J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The singular value expansion of the Volterra integral equation associated to a numerical differentiation problem

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ARTICLE INFO

Article history:
Received 19 July 2017
Available online xxxx
Submitted by B. Kaltenbacher

Keywords:
Numerical differentiation
Singular value expansion
Volterra integral equation

ABSTRACT

We consider the Volterra integral equation of the first kind for the derivative of a given function with one-side boundary conditions. We give a method to obtain the singular value expansion for the corresponding integral kernel. This singular value expansion can be used to give algorithms for the solution of the numerical differentiation problem. A numerical experiment shows the results obtained by a simple version of such algorithms.

1. Introduction

We consider the numerical differentiation problem, where the derivatives of a function are approximated by using values of the function and eventually other knowledge about the function itself.

The direct use of numerical differentiation methods can support applications where functions are known only on discrete sets, such as in the context of sampling processes, or applications where the derivatives computation involves too complex formulas. A usual situation where such applications require the computation of numerical differentiation is the solution of optimization problems by derivative-free methods, see [4] for details.

A problem of slightly different nature is the solution of differential equations giving a relation between the unknown function and its derivatives. For such problems the approximation of derivatives plays a central role and allows the definition of algebraic equations for the corresponding numerical solution [6].

The simplest method for the numerical differentiation is given by the finite difference approximations. Despite their popularity, finite difference methods for the evaluation of the derivative have low accuracy and stability properties [7], [13]. Nevertheless, if the function is analytic on a neighborhood of the derivation point and it can be evaluated for each point of this neighborhood, the problem is well-conditioned [11] and it

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https://doi.org/10.1016/j.jmaa.2017.12.007
0022-247X/

can be efficiently solved by numerical methods. So, in scientific literature, several numerical differentiation algorithms of arbitrary analytic function are present.

For example, methods to approximate derivatives of real functions using complex variables have been studied in [1], [2], [7], [15], [18]. In particular, the method proposed in [1] is based on numerical inversion of a complex Laplace transform; the one proposed in [7] uses the Fast Fourier Transform.

Differential quadrature [22] is another well-known method where derivatives are approximated by weighted sums of function values, and it has been applied extensively in various engineering problems [17]. The weighting coefficients of the polynomial based, Fourier expansion based and exponential based differential quadrature methods can be computed explicitly.

Numerical differentiation algorithms based on polynomial interpolation approximate the function derivative by the derivative of the interpolation polynomials. Some versions of this strategy have obtained good results in terms of accuracy and stability; for example, [3] uses low-order Chebyshev interpolation polynomials to compute the derivative of noisy functions, [10] and [12] use Neville algorithm for computing the interpolating polynomial in order to compute stable approximation of function derivative. We note that this algorithm is used in the routine D04AAF of the NAG Library [14] to approximate the derivatives up to order 14.

From standard arguments on Taylor series, the differentiation problem with one-side boundary conditions can be reformulated as a Volterra integral equation of the first kind. Several authors have discussed the use of such an integral equation for the numerical differentiation problem. For example, in [9] it is used to compute the stepsize in the finite-difference methods, in [21] has been proposed a sparse discretization of this integral equation, in [20] a fast multiscale solver has been proposed for the numerical solution of the Tikhonov regularization equation.

In this paper we consider the problem of numerical differentiation reformulated as this Volterra integral equation of the first kind. We present a method for the construction of the singular value expansion of the kernel of such an integral equation; so that, it allows the definition of simple algorithms to solve this integral equation and, in turn, to compute the numerical derivatives of a given function. A numerical experiment is used to test the proposed method by comparing the corresponding results with the ones obtained by a well-established scientific software.

In Section 2 the problem of numerical differentiation is described together with the corresponding Volterra integral equation of the first kind, as well as its solution obtained by using the singular value expansion of the corresponding integral kernel K. In Section 3, the characteristic equation for the singular values of K is given together with the analytic expressions of the corresponding left-singular functions and right-singular functions. In Section 4 some numerical examples are given. Section 5 describes some conclusions and future developments.

2. The integral equation for the derivation problem

Let $\nu \geq 1$ be a given integer number, and let $f : [0, 1] \to \mathbb{R}$ be a continuously differentiable function up to order ν, and suppose that $f^{(j)}(0)$, $j = 0, 1, \dots, \nu-1$, are known or already calculated, where $f^{(j)}$ denotes the jth derivative of f. Hence, from standard arguments on Taylor formula, we have that the integral equation with unknown function $v : [0, 1] \to \mathbb{R}$

$$\mathcal{K}v(t) = f(t) - \sum_{j=0}^{\nu-1} \frac{f^{(j)}(0)}{j!} t^j, \quad t \in [0, 1],$$

where $\mathcal{K}$ is the integral operator having kernel $K : [0, 1] \times [0, 1] \to \mathbb{R}$ defined by

$$K(t,s) = \begin{cases} \frac{(t-s)^{\nu-1}}{(\nu-1)!}, & 0 \le s < t \le 1, \\ 0, & 0 \le t \le s \le 1, \end{cases} \tag{1}$$

has unique solution $v = f^{(\nu)}$ (see [21] for details).

Therefore $v = f^{(\nu)}$ is the unique solution of the following Volterra integral equation of the first kind:

$$\int_0^t \frac{(t-s)^{\nu-1}}{(\nu-1)!} v(s)ds = g(t), \tag{2}$$

where $t \in [0,1]$, and the known function

$$g(t) = f(t) - \sum_{j=0}^{\nu-1} \frac{f^{(j)}(0)}{j!} t^j, \quad t \in [0,1], \tag{3}$$

has the properties that $g^{(j)}(0) = 0$, $j = 0, 1, \ldots, \nu-1$ and $g^{(\nu)}(t) = f^{(\nu)}(t)$, $t \in [0,1]$. In particular, when g is a continuously differentiable function up to order ν such that $g^{(j)}(0) = 0$, $j = 0, 1, \ldots, \nu-1$, then $v = g^{(\nu)}$ is the solution of

$$\int_0^1 K(t,s)v(s)ds = g(t), \quad t \in [0,1]. \tag{4}$$

We note that the differentiation problem can be formulate by other different integral equations. The theory of Green's functions for ordinary differential equations [19] gives a very standard approach; for example, in the case $\nu = 2$, the operator $\mathcal{L}g = g^{(2)}$ with boundary conditions $g(0) = 0$ and $g(1) = 0$, has the following Green's function:

$$G(t,s) = \begin{cases} t(s-1), & 0 \le t < s \le 1, \\ s(t-1), & 0 \le s \le t \le 1, \end{cases}$$

so, the following Fredholm integral equation of first kind:

$$\int_0^1 G(t,s)v(s)ds = g(t), \quad t \in [0,1]$$

has solution $v = g^{(2)}$, see [5] and the references therein for details of such an integral formulation and its numerical approach. We note that for the same derivation order $\nu = 2$, but with initial conditions $g(0) = 0$ and $g'(0) = 0$ we have that $v = g^{(2)}$ is the solution of the Volterra integral equation (2), or specifically:

$$\int_0^t (t-s)v(s)ds = g(t).$$

In Section 2.1 we recall some general properties of the singular value expansion of compact operators; in Section 2.2 we introduce some notations; in Section 2.3 we give some preliminar results.

2.1. The singular value expansion

From standard arguments on integral operators with square integrable kernels, we have that there exists a Singular Value Expansion (SVE) of the kernel (1), that is

$$K(t,s) = \sum_{l=1}^{\infty} \mu_l u_l(t) v_l(s), \quad t,s \in [0,1],$$

where $\mu_1 \geq \mu_2 \geq \dots$ are the singular values of K, for $l = 1, 2, \dots$, $\mu_l > 0$, moreover u_l and v_l are the corresponding left-singular function and right-singular function, respectively, see [8] for details.

Let $\mathcal{K}^*$ be the adjoint integral operator of $\mathcal{K}$, then its kernel is

$$K^*(s,t) = K(t,s), \quad t,s \in [0,1],$$

and

$$\mathcal{K} v_l = \mu_l u_l, \quad \mathcal{K}^* u_l = \mu_l v_l, \quad l = 1, 2, \dots. \tag{5}$$

Moreover the solution of (4) is

$$g^{(\nu)}(t) = \sum_{l=1}^{\infty} \frac{\langle g, u_l \rangle}{\mu_l} v_l(t), \tag{6}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on the space of real square integrable functions on $[0,1]$.

In the distribution sense, we have

$$\frac{d^\nu}{dt^\nu} K(t,s) = \delta(t-s),$$
$$\frac{d^\nu}{ds^\nu} K^*(s,t) = (-1)^\nu \delta(t-s),$$

where δ denotes the Dirac's delta, and so for $l = 1, 2, \dots$ we have

$$v_l = \frac{d^\nu}{dt^\nu} (\mathcal{K} v_l) = \mu_l \frac{d^\nu}{dt^\nu} u_l, \tag{7}$$

$$u_l = (-1)^\nu \frac{d^\nu}{dt^\nu} (\mathcal{K}^* u_l) = (-1)^\nu \mu_l \frac{d^\nu}{dt^\nu} v_l. \tag{8}$$

2.2. Notations and properties

Let $\mathbb{R}^n$ be the n-dimensional real Euclidean space, $\iota \in \mathbb{C}$ be the imaginary unit and $\mathbb{C}^n$ be the n-dimensional complex Hilbert space. Let $x \in \mathbb{C}$, we denote with $\overline{x} \in \mathbb{C}$ the conjugate complex of x, and with $|x|$ its modulus. Let $\underline{x} = (x_1, x_2, \dots, x_n)^T \in \mathbb{C}^n$ be a column vector, where ^{T} denotes the transposition operator, we define: $\overline{\underline{x}} = (\overline{x_1}, \overline{x_2}, \dots, \overline{x_n})^T \in \mathbb{C}^n$, $\underline{x}^m = (x_1^m, x_2^m, \dots, x_n^m)^T \in \mathbb{C}^n$, $m \in \mathbb{N}$. In particular $\underline{0}_n \in \mathbb{C}^n$ is the null vector; $\underline{1}_n \in \mathbb{C}^n$ is the vector having all the components equal to 1. We denote with $\mathbb{C}^{n \times n}$ the space of complex matrices having order n, $O_n \in \mathbb{C}^{n \times n}$ denotes the null matrix, $I_n \in \mathbb{C}^{n \times n}$ denotes the identity matrix, $J_n \in \mathbb{C}^{n \times n}$ denotes the anti-diagonal matrix having the anti-diagonal entries equal to 1.

Remark 1. Let $Q \in \mathbb{C}^{n \times n}$, $n \in \mathbb{N}$, $n \geq 3$, be a matrix having non null entries only on the diagonal and on the anti-diagonal, that is $Q_{i,j} = 0$ when $i \neq j$ and $i \neq n - j + 1$. Let $R \in \mathbb{C}^{(n-2) \times (n-2)}$ be the submatrix of

Q obtained by deleting the rows 1 and n and the columns 1 and n, then it is easy to prove the following formula for the determinant of Q:

$$\det(Q) = (Q_{1,1}Q_{n,n} - Q_{1,n}Q_{n,1})\det(R). \tag{9}$$

Let $\rho_2 : \mathbb{Z} \to \{0,1\}$ be the modulo operation with divisor 2, so that $\rho_2(k)$ gives the reminder after the division of k by 2, and let $\rho_2^-(k) = 1 - \rho_2(k)$, $k \in \mathbb{Z}$. For $k, q, j \in \mathbb{Z}$, $\gamma \in \mathbb{R}$, we define

$$\theta_q = \frac{2q + \rho_2(\nu)}{2\nu}\pi = \begin{cases} \frac{q\pi}{\nu}, & \text{if } \nu \text{ is even,} \\ \frac{(2q+1)\pi}{2\nu}, & \text{if } \nu \text{ is odd,} \end{cases}$$

$$c_q = \cos\theta_q, \qquad s_q = \sin\theta_q, \qquad z_q = e^{\iota\theta_q} = c_q + \iota s_q$$

$$c_{k,q} = \cos((k+1)\theta_q), \qquad s_{k,q} = \sin((k+1)\theta_q),$$

$$c_q^{(\gamma)} = \cos(\gamma s_q), \qquad s_q^{(\gamma)} = \sin(\gamma s_q),$$

$$c_{k,q}^{(\gamma)} = \cos((k+1)\theta_q - \gamma s_q) = c_{k,q}c_q^{(\gamma)} + s_{k,q}s_q^{(\gamma)}, \tag{10}$$

$$s_{k,q}^{(\gamma)} = \sin((k+1)\theta_q - \gamma s_q) = s_{k,q}c_q^{(\gamma)} - c_{k,q}s_q^{(\gamma)}, \tag{11}$$

$$\alpha_q^{(\gamma)} = (-1)^q e^{\gamma c_q},$$

$$\eta = \frac{\nu - \rho_2(\nu)}{2},$$

$$\underline{c}_{\cdot,j}^{(\gamma)} = \left(c_{0,j}^{(\gamma)}, c_{1,j}^{(\gamma)}, \dots, c_{\nu-1,j}^{(\gamma)}\right)^T \in \mathbb{R}^\nu,$$

$$\underline{s}_{\cdot,j}^{(\gamma)} = \left(s_{0,j}^{(\gamma)}, s_{1,j}^{(\gamma)}, \dots, s_{\nu-1,j}^{(\gamma)}\right)^T \in \mathbb{R}^\nu,$$

$$\underline{c}_{\cdot,j} = \underline{c}_{\cdot,j}^{(0)} \in \mathbb{R}^\nu, \ \underline{s}_{\cdot,j} = \underline{s}_{\cdot,j}^{(0)} \in \mathbb{R}^\nu.$$

Remark 2. We have the following relations:

- $c_{0,q} = c_q$, $s_{0,q} = s_q$, $e^{\gamma z_q} = e^{\gamma c_q}e^{\iota\gamma s_q}$, $s_{k,q}^{(0)} = s_{k,q}$, $c_{k,q}^{(0)} = c_{k,q}$;
- $\theta_{\nu-\rho_2(\nu)-k} = \pi - \theta_k$, $\theta_\eta = \frac{\pi}{2}$;
- $0 \le \theta_0 < \theta_1 < \dots < \theta_{\eta-1} < \theta_\eta = \frac{\pi}{2} < \theta_{\eta+1} < \dots \theta_{\nu-\rho_2(\nu)} \le \pi$, where the first and the last inequality hold as equalities only when ν is even;
- $1 \ge c_0 > c_1 > \dots > c_{\eta-1} > c_\eta = 0 > c_{\eta+1} > \dots c_{\nu-\rho_2(\nu)} \ge -1$, where the first and the last inequality hold as equalities only when ν is even.

In particular the following four square matrices of order ν are defined: when $\nu = 1$, $T^{(1)} = (s_0)$, $T^{(2)} = (c_0)$, $U^{(1)} = \left(s_{0,0}^{(\gamma)}\right)$, $U^{(2)} = \left(c_{0,0}^{(\gamma)}\right)$; when $\nu = 2$, $T^{(1)} = (\underline{c}_{\cdot,2}, \underline{s}_{\cdot,1})$, $T^{(2)} = (\underline{c}_{\cdot,2}, \underline{c}_{\cdot,1})$, $U^{(1)} = (\underline{s}_{\cdot,1}^{(\gamma)}, \underline{c}_{\cdot,0}^{(\gamma)})$, $U^{(2)} = (\underline{c}_{\cdot,1}^{(\gamma)}, \underline{c}_{\cdot,0}^{(\gamma)})$; when $\nu \ge 3$

$$T^{(1)} = \left(\underline{c}_{\cdot,\nu-\rho_2(\nu)}, \underline{c}_{\cdot,\nu-\rho_2(\nu)-1} \cdots, \underline{c}_{\cdot,\eta+1}, \underline{s}_{\cdot,\eta}, \underline{s}_{\cdot,\eta+1} \cdots, \underline{s}_{\cdot,\nu-1}\right), \tag{12}$$

$$T^{(2)} = \left(\underline{c}_{\cdot,\nu-\rho_2(\nu)}, \underline{c}_{\cdot,\nu-\rho_2(\nu)-1} \cdots, \underline{c}_{\cdot,\eta}, \underline{s}_{\cdot,\eta+1}, \underline{s}_{\cdot,\eta+2} \cdots, \underline{s}_{\cdot,\nu-1}\right), \tag{13}$$

$$U^{(1)} = \left(\underline{s}_{\cdot,\rho_2^-(\nu)}^{(\gamma)}, \underline{s}_{\cdot,\rho_2^-(\nu)+1}^{(\gamma)} \cdots, \underline{s}_{\cdot,\eta}^{(\gamma)}, \underline{c}_{\cdot,\eta-1}^{(\gamma)}, \underline{c}_{\cdot,\eta-2}^{(\gamma)}, \dots, \underline{c}_{\cdot,0}^{(\gamma)}\right), \tag{14}$$

$$U^{(2)} = \left(\underline{s}_{\cdot,\rho_2^-(\nu)}^{(\gamma)}, \underline{s}_{\cdot,\rho_2^-(\nu)+1}^{(\gamma)} \cdots, \underline{s}_{\cdot,\eta-1}^{(\gamma)}, \underline{c}_{\cdot,\eta}^{(\gamma)}, \underline{c}_{\cdot,\eta-1}^{(\gamma)}, \dots, \underline{c}_{\cdot,0}^{(\gamma)}\right). \tag{15}$$

Please cite this article in press as: N. Egidi, P. Maponi, The singular value expansion of the Volterra integral equation associated to a numerical differentiation problem, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2017.12.007

We note that matrix $T^{(1)}$ differs from $T^{(2)}$ only at column $\eta+1$, while matrix $U^{(1)}$ differs from $U^{(2)}$ only at column $\eta+\rho_2(\nu)$.

2.3. Preliminar results

The following lemma gives some relations between the determinants of matrices $T^{(1)}$ and $T^{(2)}$.

Lemma 1. *Let $d^{(1)} = det\left(T^{(1)}\right)$, and $d^{(2)} = det\left(T^{(2)}\right)$, the following relations hold*

$$d^{(2)} + \iota d^{(1)} = (-1)^{\eta}\iota^{\nu-\eta}\left(1+\rho_2^-(\nu)\iota\right)2^{\nu-\eta-1}d \tag{16}$$

$$d^{(1)}d^{(2)} = \rho_2^-(\nu)(-1)^{\eta}4^{\eta-1}d^2 \tag{17}$$

$$\left(d^{(2)}\right)^2 - \left(d^{(1)}\right)^2 = \rho_2(\nu)(-1)^{\eta+1}4^{\eta}d^2, \tag{18}$$

$$\begin{aligned} d = &\prod_{\eta+1\le q\le\nu-1}\left(\rho_2(\nu)+2\rho_2^-(\nu)\left(1+c_q\right)\right)s_q(-c_q)\cdot \\ &\cdot\prod_{\eta+1\le p<q\le\nu-1}\left(\left|z_p-\overline{z}_q\right|^2\left|z_p-z_q\right|^2\right), \end{aligned} \tag{19}$$

where the product $\prod$ are equal to 1 when the corresponding set of indices is empty.

Proof. See Appendix A. □

We note that d is a positive real number, in fact, from Remark 2, it is the product of positive factors. Moreover, when ν is odd we have $d^{(1)}d^{(2)}=0$, $\left(d^{(2)}\right)^2-\left(d^{(1)}\right)^2=(-1)^{\eta+1}4^{\eta}d^2$ and its sign depends on η; when ν is even we have $\left(d^{(2)}\right)^2-\left(d^{(1)}\right)^2=0$, $d^{(1)}d^{(2)}=(-1)^{\eta}4^{\eta-1}d^2$ and its sign depends on η. In particular when $\nu=1$ we have $d^{(1)}=1$ and $d^{(2)}=0$, when $\nu=2$ we have $d^{(1)}=-1$ and $d^{(2)}=1$.

Let us consider the following four square matrices of order ν: when $\nu=1$, $W^{(1)}=(0)$, $W^{(2)}=(1)$, $D^{(1)}=\left(s_0^{(\gamma)}\right)$, $D^{(2)}=\left(c_0^{(\gamma)}\right)$; when $\nu=2$, $W^{(1)}=(-\underline{s}_{\cdot,1}+\underline{c}_{\cdot,1},\underline{s}_{\cdot,0})$, $W^{(2)}=(-\underline{s}_{\cdot,1}-\underline{c}_{\cdot,1},\underline{c}_{\cdot,0})$, $D^{(1)}=\mathrm{Diag}(s_1^{(\gamma)},s_0^{(\gamma)})$, $D^{(2)}=\mathrm{Diag}(c_1^{(\gamma)},c_0^{(\gamma)})$; when $\nu\ge 3$

$$\begin{aligned} W^{(1)} = \Big(&-\underline{c}_{\cdot,\rho_2^-(\nu)},-\underline{c}_{\cdot,\rho_2^-(\nu)+1},\ldots,-\underline{c}_{\cdot,\eta-1}, \\ &d^{(2)}\left(\rho_2\underline{s}_{\cdot,\eta}+\rho_2^-\underline{c}_{\cdot,\eta}\right)+d^{(1)}\left(-\rho_2\underline{c}_{\cdot,\eta}+\rho_2^-\underline{s}_{\cdot,\eta}\right), \\ &\underline{s}_{\cdot,\eta-1},\underline{s}_{\cdot,\eta-2},\ldots,\underline{s}_{\cdot,1},\underline{s}_{\cdot,0}\Big) \end{aligned} \tag{20}$$

$$\begin{aligned} W^{(2)} = \Big(&\underline{s}_{\cdot,\rho_2^-(\nu)},\underline{s}_{\cdot,\rho_2^-(\nu)+1},\ldots,\underline{s}_{\cdot,\eta-1}, \\ &d^{(2)}\left(\rho_2\underline{c}_{\cdot,\eta}-\rho_2^-\underline{s}_{\cdot,\eta}\right)+d^{(1)}\left(\rho_2\underline{s}_{\cdot,\eta}+\rho_2^-\underline{c}_{\cdot,\eta}\right), \\ &\underline{c}_{\cdot,\eta-1},\underline{c}_{\cdot,\eta-2},\ldots,\underline{c}_{\cdot,1},\underline{c}_{\cdot,0}\Big) \end{aligned} \tag{21}$$

$$D^{(1)} = \mathrm{Diag}\left(s^{(\gamma)}_{\rho_2^-(\nu)},s^{(\gamma)}_{\rho_2^-(\nu)+1},\ldots,s^{(\gamma)}_{\eta-1},s^{(\gamma)}_{\eta},s^{(\gamma)}_{\eta-1},\ldots,s^{(\gamma)}_{1},s^{(\gamma)}_{0}\right) \tag{22}$$

$$D^{(2)} = \mathrm{Diag}\left(c^{(\gamma)}_{\rho_2^-(\nu)},c^{(\gamma)}_{\rho_2^-(\nu)+1},\ldots,c^{(\gamma)}_{\eta-1},c^{(\gamma)}_{\eta},c^{(\gamma)}_{\eta-1},\ldots,c^{(\gamma)}_{1},c^{(\gamma)}_{0}\right) \tag{23}$$

For the determinants of matrices $W^{(1)}$ and $W^{(2)}$ we have the following result.

Lemma 2. *For matrices $W^{(1)}$ and $W^{(2)}$ defined in (20), (21), the following results hold*

$$det\left(W^{(1)}\right) = 0, \qquad det\left(W^{(2)}\right) = \rho_2(\nu)2^{2\eta}d^2 - \rho_2^-(\nu)2^{2\eta-1}d^2 \neq 0 \tag{24}$$

where d is given by (19).

Proof. See Appendix B.

3. The singular value expansion of K

We describe the characteristic equation for the singular values μ of K and we give the expression of the left-singular function and the right-singular function associated to a given singular value μ of K.

Theorem 1. *Let $\mu_l > 0$ be a singular value of the integral operator $\mathcal{K}$ defined by its kernel (1), and let $\gamma_l = 1/\sqrt[\nu]{\mu_l}$. Then, the singular functions corresponding to μ_l are*

$$u_l(t) = \sum_{p=0}^{\nu-\rho_2(\nu)} e^{\gamma_l c_p t}\left(C_p^{(u)}\cos(\gamma_l s_p t) + S_p^{(u)}\sin(\gamma_l s_p t)\right),\ t \in [0,1], \tag{25}$$

$$v_l(t) = \sum_{p=0}^{\nu-\rho_2(\nu)} e^{\gamma_l c_p t}\left(C_p^{(v)}\cos(\gamma_l s_p t) + S_p^{(v)}\sin(\gamma_l s_p t)\right),\ t \in [0,1], \tag{26}$$

where coefficients $C_p^{(\cdot)}, S_p^{(\cdot)} \in \mathbb{R}$, $p = 0, 1, \dots, \nu - \rho_2(\nu)$, are defined by the following relations:

- *if ν is odd*

$$C_p^{(u)} = (-1)^{p+1}S_p^{(v)},\ \ S_p^{(u)} = (-1)^p C_p^{(v)}, \tag{27}$$

$$\sum_{p=0}^{\nu-1}\left(C_p^{(v)}c_{k,p} - S_p^{(v)}s_{k,p}\right) = 0,\ \ k = 0, 1, \dots, \nu - 1, \tag{28}$$

$$\sum_{p=0}^{\nu-1}\alpha_p^{(\gamma_l)}\left(S_p^{(v)}c_{k,p}^{(\gamma_l)} + C_p^{(v)}s_{k,p}^{(\gamma_l)}\right) = 0,\ \ k = 0, 1, \dots, \nu - 1; \tag{29}$$

- *if ν is even*

$$S_0^{(v)} = S_\nu^{(v)} = S_0^{(u)} = S_\nu^{(v)} = 0, \tag{30}$$

$$C_p^{(u)} = (-1)^p C_p^{(v)},\ \ S_p^{(u)} = (-1)^p S_p^{(v)}, \tag{31}$$

$$\sum_{p=0}^{\nu}\left(C_p^{(v)}c_{k,p} - S_p^{(v)}s_{k,p}\right) = 0,\ \ k = 0, 1, \dots, \nu - 1, \tag{32}$$

$$\sum_{p=0}^{\nu}\alpha_p^{(\gamma_l)}\left(C_p^{(v)}c_{k,p}^{(\gamma_l)} - S_p^{(v)}s_{k,p}^{(\gamma_l)}\right) = 0,\ \ k = 0, 1, \dots, \nu - 1. \tag{33}$$

Proof. Let $t \in [0,1]$ and l be a positive integer. Let $\mu_l > 0$ be the lth singular value of $\mathcal{K}$ with singular functions $u_l(t)$ and $v_l(t)$, then they satisfy (7) and (8). Let

$$\sigma_l(t) = u_l(t) - (-1)^\nu \iota^{\rho_2(\nu)} v_l(t) \ \ \delta_l(t) = u_l(t) + (-1)^\nu \iota^{\rho_2(\nu)} v_l(t), \tag{34}$$

then, from (7) and (8), σ_l and δ_l are solutions of the following differential equations:

$$\mu_l \frac{d^\nu}{dt^\nu}\sigma_l = -\iota^{\rho_2(\nu)}\sigma_l, \tag{35}$$

$$\mu_l \frac{d^\nu}{dt^\nu}\delta_l = \iota^{\rho_2(\nu)}\delta_l. \tag{36}$$

In particular, the solution of equation (35) is supposed to have the form $\sigma_l(t) = \lambda^t$; then parameter λ must satisfy the following equation

$$\mu_l \log^\nu \lambda = -\iota^{\rho_2(\nu)}$$

from which we have

$$\lambda = e^{\gamma_l z_{2p+1}}, \quad p = 0, 1, \ldots, \nu - 1,$$

where $\gamma_l = \frac{1}{\sqrt[\nu]{\mu_l}}$. Hence, the ν complex linearly independent solutions of (35) are

$$\sigma_{l,p}(t) = e^{t\gamma_l z_{2p+1}}, \qquad p = 0, 1, \ldots, \nu - 1. \tag{37}$$

For equation (36) similar arguments hold, in particular the ν complex linearly independent solutions of (36) are

$$\delta_{l,p}(t) = e^{t\gamma_l z_{2p}}, \quad p = 0, 1, \ldots, \nu - 1. \tag{38}$$

We note that $\sigma_{l,p}(0) = 1$, $\sigma_{l,p}(1) = e^{\gamma_l z_{2p+1}} = e^{\gamma_l c_{2p+1}}\left(c_{2p+1}^{(\gamma_l)} + \iota s_{2p+1}^{(\gamma_l)}\right)$, $\delta_{l,p}(0) = 1$, $\delta_{l,p}(1) = e^{\gamma_l z_{2p}} = e^{\gamma_l c_{2p}}\left(c_{2p}^{(\gamma_l)} + \iota s_{2p}^{(\gamma_l)}\right)$.

From (37) and (38), it is straightforward to prove that

$$\overline{\sigma}_{l,\nu-p-1}(t) = \rho_2(\nu)\delta_{l,p}(t) + \rho_2^-(\nu)\sigma_{l,p}(t), \quad p = 0, 1, \ldots \nu - 1,$$
$$\overline{\delta}_{l,\nu-p-1}(t) = \rho_2(\nu)\sigma_{l,p}(t) + \rho_2^-(\nu)\delta_{l,p+1}(t), \quad p = 0, 1, \ldots \nu - 1.$$

In particular, we have that both sets of the complex integrals of (35) and of (36) are generated by the following 2ν linearly independent real functions

$$\sigma_{l,p}^{(1)}(t) \equiv \mathrm{Re}(\sigma_{l,p}(t)) = e^{t\gamma_l c_{2p+1}}\cos(t\gamma_l s_{2p+1}),\ p = 0, \ldots, \eta - 1, \tag{39}$$

$$\sigma_{l,p}^{(2)}(t) \equiv \mathrm{Im}(\sigma_{l,p}(t)) = e^{t\gamma_l c_{2p+1}}\sin(t\gamma_l s_{2p+1}),\ p = 0, \ldots, \eta - 1, \tag{40}$$

$$\delta_{l,p}^{(1)}(t) \equiv \mathrm{Re}(\delta_{l,p}(t)) = e^{t\gamma_l c_{2p}}\cos(t\gamma_l s_{2p}),\ p = 0, \ldots, \eta, \tag{41}$$

$$\delta_{l,p}^{(2)}(t) \equiv \mathrm{Im}(\delta_{l,p}(t)) = e^{t\gamma_l c_{2p}}\sin(t\gamma_l s_{2p}),\ p = \rho_2^-(\nu), \ldots, \eta - \rho_2^-(\nu). \tag{42}$$

Hence from (34), we have that the set of the real integrals of (7), (8) is also generated by the above functions, that is

$$v_l(t) = \sum_{q=0}^{\eta}\left(C_{2q}^{(v)}\delta_{l,q}^{(1)}(t) + S_{2q}^{(v)}\delta_{l,q}^{(2)}(t)\right) + \sum_{q=0}^{\eta-1}\left(C_{2q+1}^{(v)}\sigma_{l,q}^{(1)}(t) + S_{2q+1}^{(v)}\sigma_{l,q}^{(2)}(t)\right), \tag{43}$$

$$u_l(t) = \sum_{q=0}^{\eta}\left(C_{2q}^{(u)}\delta_{l,q}^{(1)}(t) + S_{2q}^{(u)}\delta_{l,q}^{(2)}(t)\right) + \sum_{q=0}^{\eta-1}\left(C_{2q+1}^{(u)}\sigma_{l,q}^{(1)}(t) + S_{2q+1}^{(u)}\sigma_{l,q}^{(2)}(t)\right), \tag{44}$$

Please cite this article in press as: N. Egidi, P. Maponi, The singular value expansion of the Volterra integral equation associated to a numerical differentiation problem, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2017.12.007

where $C_p^{(\cdot)}, S_p^{(\cdot)} \in \mathbb{R}$, $p = 0, 1, \ldots, 2\eta$, and $S_0^{(v)} = S_0^{(u)} = S_{2\eta}^{(v)} = S_{2\eta}^{(u)} = 0$ when $\nu = 2\eta$, and so we have (25), (26) and (30).

In Appendix C we prove that the solutions of (35) and (36) satisfy the following relations:

$$\mathcal{K}\sigma_{l,p}^{(1)}(t) = -\sum_{k=0}^{\nu-1} \frac{t^{\nu-k-1}}{(\nu-k-1)!\gamma_l^{k+1}} c_{k,2p+1} - \mu_l \sigma_{l,p}^{(1+\rho_2(\nu))}(t), \tag{45}$$

$$\mathcal{K}^*\sigma_{l,p}^{(1)}(t) = \sum_{k=0}^{\nu-1} \frac{(-1)^k(1-t)^{\nu-k-1}}{(\nu-k-1)!\gamma_l^{k+1}} e^{\gamma_l c_{2p+1}} c_{k,2p+1}^{(\gamma_l)} - (-1)^\nu \mu_l \sigma_{l,p}^{(1+\rho_2(\nu))}(t), \tag{46}$$

$$\mathcal{K}\sigma_{l,p}^{(2)}(t) = \sum_{k=0}^{\nu-1} \frac{t^{\nu-k-1}}{(\nu-k-1)!\gamma_l^{k+1}} s_{k,2p+1} - (-1)^\nu \mu_l \sigma_{l,p}^{(2-\rho_2(\nu))}(t), \tag{47}$$

$$\mathcal{K}^*\sigma_{l,p}^{(2)}(t) = -\sum_{k=0}^{\nu-1} \frac{(-1)^k(1-t)^{\nu-k-1}}{(\nu-k-1)!\gamma_l^{k+1}} e^{\gamma_l c_{2p+1}} s_{k,2p+1}^{(\gamma_l)} - \mu_l \sigma_{l,p}^{(2-\rho_2(\nu))}(t), \tag{48}$$

$$\mathcal{K}\delta_{l,p}^{(1)}(t) = -\sum_{k=0}^{\nu-1} \frac{t^{\nu-k-1}}{(\nu-k-1)!\gamma_l^{k+1}} c_{k,2p} + \mu_l \delta_{l,p}^{(1+\rho_2(\nu))}(t), \tag{49}$$

$$\mathcal{K}^*\delta_{l,p}^{(1)}(t) = \sum_{k=0}^{\nu-1} \frac{(-1)^k(1-t)^{\nu-k-1}}{(\nu-k-1)!\gamma_l^{k+1}} e^{\gamma_l c_{2p}} c_{k,2p}^{(\gamma_l)} + (-1)^\nu \mu_l \delta_{l,p}^{(1+\rho_2(\nu))}(t), \tag{50}$$

$$\mathcal{K}\delta_{l,p}^{(2)}(t) = \sum_{k=0}^{\nu-1} \frac{t^{\nu-k-1}}{(\nu-k-1)!\gamma^{k+1}} s_{k,2p} + (-1)^\nu \mu_l \delta_{l,p}^{(2-\rho_2(\nu)))}(t), \tag{51}$$

$$\mathcal{K}^*\delta_{l,p}^{(2)}(t) = -\sum_{k=0}^{\nu-1} \frac{(-1)^k(1-t)^{\nu-k-1}}{(\nu-k-1)!\gamma_l^{k+1}} e^{\gamma_l c_{2p}} s_{k,2p}^{(\gamma_l)} + \mu_l \delta_{l,p}^{(2-\rho_2(\nu)))}(t). \tag{52}$$

From (43)–(52) we have

$$\begin{aligned}
\mathcal{K}v_l(t) = \mu_l \sum_{q=0}^{\eta} &\left(C_{2q}^{(v)} \delta_{l,q}^{(1+\rho_2(\nu))}(t) + (-1)^\nu S_{2q}^{(v)} \delta_{l,q}^{(2-\rho_2(\nu))}(t) \right) + \\
&+ \mu_l \sum_{q=0}^{\eta-1} \left(-C_{2q+1}^{(v)} \sigma_{l,q}^{(1+\rho_2(\nu))}(t) - (-1)^\nu S_{2q+1}^{(v)} \sigma_{l,q}^{(2-\rho_2(\nu))}(t) \right) + \\
&- \sum_{k=0}^{\nu-1} \frac{t^{\nu-k-1}}{(\nu-k-1)!\gamma_l^{k+1}} \left(\sum_{q=0}^{\eta} \left(C_{2q}^{(v)} c_{k,2q} - S_{2q}^{(v)} s_{k,2q} \right) + \right. \\
&\left. + \sum_{q=0}^{\eta-1} \left(C_{2q+1}^{(v)} c_{k,2q+1} - S_{2q+1}^{(v)} s_{k,2q+1} \right) \right),
\end{aligned}$$

$$\begin{aligned}
\mathcal{K}^*u_l(t) = \mu_l \sum_{q=0}^{\eta} &\left((-1)^\nu C_{2q}^{(u)} \delta_{l,q}^{(1+\rho_2(\nu))}(t) + S_{2q}^{(u)} \delta_{l,q}^{(2-\rho_2(\nu))}(t) \right) + \\
&\mu_l \sum_{q=0}^{\eta-1} \left((-1)^{\nu+1} C_{2q+1}^{(u)} \sigma_{l,q}^{(1+\rho_2(\nu))}(t) - S_{2q+1}^{(u)} \sigma_{l,q}^{(2-\rho_2(\nu)}(t) \right) + \\
&+ \sum_{k=0}^{\nu-1} \frac{(-1)^k(1-t)^{\nu-k-1}}{(\nu-k-1)!\gamma_l^{k+1}} \left(\sum_{q=0}^{\eta} e^{\gamma_l c_{2q}} \left(C_{2q}^{(u)} c_{k,2q}^{(\gamma_l)} - S_{2q}^{(u)} s_{k,2q}^{(\gamma_l)} \right) + \right.
\end{aligned}$$

$$+\sum_{q=0}^{\eta-1} e^{\gamma_l c_{2q+1}} \left(C_{2q+1}^{(u)} c_{k,2q+1}^{(\gamma_l)} - S_{2q+1}^{(u)} s_{k,2q+1}^{(\gamma_l)} \right) \Bigg).$$

So that, from (5) we obtain relations (27), (31) and

$$-\sum_{k=0}^{\nu-1} \frac{t^{\nu-k-1}}{(\nu-k-1)!\gamma_l^{k+1}} \left(\sum_{q=0}^{\eta} \left(C_{2q}^{(v)} c_{k,2q} - S_{2q}^{(v)} s_{k,2q} \right) + \right.$$
$$\left. + \sum_{q=0}^{\eta-1} \left(C_{2q+1}^{(v)} c_{k,2q+1} - S_{2q+1}^{(v)} s_{k,2q+1} \right) \right) = 0,$$
$$\sum_{k=0}^{\nu-1} \frac{(1-t)^{\nu-k-1}}{(\nu-k-1)!\gamma_l^{k+1}} (-1)^k \left(\sum_{q=0}^{\eta} e^{\gamma_l c_{2q}} \left(C_{2q}^{(u)} c_{k,2q}^{(\gamma_l)} - S_{2q}^{(u)} s_{k,2q}^{(\gamma_l)} \right) + \right.$$
$$\left. + \sum_{q=0}^{\eta-1} e^{\gamma_l c_{2q+1}} \left(C_{2q+1}^{(u)} c_{k,2q+1}^{(\gamma_l)} - S_{2q+1}^{(u)} s_{k,2q+1}^{(\gamma_l)} \right) \right) = 0,$$

form which we have (28), (29) and (32), (33), this concludes the proof of the theorem. □

From the above theorem, we have that when μ is a singular value of integral operator $\mathcal{K}$, defined by its kernel (1), and $\gamma = \frac{1}{\sqrt[\nu]{\mu}}$ then the coefficients of the corresponding singular functions must satisfy: equations (27)–(29) in the case ν odd, and equations (30)–(33) in the case ν even. These equations can be written as the following linear system having 2ν equations

$$M \begin{pmatrix} \underline{S} \\ \underline{C} \end{pmatrix} = \underline{0}_{2\nu}, \tag{53}$$

where

$$\underline{C}^T = \left(C_0^{(v)}, C_1^{(v)}, \dots, C_{\nu-\rho_2(\nu)}^{(v)} \right) \in \mathbb{R}^{\nu+\rho_2^-(\nu)}$$
$$\underline{S}^T = \left(S_{\rho_2^-(\nu)}^{(v)}, S_{\rho_2^-(\nu)+1}^{(v)}, \dots, S_{\nu-1}^{(v)} \right) \in \mathbb{R}^{\nu-\rho_2^-(\nu)}$$

and matrix $M = \left(\underline{M}_{\cdot,1}, \underline{M}_{\cdot,2}, \dots, \underline{M}_{\cdot,2\nu} \right) \in \mathbb{R}^{2\nu\times 2\nu}$ has columns

$$\underline{M}_{\cdot,i} = \begin{pmatrix} -\underline{s}_{\cdot,i-\rho_2(\nu)} \\ \alpha_{i-\rho_2(\nu)}^{(\gamma)} \left(\rho_2(\nu) \underline{c}_{\cdot,i-\rho_2(\nu)}^{(\gamma)} - \rho_2^-(\nu) \underline{s}_{\cdot,i-\rho_2(\nu)}^{(\gamma)} \right) \end{pmatrix},$$
$$i = 1, 2, \dots, \nu - \rho_2^-(\nu) \tag{54}$$

$$\underline{M}_{\cdot,i+\nu-\rho_2^-(\nu)} = \begin{pmatrix} \underline{c}_{\cdot,i-1} \\ \alpha_{i-1}^{(\gamma)} \left(\rho_2(\nu) \underline{s}_{\cdot,i-1}^{(\gamma)} + \rho_2^-(\nu) \underline{c}_{\cdot,i-1}^{(\gamma)} \right) \end{pmatrix},$$
$$i = 1, 2, \dots, \nu + \rho_2^-(\nu). \tag{55}$$

Definition 1. We define

$$h_\nu(\gamma) = \det(M) \tag{56}$$

where M is the matrix of order 2ν whose columns are given by (54) and (55).

When μ_l, $l = 1, 2, \dots$, is a singular value of integral operator $\mathcal{K}$ defined by its kernel (1), then $\gamma_l = 1/\sqrt[\nu]{\mu_l}$ is a zero of h_ν and the coefficients of the singular functions associated with μ_l respect to the base functions (39)–(42) are given by a non trivial solution of the corresponding linear system (53).

Hence, in order to compute the singular values of $\mathcal{K}$ we have to compute the positive zeros of h_ν. In the following we give the expression of h_ν for $\nu = 1, 2$, and we give the asymptotic behavior of h_ν when $\nu \geq 3$.

Remark 3. When $\nu = 1$ we have that $h_1 : \mathbb{R}^+ = (0, +\infty) \to \mathbb{R}$ defined in (56) is

$$h_1(\gamma) = -\cos(\gamma), \tag{57}$$

and for $l = 1, 2, \dots$ and $t \in [0, 1]$ the SVE is

$$\gamma_l = \frac{\pi}{2} + l\pi, \quad \mu_l = \frac{1}{\gamma_l}, \quad u_l(t) = \sqrt{2}\sin(\gamma_l t), \quad v_l(t) = \sqrt{2}\cos(\gamma_l t).$$

Remark 4. When $\nu = 2$ we have that $h_2 : \mathbb{R}^+ \to \mathbb{R}$ defined in (56) is

$$h_2(\gamma) = -4\left(1 - \cos(\gamma)\cosh(\gamma)\right), \tag{58}$$

and for $l = 1, 2, \dots$ and $t \in [0, 1]$ the SVE is given by

- γ_l are the positive zeros of $h_2(\gamma)$ such that $0 < \gamma_1 < \gamma_2 < \gamma_3 < \dots$,
- $\mu_l = \frac{1}{\gamma_l^2}$,
- $u_l(t) = C\left(\beta_l e^{\gamma_l t} + (\beta_l + 1)\cos(\gamma_l t) + (\beta_l - 1)\sin(\gamma_l t) + e^{-\gamma_l t}\right)$,
- $v_l(t) = C\left(\beta_l e^{\gamma_l t} - (\beta_l + 1)\cos(\gamma_l t) - (\beta_l - 1)\sin(\gamma_l t) + e^{-\gamma_l t}\right)$,

where C is the same normalization constant and

$$\beta_l = \frac{\cos(\gamma_l) + \sin(\gamma_l) + e^{-\gamma_l}}{\cos(\gamma_l) - \sin(\gamma_l) + e^{\gamma_l}}.$$

Theorem 2. *When $\nu \geq 3$ we have that $h_\nu : \mathbb{R}^+ \to \mathbb{R}$ defined in (56) satisfies the following asymptotic relation:*

$$h_\nu(\gamma) = (-1)^{\eta+1} d^2 2^{2\eta-1}(2\rho_2(\nu) - \rho_2^-(\nu))\cos(\gamma)e^{\gamma\xi} + g_\nu(\gamma), \tag{59}$$

where $g_\nu(\gamma) = \mathcal{O}(e^{\gamma\xi_0})$ when $\gamma \to +\infty$, $d > 0$ is given by (19), and

$$\xi = 2\sum_{i=0}^{\eta-1} c_i - \rho_2^-(\nu)c_0, \tag{60}$$

$$\xi_0 = c_{\eta-1} + 2\sum_{i=0}^{\eta-2} c_i - \rho_2^-(\nu)c_0. \tag{61}$$

We note that when $\nu \geq 3$

$$\frac{e^{\gamma\xi_0}}{e^{\gamma\xi}} = e^{-\gamma c_{\eta-1}} \to 0, \qquad \gamma \to +\infty.$$

Proof. Let $\nu \geq 3$, $P_{2\nu}$ is the set of permutations of $\{1, 2, \dots, 2\nu\}$, $P^{(i)}$, $i = 1, 2, 3$, is a given partition of $P_{2\nu}$, τ_1 is the restriction of the bijection $\tau \in P_{2\nu}$ to $\{1, 2, \dots, \nu\}$ and τ_2 is the restriction of the bijection $\tau \in P_{2\nu}$

to $\{\nu+1, \nu+2, \ldots, 2\nu\}$, see Appendix D for an extensive definition of these objects and their properties. Let $\mathrm{sign}(\tau)$ be the signature of permutation $\tau \in P_{2\nu}$. From standard arguments on matrix determinant, and the fact that $\{P^{(i)},\ i=1,2,3\}$ is a partition of $P_{2\nu}$ we have

$$h_\nu(\gamma) = \det(M) = \sum_{\tau\in P_{2\nu}} \mathrm{sign}(\tau) \prod_{i=1}^{2\nu} M_{i,\tau(i)} = \sum_{\tau\in P^{(1)}} \mathrm{sign}(\tau) \prod_{i=1}^{2\nu} M_{i,\tau(i)} + \\ + \sum_{\tau\in P^{(2)}} \mathrm{sign}(\tau) \prod_{i=1}^{2\nu} M_{i,\tau(i)} + \sum_{\tau\in P^{(3)}} \mathrm{sign}(\tau) \prod_{i=1}^{2\nu} M_{i,\tau(i)}. \tag{62}$$

Appendix D defines also four sets of bijections $B_i^{(j)}$, $i, j = 1, 2$ and two special permutations $\tau^{(j)} \in P^{(j)}$, $j = 1, 2$, such that for $j = 1, 2$

$$\tau \in P^{(j)} \Leftrightarrow \tau_i \in B_i^{(j)},\ i = 1, 2, \tag{63}$$

$$\mathrm{sign}(\tau) = \mathrm{sign}(\tau^{(j)})\mathrm{sign}^{(j)}(\tau_1)\mathrm{sign}^{(j)}(\tau_2), \qquad \text{if } \tau \in P^{(j)}, \tag{64}$$

where, when $\tau \in P^{(j)}$, $\mathrm{sign}^{(j)}(\tau_i) = (-1)^k$, and k is the number of inversions necessary to obtain τ_i from $\tau_i^{(j)}$.

From the definitions of τ_1 and τ_2, and formulas (63), (64), with $j = 1$, we have

$$\sum_{\tau\in P^{(1)}} \mathrm{sign}(\tau) \prod_{i=1}^{2\nu} M_{i,\tau(i)} = \sum_{\tau\in P^{(1)}} \mathrm{sign}(\tau) \left(\prod_{i=1}^{\nu} M_{i,\tau(i)}\right) \left(\prod_{i=\nu+1}^{2\nu} M_{i,\tau(i)}\right) = \\ = \sum_{\tau_1\in B_1^{(1)}, \tau_2\in B_2^{(1)}} \mathrm{sign}(\tau^{(1)})\mathrm{sign}^{(1)}(\tau_1)\mathrm{sign}^{(1)}(\tau_2) \left(\prod_{i=1}^{\nu} M_{i,\tau_1(i)}\right) \cdot \\ \cdot \left(\prod_{i=\nu+1}^{2\nu} M_{i,\tau_2(i)}\right) = \mathrm{sign}(\tau^{(1)}) \left(\sum_{\tau_1\in B_1^{(1)}} \mathrm{sign}^{(1)}(\tau_1) \prod_{i=1}^{\nu} M_{i,\tau_1(i)}\right) \cdot \\ \cdot \left(\sum_{\tau_2\in B_2^{(1)}} \mathrm{sign}^{(1)}(\tau_2) \prod_{i=\nu+1}^{2\nu} M_{i,\tau_2(i)}\right). \tag{65}$$

The second factor in the last formula is the determinant of the matrix obtained by removing the last ν rows and the last ν columns from the matrix $M_{\tau^{(1)}}$, that is matrix M with columns permutated by $\tau^{(1)}$, see Appendix D for a precise definition. In particular, from the definition of matrix M given by (54) and (55), the definition of $\tau^{(1)}$ given in Appendix D

$$\begin{aligned}\tau^{(1)} = (&2\nu, 2\nu-1, \ldots, 2\nu-\eta+1, \\ &\eta+\rho_2(\nu), \eta+\rho_2(\nu)+1, \ldots, 2\nu-\eta, \\ &\eta+\rho_2(\nu)-1, \eta+\rho_2(\nu)-2, \ldots, 1)\end{aligned}$$

and the definition of $T^{(1)}$ given by (12), we have

$$\left(\sum_{\tau_1\in B_1^{(1)}} \mathrm{sign}^{(1)}(\tau_1) \prod_{i=1}^{\nu} M_{i,\tau_1(i)}\right) =$$

$$= \det\left(\underline{c}_{\cdot,\nu-\rho_2(\nu)}, \underline{c}_{\cdot,\nu-\rho_2(\nu)-1}\dots, \underline{c}_{\cdot,\eta+1}, -\underline{s}_{\cdot,\eta}, -\underline{s}_{\cdot,\eta+1}\dots, -\underline{s}_{\cdot,\nu-1}\right) =$$
$$= (-1)^{\nu-\eta}\det\left(T^{(1)}\right). \tag{66}$$

The third factor in (65) is the determinant of the matrix obtained from $M_{\tau^{(1)}}$ by removing the first ν rows and the first ν columns, in particular, from the definition of matrix M given by (54) and (55), the definition of $\tau^{(1)}$, and the definition of $U^{(i)}$, $i = 1, 2$, given by (14), (15), we have

$$\left(\sum_{\tau_2\in B_2^{(1)}} \mathrm{sign}^{(1)}(\tau_2)\prod_{i=\nu+1}^{2\nu} M_{i,\tau_2(i)}\right) = \rho_2(\nu)\det\left(\alpha_0^{(\gamma)}\underline{s}_{\cdot,0}^{(\gamma)}, \alpha_1^{(\gamma)}\underline{s}_{\cdot,1}^{(\gamma)},\dots\right.$$
$$\left.\dots, \alpha_{\eta-1}^{(\gamma)}\underline{s}_{\cdot,\eta-1}^{(\gamma)}, \alpha_\eta^{(\gamma)}\underline{s}_{\cdot,\eta}^{(\gamma)}, \alpha_{\eta-1}^{(\gamma)}\underline{c}_{\cdot,\eta-1}^{(\gamma)}, \alpha_{\eta-2}^{(\gamma)}\underline{c}_{\cdot,\eta-2}^{(\gamma)},\dots, \alpha_{\rho_2^-(\nu)}^{(\gamma)}\underline{c}_{\cdot,\rho_2^-(\nu)}^{(\gamma)}\right) +$$
$$+ \rho_2^-(\nu)\det\left(\alpha_0^{(\gamma)}\underline{c}_{\cdot,0}^{(\gamma)}, \alpha_1^{(\gamma)}\underline{c}_{\cdot,1}^{(\gamma)}\dots, \alpha_\eta^{(\gamma)}\underline{c}_{\cdot,\eta}^{(\gamma)},\right.$$
$$\left.-\alpha_{\eta-1}^{(\gamma)}\underline{s}_{\cdot,\eta-1}^{(\gamma)}, -\alpha_{\eta-2}^{(\gamma)}\underline{s}_{\cdot,\eta-2}^{(\gamma)},\dots, -\alpha_{\rho_2^-(\nu)}^{(\gamma)}\underline{s}_{\cdot,\rho_2^-(\nu)}^{(\gamma)}\right) =$$
$$= \rho_2(\nu)\alpha_\eta^{(\gamma)}\prod_{i=0}^{\eta-1}\left(\alpha_i^{(\gamma)}\right)^2\det\left(U^{(1)}\right) +$$
$$-\rho_2^-(\nu)\alpha_\eta^{(\gamma)}\alpha_0^{(\gamma)}\prod_{i=1}^{\eta-1}\left(\alpha_i^{(\gamma)}\right)^2\det\left(U^{(2)}\right) =$$
$$= (-1)^\eta e^{\gamma\xi}\left(\rho_2(\nu)\det\left(U^{(1)}\right) - \rho_2^-(\nu)\det\left(U^{(2)}\right)\right), \tag{67}$$

in fact $\alpha_\eta^{(\gamma)} = (-1)^\eta e^{\gamma c_\eta} = (-1)^\eta$, and

$$\rho_2(\nu)\prod_{i=0}^{\eta-1}\left(\alpha_i^{(\gamma)}\right)^2 + \rho_2^-(\nu)\alpha_0^{(\gamma)}\prod_{i=1}^{\eta-1}\left(\alpha_i^{(\gamma)}\right)^2 = e^{\gamma\xi}.$$

From (65), (66) and (67), we have

$$\sum_{\tau\in P^{(1)}}\mathrm{sign}(\tau)\prod_{i=1}^{2\nu} M_{i,\tau(i)} = (-1)^\nu \mathrm{sign}(\tau^{(1)})e^{\gamma\xi}\det\left(T^{(1)}\right)\cdot$$
$$\cdot\left(\rho_2(\nu)\det\left(U^{(1)}\right) - \rho_2^-(\nu)\det\left(U^{(2)}\right)\right). \tag{68}$$

With a similar discussion but by using $\tau^{(2)}$, defined in Appendix D, we have

$$\sum_{\tau\in P^{(2)}}\mathrm{sign}(\tau)\prod_{i=1}^{2\nu} M_{i,\tau(i)} = (-1)^{\nu-1}\mathrm{sign}(\tau^{(2)})e^{\gamma\xi}\det\left(T^{(2)}\right)\cdot$$
$$\cdot\left(\rho_2(\nu)\det\left(U^{(2)}\right) + \rho_2^-(\nu)\det\left(U^{(1)}\right)\right). \tag{69}$$

On the other hand, for the last addendum in (62) we have:

$$\left|\sum_{\tau\in P^{(3)}}\mathrm{sign}(\tau)\prod_{i=1}^{2\nu} M_{i,\tau(i)}\right| \le \sum_{\tau\in P^{(3)}}\left|\prod_{i=1}^{\nu} M_{i,\tau_1(i)}\right|\left|\prod_{i=\nu+1}^{2\nu} M_{i,\tau_2(i)}\right| \le \sum_{\tau\in P^{(3)}}\left|\prod_{i=\nu+1}^{2\nu} M_{i,\tau_2(i)}\right|,$$

where the last inequality follows from the fact that when $i \le \nu$, $M_{i,\cdot}$ is sine or cosine. In Appendix E the following formula is proved:

$$\sum_{\tau \in P^{(3)}} \left| \prod_{i=\nu+1}^{2\nu} M_{i,\tau_2(i)} \right| = \mathcal{O}(e^{\gamma \xi_0}), \tag{70}$$

where ξ_0 is defined by (61). Hence we have

$$\sum_{\tau \in P^{(3)}} \operatorname{sign}(\tau) \prod_{i=1}^{2\nu} M_{i,\tau(i)} = \mathcal{O}(e^{\gamma \xi_0}). \tag{71}$$

From (62), (68), (69), (71), and from $\operatorname{sign}(\tau^{(1)}) = -\operatorname{sign}(\tau^{(2)})$ (see Appendix D), we have:

$$\begin{aligned} h_\nu(\gamma) &= \operatorname{sign}(\tau^{(2)}) e^{\gamma\xi} \rho_2(\nu) \left(\det\left(T^{(1)}\right) \det\left(U^{(1)}\right) + \det\left(T^{(2)}\right) \det\left(U^{(2)}\right) \right) + \\ &\quad + \operatorname{sign}(\tau^{(2)}) e^{\gamma\xi} \rho_2^-(\nu) \left(\det\left(T^{(1)}\right) \det\left(U^{(2)}\right) - \det\left(T^{(2)}\right) \det\left(U^{(1)}\right) \right) + \\ &\quad + \mathcal{O}(e^{\gamma\xi_0}), \end{aligned} \tag{72}$$

where matrices $T^{(1)}$, $T^{(2)}$, $U^{(1)}$, $U^{(2)}$ are defined in (12)–(15). From $\operatorname{sign}\left(\tau^{(2)}\right) = (-1)^{\eta+1}$ (see Appendix D), formulas (10), (11), (72), and the multilinearity property of the determinant, we have:

$$\begin{aligned} h_\nu(\gamma) &= (-1)^{\eta+1} e^{\gamma\xi} \rho_2(\nu) \left(d^{(1)} \det\left(U^{(1)}\right) + d^{(2)} \det\left(U^{(2)}\right) \right) + \\ &\quad + (-1)^{\eta+1} e^{\gamma\xi} \rho_2^-(\nu) \left(d^{(1)} \det\left(U^{(2)}\right) - d^{(2)} \det\left(U^{(1)}\right) \right) + \mathcal{O}(e^{\gamma\xi_0}) = \\ &= (-1)^{\eta+1} e^{\gamma\xi} \det \Big(\underline{s}^{(\gamma)}_{\cdot,\rho_2^-(\nu)}, \underline{s}^{(\gamma)}_{\cdot,\rho_2^-(\nu)+1}, \ldots, \underline{s}^{(\gamma)}_{\cdot,\eta-1}, \\ &\qquad d^{(1)} \left(\rho_2(\nu) \underline{s}^{(\gamma)}_{\cdot,\eta} + \rho_2^-(\nu) \underline{c}^{(\gamma)}_{\cdot,\eta} \right) + d^{(2)} \left(\rho_2(\nu) \underline{c}^{(\gamma)}_{\cdot,\eta} - \rho_2^-(\nu) \underline{s}^{(\gamma)}_{\cdot,\eta} \right), \\ &\qquad \underline{c}^{(\gamma)}_{\cdot,\eta-1}, \underline{c}^{(\gamma)}_{\cdot,\eta-2}, \ldots, \underline{c}^{(\gamma)}_{\cdot,0} \Big) + \mathcal{O}(e^{\gamma\xi_0}) = \\ &= (-1)^{\eta+1} e^{\gamma\xi} \cdot \det\left(W^{(2)} D^{(2)} + W^{(1)} D^{(1)} \right) + \mathcal{O}(e^{\gamma\xi_0}), \end{aligned} \tag{73}$$

where $W^{(1)}$, $W^{(2)}$, $D^{(1)}$, $D^{(2)}$ are given by (20)–(23). In Appendix B it is proved that there exist a_q, $q = 0, 1, \ldots, \eta - 1$, and b_p, $p = \rho_2^-(\nu), \rho_2^-(\nu) + 1, \ldots, \eta - 1$, such that

$$d^{(2)} \left(\rho_2 \underline{s}_{\cdot,\eta} + \rho_2^- \underline{c}_{\cdot,\eta} \right) + d^{(1)} \left(-\rho_2 \underline{c}_{\cdot,\eta} + \rho_2^- \underline{s}_{\cdot,\eta} \right) = \sum_{p=\rho_2^-(\nu)}^{\eta-1} b_p \underline{s}_{\cdot,p} + \sum_{q=0}^{\eta-1} a_q \underline{c}_{\cdot,q}. \tag{74}$$

Let

$$\underline{a} = \left(a_{\eta-1}, a_{\eta-2}, \ldots, a_{\rho_2^-(\nu)} \right)^T \in \mathbb{R}^{\eta-\rho_2^-(\nu)}, \tag{75}$$

$$\underline{b} = \left(b_{\rho_2^-(\nu)}, b_{\rho_2^-(\nu)+1}, \ldots, b_{\eta-1} \right)^T \in \mathbb{R}^{\eta-\rho_2^-(\nu)}, \tag{76}$$

from (74), we have

$$W^{(2)}\begin{pmatrix} O_\eta & \underline{b} & J_\eta \\ \underline{0}_\eta^T & 0 & \underline{0}_\eta^T \\ -J_\eta & \underline{a} & O_\eta \end{pmatrix} = W^{(1)}, \qquad \text{when } \nu \text{ is odd,} \tag{77}$$

$$W^{(2)}\begin{pmatrix} O_{\eta-1} & \underline{b} & J_{\eta-1} & \underline{0}_{\eta-1} \\ \underline{0}_{\eta-1}^T & 0 & \underline{0}_{\eta-1}^T & 0 \\ -J_{\eta-1} & \underline{a} & O_{\eta-1} & \underline{0}_{\eta-1} \\ \underline{0}_{\eta-1}^T & a_0 & \underline{0}_{\eta-1}^T & 0 \end{pmatrix} = W^{(1)}, \quad \text{when } \nu \text{ is even.} \tag{78}$$

So $W^{(1)} = W^{(2)}J^{(\nu)}$, where $J^{(\nu)}$ is the matrix defined by relations (77), (78), and

$$\det\left(W^{(2)}D^{(2)} + W^{(1)}D^{(1)}\right) = \det\left(W^{(2)}\right)\det\left(D^{(2)} + J^{(\nu)}D^{(1)}\right).$$

From (73) we have

$$h_\nu(\gamma) = (-1)^{\eta+1}e^{\gamma\xi}\det\left(W^{(2)}\right)\det\left(D^{(2)} + J^{(\nu)}D^{(1)}\right) + \mathcal{O}(e^{\gamma\xi_0}),$$

where $\det\big(D^{(2)} + J^{(\nu)}D^{(1)}\big) = \cos(\gamma)$, see Appendix F for a proof of this last relation, and from Lemma 2 we have (59). This concludes the proof of the theorem. □

4. Numerical examples

We propose a simple algorithm to compute the ν-derivative of a function $f(t)$, $t \in [0,1]$, by knowing $f^{(j)}(0)$, $j = 0,1,\ldots,\nu-1$. This algorithm is based on the singular value expansion presented in the previous sections; so it has the following straight structure:

Algorithm 1. Given ν, $L \in \mathbb{N}$, given $f(t)$, $t \in [0,1]$ and $f^{(j)}(0)$, $j = 0,1,\ldots,\nu-1$, compute an approximation $\tilde{f}^{(\nu)}$ of $f^{(\nu)}$ by the following steps:

1. compute the first L singular values, μ_l, $l = 1,2,\ldots,L$, of K in (1), by the L lowest zeros $\gamma_l > 0$, $l = 1,2,\ldots,L$, of function h_ν defined in (56),
2. for each $l = 1,2,\ldots,L$, compute the left-singular function u_l and right-singular function v_l, by (25)–(33),
3. compute the approximation $\tilde{f}^{(\nu)}$ of $f^{(\nu)}$ by formula (6), where the sum is truncated to index $l = L$ and g is given by (3).

This algorithm is tested by two numerical experiments. In the first experiment the following two functions

$$f_1(x) = \frac{1}{1+x^2}, \tag{79}$$

$$f_2(x) = \cos\left((1+x)^2\right), \tag{80}$$

are considered with respect to derivation order $\nu = 1,2,10$. Note that, for each one of these functions, the corresponding right-hand side of integral equation (4) is computed by formal calculations. Moreover an extension of the Neville algorithm [10], [12] is used to compare the results obtained with the proposed method. In the second experiment we consider the numerical derivation of some singular functions of the kernel (1) for the case $\nu = 1,2$.

Fig. 1 shows the first L singular values, μ_l, $l = 1,2,\ldots,L$, when $\nu = 1,2,10$. Algorithm 1 is used to compute the ν-derivative of f_1, f_2 at 100 points uniformly distributed into the interval $[0,0.5]$ and the errors on these approximations are evaluated by the explicit derivatives of f_1, f_2. Table 1 shows the results of this

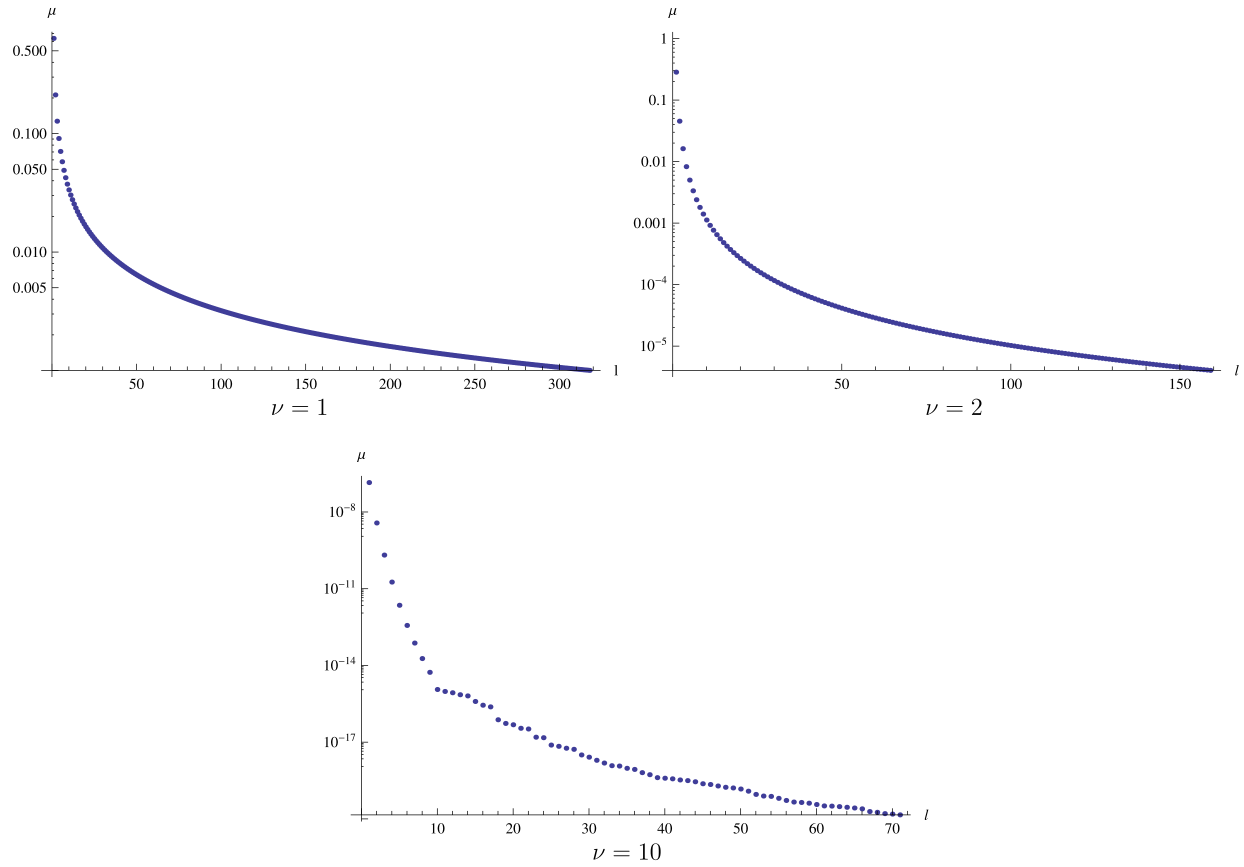

Fig. 1. The diagrams give the singular value μ_l, $l = 1, 2, \ldots, L$, of K as a function of l, for derivation orders $\nu = 1, 2, 10$ and truncation parameters: $L = 320$ for $\nu = 1$; $L = 160$ for $\nu = 2$; $L = 71$ for $\nu = 10$.

numerical simulation. In particular, it reports the errors on the approximations computed by Algorithm 1, and by the extension of Neville algorithm; moreover it reports also the difference between $||K||^2$ (the $\mathcal{L}^2$-norm of K) and the norm of the singular integral operator obtained by the first L terms of the singular value expansion of K. We note that different truncation parameters L are used for different differential orders ν. This is due to the different asymptotic behavior of the singular values μ_l when $l \to \infty$ for different orders ν, see Fig. 1, as consequence of different regularity properties of K defined in (1). In particular, from the last column of Table 1, we note that when $\nu = 10$ the truncation error for the kernel K is $1.6(-14)$ when $L = 30$, instead when $\nu = 2$ is $3.2(-5)$ when $L = 150$ and when $\nu = 1$ is $1.8(-2)$ when $L = 300$. From Table 1, we can observe that the extension of Neville algorithm is superior to Algorithm 1 for low derivation orders; on the contrary the two algorithms has similar accuracy levels for $\nu = 10$, where the truncation error is very small. We note that approximation methods based on integral equation (4) guarantee accurate approximations in a neighborhood of zero, instead the numerical error for Algorithm 1 has been computed on the interval $[0, 0.5]$.

The relevance of the second experiment is explained by the following remark.

Remark 5. The singular functions u_l and v_l corresponding to the singular value μ_l satisfy the following relations

$$u_l^{(\nu)}(t) = \frac{1}{\mu_l} v_l(t), \tag{81}$$

$$v_l^{(\nu)}(t) = \frac{(-1)^\nu}{\mu_l} u_l(t). \tag{82}$$

Please cite this article in press as: N. Egidi, P. Maponi, The singular value expansion of the Volterra integral equation associated to a numerical differentiation problem, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2017.12.007

Table 1
The second and third column report the 2-norm relative errors obtained, by applying the proposed Algorithm 1 and the extension of Neville algorithm (NAG), in the computation of the ν-derivative of f_i, $i = 1, 2$; the errors are computed by using 100 points uniformly distributed into the interval $[0, 0.5]$. The fourth columns reports the difference between the $\mathcal{L}^2$-norm of K and the norm of the corresponding singular value expansion used in the proposed algorithm.

	f_1	f_2	$\|\|K\|\|^2 - \sum_{l=1}^{L} \mu_l^2$
$\nu = 1$			
$L = 100$	$5.2(-4)$	$1.9(-4)$	$3.2(-2)$
$L = 200$	$1.9(-3)$	$2.2(-3)$	$2.2(-2)$
$L = 300$	$6.6(-5)$	$2.4(-5)$	$1.8(-2)$
NAG	$1.2(-14)$	$3.1(-15)$	
$\nu = 2$			
$L = 50$	$2.6(-3)$	$3.2(-2)$	$1.6(-4)$
$L = 100$	$1.3(-3)$	$1.6(-2)$	$5.8(-5)$
$L = 150$	$8.5(-4)$	$1.1(-2)$	$3.2(-5)$
NAG	$1.5(-13)$	$2.6(-13)$	
$\nu = 10$			
$L = 10$	$4.0(-3)$	$5.0(-1)$	$1.6(-14)$
$L = 20$	$6.3(-4)$	$3.3(-1)$	$1.6(-14)$
$L = 30$	$6.5(-4)$	$2.3(-1)$	$1.6(-14)$
NAG	$4.7(-3)$	$5.6(-2)$	

Table 2
The 2-norm relative errors, at 100 points uniformly distributed into the interval $[0, 0.5]$, obtained by applying the proposed Algorithm 1, to compute the ν-derivative, $\nu = 1, 2$ of u_1, u_5, u_{10}, v_1, v_5, v_{10}, whose analytic solutions are given by (81) and (82).

	u_1	u_5	u_{10}	v_1	v_5	v_{10}
$\nu = 1$						
$L = 100$	$2.4(-13)$	$2.1(-13)$	$9.1(-13)$	$2.1(-3)$	$3.7(-3)$	$6.3(-3)$
$L = 200$	$5.8(-13)$	$2.2(-13)$	$9.2(-13)$	$8.8(-4)$	$1.6(-3)$	$2.4(-3)$
$L = 300$	$1.3(-12)$	$2.6(-13)$	$9.2(-13)$	$5.3(-4)$	$9.6(-4)$	$1.4(-3)$
$\nu = 2$						
$L = 50$	$1.7(-9)$	$4.2(-11)$	$9.4(-12)$	$1.1(-3)$	$3.5(-3)$	$6.5(-3)$
$L = 100$	$2.1(-6)$	$5.7(-8)$	$1.3(-8)$	$2.8(-4)$	$9.8(-4)$	$2.1(-3)$
$L = 150$	$2.4(-7)$	$5.7(-8)$	$1.3(-8)$	$1.3(-4)$	$4.6(-4)$	$1.0(-3)$

These relations (see also formulas (7) and (8)) are used in the derivation of the characteristic equation $h_\nu(\gamma) = 0$ given by (56).

In particular, this experiment tests relation (81) and (82) for the cases $\nu = 1, 2$ by considering the derivation problem for the following singular functions u_1, u_5, u_{10}, v_1, v_5, v_{10}. The numerical results are obtained by applying the proposed Algorithm 1, these results are compared with the theoretical results, i.e. (81) and (82). Table 2 reports the 2-norm relative errors, at 100 points uniformly distributed into the interval $[0, 0.5]$. This table confirms the theoretical results, in fact the approximations of the ν derivatives of v_1, v_5, v_{10} have the same quality of those for general functions such as f_1 and f_2 reported in Table 1; on the contrary, for the ν derivatives of u_1, u_5, u_{10} we have good approximations because their ν derivatives are in the finite dimensional space used for the representation of the solution, i.e. $\text{span}(v_1, v_2, \ldots, v_L)$, that is the space generated by the singular functions v_l, see formula (6) for details.

These preliminar results serve to show the correctness of the proposed method; however, they also provide an interesting outcome, in fact the simple Algorithm 1 proposed in this paper can be greatly improved by different ways, such as the regularization of the inversion of singular value decomposition [16], or the joint use of operator $\mathcal{K}$ and its adjoint $\mathcal{K}^*$; in particular, with this last choice we conjecture to obtain a better approximation on all the interval $[0, 1]$ by considering a self-adjoint operator.

The numerical results shown in this section have been computed by a FORTRAN program running in an Intel Pentium D CPU 36Hz with operative system Windows 7. In particular for the Neville algorithm we have used the implementation provided by the routine D04AAF of the NAG Numerical Library [14].

5. Conclusions

The problem of numerical differentiation, reformulated by a Volterra integral equation of the first kind, has been solved through the singular value expansion of the corresponding integral kernel. This singular value expansion allows the definition of a simple algorithm to compute the numerical derivatives of a given function. The numerical results obtained with this algorithm give a numerical evidence of the correctness of the general approach proposed in the present paper. However these results can be improved by using refined algorithms taking into account the regularization techniques to deal with ill-posedness of problem (2). Another way to improve the results obtained by Algorithm 1 is the joint use of integral operator $\mathcal{K}$ and of its adjoint $\mathcal{K}^*$. Finally, an interesting future study is also the application of the proposed method in the solution of differential equations.

Appendix A

In the following Lemma 1 is proved.

When $\nu = 1$, $T^{(1)} = (s_0) = (1)$ and $T^{(2)} = (c_0) = (0)$, so $d^{(1)} = 1$ and $d^{(2)} = 0$.

When $\nu = 2$, $T^{(1)} = (\underline{c}_{\cdot,2}, \underline{s}_{\cdot,1})$ and $T^{(2)} = (\underline{c}_{\cdot,2}, \underline{c}_{\cdot,1})$, so, also in this case, it is immediate to verify that $d^{(1)} = -1$ and $d^{(2)} = 1$.

Let $\nu \geq 3$ and let

$$\underline{e}_p = \left(e^{\iota\theta_p}, e^{\iota 2\theta_p}, \ldots, e^{\iota\nu\theta_p}\right)^T = e^{\iota\theta_p}\left(1, e^{\iota\theta_p}, \ldots, e^{\iota(\nu-1)\theta_p}\right)^T \in \mathbb{C}^\nu,$$

then

$$\overline{\underline{e}}_p = \left(e^{-\iota\theta_p}, e^{-\iota 2\theta_p}, \ldots, e^{-\iota\nu\theta_p}\right)^T = e^{-\iota\theta_p}\left(1, e^{-\iota\theta_p}, \ldots, e^{-\iota(\nu-1)\theta_p}\right)^T.$$

From the multilinearity property of the determinant we have:

$$\begin{aligned}
d^{(2)} + \iota d^{(1)} &= \det\left(\underline{c}_{\cdot,\nu-\rho_2(\nu)}, \underline{c}_{\cdot,\nu-\rho_2(\nu)-1}, \ldots, \underline{c}_{\cdot,\eta+1}, \underline{c}_{\cdot,\eta} + \iota\underline{s}_{\cdot,\eta}, \underline{s}_{\cdot,\eta+1}, \underline{s}_{\cdot,\eta+2}, \ldots, \underline{s}_{\cdot,\nu-1}\right) = \\
&= \det\left(\frac{1}{2}\left(\underline{e}_{\nu-\rho_2(\nu)}, \underline{e}_{\nu-\rho_2(\nu)-1}, \ldots, \underline{e}_{\eta+1}, \underline{e}_\eta, \iota\overline{\underline{e}}_{\eta+1}, \iota\overline{\underline{e}}_{\eta+2}, \ldots, \iota\overline{\underline{e}}_{\nu-1}\right) + \right. \\
&\quad \left. + \frac{1}{2}\left(\overline{\underline{e}}_{\nu-\rho_2(\nu)}, \overline{\underline{e}}_{\nu-\rho_2(\nu)-1}, \ldots, \overline{\underline{e}}_{\eta+1}, \underline{e}_\eta, -\iota\underline{e}_{\eta+1}, -\iota\underline{e}_{\eta+1}, \ldots, -\iota\underline{e}_{\nu-1}\right)\right) = \\
&= \det\left(\frac{1}{2}\left(\underline{e}_{\nu-\rho_2(\nu)}, \underline{e}_{\nu-\rho_2(\nu)-1}, \ldots, \underline{e}_{\eta+1}, \underline{e}_\eta, \iota\overline{\underline{e}}_{\eta+1}, \iota\overline{\underline{e}}_{\eta+2}, \ldots, \iota\overline{\underline{e}}_{\nu-1}\right)(I_\nu + J)\right),
\end{aligned}$$

where $J = (J_{i,j})_{1\leq i,j\leq\nu}$ is a matrix having the following non null entries: when ν is odd, $J_{i,\nu-i+1} = -\iota$, for $i = 1, 2, \ldots, \nu$, $i \neq \eta + 1$, and $J_{\eta+1,\eta+1} = 1$; when ν is even, $J_{1,1} = 1$, $J_{i,\nu-i+2} = -\iota$, for $i = 2, 3, \ldots, \nu$, $i \neq \eta + 1$, and $J_{\eta+1,\eta+1} = 1$.

Let

$$\underline{x} = \left(e^{\iota\theta_{\nu-\rho_2(\nu)}}, e^{\iota\theta_{\nu-\rho_2(\nu)-1}}, \ldots, e^{\iota\theta_{\eta+1}}, e^{\iota\theta_\eta}, e^{-\iota\theta_{\eta+1}}, e^{-\iota\theta_{\eta+2}}, \ldots, e^{-\iota\theta_{\nu-1}}\right) \in \mathbb{C}^\nu.$$

When ν is odd

$$\begin{aligned}
&\underline{x} = \left(e^{\iota\theta_{\nu-1}}, e^{\iota\theta_{\nu-2}}, \dots, e^{\iota\theta_{\eta+1}}, e^{\iota\theta_\eta}, e^{-\iota\theta_{\eta+1}}, e^{-\iota\theta_{\eta+2}}, \dots, e^{-\iota\theta_{\nu-1}}\right),\\
&\det\left(\underline{e}_{\nu-\rho_2(\nu)}, \underline{e}_{\nu-\rho_2(\nu)-1}, \dots, \underline{e}_{\eta+1}, \underline{e}_\eta, \iota\underline{e}_{\eta+1}^{-1}, \iota\underline{e}_{\eta+2}^{-1}, \dots, \iota\underline{e}_{\nu-1}^{-1}\right) =\\
&\det\left(\underline{e}_{\nu-1}, \underline{e}_{\nu-2}, \dots, \underline{e}_{\eta+1}, \underline{e}_\eta, \iota\underline{e}_{\eta+1}^{-1}, \iota\underline{e}_{\eta+2}^{-1}, \dots, \iota\underline{e}_{\nu-1}^{-1}\right) =\\
&\iota^{\nu-\eta-1} e^{\iota\theta_\eta} \left(\prod_{p=\eta+1}^{\nu-1} e^{\iota\theta_p} e^{-\iota\theta_p}\right) \det\begin{pmatrix} \underline{1}_\nu^T \\ \underline{x} \\ \dots \\ \underline{x}^{\nu-1}\end{pmatrix},
\end{aligned}$$

moreover, from the definition of J and (9), we have

$$\det(I_\nu + J) = 2\prod_{p=1}^{\eta}(1 - (-\iota)(-\iota)) = 2^{\eta+1}.$$

When ν is even

$$\begin{aligned}
&\underline{x} = \left(e^{\iota\theta_\nu}, e^{\iota\theta_{\nu-1}}, \dots, e^{\iota\theta_{\eta+1}}, e^{\iota\theta_\eta}, e^{-\iota\theta_{\eta+1}}, e^{-\iota\theta_{\eta+2}}, \dots, e^{-\iota\theta_{\nu-1}}\right).\\
&\det\left(\underline{e}_{\nu-\rho_2(\nu)}, \underline{e}_{\nu-\rho_2(\nu)-1}, \dots, \underline{e}_{\eta+1}, \underline{e}_\eta, \iota\underline{e}_{\eta+1}^{-1}, \iota\underline{e}_{\eta+2}^{-1}, \dots, \iota\underline{e}_{\nu-1}^{-1}\right) =\\
&\det\left(\underline{e}_{\nu}, \underline{e}_{\nu-1}, \dots, \underline{e}_{\eta+1}, \underline{e}_\eta, \iota\underline{e}_{\eta+1}^{-1}, \iota\underline{e}_{\eta+2}^{-1}, \dots, \iota\underline{e}_{\nu-1}^{-1}\right) =\\
&\iota^{\nu-\eta-1} e^{\iota\theta_\eta} e^{\iota\theta_\nu} \left(\prod_{p=\eta+1}^{\nu-1} e^{\iota\theta_p} e^{-\iota\theta_p}\right) \det\begin{pmatrix} \underline{1}_\nu^T \\ \underline{x} \\ \dots \\ \underline{x}^{\nu-1}\end{pmatrix},
\end{aligned}$$

moreover, from the definition of J and (9), we have

$$\det(I_\nu + J) = 4\prod_{p=1}^{\eta-1}(1 - (-\iota)(-\iota)) = 2^{\eta+1}.$$

So that, by noting that $e^{\iota\theta_\eta} = \iota$ and when ν is even $e^{\iota\theta_\nu} = -1$, we have

$$\begin{aligned}
d^{(2)} + \iota d^{(1)} &= \left(\frac{1}{2}\right)^\nu \iota^{\nu-\eta} (-1)^{\nu+1} \det\begin{pmatrix} \underline{1}_\nu^T \\ \underline{x} \\ \dots \\ \underline{x}^{\nu-1}\end{pmatrix} 2^{\eta+1} =\\
&\left(\frac{1}{2}\right)^\nu (2)^{\eta+1} (\iota)^{\nu-\eta} (-1)^{\nu+1} \prod_{1\le i<j\le\nu} (x_j - x_i),
\end{aligned}$$

where we used the Vandermonde determinant with x_i the ith entry of $\underline{x}$.

We note that when $i \ne j$, $i, j = 1, 2, \dots, \nu$ then $x_i \ne x_j$, so that $d^{(2)} + \iota d^{(1)} \ne 0$, moreover by using the definition of x_i and by analyzing separately the two sets of indices $\{(i, j)\ :\ 1 \le i \le \eta+1,\ i < j \le \nu\}$ and $\{(i, j)\ :\ \eta+2 \le i < j \le \nu\}$, it is straightforward to prove that

$$\prod_{1\le i<j\le\nu} (x_i - x_j) = (-4)^{\nu-\eta-1}\left(1 + \rho_2^-(\nu)\iota\right) d,$$

and hence (16). Let $\zeta \in \mathbb{N}$, we have

$$d^{(2)} + \iota d^{(1)} = \begin{cases} (-1)^{\zeta} 2^{2\zeta} d\iota & \nu = 4\zeta+1,\ \eta = 2\zeta, \\ (-1)^{\zeta} 2^{2\zeta+1} d & \nu = 4\zeta+3,\ \eta = 2\zeta+1, \\ (-1)^{\zeta+1} (\iota - 1)\, 2^{2\zeta} d & \nu = 4\zeta+2,\ \eta = 2\zeta+1, \\ (-1)^{\zeta+1} (1+\iota)\, 2^{2\zeta+1} d & \nu = 4\zeta+4,\ \eta = 2\zeta+2, \end{cases}$$

$$d^{(2)} = \begin{cases} 0 & \nu = 4\zeta+1,\ \eta = 2\zeta, \\ (-1)^{\zeta} 2^{2\zeta+1} d & \nu = 4\zeta+3,\ \eta = 2\zeta+1, \\ (-1)^{\zeta} 2^{2\zeta} d & \nu = 4\zeta+2,\ \eta = 2\zeta+1, \\ (-1)^{\zeta+1} 2^{2\zeta+1} d & \nu = 4\zeta+4,\ \eta = 2\zeta+2, \end{cases}$$

$$d^{(1)} = \begin{cases} (-1)^{\zeta} 2^{2\zeta} d & \nu = 4\zeta+1,\ \eta = 2\zeta, \\ 0 & \nu = 4\zeta+3,\ \eta = 2\zeta+1, \\ (-1)^{\zeta+1} 2^{2\zeta} d & \nu = 4\zeta+2,\ \eta = 2\zeta+1, \\ (-1)^{\zeta+1} 2^{2\zeta+1} d & \nu = 4\zeta+4,\ \eta = 2\zeta+2, \end{cases}$$

from which we have (17) and (18). □

Appendix B

In the following Lemma 2 is proved. When $\nu = 1$ it is trivial. When $\nu = 2$ it follows from $\underline{s}_{\cdot,0} = \underline{0}_2$, $0 = \theta_0 < \theta_1 = \frac{\pi}{2} < \theta_2 = \pi$, $\underline{s}_{\cdot,1} = (1,0)^T$, $\underline{c}_{\cdot,1} = (0,1)^T$. When $\nu \geq 3$, for $p, k \in \mathbb{Z}$, we have

$$\begin{aligned} z_{\nu-\rho_2(\nu)-p} &= e^{\iota\theta_{\nu-\rho_2(\nu)-p}} = e^{\iota\frac{(2\nu-2\rho_2(\nu)-2p+\rho_2(\nu))}{2\nu}\pi} = e^{\iota\pi} e^{-\iota\theta_p} = -\overline{z}_p, \\ z^k_{\nu-\rho_2(\nu)-p} &= (-1)^k \overline{z}^k_p, \\ \cos(k\theta_{\nu-\rho_2(\nu)-p}) &= (-1)^k \cos(k\theta_p), \qquad \text{(B.1)} \\ \sin(k\theta_{\nu-\rho_2(\nu)-p}) &= -(-1)^k \sin(k\theta_p). \qquad \text{(B.2)} \end{aligned}$$

For $k = 1, 2, \ldots, \nu$ and $p \in \mathbb{Z}$, $\cos(k\theta_p)$ is the kth entry of vector $\underline{c}_{\cdot,p} \in \mathbb{R}^\nu$ and $\sin(k\theta_p)$ is the kth entry of vector $\underline{s}_{\cdot,p} \in \mathbb{R}^\nu$. By multiplying the $\eta + \rho_2(\nu)$ rows of the matrices $T^{(1)}$ and $T^{(2)}$ having odd indices by -1, and by using (B.1), (B.2) and the multilinearity property of the determinant, we have

$$\begin{aligned} d^{(1)} &= \det\left(\underline{c}_{\cdot,\nu-\rho_2(\nu)}, \underline{c}_{\cdot,\nu-\rho_2(\nu)-1} \ldots, \underline{c}_{\cdot,\eta+1}, \underline{s}_{\cdot,\eta}, \underline{s}_{\cdot,\eta+1} \ldots, \underline{s}_{\cdot,\nu-1}\right) = \\ &= (-1)^{\eta-\rho_2(\nu)} \det\left(\underline{c}_{\cdot,0}, \underline{c}_{\cdot,1}, \ldots, \underline{c}_{\cdot,\eta-1}, -\underline{s}_{\cdot,\eta}, -\underline{s}_{\cdot,\eta-1}, \ldots, -\underline{s}_{\cdot,\rho_2^-(\nu)}\right) = \\ &= \det\left(\underline{c}_{\cdot,0}, \underline{c}_{\cdot,1}, \ldots, \underline{c}_{\cdot,\eta-1}, \underline{s}_{\cdot,\eta}, \underline{s}_{\cdot,\eta-1}, \ldots, \underline{s}_{\cdot,\rho_2^-(\nu)}\right), \\ d^{(2)} &= \det\left(\underline{c}_{\cdot,\nu-\rho_2(\nu)}, \underline{c}_{\cdot,\nu-\rho_2(\nu)-1}, \ldots, \underline{c}_{\cdot,\eta}, \underline{s}_{\cdot,\eta+1}, \underline{s}_{\cdot,\eta+2} \ldots, \underline{s}_{\cdot,\nu-1}\right) = \\ &= (-1)^{\eta-\rho_2(\nu)} \det\left(\underline{c}_{\cdot,0}, \underline{c}_{\cdot,1}, \ldots, \underline{c}_{\cdot,\eta}, -\underline{s}_{\cdot,\eta-1}, -\underline{s}_{\cdot,\eta-2}, \ldots, -\underline{s}_{\cdot,\rho_2^-(\nu)}\right) = \\ &= -\det\left(\underline{c}_{\cdot,0}, \underline{c}_{\cdot,1}, \ldots, \underline{c}_{\cdot,\eta}, \underline{s}_{\cdot,\eta-1}, \underline{s}_{\cdot,\eta-2}, \ldots, \underline{s}_{\cdot,\rho_2^-(\nu)}\right). \end{aligned}$$

When ν is even $\underline{s}_{\cdot,0} = \underline{0}_\nu$ and so $\det(W^{(1)}) = 0$; when ν is odd

$$\det\left(W^{(1)}\right) = d^{(1)} \det\left(-\underline{c}_{\cdot,0}, -\underline{c}_{\cdot,1}, \ldots, -\underline{c}_{\cdot,\eta}, \underline{s}_{\cdot,\eta-1}, \underline{s}_{\cdot,\eta-2}, \ldots, \underline{s}_{\cdot,0}\right) +$$

$$+ d^{(2)}\det\left(-\underline{c}_{\cdot,0}, -\underline{c}_{\cdot,1}, \ldots, -\underline{c}_{\cdot,\eta-1}, \underline{s}_{\cdot,\eta}, \underline{s}_{\cdot,\eta-1}, \ldots, \underline{s}_{\cdot,0}\right) =$$
$$= -(-1)^{\eta+1} d^{(1)} d^{(2)} + (-1)^{\eta} d^{(2)} d^{(1)},$$

so that, from (17) $\det(W^{(1)}) = 0$ also for ν odd. In the same way we have

$$\begin{aligned}
\det\left(W^{(2)}\right) = d^{(2)}\det\Big(&\underline{s}_{\cdot,\rho_2^-(\nu)}, \underline{s}_{\cdot,\rho_2^-(\nu)+1}, \ldots, \underline{s}_{\cdot,\eta-1},\\
&\rho_2(\nu)\underline{c}_{\cdot,\eta} - \rho_2^-(\nu)\underline{s}_{\cdot,\eta}, \underline{c}_{\cdot,\eta-1}, \underline{c}_{\cdot,\eta-2}, \ldots, \underline{c}_{\cdot,0}\Big) +\\
&+ d^{(1)}\det\Big(\underline{s}_{\cdot,\rho_2^-(\nu)}, \underline{s}_{\cdot,\rho_2^-(\nu)+1}, \ldots, \underline{s}_{\cdot,\eta-1},\\
&\rho_2(\nu)\underline{s}_{\cdot,\eta} + \rho_2^-(\nu)\underline{c}_{\cdot,\eta}, \underline{c}_{\cdot,\eta-1}, \underline{c}_{\cdot,\eta-2}, \ldots, \underline{c}_{\cdot,0}\Big) =\\
= \rho_2(\nu)&\left(-(-1)^{\eta} d^{(2)} d^{(2)} + (-1)^{\eta} d^{(1)} d^{(1)}\right) +\\
&+ \rho_2^-(\nu)\left(-(-1)^{\eta} d^{(2)} d^{(1)} - (-1)^{\eta} d^{(1)} d^{(2)}\right),
\end{aligned}$$

from which by using Lemma 1 we have (24) and in particular $\det(W^{(2)})$ is positive when ν is odd and negative otherwise. This concludes the proof of Lemma 2. Moreover from Lemma 1 we have

$$\begin{aligned}
&\det\Big(\underline{s}_{\cdot,\rho_2^-(\nu)}, \underline{s}_{\cdot,\rho_2^-(\nu)+1}, \ldots, \underline{s}_{\cdot,\eta-1},\\
&d^{(2)}\left(\rho_2(\nu)\underline{s}_{\cdot,\eta} + \rho_2^-(\nu)\underline{c}_{\cdot,\eta}\right) + d^{(1)}\left(-\rho_2(\nu)\underline{c}_{\cdot,\eta} + \rho_2^-(\nu)\underline{s}_{\cdot,\eta}\right)\\
&, \underline{c}_{\cdot,\eta-1}, \underline{c}_{\cdot,\eta-2}, \ldots, \underline{c}_{\cdot,0}\Big) =\\
&= \rho_2(\nu)\left((-1)^{\eta} d^{(2)} d^{(1)} + (-1)^{\eta} d^{(1)} d^{(2)}\right) +\\
&+ \rho_2^-(\nu)\left(-(-1)^{\eta} d^{(2)} d^{(2)} + (-1)^{\eta} d^{(1)} d^{(1)}\right) = 0,
\end{aligned}$$

hence, because $\det W^{(2)} \neq 0$ there exist a_q, $q = 0, 1, \ldots, \eta-1$, and b_p, $p = \rho_2^-(\nu), \rho_2^-(\nu)+1, \ldots, \eta-1$, such that

$$d^{(2)}\left(\rho_2(\nu)\underline{s}_{\cdot,\eta} + \rho_2^-(\nu)\underline{c}_{\cdot,\eta}\right) + d^{(1)}\left(-\rho_2(\nu)\underline{c}_{\cdot,\eta} + \rho_2^-(\nu)\underline{s}_{\cdot,\eta}\right) = \sum_{p=\rho_2^-(\nu)}^{\eta-1} b_p \underline{s}_{\cdot,p} + \sum_{q=0}^{\eta-1} a_q \underline{c}_{\cdot,q}.$$

Appendix C

In the following formulas (45)–(52) are proved. Let

$$f_\alpha : [0,\infty) \to \mathbb{C},\ f_\alpha(t) = e^{t\alpha},\ t \geq 0,\ \alpha \in \mathbb{C},$$

by applying iteratively integration by parts we have that

$$(Kf_\alpha)(t) = -\sum_{k=0}^{\nu-1} \frac{t^{\nu-k-1}}{(\nu-k-1)!}\alpha^{-(k+1)} + \alpha^{-\nu} f_\alpha(t),$$
$$(K^* f_\alpha)(t) = e^{\alpha}\sum_{k=0}^{\nu-1} \frac{(-1)^k(1-t)^{\nu-k-1}}{(\nu-k-1)!}\alpha^{-(k+1)} + (-1)^{\nu}\alpha^{-\nu} f_\alpha(t).$$

Moreover, by using the notations introduced in Theorem 1, we have

$$\begin{aligned}
&\sigma_{l,p}(t) = e^{\gamma_l z_{2p+1} t} = f_{\gamma_l z_{2p+1}}(t),\\
&\delta_{l,p}(t) = e^{\gamma_l z_{2p} t} = f_{\gamma_l z_{2p}}(t),\\
&(\gamma_l z_q)^{-k-1} = \frac{e^{-\iota(k+1)\theta_q}}{\gamma_l^{k+1}},\\
&(\gamma_l z_q)^{-\nu} = \frac{e^{-\iota\pi(2q+\rho_2(\nu))/2}}{\gamma_l^{\nu}} = (-1)^q \mu_l e^{-\iota\rho_2(\nu)\pi/2},\\
&\mathrm{Re}\left(e^{-\iota\rho_2(\nu)\pi/2}\sigma_{l,p}(t)\right) = \sigma_{l,p}^{(1+\rho_2(\nu))}(t),\\
&\mathrm{Im}\left(e^{-\iota\rho_2(\nu)\pi/2}\sigma_{l,p}(t)\right) = (-1)^{\rho_2(\nu)}\sigma_{l,p}^{(2-\rho_2(\nu))}(t),\\
&\mathrm{Re}\left(e^{-\iota\rho_2(\nu)\pi/2}\delta_{l,p}(t)\right) = \delta_{l,p}^{(1+\rho_2(\nu))}(t),\\
&\mathrm{Im}\left(e^{-\iota\rho_2(\nu)\pi/2}\delta_{l,p}(t)\right) = (-1)^{\rho_2(\nu)}\delta_{l,p}^{(2-\rho_2(\nu))}(t),
\end{aligned}$$

and by using the above formulas we have

$$\begin{aligned}
K\sigma_{l,p}^{(1)}(t) &= \left(K\left(\mathrm{Re}\left(\sigma_{l,p}\right)\right)\right)(t) = \mathrm{Re}\left(\left(K\sigma_{l,p}\right)(t)\right) =\\
&= -\sum_{k=0}^{\nu-1}\frac{t^{\nu-k-1}}{(\nu-k-1)!\gamma_l^{k+1}}\mathrm{Re}\left(e^{-\iota(k+1)\theta_{2p+1}}\right) +\\
&\qquad + (-1)^{2p+1}\mu_l \mathrm{Re}\left(e^{-\iota\rho_2(\nu)\pi/2}\sigma_{l,p}(t)\right) =\\
&= -\sum_{k=0}^{\nu-1}\frac{t^{\nu-k-1}}{(\nu-k-1)!\gamma_l^{k+1}}c_{k,2p+1} - \mu_l\sigma_{l,p}^{(1+\rho_2(\nu))}(t),
\end{aligned}$$

that gives formula (45). Instead

$$\begin{aligned}
K^*\sigma_{l,p}^{(1)}(t) &= \sum_{k=0}^{\nu-1}\frac{(-1)^k(1-t)^{\nu-k-1}}{(\nu-k-1)!\gamma_l^{k+1}}\mathrm{Re}\left(e^{\gamma_l z_{2p+1}}e^{-\iota(k+1)\theta_{2p+1}}\right) +\\
&\qquad + (-1)^{\nu+2p+1}\mu_l\mathrm{Re}\left(e^{-\iota\rho_2(\nu)\pi/2}\sigma_{l,p}(t)\right) =\\
&= \sum_{k=0}^{\nu-1}\frac{(-1)^k(1-t)^{\nu-k-1}}{(\nu-k-1)!\gamma_l^{k+1}}e^{\gamma_l c_{2p+1}}c_{k,2p+1}^{(\gamma_l)} + (-1)^{\nu+1}\mu_l\sigma_{l,p}^{(1+\rho_2(\nu))}(t),
\end{aligned}$$

that is formula (46). The remaining formulas (47)–(52) arise in a similar way.

Appendix D

In the following we introduce some notations and properties of permutations, required in the proof of Theorem 2.

For $n, m \in \mathbb{Z}$, $n \le m$, let $I_{n,m} = \{n, n+1, \ldots, m\}$, $B(H,K)$ be the set of bijective functions from $H \subset \mathbb{Z}$ to $K \subset \mathbb{Z}$ and P_m be the set of permutations of $I_{1,m}$, that is $P_m = B(I_{1,m}, I_{1,m})$. We can denote a permutation $\tau \in P_m$, $m \ge 1$, also in the following way

$$\tau = (\tau(1), \tau(2), \ldots, \tau(m)), \qquad \tau(i) \in I_{1,m},\ i \in I_{1,m}.$$

We consider the following disjoint sets of bijections

$$B_1^{(1)} = B\left(I_{1,\nu}, I_{\eta+\rho_2(\nu),\nu+\rho_2(\nu)-1} \cup I_{\nu+\eta+\rho_2(\nu)+1,2\nu}\right), \tag{D.1}$$

$$B_1^{(2)} = B\left(I_{1,\nu}, I_{\eta+\rho_2(\nu)+1,\nu+\rho_2(\nu)-1} \cup I_{\nu+\eta+\rho_2(\nu),2\nu}\right), \tag{D.2}$$

$$B_2^{(1)} = B\left(I_{\nu+1,2\nu}, I_{1,\eta+\rho_2(\nu)-1} \cup I_{\nu+\rho_2(\nu),\nu+\rho_2(\nu)+\eta}\right), \tag{D.3}$$

$$B_2^{(2)} = B\left(I_{\nu+1,2\nu}, I_{1,\eta+\rho_2(\nu)} \cup I_{\nu+\rho_2(\nu),\nu+\eta+\rho_2(\nu)-1}\right), \tag{D.4}$$

bijections in $B_1^{(j)}$, $j = 1, 2$, have the same domain $I_{1,\nu}$ and different codomains, instead bijections in $B_2^{(j)}$, $j = 1, 2$, have the same domain $I_{\nu+1,2\nu}$ and different codomains. Let $\tau \in P_{2\nu}$ we define $\tau_1 = \tau|_{I_{1,\nu}}$, $\tau_2 = \tau|_{I_{\nu+1,2\nu}}$, it is easy to see that $P^{(1)}, P^{(2)}, P^{(3)} \subset P_{2\nu}$ such that

$$\tau \in P^{(1)} \Leftrightarrow \tau_1 \in B_1^{(1)} \Leftrightarrow \tau_2 \in B_2^{(1)},$$
$$\tau \in P^{(2)} \Leftrightarrow \tau_1 \in B_1^{(2)} \Leftrightarrow \tau_2 \in B_2^{(2)},$$
$$P^{(3)} = P \setminus \left(P^{(1)} \cup P^{(2)}\right),$$

is a particular partition of $P_{2\nu}$, moreover

$$\tau \in P^{(3)} \Leftrightarrow \tau_1 \notin B_1^{(1)} \cup B_1^{(2)} \Leftrightarrow \tau_2 \notin B_2^{(1)} \cup B_2^{(2)}. \tag{D.5}$$

Let $\tau^{(j)} \in P^{(j)}$, $j = 1, 2$, be the following two particular permutations in $P_{2\nu}$

$$\tau^{(1)}(i) = \begin{cases} i + \rho_2(\nu) - 1, & \eta + 1 \le i \le \nu + \eta + 1, \\ 2\nu - i + 1, & \text{otherwise}, \end{cases}$$

$$\tau^{(2)}(i) = \begin{cases} i + \rho_2(\nu) - 1, & \eta + 2 \le i \le \nu + \eta, \\ 2\nu - i + 1, & \text{otherwise}. \end{cases}$$

We note that: $\tau^{(1)}(i) = \tau^{(2)}(i)$ when $i \notin \{\eta + 1, \nu + \eta + 1\}$; $\tau^{(1)}(\eta + 1) = \tau^{(2)}(\nu + \eta + 1) = \eta + \rho_2(\nu)$; $\tau^{(1)}(\nu + \eta + 1) = \tau^{(2)}(\eta + 1) = \nu + \eta + \rho_2(\nu) = 2\nu - \eta$; $\operatorname{sign}(\tau^{(1)}) = -\operatorname{sign}(\tau^{(2)}) = -(-1)^{\eta+1}$, where $\operatorname{sign}(\cdot)$ denotes the signature of the permutation.

When $\tau \in P^{(j)}$, $j = 1, 2$, from the definition of $P^{(j)}$ we have that

$$(\tau_1(1), \tau_1(2), \ldots, \tau_1(\nu)) \text{ and } \left(\tau_1^{(j)}(1), \tau_1^{(j)}(2), \ldots, \tau_1^{(j)}(\nu)\right)$$

are permutations of the same ν distinct elements that depends on j, and also

$$(\tau_2(\nu + 1), \tau_2(\nu + 2), \ldots, \tau_2(2\nu)) \text{ and } \left(\tau_2^{(j)}(\nu + 1), \tau_2^{(j)}(\nu + 2), \ldots, \tau_2^{(j)}(2\nu)\right)$$

are permutations of the same ν distinct elements that depends on j, so the following quantities are well defined:

$$\operatorname{sign}^{(j)}(\tau_i) = (-1)^k, \qquad \tau \in P^{(j)},\ j = 1, 2,$$

where k is the number of inversions necessary to obtain τ_i from $\tau_i^{(j)}$, and it is easy to prove the following

$$\operatorname{sign}(\tau) = \operatorname{sign}(\tau^{(j)})\operatorname{sign}^{(j)}(\tau_1)\operatorname{sign}^{(j)}(\tau_2), \qquad \text{if } \tau \in P^{(j)},\ j = 1, 2.$$

Let $\tau \in P_{2\nu}$, we define A_τ the matrix obtained from $A = (\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_{2\nu}) \in \mathbb{C}^{2\nu\times 2\nu}$ by permuting its columns by τ, that is

$$A_\tau = A(\underline{v}_{\tau(1)}, \underline{v}_{\tau(2)}, \ldots, \underline{v}_{\tau(2\nu)}).$$

Appendix E

In this appendix relation (70) is proved. Let $\tau \in P^{(3)}$ and let M be the matrix defined by (54), (55); for $i = \nu+1, \nu+2, \ldots, 2\nu$, we have

$$M_{i,\tau_2(i)} = \begin{cases} \alpha^{(\gamma)}_{\tau_2(i)-\rho_2(\nu)} \left(\rho_2(\nu) c^{(\gamma)}_{i-\nu-1,\tau_2(i)-\rho_2(\nu)} - \rho_2^-(\nu) s^{(\gamma)}_{i-\nu-1,\tau_2(i)-\rho_2(\nu)}\right), \\ \qquad\qquad \tau_2(i) \le \nu - \rho_2^-(\nu), \\ \alpha^{(\gamma)}_j \left(\rho_2(\nu) s^{(\gamma)}_{i-\nu-1,j} + \rho_2^-(\nu) c^{(\gamma)}_{i-\nu-1,j}\right), \\ \qquad\qquad j = \tau_2(i) - \rho_2(\nu) - \nu,\ \tau_2(i) > \nu - \rho_2^-(\nu), \end{cases}$$

and so

$$\sum_{\tau\in P^{(3)}} \left|\prod_{i=\nu+1}^{2\nu} M_{i,\tau_2(i)}\right| \le \sum_{\tau\in P^{(3)}} \left(\prod_{\substack{i=\nu+1 \\ \tau_2(i) \le \nu-\rho_2^-(\nu)}}^{2\nu} |\alpha^{(\gamma)}_{\tau_2(i)-\rho_2(\nu)}| \prod_{\substack{i=\nu+1 \\ \tau_2(i) > \nu-\rho_2^-(\nu)}}^{2\nu} |\alpha^{(\gamma)}_{\tau_2(i)-\rho_2(\nu)-\nu}| \right) =$$

$$= \sum_{\tau\in P^{(3)}} \left(\prod_{\substack{i=\nu+1 \\ \tau_2(i) \le \nu-\rho_2^-(\nu)}}^{2\nu} e^{\gamma c_{\tau_2(i)-\rho_2(\nu)}} \prod_{\substack{i=\nu+1 \\ \tau_2(i) > \nu-\rho_2^-(\nu)}}^{2\nu} e^{\gamma c_{\tau_2(i)-\rho_2(\nu)-\nu}} \right) = \sum_{\tau\in P^{(3)}} e^{\gamma(a(\tau)+b(\tau))},$$

where

$$a(\tau) = \sum_{\substack{i=\nu+1 \\ \tau_2(i) \le \nu-\rho_2^-(\nu)}}^{2\nu} c_{\tau_2(i)-\rho_2(\nu)},$$

$$b(\tau) = \sum_{\substack{i=\nu+1 \\ \tau_2(i) > \nu-\rho_2^-(\nu)}}^{2\nu} c_{\tau_2(i)-\rho_2(\nu)-\nu}.$$

So, relation (70) is proved if $a(\tau) + b(\tau) \le \xi_0$ when $\tau \in P^{(3)}$ and ξ_0 is defined by (61). From Remark 2 we have $1 \ge c_0 > c_i > c_{i+1} \ge -1$, $i = 1, 2, \ldots, \nu - \rho_2(\nu) - 1$, $c_i = -c_{\nu-\rho_2(\nu)-i}$, $c_{\eta+i} = -c_{\eta-i}$, $i \in \mathbb{Z}$, and $c_\eta = 0$. The value $a(\tau)$ is the sum of distinct terms chosen into $\{c_{1-\rho_2(\nu)}, c_{2-\rho_2(\nu)}, \ldots, c_{\nu-1}\}$; the value $b(\tau)$ is the sum of distinct terms chosen into $\{c_0, c_1, \ldots, c_{\nu-\rho_2(\nu)}\}$; $a(\tau) + b(\tau)$ is the sum of ν terms chosen into $\{c_0, c_1, \ldots, c_{\nu-\rho_2(\nu)}\}$; each c_i, $i = 1, 2, \ldots, \nu - 1$, can appear at most two times into $a(\tau) + b(\tau)$; c_0 can appear at most $1 + \rho_2(\nu)$ times into $a(\tau) + b(\tau)$. So that, from (60) we have

$$a(\tau) + b(\tau) \le c_\eta + 2\sum_{i=1}^{\eta-1} c_i + (1 + \rho_2(\nu))c_0 = \xi.$$

If $\tau \in P^{(3)}$, from (D.5) $\tau_2 \notin B_2^{(1)} \cup B_2^{(2)}$ and so from (D.3) and (D.4) there exist $\bar{i}$, $\bar{j} \in I_{\nu+1,2\nu}$ such that

$$\tau_2(\bar{i}) \notin I_{1,\eta+\rho_2(\nu)-1} \cup I_{\nu+\rho_2(\nu),\nu+\rho_2(\nu)+\eta},$$
$$\tau_2(\bar{j}) \notin I_{1,\eta+\rho_2(\nu)} \cup I_{\nu+\rho_2(\nu),\nu+\eta+\rho_2(\nu)-1}.$$

If $\bar{i} = \bar{j}$ then $\tau_2(\bar{i}) \in I_{\eta+\rho_2(\nu)+1,\nu+\rho_2(\nu)-1} \cup I_{\nu+\eta+\rho_2(\nu)+1,2\nu}$ and $a(\tau) + b(\tau)$ contains a term lesser than c_η that is

$$a(\tau) + b(\tau) \le c_{\eta+1} + 2\sum_{i=1}^{\eta-1} c_i + (1 + \rho_2(\nu))c_0 = \xi_0.$$

If $\bar{i} \ne \bar{j}$ then $a(\tau) + b(\tau)$ contains two terms lesser than $c_{\eta-1}$ that is

$$a(\tau) + b(\tau) \le c_\eta + c_\eta + c_{\eta-1} + 2\sum_{i=1}^{\eta-2} c_i + (1 + \rho_2(\nu))c_0 = \xi_0.$$

This concludes the proof of (70).

Appendix F

We prove that $\det\big(D^{(2)} + J^{(\nu)}D^{(1)}\big) = \cos(\gamma)$ that is required for the proof of Theorem 2, so in the following we use the notation introduced in this theorem. In particular $D^{(\cdot)}$ are defined by (22), (23), and $J^{(\nu)}$ is defined by (77) and (78). When ν is odd, from (22), (23), (77), (78), (74), (75) and (76) we have

$$\det\left(D^{(2)} + J^{(\nu)}D^{(1)}\right) = \det\left(D^{(2)} + \begin{pmatrix} O_\eta & \underline{b} & J_\eta \\ \underline{0}_\eta^T & 0 & \underline{0}_\eta^T \\ -J_\eta & \underline{a} & O_\eta \end{pmatrix} D^{(1)}\right) =$$

$$= \det \begin{pmatrix}
c_0^{(\gamma)} & 0 & \dots & 0 & b_0 s_\eta^{(\gamma)} & 0 & \dots & 0 & s_0^{(\gamma)} \\
0 & c_1^{(\gamma)} & \dots & 0 & b_1 s_\eta^{(\gamma)} & 0 & \dots & s_1^{(\gamma)} & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & c_{\eta-1}^{(\gamma)} & b_{\eta-1} s_\eta^{(\gamma)} & s_{\eta-1}^{(\gamma)} & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & c_\eta^{(\gamma)} & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & -s_{\eta-1}^{(\gamma)} & a_{\eta-1} s_\eta^{(\gamma)} & c_{\eta-1}^{(\gamma)} & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & -s_1^{(\gamma)} & \dots & 0 & a_1 s_\eta^{(\gamma)} & 0, & \dots & c_1^{(\gamma)} & 0 \\
-s_0^{(\gamma)} & 0 & \dots & 0 & a_0 s_\eta^{(\gamma)} & 0 & \dots & 0 & c_0^{(\gamma)}
\end{pmatrix} =$$

$$= c_\eta^{(\gamma)} \det \begin{pmatrix}
c_0^{(\gamma)} & 0 & \dots & 0 & 0 & \dots & 0 & s_0^{(\gamma)} \\
0 & c_1^{(\gamma)} & \dots & 0 & 0 & \dots & s_1^{(\gamma)} & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & c_{\eta-1}^{(\gamma)} & s_{\eta-1}^{(\gamma)} & \dots & 0 & 0 \\
0 & 0 & \dots & -s_{\eta-1}^{(\gamma)} & c_{\eta-1}^{(\gamma)} & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
0 & -s_1^{(\gamma)} & \dots & 0 & 0 & \dots & c_1^{(\gamma)} & 0 \\
-s_0^{(\gamma)} & 0 & \dots & 0 & 0 & \dots & 0 & c_0^{(\gamma)}
\end{pmatrix} =$$

$$= c_\eta^{(\gamma)} = \cos(\gamma),$$

where we have used (9).

Please cite this article in press as: N. Egidi, P. Maponi, The singular value expansion of the Volterra integral equation associated to a numerical differentiation problem, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2017.12.007

When ν is even

$$\det\left(D^{(2)} + J^{(\nu)} D^{(1)}\right) = \det\left(D^{(2)} + \begin{pmatrix} O_{\eta-1} & \underline{b} & J_{\eta-1} & \underline{0}_{\eta-1} \\ \underline{0}^T_{\eta-1} & 0 & \underline{0}^T_{\eta-1} & 0 \\ -J_{\eta-1} & \underline{a} & O_{\eta-1} & \underline{0}_{\eta-1} \\ \underline{0}^T_{\eta-1} & a_0 & \underline{0}^T_{\eta-1} & 0 \end{pmatrix} D^{(1)}\right) =$$

$$= \det \begin{pmatrix} c_1^{(\gamma)} & 0 & \dots & 0 & b_1 s_\eta^{(\gamma)} & 0 & \dots & 0 & s_1^{(\gamma)} & 0 \\ 0 & c_2^{(\gamma)} & \dots & 0 & b_2 s_\eta^{(\gamma)} & 0 & \dots & s_2^{(\gamma)} & 0 & 0 \\ .. & .. & \dots & .. & .. & .. & \dots & .. & .. & .. \\ 0 & 0 & 0\dots & c_{\eta-1}^{(\gamma)} & b_{\eta-1} s_\eta^{(\gamma)} & s_{\eta-1}^{(\gamma)} & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & c_\eta^{(\gamma)} & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \dots & -s_{\eta-1}^{(\gamma)} & a_{\eta-1} s_\eta^{(\gamma)} & c_{\eta-1}^{(\gamma)} & \dots & 0 & 0 & 0 \\ .. & .. & \dots & .. & .. & .. & \dots & .. & .. & \\ -s_1^{(\gamma)} & 0 & \dots & 0 & a_1 s_\eta^{(\gamma)} & 0 & \dots & 0 & c_1^{(\gamma)} & 0 \\ 0 & 0 & \dots & 0 & a_0 s_\eta^{(\gamma)} & 0 & \dots & 0 & 0 & c_0^{(\gamma)} \end{pmatrix}$$

and so by (9) we have

$$\det\left(D^{(2)} + J^{(\nu)} D^{(1)}\right) = c_\eta^{(\gamma)} \cdot c_0^{(\gamma)} = \cos(\gamma).$$

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