(\frac{1}{s})}\right) \quad (3.4)$$

$$\frac{\zeta^{\frac{1}{2}} g_3(z; s)}{1 - \zeta} = O\left(|s|^{-1} e^{-\frac{11}{2}\pi^2 \operatorname{Re}(\frac{1}{s})}\right) \quad (3.5)$$

and

$$\frac{\zeta^{\frac{1}{2}} g_4(z; s)}{1 - \zeta} = O\left(|s|^{-1/2} e^{-\frac{\pi^2}{6} \operatorname{Re}(\frac{1}{s})}\right). \quad (3.6)$$

**Proof.** First, we consider  $\frac{\zeta^{\frac{1}{2}} g_2(z; s)}{1 - \zeta}$ . When  $z$  is away from 0, it is easy to see that

$$\begin{aligned} \frac{\zeta^{\frac{1}{2}} g_2(z; s)}{1 - \zeta} &= \frac{2\pi \zeta^{\frac{1}{2}} a^{\frac{1}{2}} e^{\frac{2\pi^2 z^2}{s}} (1 - a^{-1})}{is(1 - \zeta)} \sum_{n=1}^{\infty} \frac{(-1)^n q_1^{3(n^2+n)/2} (1 + q_1^{3n})}{(1 - q_1^{3n} a)(1 - q_1^{3n}/a)} \\ &= \frac{2\pi \zeta^{\frac{1}{2}} a^{\frac{1}{2}} e^{\frac{2\pi^2 z^2}{s}} (1 - a^{-1})}{is(1 - \zeta)} (q_1^3 + O(|q_1|^6)) \\ &= O\left(|s|^{-1} |q_1|^3 \left| e^{\frac{2\pi^2 z^2}{s} + \frac{2\pi^2 z}{s}} \right|\right). \end{aligned}$$

Noting that  $z^2 + z \leq \frac{3}{4}$ , we have

$$\frac{\zeta^{\frac{1}{2}} g_2(z; s)}{1 - \zeta} = O\left(|s|^{-1} e^{-\frac{5}{2}\pi^2 \operatorname{Re}(\frac{1}{s})}\right). \quad (3.7)$$

Since  $\lim_{z \rightarrow 0^+} \frac{\zeta^{\frac{1}{2}} a^{\frac{1}{2}} e^{\frac{2\pi^2 z^2}{s}} (1 - a^{-1})}{1 - \zeta} = \frac{2\pi}{is}$ , when  $z$  approaches 0, we have

$$\begin{aligned} \frac{\zeta^{\frac{1}{2}} g_2(z; s)}{1 - \zeta} &= \frac{2\pi \zeta^{\frac{1}{2}} a^{\frac{1}{2}} e^{\frac{2\pi^2 z^2}{s}} (1 - a^{-1})}{is(1 - \zeta)} \sum_{n=1}^{\infty} \frac{(-1)^n q_1^{3(n^2+n)/2} (1 + q_1^{3n})}{(1 - q_1^{3n} a)(1 - q_1^{3n}/a)} \\ &= O(|s|^{-2} |q_1|^3) \\ &= O\left(|s|^{-2} e^{-4\pi^2 \operatorname{Re}(\frac{1}{s})}\right). \end{aligned}$$

This together with (3.7) gives (3.4).

Proceeding as above, we can prove (3.5).

Next, we consider  $\frac{\zeta^{\frac{1}{2}} g_4(z; s)}{1 - \zeta}$ . By [26, Proposition 2.1], we know that  $h(z; \tau)$  is an even holomorphic function of  $z$ . Thus we have

$$\lim_{z \rightarrow 0^+} h\left(3z \mp \frac{is}{2\pi}; \frac{3is}{2\pi}\right) = h\left(\mp \frac{is}{2\pi}; \frac{3is}{2\pi}\right) = h\left(\frac{is}{2\pi}; \frac{3is}{2\pi}\right),$$

which gives

$$\begin{aligned} \lim_{z \rightarrow 0^+} \frac{\zeta^{\frac{1}{2}} g_4(z; s)}{1 - \zeta} &= \lim_{z \rightarrow 0^+} \frac{\zeta^{\frac{1}{2}} e^{\frac{\pi}{6}} \eta\left(\frac{is}{2\pi}\right)}{2(1 - \zeta)} \left[ \zeta^{-1} h\left(3z + \frac{is}{2\pi}; \frac{3is}{2\pi}\right) - \zeta h\left(3z - \frac{is}{2\pi}; \frac{3is}{2\pi}\right) \right] \\ &= e^{\frac{\pi}{6}} \eta\left(\frac{is}{2\pi}\right) h\left(\frac{is}{2\pi}; \frac{3is}{2\pi}\right). \end{aligned}$$

This means that, as  $z \rightarrow 0^+$ , we have

$$\frac{\zeta^{\frac{1}{2}} g_4(z; s)}{1 - \zeta} = O \left( \left| e^{\frac{s}{6}} \eta \left( \frac{is}{2\pi} \right) h \left( \frac{is}{2\pi}; \frac{3is}{2\pi} \right) \right| \right). \quad (3.8)$$

By [12, Lemma 3.4] or [15, Lemma 3.4], we obtain

$$h \left( 3z \pm \frac{is}{2\pi}; \frac{3is}{2\pi} \right) = O \left( e^{\frac{-\beta}{6}} \right).$$

This together with (3.8) and (2.5) gives

$$\frac{\zeta^{\frac{1}{2}} g_4(z; s)}{1 - \zeta} = O \left( |s|^{-1/2} e^{-\frac{\pi^2}{6} \operatorname{Re}(\frac{1}{s})} \right),$$

as  $z \rightarrow 0^+$ .

When  $z$  is away from 0, there holds

$$\frac{\zeta^{\frac{1}{2}} g_4(z; s)}{1 - \zeta} = O \left( \left| e^{\frac{s}{6}} \eta \left( \frac{is}{2\pi} \right) h \left( 3z \pm \frac{is}{2\pi}; \frac{3is}{2\pi} \right) \right| \right) = O \left( |s|^{-1/2} e^{-\frac{\pi^2}{6} \operatorname{Re}(\frac{1}{s})} \right).$$

This proves (3.6).  $\square$

Lemma 3.1 and equation (3.3) give the following estimation:

$$S(z; \tau) = \frac{q^{\frac{1}{24}}}{\eta(\tau)} \left( \frac{-\zeta^{\frac{1}{2}} g_1(z; s)}{1 - \zeta} + O \left( |s|^{-1/2} e^{-\frac{\pi^2}{6} \operatorname{Re}(\frac{1}{s})} \right) \right), \quad (3.9)$$

with  $|x| \leq 1$  and  $n$  approaching  $\infty$ .

### 3.2. Bounds near the dominant pole

In this section, we study asymptotic behavior of  $S_m(q)$  as  $q \rightarrow 1$ . Thus we assume  $|x| \leq 1$  throughout this section.

In view of equations (3.1), (3.3) and (3.9), we define

$$\mathcal{G}_{m,1}(s) := \int_0^{\frac{1}{2}} \frac{-\zeta^{\frac{1}{2}} g_1(z; s)}{1 - \zeta} \cos(2m\pi z) dz = \frac{-\pi}{2s} \int_0^{\frac{1}{2}} \frac{e^{\frac{2\pi^2 z^2}{s}} - e^{\frac{6\pi^2 z^2}{s}}}{\sin(\pi z) \sinh \left( \frac{2\pi^2 z}{s} \right)} \cos(2m\pi z) dz$$

and

$$\mathcal{G}_{m,2}(s) := \int_0^{\frac{1}{2}} \frac{-\zeta^{\frac{1}{2}} (g_2(z; s) + g_3(z; s) + g_4(z; s))}{1 - \zeta} \cos(2m\pi z) dz.$$

Thus

$$S_m(q) = \frac{2q^{\frac{1}{24}}}{\eta(\tau)} (\mathcal{G}_{m,1}(s) + \mathcal{G}_{m,2}(s)). \quad (3.10)$$

**Lemma 3.2.** Assume that  $|x| \leq 1$  and  $m \leq n^{\frac{3}{8}}$ . Then we have  $n \rightarrow \infty$

$$\mathcal{G}_{m,1}(s) = \frac{1}{8} \operatorname{sech}^2\left(\frac{m\beta}{2}\right) + O\left(\operatorname{sech}^2\left(\frac{m\beta}{2}\right) \beta m^{\frac{2}{3}}\right). \quad (3.11)$$

**Proof.** Inserting the Taylor expansions of  $\cos$  and  $\exp$ , we obtain

$$\left(e^{\frac{2\pi^2 z^2}{s}} - e^{\frac{6\pi^2 z^2}{s}}\right) \cos(2m\pi z) = \sum_{k \geq 1, j \geq 0} \frac{(-1)^j \pi^{2k+2j} (2^k - 6^k) (2m)^{2j}}{k!(2j)!s^k} z^{2k+2j}.$$

This gives

$$\mathcal{G}_{m,1}(s) = \frac{-1}{2s} \sum_{k \geq 1, j \geq 0} \frac{(-1)^j \pi^{2k+2j} (2^k - 6^k) (2m)^{2j}}{k!(2j)!s^k} \mathcal{J}_{k+j}, \quad (3.12)$$

where

$$\mathcal{J}_{k+j} := \int_0^{\frac{1}{2}} \frac{\pi z^{2k+2j}}{\sin(\pi z) \sinh\left(\frac{2\pi^2 z}{s}\right)} dz.$$

Applying the Mittag-Leffler partial fraction decomposition, we find that

$$\frac{\pi}{\sin(\pi z)} = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \left( \frac{1}{z-n} + \frac{1}{z+n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n \left( \frac{2z}{z^2 - n^2} \right).$$

Then, for  $0 \leq z \leq \frac{1}{2}$ , we have

$$\frac{\pi}{\sin(\pi z)} = \frac{1}{z} + O(1).$$

Thus

$$\mathcal{J}_{k+j} = \int_0^{\frac{1}{2}} \frac{z^{2k+2j-1} (1 + O(z))}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz. \quad (3.13)$$

Proceeding as in the proof of [10, Lemma 3.2], we can show that

$$\begin{aligned} \int_{\sqrt{|s|}}^{\frac{1}{2}} \frac{z^l}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz &\ll \int_{\sqrt{|s|}}^{\infty} \left| \frac{z^l}{\sinh\left(\frac{2\pi^2 z}{s}\right)} \right| dz \\ &\ll \operatorname{Re} \left( \frac{1}{s} \right)^{-1} |s|^{\frac{l}{2}} e^{-2\pi^2 \sqrt{|s|} \operatorname{Re}(\frac{1}{s})} \\ &\ll e^{-\pi^2 \sqrt{|s|} \operatorname{Re}(\frac{1}{s})}, \end{aligned} \quad (3.14)$$

as  $n \rightarrow \infty$ . This together with (3.13) implies that

$$\begin{aligned}
\mathcal{J}_{k+j} &= \int_0^{\sqrt{|s|}} \frac{z^{2k+2j-1}(1+O(z))}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz + \int_{\sqrt{|s|}}^{\frac{1}{2}} \frac{z^{2k+2j-1}(1+O(z))}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz \\
&= \int_0^{\sqrt{|s|}} \frac{z^{2k+2j-1}(1+O(z))}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz + O\left(\operatorname{Re}\left(\frac{1}{s}\right)^{-1} |s|^{k+j-1/2} e^{-2\pi^2 \sqrt{|s|} \operatorname{Re}(\frac{1}{s})}\right) \\
&= \int_0^{\sqrt{|s|}} \frac{z^{2k+2j-1}(1+O(z))}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz + O\left(|s|^k e^{-\pi^2 \sqrt{|s|} \operatorname{Re}(\frac{1}{s})}\right). \tag{3.15}
\end{aligned}$$

Substituting (3.15) into (3.12), we obtain

$$\begin{aligned}
\mathcal{G}_{m,1}(s) &= \frac{-1}{2s} \sum_{k \geq 1, j \geq 0} \frac{(-1)^j \pi^{2k+2j} (2^k - 6^k) (2m)^{2j}}{k!(2j)!s^k} \left\{ \int_0^{\sqrt{|s|}} \frac{z^{2k+2j-1}(1+O(z))}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz \right. \\
&\quad \left. + O\left(|s|^k e^{-\pi^2 \sqrt{|s|} \operatorname{Re}(\frac{1}{s})}\right) \right\} \\
&= \frac{-1}{2s} \sum_{j \geq 0} \frac{(-1)^j (2\pi m)^{2j}}{(2j)!} \left\{ \frac{-4\pi^2}{s} \left( \int_0^{\sqrt{|s|}} \frac{z^{2j+1}(1+O(z))}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz + O\left(|s| e^{-\pi^2 \sqrt{|s|} \operatorname{Re}(\frac{1}{s})}\right) \right) \right. \\
&\quad \left. + \sum_{k=2}^{\infty} \frac{\pi^{2k} (2^k - 6^k)}{k!s^k} \left( \int_0^{\sqrt{|s|}} \frac{z^{2k+2j-1}(1+O(z))}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz + O\left(|s|^k e^{-\pi^2 \sqrt{|s|} \operatorname{Re}(\frac{1}{s})}\right) \right) \right\} \tag{3.16}
\end{aligned}$$

Next, we note that

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{\pi^{2k} (2^k - 6^k)}{k!s^k} \left( \int_0^{\sqrt{|s|}} \frac{z^{2k+2j-1}(1+O(z))}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz + O\left(|s|^k e^{-\pi^2 \sqrt{|s|} \operatorname{Re}(\frac{1}{s})}\right) \right) \\
&= \int_0^{\sqrt{|s|}} \left( \sum_{k=2}^{\infty} \frac{\pi^{2k} (2^k - 6^k) z^{2k}}{k!s^k} \right) \frac{z^{2j-1}(1+O(z))}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz \\
&\quad + \sum_{k=2}^{\infty} \frac{\pi^{2k} (2^k - 6^k)}{k!s^k} O\left(|s|^k e^{-\pi^2 \sqrt{|s|} \operatorname{Re}(\frac{1}{s})}\right) \\
&= \int_0^{\sqrt{|s|}} \left( \sum_{k=0}^{\infty} \frac{\pi^{2(k+2)} (2^{k+2} - 6^{k+2}) z^{2k}}{(k+2)!s^k} \right) \frac{z^{2j+3}(1+O(z))}{s^2 \sinh\left(\frac{2\pi^2 z}{s}\right)} dz + O\left(e^{-\pi^2 \sqrt{|s|} \operatorname{Re}(\frac{1}{s})}\right) \\
&= O\left(|s|^{-2} \int_0^{\sqrt{|s|}} \frac{z^{2j+3}(1+O(z))}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz\right) + O\left(e^{-\pi^2 \sqrt{|s|} \operatorname{Re}(\frac{1}{s})}\right). \tag{3.17}
\end{aligned}$$

Substituting (3.17) into (3.16), we have

$$\begin{aligned} \mathcal{G}_{m,1}(s) = & \frac{-1}{2s} \sum_{j \geq 0} \frac{(-1)^j (2\pi m)^{2j}}{(2j)!} \left\{ \frac{-4\pi^2}{s} \left( \int_0^{\sqrt{|s|}} \frac{z^{2j+1} (1 + O(z))}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz + O\left(|s| e^{-\pi^2 \sqrt{|s|} \operatorname{Re}(\frac{1}{s})}\right) \right) \right. \\ & \left. + O\left(|s|^{-2} \int_0^{\sqrt{|s|}} \frac{z^{2j+3} (1 + O(z))}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz\right) + O\left(e^{-\pi^2 \sqrt{|s|} \operatorname{Re}(\frac{1}{s})}\right) \right\} \end{aligned}$$

Thus, by (3.14), we find that

$$\begin{aligned} \mathcal{G}_{m,1}(s) = & \frac{-1}{2s} \sum_{j \geq 0} \frac{(-1)^j (2\pi m)^{2j}}{(2j)!} \left\{ \frac{-4\pi^2}{s} \left( \int_0^{\infty} \frac{z^{2j+1} (1 + O(z))}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz + O\left(|s| e^{-\pi^2 \sqrt{|s|} \operatorname{Re}(\frac{1}{s})}\right) \right) \right. \\ & \left. + O\left(|s|^{-2} \int_0^{\infty} \frac{z^{2j+3} (1 + O(z))}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz\right) + O\left(e^{-\pi^2 \sqrt{|s|} \operatorname{Re}(\frac{1}{s})}\right) \right\} \\ = & \frac{-1}{2s} \sum_{j \geq 0} \frac{(-1)^j (2\pi m)^{2j}}{(2j)!} \left\{ \frac{-4\pi^2}{s} \int_0^{\infty} \frac{z^{2j+1} (1 + O(z))}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz \right. \\ & \left. + O\left(|s|^{-2} \int_0^{\infty} \frac{z^{2j+3} (1 + O(z))}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz\right) + O\left(e^{-\pi^2 \sqrt{|s|} \operatorname{Re}(\frac{1}{s})}\right) \right\} \end{aligned} \quad (3.18)$$

Next, recalling the definition of  $\mathcal{I}_l$  in Section 2.2 and Lemma 2.5, we get that

$$\int_0^{\infty} \frac{z^l}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz = \frac{l! (2 - 2^{-l}) s^{l+1}}{2^{l+1} \pi^{2l+2}} \zeta(l+1).$$

This implies that

$$\begin{aligned} \int_0^{\infty} \frac{z^{l+1}}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz &= \frac{(l+1)! (2 - 2^{-l-1}) s^{l+2}}{2^{l+2} \pi^{2l+4}} \zeta(l+2) \\ &= O\left((l+1)|s| \int_0^{\infty} \frac{z^l}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz\right). \end{aligned}$$

Hence, we have

$$\int_0^{\infty} \frac{z^{2j+1} (1 + O(z))}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz = (1 + O((2+2j)|s|)) \int_0^{\infty} \frac{z^{2j+1}}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz$$

and

$$\begin{aligned} |s|^{-2} \int_0^{\infty} \frac{z^{2j+3} (1 + O(z))}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz &= O\left((1 + (4+2j)|s|) (2j+2)(2j+3) \int_0^{\infty} \frac{z^{2j+1}}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz\right) \\ &\gg O\left(e^{-\pi^2 \sqrt{|s|} \operatorname{Re}(\frac{1}{s})}\right) \end{aligned}$$

Substituting the above equations into (3.18) and simplifying, we find that

$$\mathcal{G}_{m,1}(s) = \frac{-1}{2s} \sum_{j \geq 0} \frac{(-1)^j (2\pi m)^{2j}}{(2j)!} \left\{ \frac{-4\pi^2}{s} \int_0^\infty \frac{z^{2j+1}}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz \right. \\ \left. \times \left( 1 + O\left( (2+2j)(3+2j) (1 + (4+2j)|s|) |s| \right) \right) \right\}$$

By (2.16), we have

$$\int_0^\infty \frac{z^{2j+1}}{\sinh\left(\frac{2\pi^2 z}{s}\right)} dz = \frac{(-1)^{j+1} s^{2j+2} E_{2j+1}(0)}{2^{2j+3} \pi^{2j+2}}.$$

Thus

$$\mathcal{G}_{m,1}(s) = \frac{-1}{4} \sum_{j \geq 0} \frac{(sm)^{2j} E_{2j+1}(0)}{(2j)!} \times \left( 1 + O\left( (2+2j)(3+2j) (1 + (4+2j)|s|) |s| \right) \right) \\ = \frac{1}{8} \operatorname{sech}^2\left(\frac{ms}{2}\right) (1 + O(|s|)),$$

where the last equality follows from Lemma 2.4. We note that as  $n \rightarrow \infty$

$$\operatorname{sech}^2\left(\frac{ms}{2}\right) = \operatorname{sech}^2\left(\frac{m\beta}{2}\right) \left( 1 + O\left(\beta m^{\frac{2}{3}}\right) \right),$$

for  $m \leq n^{\frac{3}{8}}$ . Thus

$$\mathcal{G}_{m,1}(s) = \frac{1}{8} \operatorname{sech}^2\left(\frac{m\beta}{2}\right) \left( 1 + O\left(\beta m^{\frac{2}{3}}\right) \right). \quad \square$$

Next, we bound  $\mathcal{G}_{m,2}(s)$  as follows.

**Lemma 3.3.** *Let  $|x| \leq 1$ . Then, as  $n \rightarrow \infty$ , we have*

$$\mathcal{G}_{m,2}(s) = O\left(\beta^{-1} e^{-\frac{\pi^2}{16\beta}}\right).$$

**Proof.** By Lemma 3.1, we have

$$\mathcal{G}_{m,2}(s) \ll \int_0^{\frac{1}{2}} |s|^{-1} e^{-\frac{\pi^2}{6} \operatorname{Re}(\frac{1}{s})} |\cos(2m\pi z)| dz \ll |s|^{-1} e^{-\frac{\pi^2}{6} \operatorname{Re}(\frac{1}{s})}.$$

Since  $s = \beta \left( 1 + i x m^{-\frac{1}{3}} \right)$  and  $|x| \leq 1$ , we have  $|s|^{-1} \leq \frac{1}{\beta}$  and  $\frac{1}{2\beta} \leq \operatorname{Re}\left(\frac{1}{s}\right)$ . Thus

$$\mathcal{G}_{m,2}(s) \ll \beta^{-1} e^{-\frac{\pi^2}{12\beta}}. \quad \square \tag{3.19}$$

Applying Lemma 3.2 and 3.3, we obtain the following asymptotic behavior of  $S_m(\tau)$  near the dominant pole  $q = 1$ .

**Proposition 3.4.** *Let  $|x| \leq 1$ . Then we have as  $n \rightarrow \infty$*

$$S_m(\tau) = \frac{\sqrt{2}s^{\frac{1}{2}}}{8\sqrt{\pi}} e^{\frac{\pi^2}{6s}} \operatorname{sech}^2\left(\frac{m\beta}{2}\right) + O\left(\beta^{\frac{3}{2}} m^{\frac{2}{3}} \operatorname{sech}^2\left(\frac{m\beta}{2}\right) e^{\pi\sqrt{\frac{\pi}{6}}}\right).$$

**Proof.** By (2.5), we find that

$$\frac{q^{\frac{1}{24}}}{\eta(\tau)} = \sqrt{\frac{s}{2\pi}} e^{\frac{\pi^2}{6s}} \{1 + O(\beta)\}.$$

Substituting the above equation, (3.11) and (3.19) into (3.10) and noting that  $\mathcal{G}_{m,1}(s)$  gives the main error term, we find that

$$S_m(\tau) = \frac{\sqrt{2}s^{\frac{1}{2}}}{8\sqrt{\pi}} e^{\frac{\pi^2}{6s}} \operatorname{sech}^2\left(\frac{m\beta}{2}\right) + O\left(\sqrt{|s|}\beta m^{\frac{2}{3}} \operatorname{sech}^2\left(\frac{m\beta}{2}\right) e^{\frac{\pi^2}{6}\operatorname{Re}(\frac{1}{s})}\right).$$

This proves the claim since we have  $|s| = O(\beta)$  and  $\operatorname{Re}(\frac{1}{s}) \leq \frac{1}{\beta} = \frac{\sqrt{6n}}{\pi}$  when  $|x| \leq 1$ .  $\square$

### 3.3. Bounds away from the dominant pole

In this section, we study the asymptotic behavior of  $S_m(\tau)$  away from the dominant pole  $q = 1$ . We assume that  $1 \leq |x| \leq \frac{\pi m^{\frac{1}{3}}}{\beta}$ .

**Proposition 3.5.** *Assume that  $1 \leq |x| \leq \frac{\pi m^{\frac{1}{3}}}{\beta}$ . Then, as  $n \rightarrow \infty$ , we have*

$$S_m(\tau) \ll n^{\frac{5}{4}} \exp\left(\pi\sqrt{\frac{n}{6}} - \frac{\sqrt{6n}}{2\pi} m^{-\frac{2}{3}}\right). \quad (3.20)$$

**Proof.** By [5, Corollary 2.5], we have

$$S(z; \tau) = \frac{1}{(q; q)_{\infty}} \left( \sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2}}{(1 - \zeta q^n)(1 - q^n/\zeta)} - \sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1} q^{n(3n+1)/2}}{(1 - \zeta q^n)(1 - q^n/\zeta)} \right),$$

which gives

$$S_m(q) = \frac{2}{(q; q)_{\infty}} \int_0^{\frac{1}{2}} \left( \sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2} (1 - q^{n^2})}{(1 - \zeta q^n)(1 - q^n/\zeta)} \right) \cos(2\pi m z) dz \quad (3.21)$$

We bound the integral as follows:

$$\begin{aligned} & \left| \int_0^{\frac{1}{2}} \left( \sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2} (1 - q^{n^2})}{(1 - \zeta q^n)(1 - q^n/\zeta)} \right) \cos(2\pi m z) dz \right| \\ & \leq \int_0^{\frac{1}{2}} \left| \sum_{n \neq 0} \frac{(-1)^{n-1} q^{n(n+1)/2} (1 - q^{n^2})}{(1 - \zeta q^n)(1 - q^n/\zeta)} \right| dz \end{aligned}$$

$$\begin{aligned}
&\ll \frac{1}{(1-|q|)^2} \sum_{n=-\infty}^{\infty} |q|^{n(n+1)/2} (1+|q|^{n^2}) \\
&\ll \frac{1}{(1-|q|)^3} = O(\beta^{-3}) = O\left(n^{\frac{3}{2}}\right).
\end{aligned} \tag{3.22}$$

Next, applying [15, Lemma 4.6] with  $v = \frac{\beta}{2\pi}$ ,  $u = \frac{\beta m^{-\frac{1}{3}}x}{2\pi}$  and  $M = m^{-\frac{1}{3}}$ , we obtain

$$\frac{1}{(q; q)_{\infty}} \ll n^{-\frac{1}{4}} \exp \left[ \frac{2\pi}{\beta} \left( \frac{\pi}{12} - \frac{1}{2\pi} \left( 1 - \frac{1}{\sqrt{1+m^{-\frac{2}{3}}}} \right) \right) \right].$$

Thus

$$\begin{aligned}
S_m(\tau) &\ll n^{\frac{5}{4}} \exp \left[ \frac{2\pi}{\beta} \left( \frac{\pi}{12} - \frac{1}{2\pi} \left( 1 - \frac{1}{\sqrt{1+m^{-\frac{2}{3}}}} \right) \right) \right] \\
&\ll n^{\frac{5}{4}} \exp \left[ \pi \sqrt{\frac{n}{6}} - \frac{\sqrt{6n}}{\pi} \left( 1 - \frac{1}{\sqrt{1+m^{-\frac{2}{3}}}} \right) \right]
\end{aligned}$$

This proves (3.20) since  $\frac{1}{\sqrt{1+m^{-\frac{2}{3}}}} \leq 1 - \frac{m^{-\frac{2}{3}}}{2}$ .  $\square$

#### 4. The Circle Method

We use the Circle Method [24] to prove Theorem 1.2. Applying Cauchy's residue theorem, we have the following representation of the coefficients of  $S_m(\tau)$ :

$$N_S(m, n) = \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{S_m(\tau)}{q^{n+1}} dq$$

where the contour is the counterclockwise traversal of the circle  $\mathcal{C} := \{|q| = e^{-\beta}\}$ . Note that  $s = \beta \left( 1 + ixm^{-\frac{1}{3}} \right)$ . Changing variables we write

$$N_S(m, n) = \frac{\beta}{2\pi m^{\frac{1}{3}}} \int_{|x| \leq \frac{\pi m^{\frac{1}{3}}}{\beta}} S_m \left( \frac{is}{2\pi} \right) e^{ns} dx.$$

We split the integral into two pieces  $N_S(m, n) = M + E$  with

$$M := \frac{\beta}{2\pi m^{\frac{1}{3}}} \int_{|x| \leq 1} S_m \left( \frac{is}{2\pi} \right) e^{ns} dx$$

and

$$E := \frac{\beta}{2\pi m^{\frac{1}{3}}} \int_{1 \leq |x| \leq \frac{\pi m^{\frac{1}{3}}}{\beta}} S_m \left( \frac{is}{2\pi} \right) e^{ns} dx.$$

We show later that the main term in the asymptotic formula of  $N_S(m, n)$  comes only from  $M$ , however, the integral  $E$  is part of the error term.



#### 4.1. Main arc

We first consider the integral  $M$ .

**Proposition 4.1.** *We have, as  $n \rightarrow \infty$ ,*

$$M = \frac{\beta}{4} \operatorname{sech}^2 \left( \frac{m\beta}{2} \right) \operatorname{spt}(n) \left( 1 + O \left( m^{\frac{1}{3}} \beta^{\frac{1}{2}} \right) \right).$$

**Proof.** Applying Proposition 3.4, we have

$$M = \frac{\sqrt{2}\beta \operatorname{sech}^2 \left( \frac{m\beta}{2} \right)}{16\pi^{\frac{3}{2}} m^{\frac{1}{3}}} \int_{|x| \leq 1} s^{\frac{1}{2}} e^{\frac{\pi^2}{6s} + ns} dx + O \left( \beta^{\frac{5}{2}} m^{\frac{1}{3}} \operatorname{sech}^2 \left( \frac{m\beta}{2} \right) e^{\pi \sqrt{\frac{2n}{3}}} \right).$$

Making the change of variables  $v = 1 + xim^{-\frac{1}{3}}$  and noting that  $s = \beta v$ , we find that

$$\int_{|x| \leq 1} s^{\frac{1}{2}} e^{\frac{\pi^2}{6s} + ns} dx = -im^{\frac{1}{3}} \beta^{\frac{1}{2}} \int_{1-im^{-\frac{1}{3}}}^{1+im^{-\frac{1}{3}}} v^{\frac{1}{2}} e^{\pi \sqrt{\frac{\pi}{6}}(v + \frac{1}{v})} dv.$$

Thus

$$M = \frac{\sqrt{2}\beta^{\frac{3}{2}} \operatorname{sech}^2 \left( \frac{m\beta}{2} \right)}{16i\pi^{\frac{3}{2}}} \int_{1-im^{-\frac{1}{3}}}^{1+im^{-\frac{1}{3}}} v^{\frac{1}{2}} e^{\pi \sqrt{\frac{\pi}{6}}(v + \frac{1}{v})} dv + O \left( \beta^{\frac{5}{2}} m^{\frac{1}{3}} \operatorname{sech}^2 \left( \frac{m\beta}{2} \right) e^{\pi \sqrt{\frac{2n}{3}}} \right).$$

By [10, Lemma 42], we find that

$$\frac{1}{2\pi i} \int_{1-im^{-\frac{1}{3}}}^{1+im^{-\frac{1}{3}}} v^{\frac{1}{2}} e^{\pi \sqrt{\frac{\pi}{8}}(v + \frac{1}{v})} dv = I_{-\frac{3}{2}} \left( \pi \sqrt{\frac{2n}{3}} \right) + O \left( \exp \left( \pi \sqrt{\frac{n}{6}} \left( 1 + \frac{1}{1 + m^{-\frac{2}{3}}} \right) \right) \right),$$

where the  $I_l$  is the usual  $I$ -Bessel function of order  $l$ . Thus

$$\begin{aligned} M &= \frac{\sqrt{2}\beta^{\frac{3}{2}} \operatorname{sech}^2 \left( \frac{m\beta}{2} \right)}{8\pi^{\frac{1}{2}}} I_{-\frac{3}{2}} \left( \pi \sqrt{\frac{2n}{3}} \right) + O \left( \beta^{\frac{3}{2}} \operatorname{sech}^2 \left( \frac{m\beta}{2} \right) \exp \left( \pi \sqrt{\frac{n}{6}} \left( 1 + \frac{1}{1 + m^{-\frac{2}{3}}} \right) \right) \right) \\ &\quad + O \left( \beta^{\frac{5}{2}} m^{\frac{1}{3}} \operatorname{sech}^2 \left( \frac{m\beta}{2} \right) e^{\pi \sqrt{\frac{2n}{3}}} \right). \end{aligned}$$

Noting that

$$I_{-\frac{3}{2}} \left( \pi \sqrt{\frac{2n}{3}} \right) = \frac{e^{\pi \sqrt{\frac{2n}{3}}}}{\pi \sqrt{2} \left( \frac{2n}{3} \right)^{\frac{1}{4}}} + O \left( \frac{e^{\pi \sqrt{\frac{2n}{3}}}}{n^{\frac{3}{4}}} \right)$$

(by [6, Eq. (4.12.7)]), we have

$$\begin{aligned}
M &= \frac{\sqrt{2}\beta^{\frac{3}{2}}\operatorname{sech}^2\left(\frac{m\beta}{2}\right)}{8\pi^{\frac{1}{2}}}\left(\frac{e^{\pi\sqrt{\frac{2n}{3}}}}{\pi\sqrt{2}\left(\frac{2n}{3}\right)^{\frac{1}{4}}}+O\left(\frac{e^{\pi\sqrt{\frac{2n}{3}}}}{n^{\frac{3}{4}}}\right)\right) \\
&\quad +O\left(\beta^{\frac{3}{2}}\operatorname{sech}^2\left(\frac{m\beta}{2}\right)\exp\left(\pi\sqrt{\frac{n}{6}}\left(1+\frac{1}{1+m^{-\frac{2}{3}}}\right)\right)\right)+O\left(\beta^{\frac{5}{2}}m^{\frac{1}{3}}\operatorname{sech}^2\left(\frac{m\beta}{2}\right)e^{\pi\sqrt{\frac{2n}{3}}}\right) \\
&= \frac{\beta^{\frac{3}{2}}\operatorname{sech}^2\left(\frac{m\beta}{2}\right)e^{\pi\sqrt{\frac{2n}{3}}}}{8\pi^{\frac{3}{2}}\left(\frac{2n}{3}\right)^{\frac{1}{4}}}\left(1+O\left(m^{\frac{1}{3}}n^{-\frac{1}{4}}\right)\right), \tag{4.1}
\end{aligned}$$

where in the second equality we note that the last error term on the left-hand side is the dominant one. It is shown in [9] that

$$\operatorname{spt}(n) \sim \frac{1}{2\sqrt{2}\pi\sqrt{n}}e^{\pi\sqrt{\frac{2n}{3}}}\left(1+O\left(n^{-\frac{1}{2}}\right)\right),$$

as  $n \rightarrow \infty$ . Thus, the claim follows.  $\square$

#### 4.2. Error arc

**Proposition 4.2.** *As  $n \rightarrow \infty$ , we have*

$$E \ll n^{\frac{5}{4}}\exp\left(\pi\sqrt{\frac{2n}{3}}-\frac{\sqrt{6n}}{2\pi}m^{-\frac{2}{3}}\right).$$

**Proof.** By (3.20), we have

$$\begin{aligned}
E &\ll \frac{\beta}{m^{\frac{1}{3}}}\int_{1\leq|x|\leq\frac{\pi m^{\frac{1}{3}}}{\beta}}n^{\frac{5}{4}}\exp\left(\pi\sqrt{\frac{n}{6}}-\frac{\sqrt{6n}}{2\pi}m^{-\frac{2}{3}}\right)e^{\beta n}dx \\
&\ll n^{\frac{5}{4}}\exp\left(\pi\sqrt{\frac{2n}{3}}-\frac{\sqrt{6n}}{2\pi}m^{-\frac{2}{3}}\right). \quad \square
\end{aligned}$$

Thus  $E$  is exponentially smaller than  $M$  and Theorem 1.2 follows.

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