J. Math. Anal. Appl. 477 (2019) 357–379

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Mathematical analysis of a parabolic-elliptic problem with moving parabolic subdomain through a Lagrangian approach

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A R T I C L E   I N F O

Article history:
Received 29 May 2018
Available online 18 April 2019
Submitted by J. Guermond

Keywords:
Parabolic-elliptic problem
Moving parabolic subdomain
Regularity
Lagrangian formulation

A B S T R A C T

The aim of this paper is to study the regularity of the solution of some linear parabolic-elliptic problems in which parabolicity region depends on time. More specifically, this region is the position occupied by a body undergoing a motion (a deformation smoothly evolving in time). The main tool we introduce is a suitable extension of the motion to the entire spatial domain of the PDE. This enables us to reduce the original problem to a parabolic-elliptic problem with variable coefficients and with a parabolicity region independent of time. This problem can be seen as a Lagrangian formulation of our original problem. Next, we obtain regularity results for a class of parabolic-elliptic problems with variable coefficients and fixed parabolicity region. We apply these results to the Lagrangian formulation and, finally, we obtain a regularity result for our original problem.

1. Introduction

The goal of this paper is to study the regularity of the following parabolic-elliptic initial-boundary value problem: find a function $A = A(t, x)$ such that

$$\begin{cases} \sigma \frac{\partial A}{\partial t} - \rho \triangle A = f & \text{in } (0,T) \times \Omega, \\ A = 0 & \text{on } (0,T) \times \partial\Omega, \\ A(0,\cdot) = A^0 & \text{in } \widehat{\Omega}, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with Lipschitz boundary, $\widehat{\Omega} \subset\subset \Omega$ a subdomain with smooth enough boundary, $T > 0$ a given number, $\rho > 0$ is a constant and $\sigma \geq 0$ is defined through a motion $\boldsymbol{X}$: $[0,T] \times \overline{\widehat{\Omega}} \mapsto \mathbb{R}^n$ in such a way that $\{x \in \Omega; \sigma(t,x) > 0\} = \Omega_t$ where $\Omega_t = \boldsymbol{X}(t,\cdot)(\widehat{\Omega})$. (For $n = 3$, $\boldsymbol{X}$ represents a deformation of the body $\widehat{\Omega}$ evolving smoothly with time and Ω_t is the region occupied by the

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https://doi.org/10.1016/j.jmaa.2019.04.035

body at instant t.) We assume that $f(t,x)$ vanishes for $x \notin \Omega_{\mathrm{S}}$, where $\Omega_{\mathrm{S}} \subset \Omega$ is open and $\overline{\Omega}_{\mathrm{S}} \cap \overline{\Omega_t} = \emptyset$ for all $t \in [0,T]$. The detailed description of the problem is given in Section 2.

Our main goal is to find a condition on A^0 and f ensuring that $\frac{\partial^2 A}{\partial t^2} \in L^2(Q)$, where

$$Q := \{(t,x) \in \mathbb{R}^{n+1}, t \in (0,T), x \in \Omega_t\}. \tag{1.2}$$

A motivation for this study comes from an eddy current model with moving conductors considered in [3] and [4] in the context of Electromagnetic Forming (EMF). This model allows to compute the electromagnetic field produced by a coil in a cylindrical metallic moving workpiece under axisymmetric assumptions. To reduce the problem to a bounded domain, the authors introduce a three-dimensional cylinder $\widetilde{\Omega}$ containing the coil and the workpiece, with its boundary sufficiently far from them at all times $t \in [0,T]$. Because of the cylindrical symmetry, the problem is posed in $\Omega = \{(r,z); 0 < r < R, 0 < z < L\}$, which is a meridian section of $\widetilde{\Omega}$. Under suitable axisymmetry assumptions on the motion, this one is determined by its description $\boldsymbol{X}$ in this meridian section, with $\boldsymbol{X} : [0,T] \times \overline{\widehat{\Omega}} \mapsto \overline{\Omega}$, where $\overline{\widehat{\Omega}} \subset \Omega$ is a reference configuration. The meridian section of the workpiece at time will be $\Omega_t = \boldsymbol{X}(t,\cdot)(\widehat{\Omega})$. To ensure cylindrical symmetry, the coil is modeled by several concentric rings with toroidal geometry. The open set Ω_{S} is the union of the meridian sections of the rings. The assumption $\overline{\Omega}_{\mathrm{S}} \cap \overline{\Omega_t} = \emptyset$ for all $t \in [0,T]$ means that the workpiece never touches the coil. Let Γ_0 denote the intersection between $\overline{\Omega}$ and the axis $r = 0$, and $\Gamma_{\mathrm{D}} := \partial\Omega \setminus \Gamma_0$.

The density current in the coil $\vec{J}_S := J_S(t,r,z)\,\vec{e}_\theta$ is taken as a data. Then, under suitable axisymmetry assumptions on the data (see [3], [1] and [2]) there exists a divergence-free magnetic vector potential $\vec{A}$ of the form

$$\vec{A}(t,x,y,z) = A_{cyl}(t,r,z)\,\vec{e}_\theta, \tag{1.3}$$

and $\vec{A}$ satisfies

$$\sigma \frac{\partial \vec{A}}{\partial t} + \frac{1}{\mu}\ \mathrm{rot}(\mathrm{rot}\,\vec{A}) = \vec{J}_S \qquad \text{in} \ \ (0,T) \times \widetilde{\Omega}, \tag{1.4}$$

where μ is the magnetic permeability and σ is the electric conductivity, which vanishes outside the workpiece. Function σ is taken such that

$$\sigma(t,r,z) = \widehat{\sigma}(\widehat{r},\widehat{z}), \qquad \text{with } (\widehat{r},\widehat{z}) \in \overline{\widehat{\Omega}} : \ (r,z) = \boldsymbol{X}(t,\widehat{r},\widehat{z}),$$

where $\widehat{\sigma}$ is the conductivity in the reference domain $\widehat{\Omega}$. Equation (1.4) can be rewritten in cylindrical coordinates as

$$\sigma \frac{\partial A_{cyl}}{\partial t} - \frac{1}{\mu}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial(rA_{cyl})}{\partial r}\right] - \frac{1}{\mu}\frac{\partial^2 A_{cyl}}{\partial z^2} = J_S \qquad \text{in} \ \ (0,T) \times \Omega. \tag{1.5}$$

This parabolic-elliptic equation is complemented with the initial condition

$$A_{cyl}(0,\cdot) = A^0_{cyl} \text{ in } \Omega_0$$

and homogeneous Dirichlet boundary conditions for A_{cyl} on $(0,T) \times \Gamma_{\mathrm{D}}$ (see [3] for details).

For the model above, some results of existence, uniqueness and regularity were obtained in [3], and a fully discrete Euler implicit/continuous piecewise linear discretization for problem (1.5) was described and analyzed in [4]. Convergence of the solution of the discrete problem to the solution of the continuous one was obtained by assuming $A_{cyl} \in H^2(0,T; L^2_r(\Omega)))$, where

$$L_r^2(\Omega) := \left\{ Z : \Omega \mapsto \mathbb{R} \text{ measurable } ; \|Z\|^2_{L_r^2(\Omega)} := \int_\Omega |Z|^2 \, r \, dr \, dz < \infty \right\}.$$

However, the above regularity for A_{cyl} has not been previously obtained.

We notice that (1.1) and (1.5) have analogous structure. The weak formulation of the latter involves weighted Sobolev spaces, whereas the weak formulation of the former involves standard (unweighted) Sobolev spaces. Moreover, equation (1.3), $\operatorname{div} \vec{A} = 0$ and the identity

$$\Delta \vec{A} = \nabla(\operatorname{div} \vec{A}) - \operatorname{rot}(\operatorname{rot} \vec{A})$$

imply that $\vec{A}$ satisfies (1.4) if and only if its Cartesian coordinates A_i satisfy the partial differential equation (PDE) in (1.1) with $\rho = \mu^{-1}$ and right-hand side $(\vec{J}_S)_i$. Hence it can be expected to obtain regularity results for (1.5) from the regularity results for (1.1).

For both problems, the fact that the PDE is parabolic only in the time-dependent set Ω_t makes difficult to study the regularity by using the approach of [3], which is based in the Eulerian coordinates x. For this reason, we develop in this paper an alternative approach, which is based on the Lagrangian coordinate p (x is the position of the material point p at instant t). As the motion is only defined in $[0, T] \times \overline{\widehat{\Omega}}$, we construct a suitable extension $\tilde{\boldsymbol{X}}$ of the motion $\boldsymbol{X}$ to the entire $[0, T] \times \overline{\Omega}$. This enables us to reduce problem (1.1) to a parabolic-elliptic problem with variable coefficients (for the function $\hat{A}(t, p) = A(t, \tilde{\boldsymbol{X}}(t, p))$) whose region of parabolicity is $\widehat{\Omega}$, hence independent of time. This problem can be seen as a Lagrangian formulation of our original problem. We obtain regularity results for a class of parabolic-elliptic problems with variable coefficients and fixed parabolicity region. Since the Lagrangian formulation fits into this class, we obtain a regularity result for $\hat{A}$. Finally, we obtain regularity results for the solution of our original problem by changing Lagrangian coordinates back to Eulerian coordinates.

Firstly, let us comment on previous related work. Some authors deal with parabolic-elliptic problems whose structure is similar to problem (1.1). In [8], results of existence, uniqueness and regularity for some degenerate linear evolution equations are obtained. Degenerate parabolic equations are considered in Section 6 of [8], but its regularity results are not applicable to problem (1.1) because our coefficient $\sigma(t, x)$ does not have the regularity required in [8]. Besides, in the present case $\sigma(0, x)$ vanishes outside $\widehat{\Omega}$ and hence equation (6.4) of [8] does not hold true.

Article [13] includes existence, uniqueness and regularity results for an abstract differential equation of the form

$$Bu' + Au = f,$$

where B can be a non-invertible linear operator and A can be a non-linear operator, both independent of time. So they are not applicable to problem (1.1).

Existence and uniqueness results for degenerate evolutionary equations of the form

$$\frac{d}{dt}(B(t)u(t)) + A(t)u(t) = f(t),$$

where $B(t)$ can be a non-invertible linear operator are obtained in [20] and [21, Chapter III] (for linear $A(t)$), and in [12], [14] and [17] (where $A(t)$ can be non-linear). Among the last five cited references, regularity is only addressed in [17], where a result related to the first order time derivative is obtained.

Moreover, Pluschke [18] studies an initial boundary-value problem for parabolic-elliptic equations of the form

$$g(x, t, u)\frac{\partial u}{\partial t} - \operatorname{div}(a(x, t)\nabla u) + a_0(x, t)u = f(x, t, u),$$

where $a(x,t)$ is matrix-valued, $g = g(x,t,u)$ is nonnegative and it is allowed to vanish for certain values of (x,t,u). The author obtains existence, uniqueness and some regularity results. It is assumed that the region of parabolicity is independent of t. So problem (1.1) does not fit into his framework.

In [19], the authors prove the existence of a weak local solution to a parabolic-elliptic problem where the time derivative is multiplied by a coefficient which may vanish on time-dependent spatial subdomains. Problem (1.1) does not satisfy the assumptions about the parabolic region made in [19].

Consequently, none of the aforementioned articles can be applied to prove the regularity that we propose to undertake in the present article.

The article is organized as follows: in Section 2 we formulate the problem and comment about the results which can be obtained with an Eulerian approach by an easy adaptation of those of [3]. Section 3 is dedicated to the motion extension. In Section 4 we reduce our original parabolic-elliptic problem to a parabolic-elliptic problem with fixed region of parabolicity, $\widehat{\Omega}$. In Section 5 we obtain existence, uniqueness and regularity results for a class of parabolic-elliptic problems with fixed region of parabolicity by using parabolic regularization. The regularity results involve compatibility conditions between the initial condition and the initial value of the source term. In Section 6 we apply to the reduced problem a regularity result obtained in Section 5 and we derive the corresponding result for our original problem. Section 7 contains two examples. The first one concerns the spatial regularity of A and the second one illustrates the main features considered in this paper.

2. Statement of the problem

Let Ω be a bounded domain of $\mathbb{R}^n$ with Lipschitz boundary, $\widehat{\Omega} \subset\subset \Omega$ a subdomain with Lipschitz boundary, $T \in \mathbb{R}, T > 0$ the (given) final time and $\boldsymbol{X} : [0,T] \times \overline{\widehat{\Omega}} \mapsto \mathbb{R}^n$ be a given mapping such that:

(i) $\boldsymbol{X} \in \mathcal{C}^1([0,T]; [\mathcal{C}^1(\overline{\widehat{\Omega}})]^n)$;
(ii) $\det(D_p\boldsymbol{X})(t,p) > 0$ for all $(t,p) \in [0,T] \times \overline{\widehat{\Omega}}$;
(iii) $\boldsymbol{X}(t,\cdot)$ is injective $\forall t \in [0,T]$; ($\Leftrightarrow \boldsymbol{X}(t,\cdot) : \overline{\widehat{\Omega}} \mapsto \overline{\Omega_t}$ is bijective, where $\Omega_t := \boldsymbol{X}(t,\cdot)(\widehat{\Omega})$);
(iv) $\boldsymbol{X}([0,T]) \times \overline{\widehat{\Omega}}) \subset \Omega$ (In fact $\subset\subset$ holds true);
(v) $\boldsymbol{X}(0,p) = p$ for all $p \in \overline{\widehat{\Omega}}$.

Our notations are standard. In assumption (i) and thereafter, for any bounded open set $G \subset \mathbb{R}^n$, we denote by $\mathcal{C}^1(\overline{G})$ the set of functions in $\mathcal{C}(\overline{G}) \cap \mathcal{C}^1(G)$ such that all its first-order partial derivatives have continuous extensions to $\overline{G}$. Moreover, as it is usually done in the theory of time-dependent PDEs, we identify a function having independent variables (t,x) (resp. (t,p)) with a function of the time variable t taking values in a suitable function space of variable x (resp. p). For instance, if $g : (t,p) \in [0,T] \times \overline{\widehat{\Omega}} \mapsto \mathbb{R}$ is a continuous mapping, we identify it to the function $t \in [0,T] \mapsto g(t) \in \mathcal{C}(\overline{\widehat{\Omega}})$, where $g(t)(p) = g(t,p)$. Sometimes we will use the notation $g(t,\cdot)$ instead of $g(t)$. In particular, we make the identifications $\mathcal{C}([0,T]\times\overline{\widehat{\Omega}}) \equiv \mathcal{C}([0,T]; \mathcal{C}(\overline{\widehat{\Omega}}))$ and $L^2((0,T) \times \widehat{\Omega}) \equiv L^2(0,T; L^2(\widehat{\Omega}))$.

If $n = 3$, $\boldsymbol{X}$ is essentially a motion ([11]) of a body , $\widehat{\Omega} = \Omega_0$ is the region occupied by the body at $t = 0$, which is taken as the reference configuration.

Let $\widehat{\sigma} \in L^\infty(\widehat{\Omega})$ be such that

$$\widehat{\sigma}(p) \geq \underline{\sigma} > 0 \quad \text{a.e. } p \in \widehat{\Omega}. \tag{2.1}$$

We define

$$\sigma(t,x) = \begin{cases} \widehat{\sigma}(p) & \text{if } x = \boldsymbol{X}(t,p) \text{ with } p \in \overline{\widehat{\Omega}}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Finally, we are given an initial data $A^0 \in H^1(\widehat{\Omega})$ and a right-hand side $f \in H^1(0,T;L^2(\Omega_S))$, where the open set $\Omega_S \subset \Omega$ and $\overline{\Omega}_S \cap \overline{\Omega_t} = \emptyset$ for all $t \in [0,T]$ (i.e.,$\boldsymbol{X}([0,T] \times \overline{\widehat{\Omega}}) \cap \overline{\Omega}_S = \emptyset$). We extend f by zero outside $[0,T] \times \Omega_S$ and still denote this extension by f.

Let $\boldsymbol{\Psi} : [0,T] \times \overline{\widehat{\Omega}} \mapsto [0,T] \times \Omega$ be defined by $\boldsymbol{\Psi}(t,p) = (t, \boldsymbol{X}(t,p))$. We recall that Q denotes the noncylindrical open subset of $(0,T) \times \Omega$ defined in (1.2) and we notice that $Q = \boldsymbol{\Psi}((0,T) \times \widehat{\Omega})$. Moreover, the inverse $\boldsymbol{\Psi}^{-1} : \overline{Q} \mapsto [0,T] \times \overline{\widehat{\Omega}}$ has the form

$$\boldsymbol{\Psi}^{-1}(t,x) = (t, \boldsymbol{P}(t,x)) ,$$

where $\boldsymbol{P}$ is the reference map, that is, for all $t \in [0,T]$, $\boldsymbol{P}(t,\cdot) : \overline{\Omega_t} \mapsto \overline{\widehat{\Omega}}$ is the inverse of $\boldsymbol{X}(t,\cdot)$. In other words, $p = \boldsymbol{P}(t,x)$ is the material point occupying the position x at time t.

Problem (1.1) is parabolic only in the subdomain Q. For each $t \in (0,T)$, we have an elliptic PDE in $\Omega \setminus \overline{\Omega_t}$.

The initial condition is only given in $\Omega_0 = \widehat{\Omega}$, not in the entire Ω.

The weak formulation of problem (1.1) is:

$$\begin{cases} \text{Find } \; A \in L^2(0,T;H_0^1(\Omega)), \text{ with } \frac{\partial A}{\partial t} \in L^2(Q) \quad \text{such that} \\ \int_{\Omega_t} \sigma \frac{\partial A}{\partial t} z\,dx + \rho \int_\Omega \nabla A \cdot \nabla z\,dx = \int_{\Omega_S} f z\,dx \quad \forall z \in H_0^1(\Omega) \quad \text{a.e. } t \in (0,T), \\ A(0) = A^0 \quad \text{in } \widehat{\Omega}. \end{cases} \tag{2.3}$$

2.1. Results that can be obtained through an Eulerian approach

In [3] some results of existence, uniqueness and regularity were obtained for (1.5) by using an Eulerian approach (strictly speaking, the approach was based in the cylindrical coordinates (r,z) associated to x). Thus, before developing the Lagrangian approach for problem (2.3), it seems natural to state the results obtained for this problem through the Eulerian approach. It is easy to adapt the results of [3] to the analysis of problem (2.3). This amounts essentially to replace the weighted Sobolev spaces used in [3] by standard (unweighted) Sobolev spaces.

The following theorem summarizes the analogues to results [3, Theorems 4.2, 4.3 and 5.1].

Theorem 2.1. *Assume that $A^0 \in H^1(\widehat{\Omega})$ and $f \in H^1(0,T;L^2(\Omega_S))$. There exists a unique solution to Problem (2.3). Furthermore, it satisfies $A \in L^\infty(0,T;H_0^1(\Omega))$, $\sqrt{t}\partial_t A \in L^2(0,T;H_0^1(\Omega))$ and $\chi_Q \sqrt{t}\partial_t A \in L^\infty(0,T;L^2(\Omega))$.*

Here and in the sequel χ_S will stand for the characteristic function of a measurable set S and ∂_t will be often used to denote the derivative with respect to time.

The solution $A : [0,T] \to H_0^1(\Omega)$ is weakly continuous at $t = 0$. Indeed, the values of $A(0)$ in $\Omega^e := \Omega \setminus \overline{\widehat{\Omega}}$ are completely determined by the initial data of (2.3), A^0 $(= A(0)|_{\widehat{\Omega}})$, and the initial value of the right hand side, $f(0)$. Indeed, let $\Gamma_e := \partial\Omega^e \bigcap \partial\widehat{\Omega}$. We notice that, since $\widehat{\Omega} \subset\subset \Omega$, we have: $\Gamma_e = \partial\widehat{\Omega}$ and $\partial\Omega^e = \Gamma_e \bigcup \partial\Omega$ (disjoint union). Let $A^e \in H^1(\Omega^e)$ be the (unique) weak solution of

$$\begin{cases} -\rho\Delta A^e = f(0) & \text{in } \; \Omega^e, \\ A^e_{|\Gamma_e} = A^0_{|\Gamma_e}, \\ A^e_{|\partial\Omega} = 0, \end{cases} \tag{2.4}$$

and $\tilde{A}^0 \in H_0^1(\Omega)$ be the extension of A^0 to the entire Ω, defined by

$$\tilde{A}^0 := \begin{cases} A^0 & \text{in } \widehat{\Omega}, \\ A^e & \text{in } \Omega^{\mathrm{e}}. \end{cases}$$

The analogue to [3, Theorem 5.3]) reads as follows:

Theorem 2.2. *Under the assumptions of Theorem 2.1, $A(t) \rightharpoonup \tilde{A}^0$ weakly in $H_0^1(\Omega)$ as $t \to 0^+$.*

Remark 2.1. Concerning space regularity, on the grounds of elliptic regularity ([10, Theorems 3.2.1.2 and 2.2.2.3]), we have the analogue to [3, Remark 5.1]: if Ω is convex or it has a boundary of class $\mathcal{C}^{1,1}$ then $A \in L^2(0,T;H^2(\Omega))$ and $A \in L^\infty(\epsilon,T;H^2(\Omega))$ for $\epsilon > 0$.

The following regularity result is the analogue to [3, Theorems 5.4 and 5.5].

Theorem 2.3. *Under the assumptions of Theorem 2.1:*

(i) *If $A \in H^1(0,T;H_0^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega))$, then $\tilde{A}^0 \in H^2(\Omega)$.*
(ii) *If*

$$\triangle A^0 \in L^2(\widehat{\Omega}) \text{ and } \frac{\partial A^0}{\partial \nu} = \frac{\partial A^e}{\partial \nu} \text{ on } \Gamma_e\,, \tag{2.5}$$

then $A \in H^1(0,T;H_0^1(\Omega))$ and $\sigma \frac{\partial A}{\partial t} \in L^\infty(0,T;L^2(\Omega))$. If, further, Ω is convex or it has a boundary of class $\mathcal{C}^{1,1}$, then $A \in L^\infty(0,T;H^2(\Omega))$.

Remark 2.2. Condition $\tilde{A}^0 \in H^2(\Omega)$ always implies condition (2.5). The converse holds true when Ω is convex or it has a boundary of class $\mathcal{C}^{1,1}$. Note that equation (2.5) includes a regularity requirement on A^0 a non-trivial compatibility condition between A^0 and the initial value of the right hand side, $f(0)$.

The last implication in part (ii) of Theorem 2.3 and the converse implication in this remark make use of elliptic regularity ([10, Theorems 3.2.1.2 and 2.2.2.3]).

We point out that we cannot obtain further regularity through the Eulerian framework because obtaining energy estimates for $\|\frac{\partial^2 A}{\partial t^2}\|_{L^2(Q)}$ does not work. This is why we develop a Lagrangian approach.

3. Extension theorems

The aim of this section is to build a suitable extension of mapping $\boldsymbol{X}$ (defined in $[0,T] \times \overline{\widehat{\Omega}}$) to the entire $[0,T] \times \overline{\Omega}$.

In the sequel, we consider that the spaces $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ are equipped with the Euclidean norm. The distance between two subsets of this spaces must be understood according to this.

Theorem 3.1. *We assume the hypothesis about Ω, $\widehat{\Omega}$ and the mapping $\boldsymbol{X}$ stated in Section 2. We assume further that $\widehat{\Omega}$ has a boundary of class $\mathcal{C}^1$. Let K be a compact subset of $\overline{\Omega}$ such that $\boldsymbol{X}([0,T] \times \overline{\widehat{\Omega}}) \bigcap K = \emptyset$ and let ϵ, $0 < \epsilon < dist(\boldsymbol{X}([0,T] \times \overline{\widehat{\Omega}}), K)$. Then, there exists a mapping $\tilde{\boldsymbol{X}} : [0,T] \times \overline{\Omega} \mapsto \mathbb{R}^n$ such that:*

(a) *$\tilde{\boldsymbol{X}}$ is an extension of $\boldsymbol{X}$.*
(b) *$\tilde{\boldsymbol{X}} \in \mathcal{C}^1([0,T];[\mathcal{C}^1(\overline{\Omega}]^n)$.*
(c) *$\det(D_p\tilde{\boldsymbol{X}})(t,p) > 0$ for all $(t,p) \in [0,T] \times \overline{\Omega}$.*

(d) *For all* $t \in [0,T]$, $\tilde{\boldsymbol{X}}(t,\cdot) : \overline{\Omega} \mapsto \overline{\Omega}$ *is an homeomorphism.*
(e) $\tilde{\boldsymbol{X}}(0,p) = p$, $\forall p \in \overline{\Omega}$.
(f) $\tilde{\boldsymbol{X}}(t,p) = p$, $\forall p \in \partial\Omega$, $\forall t \in [0,T]$.
(g) $\tilde{\boldsymbol{X}}(t,p) = p$, $\forall p \in \mathcal{N}_\epsilon(K)$, $\forall t \in [0,T]$, *where* $\mathcal{N}_\epsilon(K) = \{p \in \overline{\Omega}; \, dist(p,K) < \epsilon\}$.

Proof. 1st step: Extension in space and time.

Let K_1 be a compact set such that $\overline{\widehat{\Omega}} \subset int(K_1) \subset K_1 \subset \Omega$ and let $\mathcal{C}^1_{K_1}(\Omega)$ be the space of functions in $\mathcal{C}^1(\Omega)$ having support contained in K_1. $\mathcal{C}^1_{K_1}(\Omega)$ is a Banach space which can be identified to $\mathcal{C}^1_{K_1}(\mathbb{R}^n)$ (analogous notation). Since $\widehat{\Omega}$ has a $\mathcal{C}^1$ boundary and $\widehat{\Omega} \subset\subset \Omega$, in virtue of [9, Lemma 6.37] and its proof, there exists a linear bounded extension operator $E : [\mathcal{C}^1(\overline{\widehat{\Omega}})]^n \mapsto [\mathcal{C}^1_{K_1}(\Omega)]^n$. Because of assumption (i), the displacement field $\boldsymbol{u}(t,p) := \boldsymbol{X}(t,p) - p$ belongs to $\mathcal{C}^1([0,T]; [\mathcal{C}^1(\overline{\widehat{\Omega}})]^n)$. We define $\tilde{\boldsymbol{u}}$ by $\tilde{\boldsymbol{u}}(t) = E(\boldsymbol{u}(t))$. Since E is linear and bounded, $\tilde{\boldsymbol{u}} \in \mathcal{C}^1([0,T]; [\mathcal{C}^1_{K_1}(\Omega)]^n)$. By using an standard procedure, we construct an extension of $\tilde{\boldsymbol{u}}$, denoted $\hat{\boldsymbol{u}}$, which belongs to $\mathcal{C}^1([-T,2T]; [\mathcal{C}^1_{K_1}(\Omega)]^n)$. Next, we define the mapping $\hat{\boldsymbol{X}} : [-T,2T] \times \mathbb{R}^n \mapsto \mathbb{R}^n$ by

$$\hat{\boldsymbol{X}}(t,p) = p + \hat{\boldsymbol{u}}(t)(p) \quad \forall t \in [-T,2T] \quad \forall p \in \mathbb{R}^n .$$

Mapping $\hat{\boldsymbol{X}}$ is an extension of $\boldsymbol{X}$. From assumption (v), we have

$$\hat{\boldsymbol{X}}(0,p) = p \quad \forall p \in \mathbb{R}^n. \tag{3.1}$$

2nd step: From the inclusion $\widehat{\Omega} \subset\subset \Omega$, assumptions (ii) and (iv), and uniform continuity arguments we deduce that $\exists \delta_1 > 0$ such that $\overline{\mathcal{N}_{\delta_1}(\widehat{\Omega})} \subset \Omega$,

$$\hat{\boldsymbol{X}}([-\delta_1, T+\delta_1] \times \overline{\mathcal{N}_{\delta_1}(\widehat{\Omega})}) \subset \Omega \tag{3.2}$$

and

$$det(D_p\hat{\boldsymbol{X}})(t,p) > 0 \text{ for all } (t,p) \in [-\delta_1, T+\delta_1] \times \overline{\mathcal{N}_{\delta_1}(\widehat{\Omega})}). \tag{3.3}$$

Here and in the sequel notation $\mathcal{N}$ has the meaning explained in item (g).

3rd step: Now we claim that there exists δ_2, $0 < \delta_2 < \delta_1$ such that for all $t \in [-\delta_2, T+\delta_2]$, the mapping $\hat{\boldsymbol{X}}(t,\cdot) : \overline{\mathcal{N}_{\delta_2}(\widehat{\Omega})} \mapsto \mathbb{R}^n$ is injective. We argue by contradiction. Let us assume that the claim is not true. Then, for all integer $m > 1/\delta_1$ there exists $t_m \in [-\frac{1}{m}, T+\frac{1}{m}]$ and $p_m, \tilde{p}_m \in \overline{\mathcal{N}_{1/m}(\widehat{\Omega})}$, $p_m \neq \tilde{p}_m$ such that

$$\hat{\boldsymbol{X}}(t_m, p_m) = \hat{\boldsymbol{X}}(t_m, \tilde{p}_m) . \tag{3.4}$$

After extracting subsequences, we have

$$p_m \to \overline{p} \in \overline{\mathcal{N}_{\delta_1}(\widehat{\Omega})}, \quad \tilde{p}_m \to \tilde{p} \in \overline{\mathcal{N}_{\delta_1}(\widehat{\Omega})}, \quad t_m \to \overline{t} \in [-\delta_1, T+\delta_1] .$$

Since $dist(p_m, \widehat{\Omega}) \leq 1/m$ for all m, we have $\overline{p} \in \overline{\widehat{\Omega}}$. In the same manner, $\tilde{p} \in \overline{\widehat{\Omega}}$ and $\overline{t} \in [0,T]$. Besides, from (3.4) and the continuity of $\hat{\boldsymbol{X}}$, we have

$$\boldsymbol{X}(\overline{t}, \overline{p}) = \hat{\boldsymbol{X}}(\overline{t}, \overline{p}) = \hat{\boldsymbol{X}}(\overline{t}, \tilde{p}) = \boldsymbol{X}(\overline{t}, \tilde{p}) .$$

If $\overline{p} \neq \tilde{p}$, this contradicts assumption (iii). Now we address the case $\overline{p} = \tilde{p}$. Let $\hat{\boldsymbol{\Psi}}$ be the mapping defined by $\hat{\boldsymbol{\Psi}}(t,p) := (t, \hat{\boldsymbol{X}}(t,p))$. Owing to (3.3) and the inverse function theorem, $\hat{\boldsymbol{\Psi}}$ is locally injective at $(\overline{t}, \overline{p})$. But, on the other hand, both sequences (t_m, p_m), $(t_m, \tilde{p}_m)$ converge to the same point $(\bar{t}, \bar{p})$, $(t_m, p_m) \neq (t_m, \tilde{p}_m)$ for all m and (3.4) implies that $\hat{\boldsymbol{\Psi}}(t_m, p_m) = \hat{\boldsymbol{\Psi}}(t_m, \tilde{p}_m)$. This contradicts the local injectivity of $\hat{\boldsymbol{\Psi}}$ at $(\bar{t}, \bar{p})$. Hence, the claim holds true.

4th step: mapping $\hat{\boldsymbol{\Psi}} : [-\delta_2, T + \delta_2] \times \overline{\mathcal{N}_{\delta_2}(\widehat{\Omega})} \mapsto \mathbb{R}^{n+1}$ is injective and continuous, so it is an homeomorphism between the compact set $[-\delta_2, T + \delta_2] \times \overline{\mathcal{N}_{\delta_2}(\widehat{\Omega})}$ and its image. In the sequel, for any $0 < \delta \leq \delta_2$, we will use the notation $Q_\delta := \hat{\boldsymbol{\Psi}}((-\delta, T + \delta) \times \mathcal{N}_\delta(\widehat{\Omega}))$. Note that, because of (3.2) and $\delta_2 < \delta_1$, we have

$$(t, x) \in Q_\delta \Rightarrow x \in \Omega. \tag{3.5}$$

We have $\hat{\boldsymbol{\Psi}}([-\delta_2, T + \delta_2] \times \overline{\mathcal{N}_{\delta_2}(\widehat{\Omega})}) = \overline{Q_{\delta_2}}$ and, from (3.3) and the inverse function theorem, the set Q_{δ_2} is open, $\hat{\boldsymbol{\Psi}} : (-\delta_2, T + \delta_2) \times \mathcal{N}_{\delta_2}(\widehat{\Omega}) \mapsto Q_{\delta_2}$ is a diffeomorphism and $\hat{\boldsymbol{\Psi}}^{-1} \in [\mathcal{C}^1(\overline{Q_{\delta_2}})]^{n+1}$. Moreover, function $\hat{\boldsymbol{\Psi}}^{-1}$ has the form $\hat{\boldsymbol{\Psi}}^{-1}(t, x) = (t, \hat{\boldsymbol{P}}(t, x))$, where $\hat{\boldsymbol{P}} : \overline{Q_{\delta_2}} \mapsto \overline{\mathcal{N}_{\delta_2}(\widehat{\Omega})}$ is an extension of $\boldsymbol{P}$.

5th step: We define the velocity field $\boldsymbol{v} : \overline{Q_{\delta_2}} \mapsto \mathbb{R}^n$ by

$$\boldsymbol{v}(t, x) = \frac{\partial \hat{\boldsymbol{X}}}{\partial t}(t, \hat{\boldsymbol{P}}(t, x)) \ \left(= \frac{\partial \hat{\boldsymbol{X}}}{\partial t}(\hat{\boldsymbol{\Psi}}^{-1}(t, x)) \right). \tag{3.6}$$

From the regularity of $\hat{\boldsymbol{X}}$ and $\hat{\boldsymbol{\Psi}}^{-1}$ we deduce that $\boldsymbol{v} \in [\mathcal{C}(\overline{Q_{\delta_2}})]^n$, $D_x \boldsymbol{v}$ exists in Q_{δ_2} and

$$\frac{\partial v_k}{\partial x_i}(t, x) = \sum_{j=1}^{n} \frac{\partial^2 \hat{X}_k}{\partial p_j \partial t}(\hat{\boldsymbol{\Psi}}^{-1}(t, x)) \frac{\partial \hat{\boldsymbol{P}}_j}{\partial x_i}(t, x) \quad \forall (t, x) \in Q_{\delta_2}\,, \tag{3.7}$$

so $D_x \boldsymbol{v} \in \mathcal{C}(Q_{\delta_2}; M_{n \times n}(\mathbb{R}))$.

6th step: Since $\overline{Q} = \hat{\boldsymbol{\Psi}}([0, T] \times \overline{\widehat{\Omega}})$ and $0 < \epsilon < dist(\boldsymbol{X}([0, T] \times \overline{\widehat{\Omega}}), K)$, we have $dist(\overline{Q}, \overline{\mathcal{N}_\epsilon([0, T] \times K)}) > 0$. Because of the uniform continuity $\hat{\boldsymbol{\Psi}}$, we can pick δ_3, $0 < \delta_3 < \delta_2$, such that $\overline{Q_{\delta_3}} \cap \overline{\mathcal{N}_\epsilon([0, T] \times K)} = \emptyset$.

Moreover, since Q_{δ_3} is an open neighborhood of the compact set $\overline{Q}$, there exists δ_4, $0 < \delta_4 < \delta_3$ such that we can construct a cut-off function $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^{n+1})$ with $supp\, \phi \subset Q_{\delta_3}$ satisfying $\phi = 1$ in Q_{δ_4}. Note that ϕ vanishes in $\mathcal{N}_\epsilon([0, T] \times K)$ and on $[0, T] \times \partial\Omega$ (this set is disjoint from Q_{δ_3} because of (3.5)).

Let $\boldsymbol{w} : \overline{Q_{\delta_3}} \mapsto \mathbb{R}^n$ be defined by $\boldsymbol{w}(t, x) := \phi(t, x)\boldsymbol{v}(t, x)$. Since $\boldsymbol{w}$ has compact support contained in Q_{δ_3}, its extension by zero to the entire $\mathbb{R}^{n+1}$, still denoted $\boldsymbol{w}$, is continuous and $D_x \boldsymbol{w}$ exists and is continuous and bounded in $\mathbb{R}^{n+1}$. Hence $\boldsymbol{w}$ is globally Lipschitz-continuous with respect to x uniformly in t.

7th step: Let $\tilde{\boldsymbol{X}}$ be the (unique) solution of the initial value problem

$$\begin{cases} \dfrac{d\tilde{\boldsymbol{X}}}{dt} = \boldsymbol{w}(t, \tilde{\boldsymbol{X}}), \\ \tilde{\boldsymbol{X}}(0, p) = p. \end{cases} \tag{3.8}$$

The function $\tilde{\boldsymbol{X}}$ is defined in fact in the entire $\mathbb{R}^{n+1}$. Now we check that $\tilde{\boldsymbol{X}}$ fulfills all the assertions (a)-(g). From (3.6), the fact that $\boldsymbol{w} = \boldsymbol{v}$ in $Q_{\delta_4} = \hat{\boldsymbol{\Psi}}((-\delta_4, T + \delta_4) \times \mathcal{N}_{\delta_4}(\widehat{\Omega}))$ and (3.1), we deduce that $\hat{\boldsymbol{X}}$ satisfies

$$\begin{cases} \frac{\partial \hat{\boldsymbol{X}}}{\partial t}(t, p) = \boldsymbol{v}(t, \hat{\boldsymbol{X}}(t, p)) = \boldsymbol{w}(t, \hat{\boldsymbol{X}}(t, p)) \quad \forall (t, p) \in (-\delta_4, T + \delta_4) \times \mathcal{N}_{\delta_4}(\widehat{\Omega}), \\ \hat{\boldsymbol{X}}(0, p) = p \quad \forall p \in \mathcal{N}_{\delta_4}(\widehat{\Omega}). \end{cases}$$

Now, by the uniqueness of the solution of problem (3.8), we have

$$\hat{\boldsymbol{X}}(t, p) = \tilde{\boldsymbol{X}}(t, p) \quad \forall (t, p) \in (-\delta_4, T + \delta_4) \times \mathcal{N}_{\delta_4}(\widehat{\Omega}).$$

This, together with the fact that $\hat{\boldsymbol{X}}$ is an extension of $\boldsymbol{X}$, proves assertion (a).

Statement (b) is a consequence of classical results about the differentiability of the solution of an ODE with respect to initial conditions ([6]). Jacobi's formula yields to

$$det(D_p\tilde{\boldsymbol{X}})(t,p) = exp\left(\int_0^t \sum_{i=1}^n \frac{\partial w_i}{\partial x_i}(\tau, \tilde{\boldsymbol{X}}(\tau,p))\,d\tau\right) > 0 \quad \forall (t,p) \in \mathbb{R}^{n+1},$$

so we have statement (c).

Assertion (e) is obvious. The fact that $\boldsymbol{w}$ vanishes on $[0,T]\times\partial\Omega$ and in $\mathcal{N}_\epsilon([0,T]\times K)\supset[0,T]\times\mathcal{N}_\epsilon(K)$ together with (e) and the uniqueness of the solution of (3.8) imply assertions (f) and (g).

From classical results, $\tilde{\boldsymbol{X}}(t,\cdot):\mathbb{R}^n\mapsto\mathbb{R}^n$ is a diffeomorphism. If $p\in\Omega$, then the orbit $\{\tilde{\boldsymbol{X}}(t,p), t\in\mathbb{R}\}$ cannot traverse $\partial\Omega$ (again by an uniqueness argument). This, together with (e) implies that $\tilde{\boldsymbol{X}}(t,\cdot)(\Omega)\subset\Omega$. Given a time $\bar{t}\in[0,T]$ and point $\overline{x}\in\Omega$, we consider the solution $\tilde{\boldsymbol{Y}}$ of the ODE in (3.8) with initial condition $\tilde{\boldsymbol{Y}}(\bar{t})=\overline{x}$. In the same manner, the orbit $\{\tilde{\boldsymbol{Y}}(t), t\in\mathbb{R}\}\subset\Omega$, in particular $\tilde{\boldsymbol{Y}}(0)\in\Omega$ and we have $\tilde{\boldsymbol{X}}(\bar{t},\tilde{\boldsymbol{Y}}(0))=\overline{x}$, so $\tilde{\boldsymbol{X}}(t,\cdot):\Omega\mapsto\Omega$ is onto and therefore a bijection. Then assertion (d) holds true. □

Remark 3.1. Let $\tilde{\boldsymbol{\Psi}}$ be the extension of $\boldsymbol{\Psi}$ defined by

$$\tilde{\boldsymbol{\Psi}}(t,p) = (t, \tilde{\boldsymbol{X}}(t,p)).$$

$\tilde{\boldsymbol{\Psi}}$ is a homeomorphism from $[0,T]\times\overline{\Omega}$ onto itself and a diffeomorphism from $(0,T)\times\Omega$ onto itself. Besides, the mapping $\tilde{\boldsymbol{\Psi}}^{-1}$ has the form

$$\tilde{\boldsymbol{\Psi}}^{-1}(t,x) = (t, \tilde{\boldsymbol{P}}(t,x)), \tag{3.9}$$

where $\tilde{\boldsymbol{P}}(t,\cdot)=\tilde{\boldsymbol{X}}(t,\cdot)^{-1}$ for all $t\in[0,T]$, and the mapping $\tilde{\boldsymbol{P}}$ is an extension of $\boldsymbol{P}$.

Theorem 3.2. *We make the assumptions of Theorem 3.1. If we have further that $\boldsymbol{X}\in\mathcal{C}^1([0,T];[\mathcal{C}^2(\overline{\widehat{\Omega}}]^n)$ and $\widehat{\Omega}$ has a boundary of class $\mathcal{C}^2$, then we can build a mapping $\tilde{\boldsymbol{X}}$ satisfying all properties (a)-(g) in Theorem 3.1 and also $\tilde{\boldsymbol{X}}\in\mathcal{C}^1([0,T];[\mathcal{C}^2(\overline{\Omega}]^n)$.*

Proof. The proof is a modification of the proof of Theorem 3.1. In the first step, we take now a $\mathcal{C}^2$ extension operator E, so that we obtain $\hat{\boldsymbol{X}}\in\mathcal{C}^1([-T,2T];[\mathcal{C}^2(\overline{G})]^n)$ for all open bounded set $G\subset\mathbb{R}^n$. Recalling that $\hat{\boldsymbol{P}}(t,\cdot)=(\hat{\boldsymbol{X}}(t,\cdot))^{-1}$, the inverse function theorem gives the formula

$$D_x\hat{\boldsymbol{P}}(t,x) = \left(D_p\hat{\boldsymbol{X}}(t,\hat{\boldsymbol{P}}(t,x))\right)^{-1} = \left(D_p\hat{\boldsymbol{X}}(\hat{\boldsymbol{\Psi}}^{-1}(t,x))\right)^{-1} \quad \forall (t,x)\in Q_{\delta_2}.$$

Using this, it can be inferred that $D_x\hat{\boldsymbol{P}}\in\mathcal{C}^1(Q_{\delta_2})$. From this, the regularity of $\hat{\boldsymbol{X}}$ and (3.7), we deduce that $D_x^2\boldsymbol{v}$ exists and is continuous in Q_{δ_2} and then $D_x^2\boldsymbol{w}$ exists and is continuous and bounded in $\mathbb{R}^{n+1}$. Now, by applying classical results about the differentiability of the solution of an ODE with respect to parameters and initial conditions ([6]), we readily obtain that $\tilde{\boldsymbol{X}}\in\mathcal{C}^1([0,T];[\mathcal{C}^2(\overline{\Omega}]^n)$. □

Theorem 3.3. *We make the assumptions of Theorem 3.1. Let us further assume that $\boldsymbol{X}\in\mathcal{C}^1([0,T];[\mathcal{C}^2(\overline{\widehat{\Omega}}]^n)\cap\mathcal{C}^2([0,T];[\mathcal{C}^1(\overline{\widehat{\Omega}}]^n)$ and $\widehat{\Omega}$ has a boundary of class $\mathcal{C}^2$. Then we can build a mapping $\tilde{\boldsymbol{X}}$ satisfying all properties (a)-(g) in Theorem 3.1 and also $\tilde{\boldsymbol{X}}\in\mathcal{C}^1([0,T];[\mathcal{C}^2(\overline{\Omega}]^n)\cap\mathcal{C}^2([0,T];[\mathcal{C}^1(\overline{\Omega}]^n)$.*

Proof. Now, we take an operator $E\in\mathcal{L}([\mathcal{C}^1(\overline{\widehat{\Omega}})]^n,[\mathcal{C}^1_{K_1}(\Omega)]^n)\cap\mathcal{L}([\mathcal{C}^2(\overline{\widehat{\Omega}})]^n,[\mathcal{C}^2_{K_1}(\Omega)]^n)$ and use a $\mathcal{C}^2$ extension operator in the time variable, so that we obtain $\hat{\boldsymbol{X}}\in\mathcal{C}^1([-T,2T];[\mathcal{C}^2(\overline{G})]^n)\cap\mathcal{C}^2([-T,2T];[\mathcal{C}^1(\overline{G})]^n)$ for all open bounded set $G \subset \mathbb{R}^n$. Now we have $\hat{\boldsymbol{\Psi}}^{-1} \in \mathcal{C}^2(Q_{\delta_2})$ and we deduce that the partial derivatives $\frac{\partial v_i}{\partial t}$, $\frac{\partial^2 v_i}{\partial t \partial x_j}$, $\frac{\partial^2 v_i}{\partial x_j \partial t}$ exist and are continuous in Q_{δ_2}. Hence, the analogous property hold true for $\boldsymbol{w}$ in the entire $\mathbb{R}^{n+1}$. This regularity, the continuity of $D_x^2 \boldsymbol{w}$ (see Theorem 3.2), and classical results on ODEs imply that $\tilde{\boldsymbol{X}} \in \mathcal{C}^2([0,T]; [\mathcal{C}^1(\overline{\Omega}]^n)$. □

4. Reduction to a parabolic-elliptic problem with fixed parabolic spatial subdomain $\hat{\boldsymbol{\Omega}}$

In this section we assume the hypothesis about Ω, $\widehat{\Omega}$ and mapping $\boldsymbol{X}$ stated in Section 2. We also assume that $\widehat{\Omega}$ has a $\mathcal{C}^1$ boundary, adopt the assumptions of Theorem 2.1 regarding the data A^0 and f, and take for simplicity $\rho = 1$.

The aim of this section is to make a change of variables which transforms problem (2.3) into a parabolic-elliptic problem with fixed parabolic spatial domain $\widehat{\Omega}$.

We apply extension Theorem 3.1 (or 3.3 when we have enough regularity of $\widehat{\Omega}$ and $\boldsymbol{X}$) taking $K = \overline{\Omega_S}$, and we consider the mapping $\tilde{\boldsymbol{X}}$ given by the applied theorem.

Let A be the solution of problem (2.3). We introduce the function $\hat{A}$ defined by

$$\hat{A}(t,p) = A(t, \tilde{\boldsymbol{X}}(t,p)), \quad t \in [0,T], p \in \overline{\Omega}. \tag{4.1}$$

Note that variable p can be considered as a Lagrangian coordinate so that the field $\hat{A}$ can be considered as a material description of spatial field A. (We follow here the terminology of [11].)

Lemma 4.1. *Let $A \in L^2(0,T; L^2(\Omega)) \equiv L^2((0,T) \times \Omega)$ and let $\hat{A}$ be defined by (4.1). Then $\hat{A} \in L^2((0,T) \times \Omega)$. Besides,*

(i) *If $A \in L^2(0,T; H_0^1(\Omega))$, then $\hat{A} \in L^2(0,T; H_0^1(\Omega))$.*
(ii) *If $A \in L^2(0,T; H^1(\Omega))$ and $\frac{\partial A}{\partial t} \in L^2(Q)$, then $\hat{A} \in L^2(0,T; H^1(\widehat{\Omega})) \cap H^1(0,T; L^2(\widehat{\Omega})) \equiv H^1((0,T) \times \widehat{\Omega})$.*

This lemma can be easily proved by using [5, Proposition IX.6] and the properties of the mapping $\tilde{\boldsymbol{X}}$. The strong measurability $\hat{A} : (0,T) \mapsto H_0^1(\Omega)$ of item (i) follows because $\hat{A} \in L^2(0,T; L^2(\Omega))$ and $\hat{A}(t) \in H_0^1(\Omega)$ a.e. $t \in [0,T]$ (see also Proposition A.1.) Moreover, under the assumptions of the third assertion, using again [5, Proposition IX.6] we obtain

$$\nabla_x A(t,x) = [D_p \tilde{\boldsymbol{X}}(t,p)]^{-T} \nabla_p \hat{A}(t,p) \qquad \text{with } x = \tilde{\boldsymbol{X}}(t,p), \tag{4.2}$$

and

$$\frac{\partial A}{\partial t}(t,x) = \frac{\partial \hat{A}}{\partial t}(t,p) - \nabla_x A(t,x) \cdot \frac{\partial \boldsymbol{X}}{\partial t}(t,p) = \frac{\partial \hat{A}}{\partial t}(t,p) - [D_p \boldsymbol{X}(t,p)]^{-1} \frac{\partial \boldsymbol{X}}{\partial t}(t,p) \cdot \nabla_p \hat{A}(t,p). \tag{4.3}$$

Remark 4.1. For all $t \in [0,T]$, the mapping $z \in H_0^1(\Omega) \mapsto \widehat{z} = z \circ \tilde{\boldsymbol{X}}(t,\cdot) \in H_0^1(\Omega)$ is an isomorphism.

We denote $\tilde{J}(t,p) = det(D_p \tilde{\boldsymbol{X}})(t,p)$ and $J = \tilde{J}_{|[0,T] \times \widehat{\Omega}}$ $(= det(D_p \boldsymbol{X}))$. Note that J only involves $\boldsymbol{X}$ but not its extension.

Using Lemma 4.1, Remark 4.1, and equations (4.2), (4.3) and (2.2), and taking into account property (g) of Theorem 3.1 and the fact that $supp\, f \subset [0,T] \times \overline{\Omega}_{\mathrm{S}} = [0,T] \times K$, we deduce that function $\hat{A}$ is a solution of the following problem:

$$\begin{cases} \text{Find } \ \hat{A} \in L^2(0,T;H_0^1(\Omega)) \text{ with } \frac{\partial \hat{A}}{\partial t} \in L^2(0,T;L^2(\widehat{\Omega})) \quad \text{ s.t.} \\ \int_{\widehat{\Omega}} \widehat{\sigma} J \frac{\partial \hat{A}}{\partial t} \widehat{z}\, dp - \int_{\widehat{\Omega}} \widehat{\sigma} J (D_p \boldsymbol{X})^{-1} \frac{\partial \boldsymbol{X}}{\partial t} \cdot \nabla_p \hat{A}\, \widehat{z}\, dp + \int_{\Omega} \tilde{J} (D_p \tilde{\boldsymbol{X}})^{-T} \nabla_p \hat{A} \cdot (D_p \tilde{\boldsymbol{X}})^{-T} \nabla_p \widehat{z}\, dp \\ = \int_{\Omega_S} f \widehat{z}\, dp \quad \forall \widehat{z} \in H_0^1(\Omega) \quad a.e.\ t \in (0,T), \\ \hat{A}(0) = A^0 \quad \text{ in } \widehat{\Omega}. \end{cases} \tag{4.4}$$

This is the weak formulation of the parabolic-elliptic PDE

$$J\widehat{\sigma}\frac{\partial \hat{A}}{\partial t} - J\widehat{\sigma}(D_p\boldsymbol{X})^{-1}\frac{\partial \boldsymbol{X}}{\partial t}\cdot\nabla_p\hat{A} - \mathrm{div}_p(\tilde{J}(t)(D_p\tilde{\boldsymbol{X}}(t))^{-1}(D_p\tilde{\boldsymbol{X}}(t))^{-T}\nabla_p\hat{A}) = f \text{ in } (0,T)\times\Omega, \tag{4.5}$$

where $\widehat{\sigma}$ has been extended by zero outside $\widehat{\Omega}$ and the product $\widehat{\sigma}\frac{\partial \hat{A}}{\partial t}$ is understood to be zero outside $(0,T)\times\widehat{\Omega}$. This PDE has variable coefficients, but now the parabolicity domain is fixed. Note also that the first term of the left hand side involves now the function $\widehat{\sigma}$, which depends only on p. This PDE holds in the sense of $\mathcal{D}'(\Omega)$ a.e. $t \in [0,T]$.

5. Parabolic-elliptic problems with variable coefficients and fixed parabolicity domain $(0,T)\times\hat{\Omega}$

In this section we consider parabolic-elliptic problems of the following form:

Find a function $u = u(t,p)$ such that:

$$u \in L^2(0,T;H_0^1(\Omega))\,,\ \frac{\partial u}{\partial t} \in L^2(0,T;L^2(\widehat{\Omega}))\,, \tag{5.1}$$

$$\widehat{\sigma}\alpha\frac{\partial u}{\partial t} - \mathrm{div}_p(\beta\nabla_p u) + \widehat{\sigma}\vec{b}\cdot\nabla_p u + \widehat{\sigma}a u = F \text{ in } (0,T)\times\Omega, \tag{5.2}$$

$$u(0) = u^0 \quad \text{ in } \widehat{\Omega}. \tag{5.3}$$

Here Ω is a bounded domain of $\mathbb{R}^n$ with Lipschitz boundary, $\widehat{\Omega} \subset\subset \Omega$ a subdomain with Lipschitz boundary, $\beta = \beta(t,p)$ is a matrix-valued function defined in $[0,T]\times\overline{\Omega}$, $\beta = (\beta_{ij})_{1\le i,j\le n}$, $\vec{b} = \vec{b}(t,p)$ is vector valued, $\vec{b} = (b_i)_{1\le i\le n}$ and $\alpha = \alpha(t,p)$ and $a = a(t,p)$ are scalar valued. Functions $\vec{b}$, α and a are defined only in $[0,T]\times\overline{\widehat{\Omega}}$ and $\widehat{\sigma}$ has the same meaning as above. In particular, the terms containing $\widehat{\sigma}$ as a factor are understood to be zero outside $[0,T]\times\widehat{\Omega}$.

Throughout this section, $\nabla = \nabla_p$ and $div = div_p$. We make the following assumptions on the given coefficients: the matrix $\beta(t,p)$ is symmetric for all $(t,p) \in [0,T]\times\overline{\Omega}$, $\alpha \in \mathcal{C}^1([0,T];\mathcal{C}(\overline{\widehat{\Omega}}))$, the $\beta_{ij} \in \mathcal{C}^1([0,T];\mathcal{C}(\overline{\Omega}))$, the $b_i \in \mathcal{C}([0,T];\mathcal{C}(\overline{\widehat{\Omega}}))$, $a \in \mathcal{C}([0,T];\mathcal{C}(\overline{\widehat{\Omega}}))$,

$$\alpha_1 := \min_{(t,p)\in[0,T]\times\overline{\widehat{\Omega}}} \alpha(t,p) > 0\,, \tag{5.4}$$

and there exists a constant $\nu > 0$ such that

$$\nu|\xi|^2 \le \sum_{i=1}^n\sum_{j=1}^n \beta_{ij}(t,p)\xi_i\xi_j \quad \forall\xi\in\mathbb{R}^n \quad \forall(t,p)\in[0,T]\times\overline{\Omega}. \tag{5.5}$$

We denote

$$M := \max_{(t,p)\in[0,T]\times\overline{\Omega}}\left[\sum_{i=1}^n\sum_{j=1}^n|\beta_{ij}(t,p)|^2\right]^{1/2},\ M_1 := \max_{(t,p)\in[0,T]\times\overline{\Omega}}\left[\sum_{i=1}^n\sum_{j=1}^n|\partial_t\beta_{ij}(t,p)|^2\right]^{1/2}. \tag{5.6}$$

Poincaré inequality and (5.5) imply that there is a constant $\gamma > 0$ such that

$$\gamma \|z\|^2_{H^1(\Omega)} \leq \int_\Omega \sum_{i=1}^n \sum_{j=1}^n \beta_{ij}(t,p) \frac{\partial z}{\partial p_j} \frac{\partial z}{\partial p_i} \; dp \quad \forall z \in H^1_0(\Omega) \quad \forall t \in [0,T]. \tag{5.7}$$

Let us assume for the moment $F \in L^2(0,T;H^{-1}(\Omega))$ and $u^0 \in L^2(\widehat{\Omega})$. The weak form of problem (5.1), (5.2), (5.3) is given by

$$\int_{\widehat{\Omega}} \widehat{\sigma}\alpha(t)\partial_t u(t) z \; dp + \int_\Omega \beta(t)\nabla u(t) \cdot \nabla z \; dp + \int_{\widehat{\Omega}} \widehat{\sigma} \left[\vec{b}(t) \cdot \nabla u(t) + a(t)u(t)\right] z \; dp$$
$$= < F(t), z > \quad \forall z \in H^1_0(\Omega) \quad \text{a.e. } t \in [0,T] \tag{5.8}$$

together with (5.1) and (5.3). (In this section $< \cdot, \cdot > = < \cdot, \cdot >_{H^{-1}(\Omega), H^1_0(\Omega)}$.) From now on, in any inequality, C will denote a strictly positive constant not necessarily the same at each occurrence, depending only on the functions $\widehat{\sigma}$, α, β, $\vec{b}$, a, the final time T and domains Ω and $\widehat{\Omega}$ but independent of F and u^0. Later we will deal with problems depending on a small parameter ε. Constants denoted by C will be also independent of ε.

Theorem 5.1. *We make the above assumptions on α, β, $\vec{b}$, a, F and u^0. Problem (5.1), (5.8), (5.3) has at most one solution u and the following a priori estimate holds*

$$\sup_{t\in[0,T]} \left[\int_{\widehat{\Omega}} \widehat{\sigma}|u(t)|^2 \; dp\right]^{1/2} + \|u\|_{L^2(0,T;H^1(\Omega))} \leq C\left(\|u^0\|_{L^2(\widehat{\Omega})} + \|F\|_{L^2(0,T;H^{-1}(\Omega))}\right). \tag{5.9}$$

The proof follows essentially the same lines of the obtention of inequality (6.3) of [15]. We just point out that the integral $\int_{\widehat{\Omega}} \widehat{\sigma}\vec{b}(t) \cdot \nabla u(t)\, u(t) \; dp$ is bounded by using a Young inequality.

Now we derive an existence result for problem (5.1), (5.8), (5.3).

Theorem 5.2. *We make the assumptions of Theorem 5.1 and also*

$$u^0 \in H^1(\widehat{\Omega}), \tag{5.10}$$

$$i) F \in H^1(0,T;H^{-1}(\Omega)) \text{ or } ii) \; F \in L^2(0,T;L^2(\Omega)) \text{ and } supp(F) \subset [0,T] \times \overline{\widehat{\Omega}}. \tag{5.11}$$

Then problem (5.1), (5.8), (5.3) has a unique solution u. Besides $u \in L^\infty(0,T;H^1_0(\Omega))$ and satisfies the estimate

$$\|u\|_{L^\infty(0,T;H^1_0(\Omega))} + \|\partial_t u\|_{L^2(0,T;L^2(\widehat{\Omega}))} \leq C\left(\|u^0\|_{H^1(\widehat{\Omega})} + |||F|||\right), \tag{5.12}$$

where $|||F|||$ stands for $\|F\|_{H^1(0,T;H^{-1}(\Omega))}$ in case i) of assumption (5.11) and $\|F\|_{L^2(0,T;L^2(\widehat{\Omega}))}$ in case ii).

Proof. Let $\tilde{u}^0 \in H^1_0(\Omega)$ be an extension of u^0 to the whole Ω. We choose $\tilde{u}^0$ so that $\|\tilde{u}^0\|_{H^1(\Omega)} \leq C\|u^0\|_{H^1(\widehat{\Omega})}$. We introduce the following regularized problem, where $\varepsilon > 0$ is a small parameter: Find u^ε such that

$$u^\varepsilon \in L^2(0,T;H^1_0(\Omega)) \cap H^1(0,T;L^2(\Omega)), \tag{5.13}$$

$$(\varepsilon\chi_{\Omega^e} + \widehat{\sigma}\alpha)\frac{\partial u^\varepsilon}{\partial t} - \operatorname{div}(\beta\nabla u^\varepsilon) + \widehat{\sigma}\vec{b} \cdot \nabla u^\varepsilon + \widehat{\sigma} a u^\varepsilon = F \text{ in } (0,T) \times \Omega, \tag{5.14}$$

$$u^\varepsilon(0) = \tilde{u}^0 \quad \text{ in } \Omega. \tag{5.15}$$

We recall that $\Omega^e = \Omega \setminus \overline{\widehat{\Omega}}$. Of course, a weak form of this problem can be written just by adding the term $\varepsilon \int_{\Omega^e} \partial_t u^\varepsilon \ zdp$ in the left-hand side of (5.8). This regularized problem is a special case of the diffraction problem formed by equations (13.10), (13.2)-(13.4) and (13.11) of ([15, Chapter III]).

In order to pass to the limit as $\varepsilon \to 0^+$ we need to obtain estimates on u^ε independent of ε. To do this, we introduce the Galerkin approximation of the regularized problem. Let $\{\phi_n\}_{n\in\mathbb{N}}$ a "basis" of $H_0^1(\Omega)$ in the following sense: $\{\phi_n\}_{n\in\mathbb{N}}$ is a set of linear independent elements of $H_0^1(\Omega)$ whose linear span is dense in $H_0^1(\Omega)$. We consider the subspaces $V_N :=< \{\phi_1, \cdots, \phi_N\} >$. We take $u_N^0 \in V_N$ such that $u_N^0 \to \tilde{u}^0$ strongly in $H_0^1(\Omega)$ and $\|u_N^0\|_{H^1(\Omega)} \le C\|\tilde{u}^0\|_{H^1(\Omega)}$. The Galerkin approximation of the regularized problem reads: Find a function of the form $u_N^\varepsilon(t,p) = \sum_{j=1}^N u_{jN}^\varepsilon(t)\phi_j(p)$ such that

$$\int_\Omega (\varepsilon\chi_{\Omega^e} + \widehat{\sigma}\alpha(t))\partial_t u_N^\varepsilon(t)\ \phi_i\ dp + \int_\Omega \beta(t)\nabla u_N^\varepsilon(t)\cdot\nabla\phi_i\ dp + \int_{\widehat{\Omega}} \widehat{\sigma}\vec{b}(t)\cdot\nabla u_N^\varepsilon(t)\phi_i\ dp$$
$$+ \int_{\widehat{\Omega}} \widehat{\sigma}a(t)u_N^\varepsilon(t)\phi_i\ dp =< F(t), \phi_i > \quad i = 1,\dots,N \quad \text{a.e. } t\in[0,T], \tag{5.16}$$

$$u_N^\varepsilon(0) = u_N^0 \text{ in } \Omega \tag{5.17}$$

This problem has a unique solution in $H^1(0,T;V_N)$. To obtain a priori estimates, we multiply (5.16) by $\partial_t u_{iN}^\varepsilon(t)$ and sum up from $i=1$ to N. This yields an equation (like (5.16) itself with ϕ_i replaced by $\partial_t u_N^\varepsilon(t)$), which can be rewritten as

$$\int_\Omega (\varepsilon\chi_{\Omega^e} + \widehat{\sigma}\alpha(t))|\partial_t u_N^\varepsilon(t)|^2\ dp + \frac{1}{2}\frac{d}{dt}\int_\Omega \beta(t)\nabla u_N^\varepsilon(t)\cdot\nabla u_N^\varepsilon(t)\ dp =< F(t), \partial_t u_N^\varepsilon(t) > +$$
$$\frac{1}{2}\int_\Omega \partial_t\beta(t)\nabla u_N^\varepsilon(t)\cdot\nabla u_N^\varepsilon(t)\ dp - \int_{\widehat{\Omega}} \widehat{\sigma}\left[\vec{b}(t)\cdot\nabla u_N^\varepsilon(t) + a(t)u_N^\varepsilon(t)\right]\partial_t u_N^\varepsilon(t)\ dp.$$

We integrate in time from 0 to τ $(0 < \tau \le T)$. By using the inequality

$$|\widehat{\sigma}\vec{b}\cdot\nabla u_N^\varepsilon\,\partial_t u_N^\varepsilon| \le \frac{\alpha_1}{4}\widehat{\sigma}|\partial_t u_N^\varepsilon|^2 + \frac{1}{\alpha_1}\|\widehat{\sigma}\|_{L^\infty(\widehat{\Omega})}\|\ |\vec{b}|\ \|^2_{L^\infty((0,T)\times\widehat{\Omega}))}|\nabla u_N^\varepsilon|^2,$$

handling the term $\widehat{\sigma}a(t)u_N^\varepsilon(t)\partial_t u_N^\varepsilon(t)$ in a similar way and using (5.4), (5.6) and (5.7), we arrive to

$$\varepsilon\int_0^\tau\int_{\Omega^e} |\partial_t u_N^\varepsilon(t)|^2\ dp\ dt + \frac{\alpha_1}{2}\int_0^\tau\int_{\widehat{\Omega}} \widehat{\sigma}|\partial_t u_N^\varepsilon(t)|^2\ dp\ dt + \frac{\gamma}{2}\|u_N^\varepsilon(\tau)\|^2_{H^1(\Omega)}$$
$$\le \frac{M}{2}\|u_N^0\|^2_{H^1(\Omega)} + \int_0^\tau < F(t), \partial_t u_N^\varepsilon(t) > dt + C\int_0^\tau \|u_N^\varepsilon(t)\|^2_{H^1(\Omega)}\ dt. \tag{5.18}$$

In case i) of assumption (5.11), we integrate by parts in time and apply a Young inequality to get the bound

$$|\int_0^\tau < F(t), \partial_t u_N^\varepsilon(t) > dt| \le \frac{1}{2}\int_0^\tau \|u_N^\varepsilon(t)\|^2_{H^1(\Omega)}\ dt + \frac{1}{2}\int_0^\tau \|\frac{dF}{dt}(t)\|^2_{H^{-1}(\Omega)}\ dt$$
$$+ \frac{\gamma}{4}\|u_N^\varepsilon(\tau)\|^2_{H^1(\Omega)} + \frac{1}{\gamma}\|F(\tau)\|^2_{H^{-1}(\Omega)} + \frac{1}{2}\|u_N^0\|^2_{H^1(\Omega)} + \frac{1}{2}\|F(0)\|^2_{H^{-1}(\Omega)}. \tag{5.19}$$

By inserting this in (5.18), applying Gronwall's lemma and recalling the bound of $\|u_N^0\|_{H^1(\Omega)}$, we arrive to the estimate

$$\varepsilon \int_0^\tau \int_{\Omega^e} |\partial_t u_N^\varepsilon(t)|^2 \, dp \, dt + \frac{\alpha_1}{2} \int_0^\tau \int_{\widehat{\Omega}} \widehat{\sigma} |\partial_t u_N^\varepsilon(t)|^2 \, dp \, dt + \frac{\gamma}{4} \|u_N^\varepsilon(\tau)\|^2_{H^1(\Omega)}$$
$$\leq C \left(\|\tilde{u}^0\|^2_{H^1(\Omega)} + \|F\|^2_{H^1(0,T;H^{-1}(\Omega))} \right) \quad \forall \tau \in [0,T]. \tag{5.20}$$

In case ii) of assumption (5.11), we have

$$\int_0^\tau < F(t), \partial_t u_N^\varepsilon(t) > \, dt = \int_0^\tau \int_{\widehat{\Omega}} F(t) \partial_t u_N^\varepsilon(t) \, dp \, dt$$
$$\leq \frac{\alpha_1}{4} \int_0^\tau \int_{\widehat{\Omega}} \widehat{\sigma} |\partial_t u_N^\varepsilon(t)|^2 \, dp \, dt + \frac{1}{\alpha_1 \underline{\sigma}} \int_0^\tau \int_{\widehat{\Omega}} |F(t)|^2 \, dp \, dt. \tag{5.21}$$

The same steps as above yield to an analogous estimate to (5.20) but with $\|F\|_{L^2(0,T;L^2(\widehat{\Omega}))}$, $\alpha_1/4$ and $\gamma/2$ instead of $\|F\|_{H^1(0,T;H^{-1}(\Omega))}$, $\alpha_1/2$ and $\gamma/4$.

Thus, we have proved the following a priori estimates:

- u_N^ε is bounded in $L^\infty(0,T;H_0^1(\Omega))$,
- $\partial_t u_N^\varepsilon$ is bounded in $L^2(0,T;L^2(\widehat{\Omega}))$,
- $\sqrt{\varepsilon}\, \partial_t u_N^\varepsilon$ is bounded in $L^2(0,T;L^2(\Omega^e))$.

Therefore, for fixed ε, there exists $u^\varepsilon \in L^\infty(0,T;H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$ and a subsequence $\{u_{N_m}^\varepsilon\}$ such that $u_{N_m}^\varepsilon \rightharpoonup u^\varepsilon$ weakly-star in $L^\infty(0,T;H_0^1(\Omega))$ and $\partial_t u_{N_m}^\varepsilon \rightharpoonup \partial_t u^\varepsilon$ weakly in $L^2(0,T;L^2(\Omega))$. We take any fixed $i \in \mathbb{N}$, so that for $N_m \geq i$, $\phi_i \in V_{N_m}$. We multiply (5.16) by an arbitrary function in $\mathcal{D}((0,T))$, integrate in time, and pass to the limit as $m \to \infty$. This allows to show that u^ε satisfies an equation like (5.16) but with u_N^ε replaced by u^ε. This equation holds for all $i \in N$ a.e. $[0,T]$. Since the linear combinations of functions ϕ_i are dense in $H_0^1(\Omega)$, we deduce that u^ε is a weak solution of (5.14). Besides, the above convergences imply $u_{N_m}^\varepsilon(0) \rightharpoonup u^\varepsilon(0)$ weakly in $L^2(\Omega)$. From this, (5.17) and the fact that $u_N^0 \to \tilde{u}^0$ strongly in $H_0^1(\Omega)$, we get $u^\varepsilon(0) = \tilde{u}^0$. Therefore u^ε is the weak solution of the regularized problem (5.13)-(5.15). Besides, it is also possible to pass to the limit in estimate (5.20) (or in its analogue in case ii) of assumption (5.11)) (see, for instance, [5, Prop. III.5 & III.12]) to obtain

$$\varepsilon \int_0^\tau \int_{\Omega^e} |\partial_t u^\varepsilon|^2 \, dp \, dt + \frac{\alpha_1}{4} \int_0^\tau \int_{\widehat{\Omega}} \widehat{\sigma} |\partial_t u^\varepsilon|^2 \, dp \, dt + \frac{\gamma}{4} \|u^\varepsilon(\tau)\|^2_{H^1(\Omega)}$$
$$\leq C \left(\|\tilde{u}^0\|^2_{H^1(\Omega)} + |||F|||^2 \right) \quad \forall \tau \in [0,T]. \tag{5.22}$$

From the above estimate, we conclude that there exists $u \in L^\infty(0,T;H_0^1(\Omega))$ with $\partial_t u \in L^2(0,T;L^2(\widehat{\Omega}))$ and a sequence $\{\varepsilon_m\}_{m\in\mathbb{N}}$ converging to 0 such that

$$u^{\varepsilon_m} \rightharpoonup u \text{ weakly-star in } L^\infty(0,T;H_0^1(\Omega)), \tag{5.23}$$

$$\partial_t u^{\varepsilon_m} \rightharpoonup \partial_t u \text{ weakly in } L^2(0,T;L^2(\widehat{\Omega})), \tag{5.24}$$

$$\sqrt{\varepsilon_m}\,\partial_t u^{\varepsilon_m} \rightharpoonup 0 \text{ weakly in } L^2(0,T;L^2(\Omega^{\mathrm{e}})). \tag{5.25}$$

This allows to pass to the limit in the weak form of (5.14) to obtain that u satisfies (5.8). Besides, the above convergences imply $u^{\varepsilon_m}(0) \rightharpoonup u(0)$ weakly in $L^2(\widehat{\Omega})$. From this and (5.15) we deduce that $u(0) = u^0$ in $\widehat{\Omega}$. Therefore u is a solution of problem (5.1), (5.8), (5.3). Finally, passing to the limit as $\varepsilon_m \to 0$ in estimate (5.22), we obtain (5.12). □

Now we derive a regularity result for the problem (5.1), (5.8), (5.3). (We restrict ourselves to the case $a = 0$ because it will be sufficient for our goal.) Let us assume (5.10) and

$$F \in H^1(0,T;L^2(\Omega)) \text{ with } supp(F) \subset [0,T] \times \overline{\Omega^{\mathrm{e}}}. \tag{5.26}$$

Let $u^e \in H^1(\Omega^{\mathrm{e}})$ be the unique weak solution of

$$\begin{cases} -div(\beta(0)\nabla u^e) = F(0) & \text{in } \Omega^{\mathrm{e}}, \\ u^e_{|\Gamma_{\mathrm{e}}} = u^0_{|\Gamma_{\mathrm{e}}}, \\ u^e_{|\partial\Omega} = 0, \end{cases} \tag{5.27}$$

and $\tilde{u}^0 \in H^1_0(\Omega)$ be the extension of u^0 to the entire Ω, defined by

$$\tilde{u}^0 := \begin{cases} u^0 & \text{in } \widehat{\Omega}, \\ u^e & \text{in } \Omega^{\mathrm{e}}. \end{cases} \tag{5.28}$$

The condition

$$div(\beta(0)\nabla u^0) \in L^2(\widehat{\Omega}) \text{ and } \beta(0)\nabla u^0 \cdot \nu = \beta(0)\nabla u^e \cdot \nu \text{ on } \Gamma_e \tag{5.29}$$

imposes not only a regularity requirement on u^0 but also a compatibility condition between u^0 and $F(0)$. Provided that the $\beta_{ij}(0) \in \mathcal{C}^1(\overline{\Omega})$, condition $\tilde{u}^0 \in H^2(\Omega)$ implies (5.29) and the converse holds true if, in addition, Ω is convex or it has a boundary of class $\mathcal{C}^{1,1}$ ([10, Theorems 3.2.1.2 and 2.2.2.3]).

Theorem 5.3. *We make the assumptions of Theorem 5.1. We further assume: $a = 0$, $b_i \in \mathcal{C}^1([0,T];\mathcal{C}(\overline{\widehat{\Omega}}))$, $1 \le i \le n$, (5.10), (5.26) and (5.29). Then $u \in H^1(0,T;H^1_0(\Omega))$ with $\partial_t u \in L^\infty(0,T;L^2(\widehat{\Omega}))$. Besides, $\partial_t u$ satisfies*

$$\frac{d}{dt}\int_{\widehat{\Omega}} \widehat{\sigma}\alpha(t)\partial_t u(t) z \, dp + \int_{\Omega} \beta(t)(\nabla\partial_t u)(t)\cdot\nabla z \, dp$$
$$=< g(t), z > \forall z \in H^1_0(\Omega) \text{ in } \mathcal{D}'((0,T)), \tag{5.30}$$
$$(\widehat{\sigma}\alpha\partial_t u)(0) = \widehat{\sigma}\alpha(0) w^0 \text{ in } H^{-1}(\Omega), \tag{5.31}$$

where

$$g(t) := \partial_t F(t) + div(\partial_t\beta(t)\nabla u(t)) - \widehat{\sigma}\vec{b}(t)\cdot(\nabla\partial_t u)(t) - \widehat{\sigma}\partial_t\vec{b}(t)\cdot\nabla u(t)\,, \tag{5.32}$$
$$w^0 := \frac{1}{\widehat{\sigma}\alpha(0)} div(\beta(0)\nabla u^0) - \frac{1}{\alpha(0)}\vec{b}(0)\cdot\nabla u^0 \quad \text{in } \widehat{\Omega}. \tag{5.33}$$

Proof. We keep on using notation and partial results from the proof of Theorem 5.2. But we apply now the Galerkin method with a "basis" $\{\phi_n\}_{n\in\mathbb{N}}$ of $H_0^1(\Omega)$ such that $\phi_1 = \tilde{u}^0$. This allows us to take $u_N^0 = \tilde{u}^0$ for all $N \in \mathbb{N}$, since $\tilde{u}^0 \in H_0^1(\Omega)$ is an extension of u^0. Note that the initial condition (5.17) of the "approximate" regularized problem reduces to $u_N^\varepsilon(0) = \tilde{u}^0$; hence, it is exact and independent of both, N and ε.

The time regularity of the coefficients and F implies that $u_N^\varepsilon \in H^2(0,T;V_N)$. Thus (5.16) holds for all $t \in [0,T]$. Besides, we are allowed to differentiate with respect to time, which leads to

$$\int_\Omega (\varepsilon\chi_{\Omega^e} + \widehat{\sigma}\alpha(t))\partial_{tt}u_N^\varepsilon(t)\phi_i \, dp + \int_{\widehat{\Omega}} \widehat{\sigma}\partial_t\alpha(t)\ \partial_t u_N^\varepsilon(t)\phi_i \, dp + \int_\Omega \beta(t)(\nabla\partial_t u_N^\varepsilon)(t)\cdot\nabla\phi_i \, dp$$
$$+ \int_\Omega \partial_t\beta(t)\nabla u_N^\varepsilon(t)\cdot\nabla\phi_i \, dp + \int_{\widehat{\Omega}} \widehat{\sigma}\left[\vec{b}(t)\cdot(\nabla\partial_t u_N^\varepsilon)(t) + \partial_t\vec{b}(t)\cdot\nabla u_N^\varepsilon(t)\right]\phi_i \, dp = < \frac{dF}{dt}(t), \phi_i > . \quad (5.34)$$

We multiply (5.34) by $\partial_t u_{iN}^\varepsilon(t)$ and sum up from $i = 1$ to N. This yields an equation (like (5.34) itself with ϕ_i replaced by $\partial_t u_N^\varepsilon(t)$) which can be rewritten as

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (\varepsilon\chi_{\Omega^e} + \widehat{\sigma}\alpha(t))|\partial_t u_N^\varepsilon(t)|^2 \, dp + \int_\Omega \beta(t)(\nabla\partial_t u_N^\varepsilon)(t)\cdot(\nabla\partial_t u_N^\varepsilon)(t) \, dp$$
$$= < \frac{dF}{dt}(t), \partial_t u_N^\varepsilon)(t) > -\frac{1}{2}\int_{\widehat{\Omega}} \widehat{\sigma}\partial_t\alpha(t)|\partial_t u_N^\varepsilon(t)|^2 \, dp - \int_\Omega \partial_t\beta(t)\nabla u_N^\varepsilon(t)\cdot(\nabla\partial_t u_N^\varepsilon)(t) \, dp$$
$$- \int_{\widehat{\Omega}} \widehat{\sigma}\vec{b}(t)\cdot(\nabla\partial_t u_N^\varepsilon)(t)\ \partial_t u_N^\varepsilon(t) \, dp - \int_{\widehat{\Omega}} \widehat{\sigma}\partial_t\vec{b}(t)\cdot\nabla u_N^\varepsilon(t)\ \partial_t u_N^\varepsilon(t) \, dp =: \mathcal{T}_1 - \mathcal{T}_2 - \mathcal{T}_3 - \mathcal{T}_4 - \mathcal{T}_5$$

with evident notations. Using (5.7) and the bounds

$$|\mathcal{T}_1| \le \frac{\gamma}{4}\|\partial_t u_N^\varepsilon(t)\|^2_{H^1(\Omega)} + \frac{1}{\gamma}\left\|\frac{dF}{dt}(t)\right\|^2_{H^{-1}(\Omega)},$$
$$|\mathcal{T}_2| \le \frac{1}{2}\|\partial_t\alpha\|_{L^\infty((0,T)\times\widehat{\Omega}))}\int_{\widehat{\Omega}} \widehat{\sigma}|\partial_t u_N^\varepsilon(t)|^2 \, dp,$$
$$|\mathcal{T}_3| \le \frac{\gamma}{4}\|\partial_t u_N^\varepsilon(t)\|^2_{H^1(\Omega)} + \frac{M_1^2}{\gamma}\|u_N^\varepsilon(t)\|^2_{H^1(\Omega)},$$
$$|\mathcal{T}_4| \le \frac{\gamma}{4}\|\partial_t u_N^\varepsilon(t)\|^2_{H^1(\Omega)} + \frac{1}{\gamma}\|\widehat{\sigma}\|_{L^\infty(\widehat{\Omega})}\|\ |\vec{b}|\ \|^2_{L^\infty((0,T)\times\widehat{\Omega}))}\int_{\widehat{\Omega}} \widehat{\sigma}|\partial_t u_N^\varepsilon(t)|^2 \, dp,$$
$$|\mathcal{T}_5| \le \frac{1}{2}\|u_N^\varepsilon(t)\|^2_{H^1(\Omega)} + \frac{1}{2}\|\widehat{\sigma}\|_{L^\infty(\widehat{\Omega})}\|\ |\partial_t\vec{b}|\ \|^2_{L^\infty((0,T)\times\widehat{\Omega}))}\int_{\widehat{\Omega}} \widehat{\sigma}|\partial_t u_N^\varepsilon(t)|^2 \, dp,$$

and integrating in $[0,\tau]$ we obtain

$$\frac{1}{2}\int_\Omega (\varepsilon\chi_{\Omega^e} + \widehat{\sigma}\alpha(\tau))|\partial_t u_N^\varepsilon(\tau)|^2 \, dp + \frac{\gamma}{4}\int_0^\tau \|u_N^\varepsilon(t)\|^2_{H^1(\Omega)}dt \le \frac{1}{2}\int_\Omega (\varepsilon\chi_{\Omega^e} + \widehat{\sigma}\alpha(0))|\partial_t u_N^\varepsilon(0)|^2 \, dp$$
$$+ \frac{1}{\gamma}\int_0^\tau \left\|\frac{dF}{dt}(t)\right\|^2_{H^{-1}(\Omega)} dt + C\left[\int_0^\tau\int_{\widehat{\Omega}} \widehat{\sigma}|\partial_t u_N^\varepsilon(t)|^2 \, dp\, dt + \int_0^\tau \|u_N^\varepsilon(t)\|^2_{H^1(\Omega)} \, dt\right].$$

Now, taking into account (5.20) and (5.4), we obtain

$$\frac{\varepsilon}{2}\int_{\Omega^e} |\partial_t u_N^\varepsilon(\tau)|^2 \, dp + \frac{\alpha_1}{2}\int_{\widehat{\Omega}} |\partial_t u_N^\varepsilon(\tau)|^2 \, dp + \frac{\gamma}{4}\int_0^\tau \|u_N^\varepsilon(t)\|^2_{H^1(\Omega)} dt$$
$$\leq \frac{\varepsilon}{2}\int_{\Omega^e} |\partial_t u_N^\varepsilon(0)|^2 \, dp + \frac{1}{2}\int_{\widehat{\Omega}} \widehat{\sigma}\alpha(0)|\partial_t u_N^\varepsilon(0)|^2 \, dp + C\left(\|\tilde{u}^0\|^2_{H^1(\Omega)} + \|F\|^2_{H^1(0,T;H^{-1}(\Omega))}\right) \quad \forall \tau \in [0,T]. \tag{5.35}$$

A key point of the proof is to obtain a priori estimates for the first two terms of the right-hand side. To do this, we use the same argument as in [3, Theorem 5.5]. Taking $t = 0$ in equation (5.16), recalling that $u_N^\varepsilon(0) = \tilde{u}^0$ and using (5.26)-(5.29), we obtain

$$\int_\Omega (\varepsilon\chi_{\Omega^e} + \widehat{\sigma}\alpha(0))\partial_t u_N^\varepsilon(0)\phi_i \, dp = \int_{\widehat{\Omega}} \left(div(\beta(0)\nabla u^0) - \widehat{\sigma}\vec{b}(0)\cdot\nabla u^0\right)\phi_i \, dp. \tag{5.36}$$

Multiplying this equation by $\partial_t u_{iN}^\varepsilon(0)$ and summing up from $i = 1$ to N yields

$$\varepsilon\int_{\Omega^e} |\partial_t u_N^\varepsilon(0)|^2 \, dp + \int_{\widehat{\Omega}} \widehat{\sigma}\alpha(0)|\partial_t u_N^\varepsilon(0)|^2 \, dp = \int_{\widehat{\Omega}} \left(div(\beta(0)\nabla u^0) - \widehat{\sigma}\vec{b}(0)\cdot\nabla u^0\right)\partial_t u_N^\varepsilon(0) \, dp, \tag{5.37}$$

from which we easily deduce

$$\varepsilon\int_{\Omega^e} |\partial_t u_N^\varepsilon(0)|^2 \, dp + \int_{\widehat{\Omega}} \widehat{\sigma}\alpha(0)|\partial_t u_N^\varepsilon(0)|^2 \, dp \leq \frac{1}{\alpha_1\underline{\sigma}}\|div(\beta(0)\nabla u^0) - \widehat{\sigma}\vec{b}(0)\cdot\nabla u^0\|^2_{L^2(\widehat{\Omega})}. \tag{5.38}$$

This, together with (5.35), implies

$$\sqrt{\varepsilon}\|\partial_t u_N^\varepsilon\|_{L^\infty(0,T;L^2(\Omega^e))} + \|\partial_t u_N^\varepsilon\|_{L^\infty(0,T;L^2(\widehat{\Omega}))} + \|\partial_t u_N^\varepsilon\|_{L^2(0,T;H^1(\Omega))}$$
$$\leq C\left(\|\tilde{u}^0\|_{H^1(\Omega)} + \|div(\beta(0)\nabla u^0) - \widehat{\sigma}\vec{b}(0)\cdot\nabla u^0\|_{L^2(\widehat{\Omega})} + \|F\|_{H^1(0,T;H^{-1}(\Omega))}\right).$$

For fixed $\varepsilon > 0$, we can extract from $\{u_{N_m}^\varepsilon\}$ a subsequence, still denoted $\{u_{N_m}^\varepsilon\}$, such that $\partial_t u_{N_m}^\varepsilon \rightharpoonup \partial_t u^\varepsilon$ weakly in $L^2(0,T;H_0^1(\Omega))$ and weakly-star in $L^\infty(0,T;L^2(\Omega))$. Hence $u^\varepsilon \in H^1(0,T;H_0^1(\Omega)) \cap W^{1,\infty}(0,T;L^2(\Omega))$. Moreover, $\partial_t u^\varepsilon$ is bounded in $L^2(0,T;H_0^1(\Omega))$ and in $L^\infty(0,T;L^2(\widehat{\Omega}))$. Hence $u \in H^1(0,T;H_0^1(\Omega))$ with $\partial_t u \in L^\infty(0,T;L^2(\widehat{\Omega}))$ and we have

$$\partial_t u^{\varepsilon_m} \rightharpoonup \partial_t u \text{ weakly in } L^2(0,T;H_0^1(\Omega)) \text{ and weakly-star in } L^\infty(0,T;L^2(\widehat{\Omega})). \tag{5.39}$$

Let $(\cdot,\cdot)_\varepsilon$ denote the scalar product in $L^2(\Omega)$ defined by

$$(v,z)_\varepsilon = \int_\Omega (\varepsilon\chi_{\Omega^e} + \widehat{\sigma}\alpha(0))vz dp \quad v,z \in L^2(\Omega).$$

Note that $\sqrt{(v,v)_\varepsilon}$ is a norm in $L^2(\Omega)$, equivalent to the standard norm (but one of the equivalence constants depends on ε.) Let $P_N^\varepsilon : L^2(\Omega) \mapsto V_N$ be the orthogonal projection with respect to $(\cdot,\cdot)_\varepsilon$. Since $\partial_t u_N^\varepsilon(0) \in V_N$ and satisfies (5.36), we have that $\partial_t u_N^\varepsilon(0) = P_N^\varepsilon w_0^0$, where

$$w_0^0 = \begin{cases} w^0 & \text{in } \widehat{\Omega}, \\ 0 & \text{in } \Omega^e. \end{cases} \tag{5.40}$$

Hence, for fixed $\varepsilon > 0$, $\partial_t u_N^\varepsilon(0) \to w_0^0$ strongly in $L^2(\Omega)$ as $N \to \infty$.

Now we will obtain an equation, in weak form, which is satisfied by $\partial_t u^\varepsilon$. We first note that the sum of the first two terms of the left-hand side of equation (5.34) is equal to

$$\frac{d}{dt}\int_\Omega (\varepsilon\chi_{\Omega^e} + \widehat{\sigma}\alpha(t))\partial_t u_N^\varepsilon(t)\phi_i \; dp$$

and we rewrite (5.34) accordingly to this. We define

$$g_\varepsilon(t) := \partial_t F(t) + div(\partial_t\beta(t)\nabla u^\varepsilon(t)) - \widehat{\sigma}\vec{b}(t)\cdot(\nabla\partial_t u^\varepsilon)(t) - \widehat{\sigma}\partial_t\vec{b}(t)\cdot\nabla u^\varepsilon(t).$$

Note that $g_\varepsilon \in L^2(0,T;H^{-1}(\Omega))$. Using that $u_{N_m}^\varepsilon \rightharpoonup u^\varepsilon$ weakly in $H^1(0,T;H_0^1(\Omega))$ as $m \to \infty$ and the above convergence of $\partial_t u_N^\varepsilon(0)$ together with standard techniques (see, for instance, [7, Chapter XVIII]), we can pass to the limit as $m \to \infty$ to obtain that equation

$$\begin{aligned} &-\int_0^T\int_\Omega (\varepsilon\chi_{\Omega^e} + \widehat{\sigma}\alpha)\partial_t u^\varepsilon \partial_t\eta \; dp\; dt + \int_0^T\int_\Omega \beta\nabla(\partial_t u^\varepsilon)\cdot\nabla\eta \; dp \; dt \\ &\qquad = \int_0^T < g_\varepsilon(t), \eta(t) > dt + \int_\Omega (\varepsilon\chi_{\Omega^e} + \widehat{\sigma}\alpha(0))w_0^0\eta(0)\; dp \end{aligned} \tag{5.41}$$

holds for all functions η which are finite sums of the form $\eta(t,p) = \sum_i \psi_i(t)\phi_i(p)$ with the $\psi_i \in \mathcal{C}^1([0,T])$ such that $\psi_i(T) = 0$. The set of such functions is dense in the space $\{\eta \in L^2(0,T;H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega)); \eta(T) = 0\}$ ([15, Chapter II, Lemma 4.12]. Hence equation (5.41) holds for all η in this space. (In fact, this equation is a weak formulation which contains the PDE obtained by deriving (5.14) (in which $a = 0$) respect to time and the initial condition $\partial_t u^\varepsilon(0) = w_0^0$.)

We are allowed to pass to the limit as $\varepsilon_m \to 0$ in equation (5.41) by using the convergences (5.23)-(5.25) and (5.39). We thus obtain that $\partial_t u \in L^2(0,T;H^1(\Omega))$ satisfies

$$\begin{aligned} &-\int_0^T\int_{\widehat{\Omega}} \widehat{\sigma}\alpha\partial_t u\partial_t\eta \; dp \; dt + \int_0^T\int_\Omega \beta\nabla(\partial_t u)\cdot\nabla\eta \; dp \; dt = \int_0^T < g(t), \eta(t) > dt \\ &\qquad + \int_{\widehat{\Omega}} \widehat{\sigma}\alpha(0)w^0\eta(0)\; dp \quad \forall\eta \in L^2(0,T;H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega)) \text{ with } \eta(T) = 0. \end{aligned} \tag{5.42}$$

This equation fits into the framework of linear degenerate parabolic equations described in [21, Section III.3]. Owing to the equivalence stated at the beginning of page 115 of this reference, we deduce that $\partial_t u$ satisfies (5.30)-(5.31). □

Theorem 5.4. *We make the assumptions of Theorem 5.3. We further assume the $\beta_{ij} \in \mathcal{C}^2([0,T];\mathcal{C}(\overline{\Omega}))$, $F \in H^2(0,T;L^2(\Omega))$ and $w^0 \in H^1(\widehat{\Omega})$. Then $\frac{\partial^2 u}{\partial t^2} \in L^2(0,T;L^2(\widehat{\Omega}))$ and $u \in W^{1,\infty}(0,T;H_0^1(\Omega))$.*

Proof. Let us consider g and w^0 as given data. We introduce the problem

$$w \in L^2(0,T;H_0^1(\Omega)), \tag{5.43}$$

$$\frac{d}{dt}\int\limits_{\widehat{\Omega}} \widehat{\sigma}\alpha(t)w(t)zdp + \int\limits_{\Omega} \beta(t)\nabla w(t)\cdot\nabla z\; dp =$$

$$< g(t), z > \quad \forall z \in H_0^1(\Omega) \text{ in } \mathcal{D}'((0,T)), \tag{5.44}$$

$$(\widehat{\sigma}\alpha w)(0) = \widehat{\sigma}\alpha(0)w^0 \quad \text{ in } H^{-1}(\Omega). \tag{5.45}$$

Owing to [21, Propositions III.3.2, III.3.3], this problem has a unique solution. Since $\partial_t u \in L^2(0,T;H_0^1(\Omega))$ and satisfies (5.30) and (5.31), we have $\partial_t u = w$. Let us recall equation (5.32). The terms $\widehat{\sigma}\vec{b}\cdot(\nabla\partial_t u)$ and $\widehat{\sigma}\partial_t\vec{b}\cdot\nabla u$ are in $L^2(0,T;L^2(\Omega))$ and vanish outside $[0,T]\times\widehat{\Omega}$, and, because of the smoothness of β and F, the terms $\frac{dF}{dt}$ and $div(\partial_t\beta\nabla u)$ belong to $H^1(0,T;H^{-1}(\Omega))$. In virtue of Theorem 5.2 the problem

$$\tilde{w} \in L^2(0,T;H_0^1(\Omega)), \text{ with } \partial_t\tilde{w} \in L^2(0,T;L^2(\widehat{\Omega}))\,, \tag{5.46}$$

$$\int\limits_{\widehat{\Omega}} \widehat{\sigma}\alpha(t)\partial_t\tilde{w}(t)zdp + \int\limits_{\widehat{\Omega}} \widehat{\sigma}\partial_t\alpha(t)\tilde{w}(t)zdp + \int\limits_{\Omega} \beta(t)\nabla\tilde{w}(t)\cdot\nabla_p z\; dp$$

$$=< g(t), z > \quad \forall z \in H_0^1(\Omega) \text{ in } \mathcal{D}'((0,T)), \tag{5.47}$$

$$\tilde{w}(0) = w^0 \quad \text{ in } \widehat{\Omega}, \tag{5.48}$$

has a unique solution $\tilde{w}$ and, besides, $\tilde{w} \in L^\infty(0,T;H_0^1(\Omega))$. Clearly $\tilde{w}$ is also a solution of problem (5.43), (5.44), (5.45). Hence $\tilde{w} = w$ and since we still have $\frac{\partial u}{\partial t} = w$, we obtain the desired result. □

6. Regularity results obtained through the Lagrangian approach

Problem (4.4) fits into the framework of Section 5, with:
$\alpha = J$, $\vec{b} = -J(D_p\boldsymbol{X})^{-1}\frac{\partial \boldsymbol{X}}{\partial t}$, $\beta = \tilde{J}(D_p\tilde{\boldsymbol{X}})^{-1}(D_p\tilde{\boldsymbol{X}})^{-T}$, $a = 0$, $F = f$, $u^0 = A^0$ and $u = \hat{A}$. Note that $D_p\tilde{\boldsymbol{X}}(0) = I$ (the identity matrix) so $\alpha(0) = J(0) = 1$, $\beta(0) = I$. Hence $\tilde{u}^0 = \tilde{A}^0$ and the compatibility condition (5.29) is exactly (2.5). Besides $w^0 = \frac{1}{\hat{\sigma}}\triangle_p A^0 + \frac{\partial \boldsymbol{X}}{\partial t}(0)\cdot\nabla_p A^0$.

Lemma 6.1. *We make the hypothesis about Ω, $\widehat{\Omega}$ and the mapping $\boldsymbol{X}$ stated in Section 2. We assume further that $\widehat{\Omega}$ has a boundary of class $\mathcal{C}^2$, $\boldsymbol{X} \in \mathcal{C}^2([0,T];[\mathcal{C}^1(\overline{\widehat{\Omega}})]^n) \cap \mathcal{C}^1([0,T];[\mathcal{C}^2(\overline{\widehat{\Omega}})]^n)$, $\rho = 1$, $f \in H^2(0,T;L^2(\Omega_{\mathrm{S}}))$, $A^0 \in H^1(\widehat{\Omega})$, and that both compatibility conditions (2.5) and*

$$\frac{1}{\hat{\sigma}}\triangle_p A^0 + \frac{\partial \boldsymbol{X}}{\partial t}(0)\cdot\nabla_p A^0 \in H^1(\widehat{\Omega}) \tag{6.1}$$

hold true. Then we have

$$\hat{A} \in W^{1,\infty}(0,T;H_0^1(\Omega)) \textit{ and } \hat{A} \in H^2(0,T;L^2(\widehat{\Omega})). \tag{6.2}$$

Proof. We construct the extension $\tilde{\boldsymbol{X}}$ of $\boldsymbol{X}$ given by Theorem 3.3, so $\tilde{\boldsymbol{X}} \in \mathcal{C}^1([0,T];[\mathcal{C}^2(\overline{\Omega}]^n) \cap \mathcal{C}^2([0,T];[\mathcal{C}^1(\overline{\Omega}]^n)$. The result follows from Theorem 5.4 by noting that the coefficient functions appearing in (4.4) have the required smoothness, the compatibility condition (5.29) holds true and assumption (6.1) is in fact $w^0 \in H^1(\widehat{\Omega})$. □

By reversing the change of variables (4.1) and recalling equation (3.9), we can write

$$A(t,x) = \hat{A}(\tilde{\boldsymbol{\Psi}}^{-1}(t,x)) = \hat{A}(t,\tilde{\boldsymbol{P}}(t,x)), \quad t \in [0,T], x \in \overline{\Omega}. \tag{6.3}$$

Theorem 6.1. *We make the assumptions of Lemma 6.1. If Ω is convex or its boundary is of class $\mathcal{C}^{1,1}$, then $A \in H^2(Q)$.*

Proof. The regularity of $\tilde{\boldsymbol{X}}$ implies that $\tilde{\boldsymbol{\Psi}} \in [\mathcal{C}^2([0,T] \times \overline{\Omega})]^{n+1}$ due to Theorem 3.3. From Theorem 2.3 and Remark 2.2, we have $A(t) \in H^2(\Omega)$ for all $t \in [0,T]$. Recalling equation (4.1) and applying (for each t) the formula of change of variables ([5, Proposition IX.6]), we deduce that $\hat{A}(t) \in H^2(\Omega)$ and $\|\hat{A}(t)\|_{H^2(\Omega)} \leq C\|A(t)\|_{H^2(\Omega)}$ for all $t \in [0,T]$. This, together with Proposition A.1, gives $\hat{A} \in L^2(0,T;H^2(\Omega))$. This regularity, together with (6.2) imply that $\hat{A} \in H^2((0,T) \times \widehat{\Omega})$. Now, applying (in both space and time) the formula of change of variables in equation (6.3) and using that $\psi^{-1} \in [\mathcal{C}^2(\overline{Q})]^{n+1}$, we deduce that $A \in H^2(Q)$. $\square$

7. Two examples

Let us notice that $H^3(\Omega)$ regularity in space cannot be obtained because of the discontinuity of σ across ∂Q.

In fact, we have the following counterexample in 1D. Let $\Omega = (0,1)$, $\widehat{\Omega} = (a,b)$ where $0 < a < b < 1$, $\boldsymbol{X}(t,p) = p$ (hence $\Omega_t = \widehat{\Omega}$ for all $t \in [0,T]$), $\widehat{\sigma} = 1$ in $\widehat{\Omega}$, $\rho = 1$ and $f = 0$. Problem (1.1) reduces to

$$\begin{cases} \chi_{\widehat{\Omega}} \dfrac{\partial A}{\partial t} - \dfrac{\partial^2 A}{\partial x^2} = 0 & \text{in } (0,T) \times \Omega, \\ A(t,0) = A(t,1) = 0\,, & t \in (0,T), \\ A(0,x) = A^0(x)\,, & a < x < b. \end{cases} \tag{7.1}$$

We can find the explicit expression of $A(t,x)$ in the elliptic regions in terms of $A(t,a)$ or $A(t,b)$. Next, from the continuity of $A(t,\cdot)$ and $\frac{\partial A}{\partial x}(t,\cdot)$ at $x = a$ and $x = b$, we obtain Robin boundary conditions at $x = a$ and $x = b$. Hence the restriction of A to $(0,T) \times (a,b)$ is the unique solution of the following problem

$$\begin{cases} \frac{\partial A}{\partial t} - \frac{\partial^2 A}{\partial x^2} = 0 & \text{in } (0,T) \times (a,b), \\ -a\frac{\partial A}{\partial x}(t,a) + A(t,a) = 0\,, & t \in (0,T), \\ (1-b)\frac{\partial A}{\partial x}(t,b) + A(t,b) = 0\,, & t \in (0,T), \\ A(0,x) = A^0(x)\,, & a < x < b. \end{cases} \tag{7.2}$$

Let us take $b = 1 - a$ with $0 < a < \frac{1}{2}$ and consider the eigenvalue problem

$$\begin{cases} -\frac{d^2\phi}{dx^2} = \lambda\phi & \text{in } (a,b), \\ -a\frac{d\phi}{dx}(a) + \phi(a) = 0\,, \\ a\frac{d\phi}{dx}(b) + \phi(b) = 0\,. \end{cases}$$

The results given in [16, Appendix B.1] motivate the choice $a = \frac{2}{\pi+4}$. For this value, we have that

$$\phi(x) = \sin(\frac{\pi+4}{2}x - 1) + \cos(\frac{\pi+4}{2}x - 1)$$

is an eigenfunction, with associate eigenvalue $\lambda = (\frac{\pi+4}{2})^2$ and $\phi(a) = \phi(b) = 1$. Thus, for the initial condition

$$A^0(x) = \phi(x), \quad a < x < b,$$

the solution of problem (7.2) is $A(t,x) = e^{-\lambda t}\phi(x)$ and that of problem (7.1) is

$$A(t,x) = \begin{cases} \dfrac{1}{a}e^{-\lambda t}x & \text{for } 0 \le x \le a, \\ e^{-\lambda t}\phi(x) & \text{for } a \le x \le b, \\ \frac{1}{a}e^{-\lambda t}(1-x) & \text{for } b \le x \le 1. \end{cases}$$

It is easy to see that the equation appearing in compatibility condition (2.5) reduces to the above Robin boundary conditions written for A^0. Hence (2.5) is fulfilled. Condition (6.1) is also satisfied and we also have that $\widehat{\sigma}^{-1}\frac{\partial^2 A^0}{\partial x^2} = -\lambda\phi$, which is the restriction of an entire function. On the other hand, $\frac{\partial^2 A}{\partial x^2}(t,\cdot)$ is discontinuous across $x = a$ and $x = b$, hence $A(t,\cdot) \notin H^3(\Omega)$ for all $t \in (0,T)$.

The following example illustrates the main features considered in this paper. Let $0 < \alpha < a < b < \beta < 1$ be given, $\Omega = (0,1)$, $\widehat{\Omega} = (a,b)$, $\boldsymbol{X}(t,p) = p$, $\widehat{\sigma} = \chi_{(a,(a+b)/2)} + 2\chi_{((a+b)/2),b)}$ (for instance), $\rho = 1$ and $f(t,x) = \hat{f}(t)(\chi_{(0,\alpha)}(x) + \chi_{(\beta,1)})(x)$, where $\hat{f} : [0,T] \mapsto \mathbb{R}$ is given. Problem (1.1) reduces to

$$\begin{cases} \widehat{\sigma}(x)\dfrac{\partial A}{\partial t} - \dfrac{\partial^2 A}{\partial x^2} = \hat{f}(t)(\chi_{(0,\alpha)}(x) + \chi_{(\beta,1)}(x)) & \text{in } (0,T)\times\Omega, \\ A(t,0) = A(t,1) = 0\,, & t \in (0,T), \\ A(0,x) = A^0(x)\,, & a < x < b. \end{cases} \tag{7.3}$$

Proceeding as in the previous example, we find that the restriction of A to $(0,T)\times(a,b)$ is the unique solution of the problem

$$\widehat{\sigma}(x)\frac{\partial A}{\partial t} - \frac{\partial^2 A}{\partial x^2} = 0 \qquad \text{in } (0,T)\times(a,b), \tag{7.4}$$

$$-a\frac{\partial A}{\partial x}(t,a) + A(t,a) = \frac{1}{2}\alpha^2\hat{f}(t)\,, \qquad t \in (0,T), \tag{7.5}$$

$$(1-b)\frac{\partial A}{\partial x}(t,b) + A(t,b) = \frac{1}{2}(1-\beta)^2\hat{f}(t)\,, \qquad t \in (0,T), \tag{7.6}$$

$$A(0,x) = A^0(x)\,, a < x < b. \tag{7.7}$$

Let us consider also the PDE

$$\frac{\partial}{\partial t}(\widehat{\sigma}(x)A) - \frac{\partial^2 A}{\partial x^2} = 0 \qquad \text{in } (0,T)\times(a,b). \tag{7.8}$$

If $A^0 \in L^2(a,b)$ and $\hat{f} \in L^2(0,T)$, the problem (7.8), (7.5)-(7.7) has a unique solution in $L^2(0,T;H^1(a,b)) \cap \mathcal{C}([0,T];L^2(a,b))$. In fact, the solution can be written as a series in terms of the eigenfunctions of the problem

$$\begin{cases} -\frac{d^2\phi}{dx^2} = \lambda\widehat{\sigma}(x)\phi & \text{in } (a,b), \\ -a\frac{d\phi}{dx}(a) + \phi(a) = 0\,, \\ (1-b)\frac{d\phi}{dx}(b) + \phi(b) = 0\,, \end{cases}$$

The eigenvalues λ_n are real and strictly positive, they can be arranged in an increasing sequence such that $\lim_{m\to\infty}\lambda_m = +\infty$, and there exists a Hilbert basis of $L^2(a,b)$, $\{\phi_m\}_{m=1}^{\infty}$ formed by eigenfunctions. Here $L^2(a,b)$ is endowed with the scalar product $(v,z)_{\widehat{\sigma}} = \int_a^b \widehat{\sigma}(x)v(x)z(x)\,dx$. Besides, $\{\lambda_m^{-1/2}\phi_m\}_{m=1}^{\infty}$ is a Hilbert basis of $H^1(a,b)$ for the scalar product $a(v,z) = \int_a^b v'(x)z'(x) + \frac{1}{a}v(a)z(a) + \frac{1}{1-b}v(b)z(b)$, which induced norm is equivalent to the norm of $H^1(a,b)$. We have the expansion

$$A(t,p) = A_1(t,p) + A_2(t,p) = \sum_{m=1}^{\infty}\mu_m(t)\phi_m(x) + \sum_{m=1}^{\infty}\nu_m(t)\phi_m(x) \tag{7.9}$$

with $\mu_m(t) = (A^0, \phi_m)_{\widehat{\sigma}}\, e^{-\lambda_m t}$ and $\nu_m(t)$ the solution of

$$\frac{d\nu_m}{dt} + \lambda_m \nu_m = \hat{f}(t) < \ell, \phi_m >, \quad \nu_m(0) = 0, \tag{7.10}$$

where $\ell \in (H^1(a,b))'$ is defined by $< \ell, z >= \frac{1}{2}(\frac{\alpha^2}{a} z(a) + \frac{(1-\beta)^2}{1-b} z(b))$. Due to the above properties of the ϕ_m, we have $\sum_{m=1}^{\infty} \lambda_m^{-1} | < \ell, \phi_m > |^2 \leq C \|\ell\|^2_{(H^1(a,b))'}$. Well-known estimates imply that the first series of (7.9) converges in $L^2(0,T;H^1(a,b)) \cap \mathcal{C}([0,T];L^2(a,b))$. It is also known that

$$\max_{[0,T]} |\nu_m(t)|^2 \leq \frac{1}{2\lambda_m} | < \ell, \phi_m > |^2 \int_0^T |\hat{f}(s)|^2 \, ds, \tag{7.11}$$

$$\int_0^T |\nu_m(t)|^2 \, dt \leq \frac{1}{\lambda_m^2} | < \ell, \phi_m > |^2 \int_0^T |\hat{f}(s)|^2 \, ds. \tag{7.12}$$

This estimates imply that the second series converges in $L^2(0,T;H^1(a,b)) \cap \mathcal{C}([0,T];L^2(a,b))$, too.

Assume now $A^0 \in H^1(a,b)$ and $\hat{f} \in H^1(0,T)$. Let $\Psi(x) := \frac{(1-\beta)^2 - \alpha^2}{2} \hat{f}(0) x + \frac{\alpha^2}{2} \hat{f}(0)$. The transformation $A^0 \leftarrow A^0 - \Psi$, $\hat{f} \leftarrow \hat{f} - \hat{f}(0)$ allows us to assume $\hat{f}(0) = 0$. Assumption $A^0 \in H^1(a,b)$ is equivalent to $\sum_{m=1}^{\infty} |(A^0, \phi_m)_{\widehat{\sigma}}|^2 \lambda_m < \infty$ and (since $\frac{1}{2}(1 - e^{-\lambda_1 T}) \leq \lambda_m \int_0^T e^{-2\lambda_m t} \, dt < \frac{1}{2}$) this in turn is equivalent to $\partial_t A_1 \in L^2(0,T;L^2(a,b))$. On the other hand, function ν_m' satisfies a differential equation like (7.10) but with right-hand side $\hat{f}'(t) < \ell, \phi_m >$. Besides, $\hat{f}(0) = 0$ and $\nu_m(0) = 0$ imply $\nu_m'(0) = 0$. Thus we have for ν_m' analogous estimates to (7.11) and (7.12) with $\hat{f}'$ in lieu of $\hat{f}$. Hence we have $\partial_t A_2 = \sum_{m=1}^{\infty} \nu_m'(t) \phi_m(x)$, where the series converges in $L^2(0,T;H^1(a,b)) \cap \mathcal{C}([0,T];L^2(a,b))$. Therefore $\partial_t A \in L^2(0,T;L^2(a,b))$ and A is the solution of problem (7.4) -(7.7).

Now we further assume $A^0 \in H^2(a,b)$ and the compatibility condition (2.5). This reduces to the Robin boundary conditions (7.5) and (7.6) written for A^0. Since $\hat{f}(0) = 0$, we have

$$(-\frac{1}{\widehat{\sigma}} \frac{\partial^2 A^0}{\partial x^2}, \phi_m)_{\widehat{\sigma}} = a(A^0, \phi_m) = \lambda_m (A^0, \phi_m)_{\widehat{\sigma}}, \tag{7.13}$$

so $\sum_{m=1}^{\infty} |(A^0, \phi_m)_{\widehat{\sigma}}|^2 \lambda_m^2 < \infty$. This implies that $\partial_t A_1 = \sum_{m=1}^{\infty} \mu_m'(t) \phi_m(x)$, where the series converges in $L^2(0,T;H^1(a,b)) \cap \mathcal{C}([0,T];L^2(a,b))$. Derivation with respect to time of the weak formulation of problem (7.8), (7.5)-(7.7) is allowed. Besides

$$\frac{\partial A}{\partial t}(0) = \sum_{m=1}^{\infty} \mu_m'(0) \phi_m = -\sum_{m=1}^{\infty} \lambda_m \mu_m(0) \phi_m = -\sum_{m=1}^{\infty} \lambda_m (A^0, \phi_m)_{\widehat{\sigma}} \phi_m = \frac{1}{\widehat{\sigma}} \frac{\partial^2 A^0}{\partial x^2}.$$

Thus, $w = \frac{\partial A}{\partial t}$ is the unique weak solution of the problem analogous to (7.8), (7.5)-(7.7) with $\hat{f}'$ instead of $\hat{f}$ and initial condition $w(0) = \frac{1}{\widehat{\sigma}} \frac{\partial^2 A^0}{\partial x^2}$.

Therefore, if we further assume $\hat{f} \in H^2(0,T)$ and $\frac{1}{\widehat{\sigma}} \frac{\partial^2 A^0}{\partial x^2} \in H^1(a,b)$, we will have $\frac{\partial^2 A}{\partial t^2} \in L^2(0,T;L^2(a,b)) \equiv L^2((0,T) \times (a,b))$. All the considerations developed in this example agree with the results of the paper.

Acknowledgments

This work has been partially supported by FEDER/Ministerio de Ciencia, Innovación y Universidades – Agencia Estatal de Investigación under the research project MTM2017-86459-R and Xunta de Galicia (Spain) under grant 2017 GRC GI-1563. Thanks are also given to the anonymous referee for some helpful suggestions.

Appendix A. An auxiliary result

Proposition A.1. *Let X be a reflexive and separable Banach space and let V be a Banach space such that $X \subset V$ with continuous and dense injection. If $f : (0,T) \mapsto V$ is measurable and $f(t) \in X$ for a.e. $t \in [0,T]$, then $f : (0,T) \mapsto X$ (defined a.e.) is measurable.*

Proof. Due to Pettis theorem and the separability of X, it is enough to prove that $f : (0,T) \mapsto X$ is weakly measurable. We have $V' \subset X'$ with continuous injection. The reflexivity of X implies that V' is dense in X'. Thus, given $\ell \in X'$, there exists a sequence $\{\ell_n\} \subset V'$ converging to ℓ in X'-strong. Since real-valued functions $< \ell_n, f(\cdot) >_{V',V} = < \ell_n, f(\cdot) >_{X',X}$ are measurable and converge a.e. to function $< \ell, f(\cdot) >_{X',X}$, this will be measurable. $\square$

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