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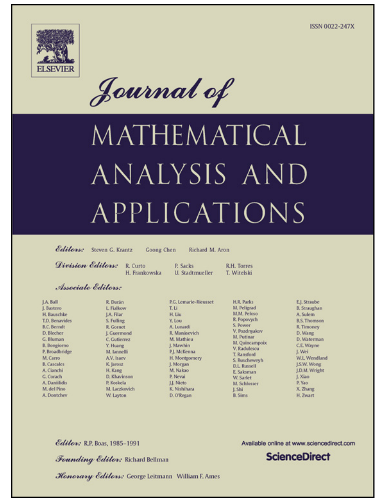


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# Performance output tracking and disturbance rejection for an Euler-Bernoulli beam equation with unmatched boundary disturbance [*]

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**Abstract**

This paper considers performance output tracking and disturbance rejection for a boundary controlled one-dimensional Euler-Bernoulli beam equation suffered unknown external bounded disturbance. We first propose a disturbance estimator and then based on this disturbance estimator, we construct a servomechanism to track the reference signal and reject the external disturbance. Four control objectives are achieved: a) the output is exponentially tracking the reference signal; b) the disturbance is rejected; c) all the states of internal-loops are bounded; d) when the disturbance and reference are disconnected, the closed-loop is exponentially stable. Finally, the state of the system is shown to be exponentially tracking the reference state.

**Keywords:** Distributed parameter system, Euler-Bernoulli beam, disturbance rejection, performance output tracking.

**AMS subject classifications:** 93C20, 93D15, 35B35, 35P10

## 1 Introduction

Up to today, the output regulation problem, or alternatively the servomechanism is still one central problem in control theory. This problem addresses designing a controller so that the output of closed-loop system asymptotically tracks a reference signal regardless of the external disturbances and the initial state. Many effort have been made to generalize classical output regulation results for finite-dimensional systems (see, e.g., [3, 4, 5, 6, 7, 13, 15]) to infinite-dimensional systems, like [1, 12, 14, 16, 17, 20, 18] and [2], among many others. On one hand, most of the above works about

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output regulation problem focus on the extension of internal model principle theory to infinite-dimensional systems where reference signal and disturbance are generated by finite-dimensional or infinite-dimensional exosystem. Moreover, the most infinite-dimensional systems considered are of bounded control and observer operators. On the other hand, the performance output tracking is not sufficiently addressed in the context of infinite-dimensional systems. Recently, the performance output tracking problem for finite-dimensional linear system ([19, p.315]) which has no disturbance is firstly generalized to a one-dimensional wave equation with unknown general harmonic disturbance in [8], where an adaptive tracking controller is designed and the error between reference signal and output is shown to be asymptotically convergent to zero as the time goes to infinity. Later, two improved results compared with that of [8] are reported in [24, 26] where an exponentially tracking controller is designed for a one-dimensional wave equation with unknown general bounded disturbance. More recently, the assumption in [8] that the wave system is exponentially stable if there is no disturbance at the boundary, is removed by paying the price that the reference is taken as the zero signal [9] and the harmonic signal [10] rather than the general signal. It is well know that both the Euler-Bernoulli beam equation and wave equation are two benchmark vibration systems. To our best knowledge, there is no report about the performance output tracking problem for Euler-Bernoulli beam equation no matter with harmonic disturbance or more general bounded disturbance.

In this paper, we consider the performance output tracking problem for the following Euler-Bernoulli beam equation :

$$\begin{cases} y_{tt}(x,t) + y_{xxxx}(x,t) = 0,\ x \in (0,1),\ t > 0, \\ y(0,t) = y_x(0,t) = 0,\ t \geq 0, \\ y_{xx}(1,t) = U(t),\ t \geq 0, \\ y_{xxx}(1,t) = q y_t(1,t) + d(t),\ t \geq 0, \\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x),\ 0 \leq x \leq 1, \\ y_{\mathrm{m}}(t) = \{y(1,t), y_{xt}(1,t)\}, \\ e(t) = y_{\mathrm{out}} - y_{\mathrm{ref}}(t) = y_x(1,t) - y_{\mathrm{ref}}(t), \end{cases} \tag{1.1}$$

where and henceforth $y'$ or $y_x$ denotes the derivative of $y$ with respect to $x$ and $\dot{y}$ or $y_t$ the derivative with respect to $t$, $U(t)$ is the input, $y_{\mathrm{out}}$ is output to be regulated which is not necessary measured. $y_m$ is measured output, $y_{\mathrm{ref}}(t)$ is a reference signal, $d(t)$ is a unknown disturbance, $e(t)$ is tracking error. $(y_0, y_1)$ is the initial state. As that in [8], we assume that $q$ is a positive constant, i.e., $q > 0$.

System (1.1) is a typical control system in which the control is unmatched with the external disturbance. That is, the control input and the disturbance are not at the same end.

We consider system (1.1) in the state Hilbert space $\mathcal{H} = H_L^2(0,1) \times L^2(0,1)$ (where $H_L^2(0,1) = \{\phi \in H^2(0,1) : \phi(0) = \phi'(0) = 0\}$) with the inner product given by

$$\langle (\phi_1, \psi_1)^\top, (\phi_2, \psi_2)^\top \rangle_{\mathcal{H}} = \int_0^1 [\phi_1''(x)\overline{\phi_2''(x)} + \psi_1(x)\overline{\psi_2(x)}]dx,\ \forall\ (\phi_i, \psi_i)^\top \in \mathcal{H}, i = 1, 2. \tag{1.2}$$

The objective of this paper is to find feedback control law for system (1.1) such that

- (i) $\lim_{t\to\infty} e(t) = 0$;
- (ii) the bounded disturbance is rejected;
- (iii) all the states of internal-loops are bounded;
- (iv) the closed-loop is exponentially convergent to zero, when the disturbance and reference are disconnected, that is, $d(t) = y_{ref}(t) \equiv 0$.

The key characteristic of our approach is to design an infinite-dimensional disturbance estimator in which there is no high gain needed. Then the servo system is constructed, which is completely determined by the reference signal to be tracked and the disturbance estimator.

The paper is organized as follows. In next section, Section 2, we give the disturbance estimator design. The servomechanism design is given in Section 3. In Section 4, we give the boundedness analysis of the reference system. We give main result of this paper in Section 5 . We present some illustrative simulation results in Section 6.

## 2 Disturbance estimator design

This section is devoted to the design of the disturbance estimator with the measured output $y_m(t) = (y(1,t), y_{xt}(1,t))$. We propose an infinite-dimensional disturbance estimator for $d(t)$ as follows:

$$\begin{cases} z_{tt}(x,t) + z_{xxxx}(x,t) = 0, \\ z(0,t) = z_x(0,t) = 0, \\ z_{xx}(1,t) = -c_0(z_{xt}(1,t) - y_{xt}(1,t)) + U(t), \\ z(1,t) = y(1,t), \\ z(x,0) = z_0(x),\ z_t(x,0) = z_1(x), \end{cases} \tag{2.1}$$

where $c_0$ is a positive tuning parameter. Here and in the rest of paper, we omit the obvious domains for $t$ and $x$. System (2.1) can be used to recover the disturbance. To this end, we are going to show that $z_{xxx}(1,t) - qz_t(1,t) \approx d(t)$. Indeed, set

$$\alpha(x,t) = z(x,t) - y(x,t). \tag{2.2}$$

Then, it is easy to check that $\alpha(x,t)$ satisfies

$$\begin{cases} \alpha_{tt}(x,t) + \alpha_{xxxx}(x,t) = 0, \\ \alpha(0,t) = \alpha_x(0,t) = 0, \\ \alpha_{xx}(1,t) = -c_0\alpha_{xt}(1,t),\ \alpha(1,t) = 0, \\ \alpha(x,0) = \alpha_0(x), \quad \alpha_t(x,0) = \alpha_1(x), \end{cases} \tag{2.3}$$

where

$$\alpha_0(x) = z_0(x) - y_0(x), \quad \alpha_1(x) = z_1(x) - y_1(x).$$

The system (2.3) can be rewritten as

$$\begin{cases} \frac{d}{dt}(\alpha(\cdot,t),\alpha_t(\cdot,t)) = A_0(\alpha(\cdot,t),\alpha_t(\cdot,t)), \\ (\alpha(\cdot,0),\alpha_t(\cdot,0)) = (\alpha_0,\alpha_1), \end{cases}$$

where the operator $A_0 : D(A_0)(\subset \mathcal{H}) \to \mathcal{H}$ is given by

$$\begin{cases} A_0(\phi,\psi) = (\psi, -\phi''''), \ \ \forall(\phi,\psi) \in D(A_0), \\ D(A_0)=\big\{(\phi,\psi) \in (H^4(0,1) \times H_L^2(0,1)) \cap \mathcal{H} : \phi''(1) = -c_0\psi'(1), \ \phi(1) = 0\big\}. \end{cases} \tag{2.4}$$

Next lemma states the properties of $A_0$.

**Lemma 2.1.** *Let $A_0$ be given by (2.4). Then, there is a sequence of generalized eigenvectors of $A_0$ which forms a Riesz basis for the state space $\mathcal{H}$. Moreover, $A_0$ generates an exponential stable $C_0$-semigroup on $\mathcal{H}$.*

*Proof.* The proof is broken into several steps as follows.
Step 1. We claim that there is a family of eigenvalues $\{\lambda_n, \overline{\lambda}_n\}$, $\lambda_n = i\tau_n^2$ of $A_0$ with the following asymptotic expression:

$$\tau_n = (n+1/2)\pi + \mathcal{O}(n^{-1}), \ \ \lambda_n = i(n+1/2)^2\pi^2 - \frac{2}{c_0} + \mathcal{O}(n^{-1}). \tag{2.5}$$

A direct computation shows that

$$\begin{aligned} A_0^{-1}(\phi,\psi)=&\bigg(\frac{(3x^2-x^3)}{12}\int_0^1(1-\xi)^3\psi(\xi)d\xi+\frac{(x^3-x^2)}{4}\Big(\int_0^1(1-\xi)\psi(\xi)d\xi-c_0\phi'(1)\Big) \\ &-\frac{1}{6}\int_0^x(x-\xi)^3\psi(\xi)d\xi, \phi(x)\bigg). \end{aligned} \tag{2.6}$$

By the Sobolev embedding theorem, $A_0^{-1}$ is compact on $\mathcal{H}$, and thus $\sigma(A_0)$ only consists of eigenvalues of $A_0$. It is easily seen that $\lambda = i\tau^2 \in \sigma(A_0)$ if and only if there exists $\phi \neq 0$ satisfying

$$\begin{cases} \phi^{(4)}(x) - \tau^4\phi(x) = 0, \\ \phi(0) = \phi'(0) = \phi(1) = 0, \ \ \phi''(1) = -ic_0\tau^2\phi'(1) \end{cases}$$

and the associated egienfunction is $(\phi, \lambda\phi)$. First, the general solution of

$$\begin{cases} \phi^{(4)}(x) - \tau^4\phi(x) = 0, \\ \phi(0) = \phi'(0) = 0 \end{cases}$$

is of the form

$$\phi(x) = a_1(\cos\tau x - \cosh\tau x) + a_2(\sin\tau x - \sinh\tau x), \tag{2.7}$$

where $a_1$, $a_2$ are constants. Next, by the condition $\phi(1) = 0$, we can take

$$a_1 = (\sin\tau - \sinh\tau), \ \ a_2 = -(\cos\tau - \cosh\tau).$$

Substituting this into (2.7) gives

$$\phi(x) = (\sin\tau - \sinh\tau)(\cos\tau x - \cosh\tau x) - (\cos\tau - \cosh\tau)(\sin\tau x - \sinh\tau x). \tag{2.8}$$

The last condition $\phi''(1) = -ic_0\tau^2\phi'(1)$ yields

$$ic_0\tau[1 - \cos\tau\cosh\tau] + \cosh\tau\sin\tau - \cos\tau\sinh\tau = 0, \tag{2.9}$$

which can be re-written asymptotically as

$$\begin{cases} \cos\tau = \mathcal{O}(|\tau|^{-1}), \text{ or} \\ \cos\tau = -\dfrac{1}{ic_0\tau}[\cos\tau\tanh\tau - \sin\tau] + \mathcal{O}(e^{-\mathrm{Re}\tau}), \text{ as Im}\tau \text{ bounded Re}\tau \to \infty. \end{cases} \tag{2.10}$$

By the first equality of (2.10), we get $\tau_n = (n + 1/2)\pi + \mathcal{O}(n^{-1})$. Substitute $\tau_n$ into the second equality of (2.10) to obtain $\mathcal{O}(n^{-1}) = -\frac{1}{[ic_0(n+1/2)\pi]} + \mathcal{O}(n^{-2})$, and so

$$\lambda_n = i(n + 1/2)^2\pi^2 - \frac{2}{c_0} + \mathcal{O}(n^{-1}).$$

Step 2. We claim that there is an eigenfunction $(\phi_n, \lambda_n\phi_n)^\top$ of $A_0$ corresponding to $\lambda_n = i\tau_n^2$ such that

$$F_n(x)=\begin{pmatrix} -(-1)^n e^{-(n+1/2)\pi(1-x)} + \cos(n+1/2)\pi x - \sin(n+1/2)\pi x + e^{-(n+1/2)\pi x} \\ -i(-1)^n e^{-(n+1/2)\pi(1-x)} - i\cos(n+1/2)\pi x + i\sin(n+1/2)\pi x + ie^{-(n+1/2)\pi x} \end{pmatrix} + \mathcal{O}(n^{-1})$$

and $\lim_{n\to\infty} \|F_n(x)\|_{[L^2(0,1)]^2} = 2$, where $F_n(x) = 2\tau_n^{-2}e^{-\tau_n}(\phi_n''(x), \lambda_n\phi_n(x))^\top$. Actually, let $(\phi_n, \lambda_n\phi_n)$ be the eigenfunction of $A_0$ corresponding to $\lambda_n$, where $\phi_n = \phi(x)$ is defined by (2.7) with $\tau = \tau_n$. By (2.8), we derive

$$\tau^{-2}\phi''(x) = (\sin\tau - \sinh\tau)(-\cos\tau x - \cosh\tau x) - (\cos\tau - \cosh\tau)(-\sin\tau x - \sinh\tau x). \tag{2.11}$$

Noticing that by (2.5), for any $y > 0$ and $0 \le x \le 1$, $e^{-\tau_n y} = e^{-(n+1/2)\pi y} + \mathcal{O}(n^{-1})$, $\sin\tau_n x = \sin(n + 1/2)\pi x + \mathcal{O}(n^{-1})$, $\cos\tau_n x = \cos(n + 1/2)\pi x + \mathcal{O}(n^{-1})$, and letting $\tau = \tau_n$ in (2.11), we obtain

$$2\tau_n^{-2}e^{-\tau_n}\phi_n''(x) = -(-1)^n e^{-(n+1/2)\pi(1-x)} + \cos(n+1/2)\pi x - \sin(n+1/2)\pi x + e^{-(n+1/2)\pi x} + \mathcal{O}(n^{-1}).$$

The estimate for $\phi_n$ is similar, we omit the detail. By using the Lebesgue's dominated convergence theorem, it is easy to verify $\lim_{n\to\infty} \|F_n(x)\|_{[L^2(0,1)]^2} = 2$.

Step 3. We claim that the eigenfunctions of $A_0$ form an Riesz basis for $\mathcal{H}$. For this purpose, we introduce the following auxiliary operator $A_a$ given by

$$\begin{cases} A_a(\phi,\psi) = (\psi, -\phi''''), \ \forall(\phi,\psi) \in D(A_a), \\ D(A_a) = \Big\{(\phi,\psi) \in (H^4(0,1) \times H_L^2(0,1)) \cap \mathcal{H} : \phi''(1) = 0, \ \phi(1) = 0\Big\}. \end{cases} \tag{2.12}$$

By letting $c_0 = 0$ in (2.6), we know that $A_a$ has compact resolvent. It is easily to verify that the operator $A_a$ is skew-adjoint in the state space $\mathcal{H}$, i.e., $A_a^* = -A_a$ and all eigenvalues of $A_a$ are located on the imaginary axis and there is a sequence of generalized eigenfunctions of $A_a$ forming a Riesz basis for $\mathcal{H}$. Let $\lambda_a = i\omega^2$ be the eigenvalue of $A_a$ and $(\phi_a, \lambda_a\phi_a)$ be the eigenfunction of $A_a$ corresponding to $\lambda_a = i\omega^2$. By letting $c_0 = 0$ in (2.9), we obtain

$$\cosh\omega\sin\omega - \cos\omega\sinh\omega = 0, \tag{2.13}$$

which gives

$$\omega_n = (n+1/2)\pi + \mathcal{O}(n^{-1}).$$

Similar to the calculation in Step 2, we obtain that the eigenfunction $(\phi_{an}, \lambda_{an}\phi_{an})$ of $A_a$ have the following asymptotical expression:

$$G_n(x)=\begin{pmatrix} -(-1)^n e^{-(n+1/2)\pi(1-x)} + \cos(n+1/2)\pi x - \sin(n+1/2)\pi x + e^{-(n+1/2)\pi x} \\ -i(-1)^n e^{-(n+1/2)\pi(1-x)} - i\cos(n+1/2)\pi x + i\sin(n+1/2)\pi x + ie^{-(n+1/2)\pi x} \end{pmatrix} + \mathcal{O}(n^{-1}),$$

where $G_n(x) = 2\omega_n^{-2}e^{-\omega_n}(\phi''_{an}(x), \lambda_n\phi_{an}(x))^\top$. It is easy to see that $\{(\phi_{an}, \lambda_{an}\phi_{an})\}_{n=1}^\infty \cup \{\text{conjugates}\}$ is a Riesz basis for $\mathcal{H}$. It follows that there is an $N > 0$ such that

$$\begin{aligned} \sum_{n>N}^{\infty} \|F_n - G_n\|_{[L^2(0,1)]^2} &= \sum_{n>N}^{\infty} \|2\tau_n^{-2}e^{-\tau_n}(\phi''_n(x), \lambda_n\phi_n(x))^\top \\ -2\omega_n^{-2}e^{-\omega_n}(\phi''_{an}(x), \lambda_{an}\phi_{an}(x))^\top\|_{[L^2(0,1)]^2} &= \sum_{n>N}^{\infty} \mathcal{O}(n^{-2}) < +\infty. \end{aligned} \tag{2.14}$$

The same thing is true for conjugates. Therefore, operator $A_0$ has a sequence of eigenfunctions which quadratically closed to a Riesz basis in the sense of (2.14). By [11, Theorem 1], we have shown that the eigenfunctions of $A_0$ form an Riesz basis for $\mathcal{H}$.

Step 4. We claim that $A_0$ generates an exponential stable $C_0$-semigroup on $\mathcal{H}$. Since the eigenfunctions of $A_0$ form an Riesz basis for $\mathcal{H}$ that is justified by Step 3, the spectrum-determined growth condition holds. In order to show that $e^{A_0 t}$ is a exponential stable semigroup, it suffices to prove that $\mathrm{Re}\lambda < 0$ for any $\lambda \in \sigma(A_0)$. Actually, a simple computation gives

$$\mathrm{Re}\langle A_0(\phi,\psi), (\phi,\psi)\rangle_{\mathcal{H}} = -c_0|\psi'(1)|^2 \le 0, \tag{2.15}$$

which implies that for any $\lambda \in \sigma(A_0)$ must satisfy $\mathrm{Re}\lambda \le 0$. Since $A_0^{-1}$ is compact, we only need to show that there is no eigenvalue on the imaginary axis. Let $\lambda = i\tau^2 \in \sigma(A_0)$ with $\tau \in \mathbb{R}^+$ and the corresponding eigenfunction $(\phi,\psi)^\top \in D(A_0)$. By (2.15),

$$\mathrm{Re}\langle A_0(\phi,\psi), (\phi,\psi)\rangle_{\mathcal{H}} = \mathrm{Re}\langle i\tau^2(\phi,\psi), (\phi,\psi)\rangle_{\mathcal{H}} = -c_0|\psi'(1)|^2 = 0, \tag{2.16}$$

and hence $\psi'(1) = 0$. Furthermore, $A(\phi,\psi) = i\tau^2(\phi,\psi)$ gives that $\psi = i\tau^2\phi$ with $\phi$ satisfying

$$\begin{cases} \phi^{(4)}(x) - \tau^4\phi(x) = 0, \\ \phi(0) = \phi'(0) = \phi(1) = \phi'(1) = \phi''(1) = 0, \end{cases} \tag{2.17}$$

Now, we show that the above equation admits only zero solution. For this, we prove that there exists at least one zero of $\phi$ in $(0,1)$. Actually, by $\phi(0) = \phi(1) = 0$, Rolle's theorem yields $\phi'(\xi_1) = 0$ for some $\xi_1 \in (0,1)$, which, jointly with $\phi'(0) = \phi'(1) = 0$, implies that $\phi''(\xi_2) = \phi''(\xi_3) = 0$ for some $\xi_2 \in (0,\xi_1)$, $\xi_3 \in (\xi_1, 1)$, and so $\phi'''(\xi_4) = \phi'''(\xi_5) = 0$ for some $\xi_4 \in (\xi_2, \xi_3)$, $\xi_5 \in (\xi_3, 1)$ by the condition $\phi''(1) = 0$. Thus, there exists a $\xi_6 \in (\xi_4, \xi_5)$ such that $\phi^{(4)}(\xi_6) = 0$, which, together with the first equation of (2.17), gives $\phi(\xi_6) = 0$. Next, we prove that if there are $n$ different zeros of $\phi$ in $(0,1)$, then there at least $n+1$ number of different zeros of $\phi$ in $(0,1)$. Indeed, suppose that $0 < \xi_1 < \xi_2 < \cdots < \xi_n < 1$, $\phi(\xi_j) = 0, j = 1,2,\ldots,n$. Since $\phi(0) = \phi(1) = 0$, if follows from Rolle's theorem that there exist $\eta_j, j = 1,2,\ldots,n+1$, $0 < \eta_1 < \xi_1 < \eta_2 < \xi < 2 \cdots < \xi_n < \eta_{n+1} < 1$ such that $\phi'(\eta_j) = 0$. By $\phi'(0) = \phi'(1) = 0$, using Rolle's theorem again, there exist $\alpha_j, j = 1,2,\ldots,n+2$, $0 < \alpha_1 < \eta_1 < \alpha_2 < \eta_2 < \cdots < \xi_{n+1} < \alpha_{n+2} < 1$ such that $\phi''(\alpha_j) = 0$. It follows from $\phi''(1) = 0$ that there exist $\beta_j, j = 1,2,\ldots,n+2$, $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_{n+2} < \beta_{n+2} < 1$ such that $\phi'''(\beta_j) = 0$. Using Rolle's theorem again, we have $\theta_j, j = 1,2,\ldots,n+1$, $\beta_1 < \theta_1 < \beta_2 < \cdots < \beta_{n+1} < \theta_{n+1} < \beta_{n+2}$ such that $\phi^{(4)}(\theta_j) = 0$. Thus, $\phi(\theta_j) = 0$, $j = 1,2,\ldots,n+1$. By mathematical induction, there is an infinite number of different zeros $\{x_j\}_{j=1}^{\infty}$ of $\phi$ in $(0,1)$. Let $x_0 \in [0,1]$ be an accumulation point of $\{x_j\}_{j=1}^{\infty}$. Obviously, $\phi^{(j)}(x_0) = 0$, $j = 0,1,2,3$. Since $\phi$ satisfies the first equation of (2.17), by the uniqueness of the solution of linear ordinary different equation, we have $\phi \equiv 0$. $\square$

By lemma 2.1, we have the following well-posedness and stability results for system (2.3).

**Lemma 2.2.** *For any initial value $(\alpha_0, \alpha_1) \in \mathcal{H} \cap [H_0^1(0,1) \times L^2(0,1)]$, system (2.3) admits a unique solution $(\alpha(\cdot,t), \alpha_t(\cdot,t)) \in C(0,\infty; \mathcal{H} \cap [H_0^1(0,1) \times L^2(0,1)])$ that satisfies $\|(\alpha(\cdot,t), \alpha_t(\cdot,t))\|_{\mathcal{H}} \leq Me^{-\mu t}$ with some $M, \mu > 0$.*

**Lemma 2.3.** *([23]) For any initial value $(\alpha_0, \alpha_1) \in D(A_0)$ with the compatibility condition $\alpha(1,0) = \alpha_t(1,0) = 0$, the classical solution of (2.3) satisfies $|\alpha_{xxx}(1,t)| \leq Me^{-\mu t}$ with some $M, \mu > 0$.*

From (1.1) and (2.1), one has that

$$\alpha_{xxx}(1,t) = z_{xxx}(1,t) - y_{xxx}(1,t) = z_{xxx}(1,t) - qz_t(1,t) - d(t).$$

By Lemma 2.3, we can regard $z_{xxx}(1,t) - qz_t(1,t)$ as an approximation of $d(t)$, that is,

$$z_{xxx}(1,t) - qz_t(1,t) \approx d(t).$$

It is worth noting that the above approximation is untraditional since the error between the estimated value and the real value tends to zero only when the initial state is sufficiently smooth. In the next two sections, we will see that this approximation is still valid when the initial state is in the state space and is possibly not smoother.

## 3 Servomechanism design

For the reference signal $y_{ref}(t)$, we design the following reference model:

$$\begin{cases} \widehat{y}_{tt}(x,t) + \widehat{y}_{xxxx}(x,t) = 0, \\ \widehat{y}(0,t) = \widehat{y}_x(0,t) = 0, \\ \widehat{y}_x(1,t) = y_{ref}(t), \\ \widehat{y}_{xxx}(1,t) = q\widehat{y}_t(1,t) + z_{xxx}(1,t) - qz_t(1,t), \\ \widehat{y}(x,0) = \widehat{y}_0(x),\ \widehat{y}_t(x,0) = \widehat{y}_1(x). \end{cases} \tag{3.1}$$

Noting that $z_{xxx}(1,t) - qz_t(1,t) \approx d(t)$, $z_{xxx}(1,t) - qz_t(1,t)$ in (3.1) plays a role of the total disturbance $d(t)$. The motivation for the design of the above reference model is that finding a controller makes reference model behaves as system (1.1), then boundary condition of (3.1) forces the output of (1.1) tacks the reference signal. Let $\beta(x,t) = y(x,t) - \widehat{y}(x,t)$ denote the error between $y(x,t)$ and $\widehat{y}(x,t)$. Then $\beta(x,t)$ is governed by

$$\begin{cases} \beta_{tt}(x,t) + \beta_{xxxx}(x,t) = 0, \\ \beta(0,t) = \beta_x(0,t) = 0, \\ \beta_{xx}(1,t) = U(t) - \widehat{y}_{xx}(1,t), \\ \beta_{xxx}(1,t) = q\beta_t(1,t) - \alpha_{xxx}(1,t), \\ \beta(x,0) = \beta_0(x), \quad \beta_t(x,0) = \beta_1(x), \end{cases} \tag{3.2}$$

where

$$\beta_0(x) = y_0(x) - \widehat{y}_0(x), \quad \beta_1(x) = y_1(x) - \widehat{y}_1(x).$$

Moreover,

$$\beta_x(1,t) = y_x(1,t) - y_{ref}(t) = e(t)$$

is the performance output tracking error.

We propose the following output feedback controller:

$$U(t) = -c_0(y_{xt}(1,t) - \widehat{y}_{xt}(1,t)) + \widehat{y}_{xx}(1,t). \tag{3.3}$$

Then the closed-loop of system (3.2) corresponding to the controller (3.3) becomes

$$\begin{cases} \beta_{tt}(x,t) + \beta_{xxxx}(x,t) = 0, \\ \beta(0,t) = \beta_x(0,t) = 0, \\ \beta_{xx}(1,t) = -c_0\beta_{xt}(1,t), \\ \beta_{xxx}(1,t) = q\beta_t(1,t) - \alpha_{xxx}(1,t). \end{cases} \tag{3.4}$$

We consider system (3.4) in the state space $\mathcal{H}$.

**Theorem 3.1.** *Suppose that $\alpha_{xxx}(1,t)$ is generated by (2.3). For any initial value $(\beta_0, \beta_1) \in \mathcal{H}$, there exists a unique solution to (3.2) such that $(\beta, \beta_t) \in C(0,\infty;\mathcal{H})$. Moreover, there exist two constants $M, \mu > 0$ such that*

$$\int_0^1 [\beta_{xx}^2(x,t) + \beta_t^2(x,t)]dx \le Me^{-\mu t} \int_0^1 \left[ [\beta_0''(x)]^2 + [\beta_1(x)]^2 \right] dx. \tag{3.5}$$

*Proof.* Following the transformation trick as indicated in [25, Remark 4.1], we introduce a new variable $p(x,t) = \beta(x,t) + \alpha(x,t)$. Then it is easy to verify that $p(x,t)$ satisfies

$$\begin{cases} p_{tt}(x,t) + p_{xxxx}(x,t) = 0, \\ p(0,t) = p_x(0,t) = 0, \\ p_{xx}(1,t) = -c_0 p_{xt}(1,t), \\ p_{xxx}(1,t) = q p_t(1,t), \end{cases} \tag{3.6}$$

which can be rewritten as the following operator form:

$$\frac{d}{dt}(p(\cdot,t), p_t(\cdot,t))^\top = A_p (p(\cdot,t), p_t(\cdot,t))^\top, \tag{3.7}$$

where the operator $A_p : D(A_p)(\subset \mathbb{H}) \to \mathbb{H}$ is defined by

$$\begin{cases} A_p(\phi,\psi)^\top = (\psi, \phi'')^\top, \ \forall (\phi,\psi)^\top \in D(A_p), \\ D(A_p) = \left\{ (\phi,\psi)^\top \in \mathbb{H} \cap H^2(0,1) \times H^1(0,1) : \phi''(1) = -c_0\psi'(1), \ \phi'''(1) = q\psi(1) \right\}. \end{cases} \tag{3.8}$$

By [11], $A_p$ generates an exponentially stable $C_0$-semigroup, which implies that system (3.7) has a unique solution that is exponentially stable. By Lemma 2.2 and noting that $\beta(x,t) = p(x,t) - \alpha(x,t)$, we have that $\beta$ is well-defined and is exponentially stable, i.e., (3.5) holds. □

## 4 Well-posed and boundness of $\widehat{y}$ system

Now we turn to the state reference model (3.1). The resulting system now reads

$$\begin{cases} \widehat{y}_{tt}(x,t) + \widehat{y}_{xxxx}(x,t) = 0, \\ \widehat{y}(0,t) = \widehat{y}_x(0,t) = 0, \\ \widehat{y}_x(1,t) = y_{\text{ref}}(t), \\ \widehat{y}_{xxx}(1,t) = q\widehat{y}_t(1,t) + \alpha_{xxx}(1,t) + d(t), \\ \widehat{y}(x,0) = \widehat{y}_0(x), \ \widehat{y}_t(x,0) = \widehat{y}_1(x). \end{cases} \tag{4.1}$$

**Theorem 4.1.** *For any initial value $(\widehat{y}_0, \widehat{y}_1) \in \mathcal{H}$, $d \in L^\infty(0,+\infty)$ and $y_{\text{ref}} \in W^{1,\infty}(0,\infty)$ satisfying the compatibility condition $\widehat{y}_0'(1) = y_{\text{ref}}(0)$, there exists a unique (weak) solution $(\widehat{y}, \widehat{y}_t) \in C(0,\infty;\mathcal{H})$ to (4.1). Moreover, the solution of (4.1) satisfies*

$$\sup_{t\ge 0} E_{\widehat{y}}(t) = \sup_{t\ge 0} \frac{1}{2}\int_0^1 [\widehat{y}_{xx}^2(x,t) + \widehat{y}_t^2(x,t)]dx < \infty.$$

*Further, if $y_{\text{ref}}(t) = d(t) \equiv 0$, the solution $(\widehat{y}, \widehat{y}_t)$ is exponentially stable.*

**Proof** We start with proving the first part. To this end, transform (4.1) into an equivalent problem by the transformation $v(x,t) = \widehat{y}(x,t) - \alpha(x,t)$ to obtain

$$\begin{cases} v_{tt}(x,t) + v_{xxxx}(x,t) = 0, \\ v(0,t) = v_x(0,t) = 0, \\ v_x(1,t) = y_{\rm ref}(t) - \alpha_x(1,t), \\ v_{xxx}(1,t) = qv_t(1,t) + d(t), \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), \end{cases} \tag{4.2}$$

where

$$v_0(x) = \widehat{y}_0(x) - \alpha_0(x),\ v_1(x) = \widehat{y}_1(x) - \alpha_1(x).$$

**Noticing that $\dot{y}_{\rm ref} \in L^\infty(0,\infty)$ and $\alpha_{xt}(1,t) \in L^2(0,\infty)$, it is seen that the solution of (4.2) is the solution of the following system:**

$$\begin{cases} v_{tt}(x,t) + v_{xxxx}(x,t) = 0, \\ v(0,t) = v_x(0,t) = 0, \\ v_{xt}(1,t) = \dot{y}_{\rm ref}(t) - \alpha_{xt}(1,t), \\ v_{xxx}(1,t) = qv_t(1,t) + d(t). \end{cases} \tag{4.3}$$

**Therefore, we show that (i) system (4.3) has a unique solution that is bounded; (ii) the solution of (4.3) is the solution of (4.2).** We define an operator $\mathbf{A}_v : D(\mathbf{A}_v) \to \mathcal{H}$ by

$$\begin{cases} \mathbf{A}_v(\phi,\psi)^\top = (\psi, -\phi'''')^\top,\ \forall (\phi,\psi) \in D(A_v), \\ D(\mathbf{A}_v) = \left\{(\phi,\psi)^\top \in \mathcal{H} \bigcap (H^4(0,1) \times H_L^2(0,1)) |\ \psi'(1) = 0,\ \phi'''(1) = q\psi(1)\right\}. \end{cases} \tag{4.4}$$

Then system (4.3) can be written as

$$\frac{d}{dt}\begin{pmatrix} v \\ v_t \end{pmatrix} = \mathbf{A}_v \begin{pmatrix} v \\ v_t \end{pmatrix} + \mathbf{B}_1[\dot{y}_{\rm ref}(t) - \alpha_{xt}(1,t)] + \mathbf{B}_2 d(t) \tag{4.5}$$

where $\mathbf{B}_1 = (0, -\delta''(x-1))^\top$ and $\mathbf{B}_2 = (0, -\delta(x-1))^\top$. It is well-known from [11] that $\mathbf{A}_v$ generates an exponential stable $C_0$-semigroup on $\mathcal{H}$.

Next, we show that $\mathbf{B}_1$ and $\mathbf{B}_2$ are admissible for $e^{\mathbf{A}_v t}$([22]). Actually, a straightforward computation gives

$$\begin{cases} \mathbf{A}_v^*(\phi,\psi)^\top = (-\psi, \phi^{(4)})^\top,\ \forall (\phi,\psi) \in D(\mathbf{A}_v^*), \\ D(\mathbf{A}_v^*) = \{(\phi,\psi)^\top \in \mathcal{H} \bigcap (H^4(0,1) \times H_L^2(0,1)) |\ \psi'(1) = 0,\ \phi'''(1) = -q\psi(1)\}. \end{cases} \tag{4.6}$$

Consider the observation problem of dual system of (4.3):

$$\begin{cases} \widehat{y}^*_{tt}(x,t) + \widehat{y}^*_{xxxx}(x,t) = 0, \\ \widehat{y}^*(0,t) = \widehat{y}^*_x(0,t) = 0, \\ \widehat{y}^*_{xt}(1,t) = 0, \\ \widehat{y}^*_{xxx}(1,t) = -q\widehat{y}^*_t(1,t), \\ y_o = \{\widehat{y}^*_t(1,t), \widehat{y}^*_{xx}(1,t)\}. \end{cases} \tag{4.7}$$

Define the energy function for (4.7) as

$$E_{\widehat{y}^*}(t) = \frac{1}{2}\int_0^1 [\widehat{y}^*_t(x,t)]^2 dx + \frac{1}{2}\int_0^1 [\widehat{y}^*_{xx}(x,t)]^2 dx.$$

Since $\mathbf{A}_v$ generates a $C_0$-semigroup solution, and so does for $\mathbf{A}^*_v$. Hence system (4.7) associates with a $C_0$-semigroup solution and there exist two constants $M_1, u_1 > 0$ such that

$$\int_0^1 [\widehat{y}^*_t(x,t)]^2 dx + \int_0^1 [\widehat{y}^*_{xx}(x,t)]^2 dx \le M_1 e^{\mu_1 t}\left[\int_0^1 [\widehat{y}^*_t(x,0)]^2 dx + \int_0^1 [\widehat{y}^*_{xx}(x,0)]^2 dx\right]. \tag{4.8}$$

Let

$$\rho(t) = \int_0^1 x\widehat{y}^*_t(x,t)\widehat{y}^*_x(x,t)dx.$$

Obviously, $|\rho(t)| \le E_{\widehat{y}^*}(t)$. Differentiate $\rho(t)$ with respect to $t$ along the solution to (4.7) to obtain

$$\begin{aligned} \dot\rho(t) &= \int_0^1 x\widehat{y}^*_t(x,t)\widehat{y}^*_{xt}(x,t)dx - \int_0^1 x\widehat{y}^*_{xxxx}(x,t)\widehat{y}^*_x(x,t)dx \\ &= \frac{1}{2}[\widehat{y}^*_t(1,t)]^2 + \frac{1}{2}[\widehat{y}^*_{xx}(1,t)]^2 + \widehat{y}^*_{xx}(1,t)\widehat{y}^*_x(1,t) - \widehat{y}^*_x(1,t)\widehat{y}^*_{xxx}(1,t) \\ &\quad -\frac{1}{2}\int_0^1 [\widehat{y}^*_t(x,t)]^2 + 3\widehat{y}^*_{xx}(x,t)]^2 dx. \end{aligned} \tag{4.9}$$

Since

$$[\widehat{y}^*_x(1,t)]^2 = \left(\int_0^1 \widehat{y}^*_{xx}(\xi,t)d\xi\right)^2 \le \int_0^1 [\widehat{y}^*_{xx}(\xi,t)]^2 d\xi, \tag{4.10}$$

and by Young's inequality, we obtain

$$|\widehat{y}^*_{xx}(1,t)\widehat{y}^*_x(1,t)| \le \varepsilon_1[\widehat{y}^*_{xx}(1,t)]^2 + \frac{1}{4\varepsilon_1}[\widehat{y}^*_x(1,t)]^2, \tag{4.11}$$

and

$$|\widehat{y}^*_x(1,t)\widehat{y}^*_{xxx}(1,t)| \le \varepsilon_2[\widehat{y}^*_{xxx}(1,t)]^2 + \frac{1}{4\varepsilon_2}[\widehat{y}^*_x(1,t)]^2 \le \varepsilon_2 q^2[\widehat{y}^*_t(1,t)]^2 + \frac{1}{4\varepsilon_2}[\widehat{y}^*_x(1,t)]^2, \tag{4.12}$$

where $\varepsilon_1, \varepsilon_2$ is chosen so that $\varepsilon_1 \in (0, 1/2)$ and $\varepsilon_2 \in (0, 1/2q^2)$. It follows from (4.9)-(4.12) that

$$\begin{aligned} \dot\rho(t) \ge &\left(\frac{1}{2} - \varepsilon_2 q^2\right)[\widehat{y}^*_t(1,t)]^2 + \left(\frac{1}{2} - \varepsilon_1\right)[\widehat{y}^*_{xx}(1,t)]^2 \\ &-\left(3 + \frac{1}{4\varepsilon_1} + \frac{1}{4\varepsilon_2}\right)\int_0^1 [\widehat{y}^*_t(x,t)]^2 + \widehat{y}^*_{xx}(x,t)]^2 dx. \end{aligned}$$

Integrating from 0 to $T$ with respect to $t$ in the above equation and noting (4.8), one has

$$\begin{aligned}
&\left(\frac{1}{2}-\varepsilon_2 q^2\right)\int_0^T [\widehat{y}_t^*(1,t)]^2 dt+\left(\frac{1}{2}-\varepsilon_1\right)\int_0^T [\widehat{y}_{xx}^*(1,t)]^2 dt \\
&\le E_{\widehat{y}^*}(T)+2\Big(3+\frac{1}{4\varepsilon_1}+\frac{1}{4\varepsilon_2}\Big)TE_{\widehat{y}^*}(T) \\
&\le M_1 e^{\mu_1 T}\left(1+2\Big(3+\frac{1}{4\varepsilon_1}+\frac{1}{4\varepsilon_2}\Big)T\right)\left[\int_0^1 [\widehat{y}_t^*(x,0)]^2 dx+\int_0^1 [\widehat{y}_{xx}^*(x,0)]^2 dx\right].
\end{aligned} \tag{4.13}$$

On the other hand, a straightforward computation shows that $\mathbf{B}_1^*(I-\mathbf{A}_v)^{*-1}$ and $\mathbf{B}_2^*(I-\mathbf{A}_v)^{*-1}$ are bounded from $\mathcal{H}$ to $\mathbb{C}$. This together with (4.13) shows that $\mathbf{B}_1^*$ and $\mathbf{B}_2^*$ are admissible for $e^{\mathbf{A}_v^* t}$, and so are $\mathbf{B}_1$ and $\mathbf{B}_2$ for $e^{\mathbf{A}_v t}$. Since $\alpha_{xt}(1,t)\in L^2(0,\infty)$ due to Lemma 2.2 and the fact $d$, $\dot{y}_{\rm ref}\in L^\infty(0,\infty)$, it follows from [25, Lemma 2.1] or [26, Lemma 1.1] that for any initial value $(v_0,v_1)\in\mathcal{H}$, system (4.3) has a unique solution that is bounded.

Next, we claim that the solution of (4.3) is the solution of (4.2). Actually, letting $(v,v_t)$ be the solution of (4.3), by the boundary condition of (4.3) $v_{xt}(1,t)=\dot{y}_{\rm ref}(t)-\alpha_{xt}(1,t)$ for all $t\ge 0$ , we get $v_x(1,t)=y_{\rm ref}(t)-\alpha_x(1,t)+C$ with $C=v_x(1,0)-[y_{\rm ref}(0)-\alpha_x(1,0)]$. From the compatibility condition $\widehat{y}_0'(1)=y_{\rm ref}(0)$ and $v(x,0)=\widehat{y}(x,0)-\alpha(x,0)$, we obtain $C=\widehat{y}_0'(1)-\alpha_x(1,0)-[y_{\rm ref}(0)-\alpha_x(1,0)]=0$. Thus, the solution of (4.3) satisfies all the boundary condition of (4.2), which implies that the solution of (4.3) is also the solution of (4.2).

Finally, Lemma 2.2 and the boundedness of the solution of system (4.3) imply that $(\widehat{y},\widehat{y}_t)$ is bounded on $\mathcal{H}$. When $y_{\rm ref}(t)=d(t)\equiv 0$, the exponential stability of the solution $(\widehat{y},\widehat{y}_t)$ follows from the fact that the operator $\mathbf{A}_v$ generates an exponential stable $C_0$-semigroup on $\mathcal{H}$. $\square$

## 5 Main result

In this section, we show the well-posedness and performance output tacking of the closed-loop system obtained from (1.1). Under the control law (3.3), the closed-loop system is composed of

(1.1), (2.1) and (3.1), that is

$$\begin{cases} y_{tt}(x,t) + y_{xxxx}(x,t) = 0, \\ y(0,t) = y_x(0,t) = 0, \\ y_{xx}(1,t) = -c_0[y_{xt}(1,t) - \widehat{y}_{xt}(1,t)] + \widehat{y}_{xx}(1,t), \\ y_{xxx}(1,t) = qy_t(1,t) + d(t), \\ z_{tt}(x,t) + z_{xxxx}(x,t) = 0, \\ z(0,t) = z_x(0,t) = 0, \quad z(1,t) = y(1,t), \\ z_{xx}(1,t) = -c_0[z_{xt}(1,t) - \widehat{y}_{xt}(1,t)] + \widehat{y}_{xx}(1,t), \\ \widehat{y}_{tt}(x,t) + \widehat{y}_{xxxx}(x,t) = 0, \\ \widehat{y}(0,t) = \widehat{y}_x(0,t) = 0, \quad \widehat{y}_x(1,t) = y_{ref}(t), \\ \widehat{y}_{xxx}(1,t) = q\widehat{y}_t(1,t) + z_{xxx}(1,t) - qz_t(1,t). \\ e(t) = y_x(1,t) - y_{\mathrm{ref}}(t) \end{cases} \tag{5.1}$$

We consider system (5.1) in the state space $\mathcal{X} = (H_L^2(0,1) \times L^2(0,1))^3$.

**Theorem 5.1.** *Suppose that $d \in L^\infty(0,+\infty)$, $y_{\mathrm{ref}} \in W^{1,\infty}(0,\infty)$. Then, for any initial value $(y_0, y_1, z_0, z_1, \widehat{y}_0, \widehat{y}_1) \in \mathcal{X}$ with compatible boundary conditions $z_0(1) - y_0(1) = 0$, $\widehat{y}_0'(1) = y_{\mathrm{ref}}(0)$, there exists a unique solution to (5.1) such that $(y, y_t, z, z_t, \widehat{y}, \widehat{y}_t) \in C(0,\infty;\mathcal{X})$. Moreover, the closed-loop system solution has the following properties:*
*(i)*

$$\sup_{t\ge 0}\left(\int_0^1 [y_{xx}^2(x,t) + y_t^2(x,t) + z_{xx}^2(x,t) + z_t^2(x,t) + \widehat{y}_{xx}^2(x,t) + \widehat{y}_t^2(x,t)]dx\right) < +\infty;$$

*(ii) there exist two constants $M, \mu > 0$ such that*

$$\int_0^1 \left([z_{xx}(x,t) - y_{xx}(x,t)]^2 + [z_t(x,t) - y_t(x,t)]^2\right)dx \le Me^{-\mu t};$$

*and*

$$\int_0^1 \left([\widehat{y}_{xx}(x,t) - y_{xx}(x,t)]^2 + [\widehat{y}_t(x,t) - y_t(x,t)]^2\right)dx \le Me^{-\mu t};$$

*(iii) there exist two constants $M, \mu > 0$ such that*

$$|e(t)| = |y_x(1,t) - y_{\mathrm{ref}}(t)| \le Me^{-\mu t}, \quad \text{for all} \quad t \ge 0.$$

*(iv) When $y_{\mathrm{ref}}(t) = d(t) \equiv 0$, the solution $(y, y_t, z, z_t, \widehat{y}, \widehat{y}_t))$ is exponentially stable on $\mathcal{X}$.*

*Proof.* Since

$$\begin{pmatrix} z(x,t) \\ y(x,t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha(x,t) \\ \beta(x,t) \end{pmatrix} + \begin{pmatrix} \widehat{y}(x,t) \\ \widehat{y}(x,t) \end{pmatrix},$$

(i) follows from Lemma 2.2, Theorem 3.1 and 4.1. Note that $\alpha(x,t) = z(x,t) - y(x,t)$, $\beta(x,t) = y(x,t) - \widehat{y}(x,t)$, (ii) follows from Lemma 2.2 and Theorem 3.1. Since $\widehat{y}_x(1,t) = y_{\text{ref}}(t)$, (ii) and Sobolev embedding theorem imply (iii). Finally, (iv) follows from (ii) and Theorem 4.1. □

**Remark 5.1.** From (ii) in Theorem 5.1, both the $z$-part and the $\widehat{y}$-part of the closed-loop system (5.1) could be regarded as the state observer of (1.1). However, they play the different roles. The main difference is that the $z$-part of (5.1) is used to estimate the disturbance while the $\widehat{y}$-part of (5.1) is used to be a servo system which is essentially a copy of the original system (1.1).

## 6 Numerical simulation

In this section, we present some numerical simulations for the closed-loop system (5.1) to illustrate the effectiveness of the proposed feedback control. For numerical computations, we take reference signal $r(t) = 2\sin(2t)$ and the external disturbance $d(t) = 2\sin(t) + 0.7\cos(6t) - 1$. The parameters are taken as $q = 2$, $c_0 = 1$. The initial value are

$$y(x,0) = 2x - x^2,\ y_t(x,0) = 0,\ z(x,0) = 0,$$

$$z_t(x,0) = 0,\ \widehat{y}(x,0) = -2x + x^2,\ \widehat{y}_t(x,0) = 0.$$

The Galerkin finite element method is adopted in computation of the displacements and velocity. The time step is chosen as $dt = 0.001$, the interval $[0,1]$ is partitioned $[(k-1)/N, k/N]$, $k = 1, 2, \ldots, N$, where $N = 4$. Hermite cubic polynomials are used as basis functions ([21]).

The solution of system is plotted in Figures 1-3. Figure 4 shows that the reference model (3.1) can be regarded as a state observer of (1.1). Figure 5 shows that the disturbance $d(t)$ and its estimate $z_{xxx}(1,t) - qz_t(1,t)$. It is seen that the disturbance is estimated effectively. The convergence is very satisfactory. Figure 6 shows that $y_x(1,t)$ and $y_{\text{ref}}(t)$. It is seen that $y_x(1,t)$ can tracks the reference signal $y_{\text{ref}}(t)$ satisfactorily. Figure 7 displays the feedback control in time.

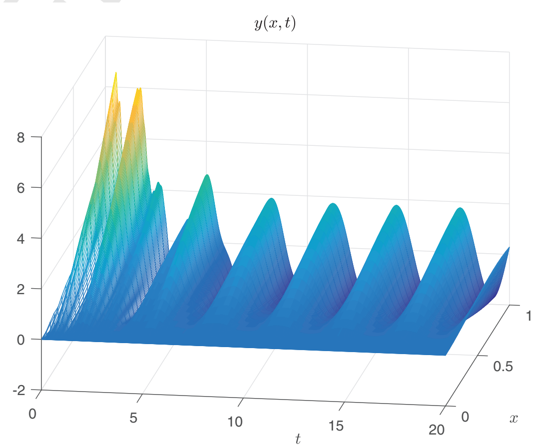


Figure 1: The state $y(x,t)$

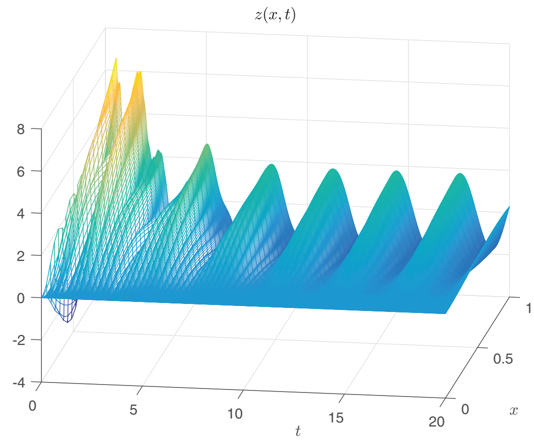


Figure 2: The state $z(x,t)$

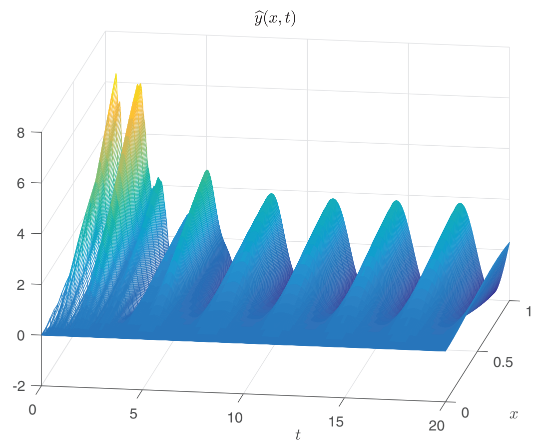


Figure 3: The state $\widehat{y}(x,t)$

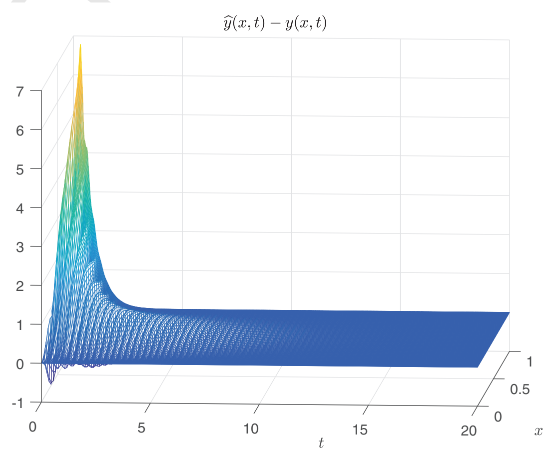


Figure 4: The error $-\beta(x,t) = \widehat{y}(x,t) - y(x,t)$

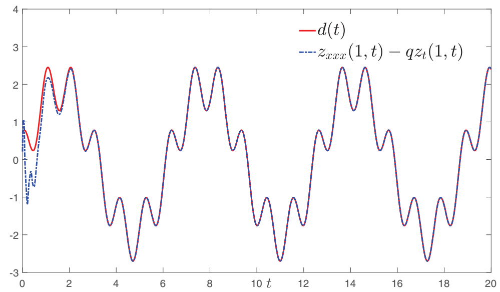


Figure 5: The disturbance and its estimation

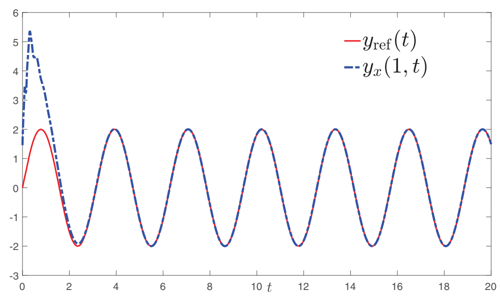


Figure 6: The reference signal $r(t)$ and the output $y_x(1,t)$

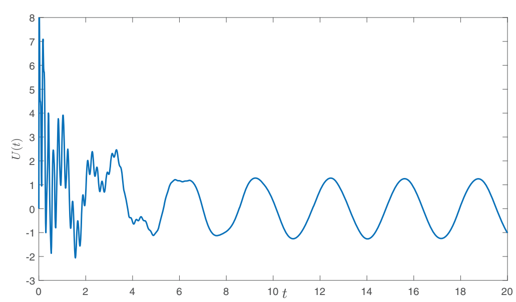


Figure 7: The control law $U(t)$

## 7 Concluding Remark

We have designed a new infinite-dimensional disturbance estimator to estimate the unknown disturbance and proposed a servomechanism by using the measured output and the reference signal where the estimation mechanism of unknown disturbance is presented. Four major control objectives are achieved. a) The performance output tracks exponentially the reference signal; b) The unknown disturbance can be estimated and thus be compensated; c) All the states of internal-loops are bounded in the energy state space; d) When the disturbance and reference signal are disconnected to the system, the closed-loop is exponentially stable.

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