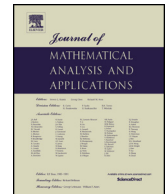




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Characterization of real inner product spaces by Hermite–Hadamard type orthogonalities

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ABSTRACT

In this study, we provide some new characterizations of the real inner product spaces using the notion of Hermite–Hadamard (HH) type orthogonalities and by considering their relationships with Birkhoff–James orthogonality. In addition, we investigate the classes of linear mappings that preserve two special types of these orthogonalities. In particular, we show that every HH-I-orthogonality preserving linear mappings is necessarily a scalar multiple of a linear isometry. Finally, we present some other characterizations of real inner product spaces in terms of HH-P- and HH-I-orthogonality preserving mappings.

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1. Introduction and preliminaries

One of the best known concepts in studies of the geometry of normed linear spaces is the notion of orthogonality. Extensive studies have investigated this concept and its connection with several geometric properties of normed linear spaces, such as strict convexity and smoothness. Further details regarding the geometry of normed linear spaces and other information about the basic theory of real normed linear spaces can be found in previous studies by [14,23] and the references therein.

Let $(X, \|\cdot\|)$ be a real normed linear space and let $S_X = \{x \in X : \|x\| = 1\}$ be the unit sphere of X . If the norm comes from an inner product $\langle \cdot, \cdot \rangle$, there is one natural orthogonality relationship: $x \perp y \Leftrightarrow \langle x, y \rangle = 0$. However, there is no unique way to define the notion of orthogonality in general normed linear spaces. Indeed, numerous notions of orthogonality in normed linear spaces have been introduced and studied in the general case. The first orthogonality type was introduced by Roberts [24]. A vector $x \in X$ is said to be Roberts orthogonal to a vector $y \in X$, which can be abbreviated as $x \perp_R y$, if the equality $\|x - \lambda y\| = \|x + \lambda y\|$

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holds for any real number λ . Birkhoff–James orthogonality is one of the most important orthogonality types, where it was introduced by Birkhoff [5] and then developed by James [17,18]. A vector $x \in X$ is said to be orthogonal to a vector $y \in X$ in the sense of Birkhoff–James, which can be written as $x \perp_B y$, if the inequality $\|x\| \leq \|x + \lambda y\|$ holds for all $\lambda \in \mathbb{R}$. The geometrical interpretation is that the line passing through x in the direction of y supports (at the point x) the ball centered at 0 and with radius $\|x\|$. Moreover, James introduced the following orthogonality relationships [16].

1. Pythagorean orthogonality:

$$x \perp_P y \text{ if and only if } \|x + y\|^2 = \|x\|^2 + \|y\|^2 \quad (x, y \in X).$$

2. Isosceles orthogonality:

$$x \perp_I y \text{ if and only if } \|x - y\| = \|x + y\| \quad (x, y \in X).$$

We note that Pythagorean and isosceles orthogonalities are both symmetric, but this is not true for Birkhoff–James orthogonality. In addition, it is known that Birkhoff–James orthogonality is homogeneous, whereas Pythagorean and isosceles orthogonalities are not homogeneous. We also recall that Pythagorean, isosceles, and Birkhoff–James orthogonalities have the existence property. In particular, considering the existence property of Birkhoff–James orthogonality, we recall the following result given by [17].

Lemma 1.1. [17, Corollary 2.2] *Let X be a real normed linear space and let $x, y \in X$ with $x \neq 0$. Then, a real number r exists such that $x \perp_B (rx + y)$.*

Let $(X, \langle \cdot, \cdot \rangle)$ be a real inner product space and let $m \in \mathbb{N}$. If α_i , β_i , and γ_i ($i = 1, \dots, m$) are real numbers that satisfy:

$$\sum_{i=1}^m \alpha_i \beta_i^2 = \sum_{i=1}^m \alpha_i \gamma_i^2 = 0 \quad \text{and} \quad \sum_{i=1}^m \alpha_i \beta_i \gamma_i = 1, \quad (1)$$

then for each $x, y \in X$, we have:

$$2\langle x, y \rangle = \sum_{i=1}^m \alpha_i \|\beta_i x + \gamma_i y\|^2.$$

According to this situation, Pythagorean and isosceles orthogonality relationships were generalized by Carlsson in 1962 (see [9]). In particular, in a real normed linear space $(X, \|\cdot\|)$, a vector $x \in X$ is said to be Carlsson's orthogonal (C-orthogonal) to a vector $y \in X$ (denoted by $x \perp_C y$) if and only if:

$$\sum_{i=1}^m \alpha_i \|\beta_i x + \gamma_i y\|^2 = 0,$$

where α_i , β_i , and γ_i are real numbers that satisfy (1). Moreover, Boussouis [8] introduced a similar but more general family of orthogonality relationships, which are referred to as Boussouis orthogonality, where they embrace Carlsson's orthogonality and they are special cases of Pythagorean and isosceles orthogonalities.

More results concerning these concepts and their main properties can be found in previous studies [1,2,9] and books [3,4].

Dragomir and Kikianty [15] introduced two notions of orthogonality (called Hermite–Hadamard (HH) type orthogonality) by employing the integral means, as follows.

3. A vector $x \in X$ is said to be HH-P-orthogonal to $y \in X$, denoted by $x \perp_{HH-P} y$, if and only if:

$$\int_0^1 \|(1-t)x + ty\|^2 dt = \frac{1}{3}(\|x\|^2 + \|y\|^2).$$

4. A vector $x \in X$ is said to be HH-I-orthogonal to $y \in X$, denoted by $x \perp_{HH-I} y$, if and only if:

$$\int_0^1 \|(1-t)x - ty\|^2 dt = \int_0^1 \|(1-t)x + ty\|^2 dt.$$

These notions of orthogonality are closely related to the classical Pythagorean and isosceles orthogonalities. In a previous study [15], it was noted that Pythagorean and isosceles orthogonalities are not equivalent to HH-P-orthogonality and HH-I-orthogonality, respectively. In general, the HH version of Carlsson's orthogonality was introduced and studied by [19]. In a real normed linear space $(X, \|\cdot\|)$, a vector $x \in X$ is called HH-C-orthogonal to a vector $y \in X$ (denoted by $x \perp_{HH-C} y$) if and only if:

$$\sum_{i=1}^m \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i y\|^2 dt = 0,$$

with the conditions (1). In addition, it was noted that HH-P-orthogonality is a particular case of HH-C-orthogonality, which is obtained by choosing $m = 3$, $\alpha_1 = -1$, $\alpha_2 = \alpha_3 = 1$, $\beta_1 = \beta_2 = 1$, $\beta_3 = 0$, $\gamma_1 = \gamma_3 = 1$, and $\gamma_2 = 0$. Similarly, HH-I-orthogonality is a particular case of HH-C-orthogonality, which is obtained by choosing $m = 2$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = -\frac{1}{2}$, $\beta_1 = \beta_2 = 1$, $\gamma_1 = 1$, and $\gamma_2 = -1$ (also see [1]).

It is obvious that for a real inner product space, all of the relationships given above coincide with the standard orthogonality given by the inner product. Furthermore, in a normed linear space, all of the relationships given above satisfy the continuity property.

The problem of finding the necessary and sufficient conditions for a normed linear space X to be an inner product has been investigated in many studies (e.g., see [2–4,15,16,19,21], and their references). One way to obtain the characterizations of inner product spaces is to force the orthogonality relationship on X to fulfill some properties of the orthogonality. For example, it is known that Roberts orthogonality lacks the existence property. However, James [16] showed that Roberts orthogonality only exists in inner product spaces. In particular, James [16] provided an example of a normed plane where at least one of any two vectors that are Roberts orthogonal to each other must be the origin, before then proving the following characterization of inner product spaces in terms of Roberts orthogonality.

Lemma 1.2. [16, Corollary 4.7] *If for every vector x of a real normed linear space X , we can find a nonzero vector orthogonal to x by the Roberts definition in each two-dimensional subspace containing x , then X is an inner product space.*

Furthermore, James [16] proved that a normed linear space is an inner product space if and only if Pythagorean and isosceles orthogonalities are homogeneous. Analogously, it was proved by [15] that HH-P- and HH-I-orthogonalities in a real normed linear space X are homogeneous if and only if X is an inner product space. In general, some characterizations of real inner product spaces were obtained by [19] after considering a property introduced by Carlsson [9], which is weaker than homogeneity, and the additivity of the orthogonality. In particular, it was proved that HH-C-orthogonality is homogeneous in a real normed linear space X if and only if X is an inner product space. It was shown by [7] that a necessary and sufficient condition for a real normed linear space being an inner product space is that Carlsson's orthogonality implies

Birkhoff–James orthogonality. Moreover, some further characterizations of inner product spaces based on the relationship between Boussois orthogonality and Birkhoff–James orthogonality were presented by [8]. Finally, according to [19], as we show in Corollary 2.6, HH-C-orthogonality and Birkhoff–James orthogonality are equivalent in a real normed linear space X if and only if X is an inner product space.

In the present study, we provide some new characterizations of real inner product spaces using the notion of HH-C-orthogonality and its special cases, i.e., HH-P- and HH-I-orthogonalities, and by considering their relationships with Birkhoff–James orthogonality. Moreover, we consider classes of linear mappings that preserve these types of orthogonalities. In particular, we show that every linear mapping that preserves HH-I-orthogonality is necessarily a scalar multiple of a linear isometry. Finally, we give some other characterizations of real inner product spaces in terms of HH-P- and HH-I-orthogonality preserving mappings.

2. Characterizations of real inner product spaces using HH-C-orthogonality relationships

In this section, we provide some new characterizations of real inner product spaces in terms of HH-C-orthogonality and its connection with Birkhoff–James orthogonality. According to [19], considering existence property of HH-C-orthogonality, we have the following result.

Theorem 2.1. [19, Theorem 2.5] *Let $(X, \|\cdot\|)$ be a real normed linear space. Then, HH-C-orthogonality exists on the right and on the left, i.e., for every $x, y \in X$ with $x \neq 0$, $r, s \in \mathbb{R}$ exist such that $x \perp_{HH-C} (rx + y)$ and $(sx + y) \perp_{HH-C} x$.*

Following Carlsson's ideas [9] and considering a condition that is weaker than the homogeneity and additivity of the orthogonality, a previous study by [19] proved that HH-C-orthogonality is homogeneous (or additive to the left) in a real normed linear space X if and only if X is an inner product space.

Definition 2.2. [19, Definition 3.2] HH-C-orthogonality is said to have property (H) in a real normed linear space $(X, \|\cdot\|)$ if $x \perp_{HH-C} y$ implies that:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \int_0^1 \sum_{i=1}^m \alpha_i \|n\beta_i(1-t)x + \gamma_i ty\|^2 dt = 0.$$

The symmetry of Birkhoff–James orthogonality characterizes the inner product spaces (see [17]) in a normed linear space X with a dimension of at least three, so the next result given by [19] provides a characterization of the real inner product spaces with a dimension of at least three.

Theorem 2.3. [19, Corollary 3.9] *Let $(X, \|\cdot\|)$ be a real normed linear space. If the HH-C-orthogonality in X has property (H), then it is symmetric and equivalent to Birkhoff–James orthogonality.*

It was proved by [19] that if X is a two-dimensional real normed linear space such that HH-C-orthogonality has property (H) in X , then HH-C-orthogonality implies Roberts orthogonality, and thus X is an inner product space because HH-C-orthogonality exists by Theorem 2.1. Combining this fact and Theorem 2.3, the following characterization of real inner product spaces was obtained by [19].

Theorem 2.4. [19] *Let $(X, \|\cdot\|)$ be a real normed linear space. Then, HH-C-orthogonality has property (H) in X if and only if X is an inner product space.*

The following characterization of inner product spaces was obtained by [19] as a direct consequence of the previous theorem.

Theorem 2.5. [19, Theorem 3.1] Let $(X, \|\cdot\|)$ be a normed linear space where HH-C-orthogonality is homogeneous. Then, X is an inner product space.

In addition, we have the following result as an immediate consequence of the homogeneity of Birkhoff–James orthogonality and Theorem 2.5.

Corollary 2.6. Let $(X, \|\cdot\|)$ be a real normed linear space. Then, HH-C-orthogonality and Birkhoff–James orthogonality are equivalent in X if and only if X is an inner product space.

Proof. If X is an inner product space, then it is clear that HH-C-orthogonality is equivalent to Birkhoff–James orthogonality (see [19]). Now, if HH-C-orthogonality is equivalent to Birkhoff–James orthogonality in X , then HH-C-orthogonality is homogeneous. Therefore, X is an inner product space by Theorem 2.5. \square

Now, according to [9], we recall that the functional equation:

$$\sum_{k=1}^r p_k \varphi(q_k \lambda) = C_1 + C_2 \lambda^2 \quad (\lambda \in \mathbb{R}), \quad (2)$$

where $r \in \mathbb{N}$, $q_k \neq 0$ ($k = 1, \dots, r$) and:

$$\sum_{k=1}^r p_k = C_1, \quad \sum_{k=1}^r p_k q_k^2 = C_2, \quad \sum_{k=1}^r p_k q_k = 1$$

is called symmetrical if it can be written in the form:

$$\sum_{k=1}^s m_k \varphi(n_k \lambda) - \sum_{k=1}^s m_k \varphi(-n_k \lambda) = C_1 + C_2 \lambda^2$$

for some $s \in \mathbb{N}$ and some real numbers C_1, C_2, m_k and n_k ; otherwise, (2) is called non-symmetrical. Carlsson [9] completely determined the behavior of the solution $\varphi(\lambda)$ of (2) for large and small values of $|\lambda|$ in the following manner.

Theorem 2.7. [9] Let $\varphi(\lambda)$ be a continuously differentiable solution of functional equation (2). Suppose that φ satisfies:

$$\varphi(\lambda) = \lambda^2 + O(\lambda), \quad \text{when } |\lambda| \rightarrow \infty$$

and:

$$\varphi(\lambda) = 1 + O(\lambda^2), \quad \text{when } \lambda \rightarrow 0.$$

Then, we have the following.

- (i) If equation (2) is non-symmetrical, then $\varphi(\lambda) = 1 + \lambda^2$ for all $\lambda \in \mathbb{R}$.
- (ii) If equation (2) is symmetrical, then $\varphi(-\lambda) = \varphi(\lambda)$ for all $\lambda \in \mathbb{R}$.

Let $(X, \|\cdot\|)$ be a real normed linear space and let $u, v \in X$. In [15, Lemma 4.1], it was noted that the function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ defined by:

$$\varphi(\lambda) = 3 \int_0^1 \|(1-t)u + t\lambda v\|^2 dt$$

is a convex function, and thus φ is continuously differentiable (almost everywhere) on \mathbb{R} (see [19, p. 867]).

The next two lemmas from [19] are employed in the proof of our main results.

Lemma 2.8. [19, Lemma 3.15] *Let $(Y, \|\cdot\|)$ be a two-dimensional real normed linear space and let $u, v \in S_Y$. Then, the function:*

$$\varphi(\lambda) = 3 \int_0^1 \|(1-t)u + t\lambda v\|^2 dt$$

satisfies the condition:

$$\varphi(\lambda) = \lambda^2 + O(\lambda) \quad \text{when } |\lambda| \rightarrow \infty.$$

Lemma 2.9. [19, Lemma 3.16] *Let $(Y, \|\cdot\|)$ be a two-dimensional real normed linear space. A dense subset \mathbb{D} of S_Y exists such that if $u \in \mathbb{D}$ and $u \perp_B v$, then the function:*

$$\varphi(\lambda) = 3 \int_0^1 \|(1-t)u + t\lambda v\|^2 dt$$

satisfies:

$$\varphi(\lambda) = 1 + O(\lambda^2) \quad \text{when } \lambda \rightarrow 0.$$

In fact, it is known that the norm of Y is twice differentiable almost everywhere in S_Y (e.g., see [4]) and \mathbb{D} is the subset of S_Y comprising all points where the norm $\|\cdot\|$ is twice differentiable.

We are now in a position to establish our main results.

Theorem 2.10. *Let $(X, \|\cdot\|)$ be a real normed linear space. Then, the following statements are equivalent.*

- (i) $\perp_B \subseteq \perp_{HH-C}$.
- (ii) X is an inner product space.

Proof. The implication of (ii) \Rightarrow (i) is clear. To prove (i) \Rightarrow (ii), it is sufficient to show that every two-dimensional subspace of X is an inner product space. Assume that Y is an arbitrary two-dimensional subspace of X containing $u \neq 0$. Let \mathbb{D} be the dense subset of S_Y , which is presented in Lemma 2.9. Then, a sequence $\{u_n\}$ in \mathbb{D} exists such that $\lim_{n \rightarrow \infty} u_n = \frac{u}{\|u\|}$. In addition, the homogeneity and existence property of Birkhoff–James orthogonality imply that for each $n \in \mathbb{N}$, $v_n \in S_Y$ exists such that $u_n \perp_B v_n$, and thus $u_n \perp_B \lambda v_n$ for all $\lambda \in \mathbb{R}$. Hence, (i) yields that $u_n \perp_{HH-C} \lambda v_n$ for all $n \in \mathbb{N}$ and all $\lambda \in \mathbb{R}$. Therefore, according to [19], the convex function:

$$\varphi(\lambda) = 3 \int_0^1 \|(1-t)u_n + t\lambda v_n\|^2 dt$$

is a continuously differentiable solution of the functional equation:

$$\sum_{k=1}^r p_k \varphi(q_k \lambda) = C_1 + C_2 \lambda^2 \quad (\lambda \in \mathbb{R}), \quad (3)$$

where $r \in \mathbb{N}$, $q_k \neq 0$ ($k = 1, \dots, r$) and $\sum_{k=1}^r p_k = C_1$, $\sum_{k=1}^r p_k q_k^2 = C_2$, $\sum_{k=1}^r p_k q_k = 1$. From Lemma 2.8 and Lemma 2.9, it follows that φ exist that satisfy:

$$\varphi(\lambda) = \lambda^2 + O(\lambda), \quad |\lambda| \rightarrow \infty$$

and

$$\varphi(\lambda) = 1 + O(\lambda^2), \quad \lambda \rightarrow 0.$$

Now, we consider two cases.

Case 1: If the functional equation (3) is non-symmetrical, then from Theorem 2.7, it follows that $\varphi(\lambda) = 1 + \lambda^2$ for all $\lambda \in \mathbb{R}$. It should be noted that in a real two-dimensional space, it is well known that the space is an inner product space if and only if the set of points of norm one is an ellipse (e.g., see [13]). Now, if we choose u_{n_0} and v_{n_0} (for some fixed $n_0 \in \mathbb{N}$) as the unit vectors of a coordinate system in Y , then every $w \in Y$ could be written as $w = \alpha u_{n_0} + \beta v_{n_0}$ for some $\alpha, \beta \in \mathbb{R}$. Then, from $\varphi(\lambda) = 1 + \lambda^2$ for all $\lambda \in \mathbb{R}$, we conclude that:

$$\begin{aligned} \|w\|^2 &= \|\alpha u_{n_0} + \beta v_{n_0}\|^2 = 3 \|\alpha u_{n_0} + \beta v_{n_0}\|^2 \int_0^1 (1-t)^2 dt \\ &= 3\alpha^2 \int_0^1 \left\| (1-t)u_{n_0} + t \frac{1-t}{t} \frac{\beta}{\alpha} v_{n_0} \right\|^2 dt \\ &= \alpha^2 \varphi\left(\frac{1-t}{t} \frac{\beta}{\alpha}\right) = \alpha^2 + \left(\frac{1-t}{t}\right)^2 \beta^2. \end{aligned}$$

Then, it follows that $\|w\| = 1$ if and only if $\alpha^2 + \left(\frac{1-t}{t}\right)^2 \beta^2 = 1$, which means that the unit sphere of Y is an ellipse. Therefore, Y is an inner product space.

Case 2: If the functional equation (3) is symmetrical, then from Theorem 2.7, it follows that $\varphi(-\lambda) = \varphi(\lambda)$ for all $\lambda \in \mathbb{R}$, i.e.:

$$\int_0^1 \|(1-t)u_n - t\lambda v_n\|^2 dt = \int_0^1 \|(1-t)u_n + t\lambda v_n\|^2 dt \quad (\forall \lambda \in \mathbb{R}).$$

Let $\lambda = \frac{1-t}{t} \beta$, where $t \in (0, 1)$ and $\beta \in \mathbb{R}$. Then,

$$\int_0^1 (1-t)^2 \|u_n - \beta v_n\|^2 dt = \int_0^1 (1-t)^2 \|u_n + \beta v_n\|^2 dt,$$

and thus $\|u_n - \beta v_n\| = \|u_n + \beta v_n\|$ for all $\beta \in \mathbb{R}$. Without loss of generality, we can assume that $v \in \mathbb{S}_Y$ exists such that $\lim_{n \rightarrow \infty} v_n = v$. Hence, $\|\frac{u}{\|u\|} - \beta v\| = \|\frac{u}{\|u\|} + \beta v\|$ for all $\beta \in \mathbb{R}$, and thus $\|u - \beta v\| = \|u + \beta v\|$ for all $\beta \in \mathbb{R}$. Therefore, we have found a vector $v \in Y$ such that $u \perp_R v$. Consequently, from Lemma 1.2, it follows that Y is an inner product space. \square

Let $(X, \|\cdot\|)$ be a real normed linear space and let $x, y \in X$. The functionals comprising:

$$\tau_{-}(x, y) := \lim_{t \rightarrow 0^{-}} \frac{\|x + ty\| - \|x\|}{t}$$

and

$$\tau_{+}(x, y) := \lim_{t \rightarrow 0^{+}} \frac{\|x + ty\| - \|x\|}{t}$$

are called Gateaux left and right derivatives of the norm at x in direction y , respectively. The following well-known result provides a good characterization of Birkhoff–James orthogonality in real normed linear spaces.

Theorem 2.11. [4, 17] *Let $(X, \|\cdot\|)$ be real a normed linear space, $x, y \in X$ and let $\alpha \in \mathbb{R}$. Then, the following conditions are equivalent.*

- (i) $x \perp_B (y - \alpha x)$.
- (ii) $\tau_{-}(x, y) \leq \alpha \|x\| \leq \tau_{+}(x, y)$.

In addition, we recall that a real normed linear space $(X, \|\cdot\|)$ is said to be strictly convex if every point of S_X is an extreme point of S_X . Thus, X is called strictly convex if S_X contains no line segment, i.e., for every $x, y \in S_X$ with $x \neq y$, we have $\|\lambda x + (1 - \lambda)y\| < 1$ for all $\lambda \in (0, 1)$. The following characterization of strictly convex normed linear spaces with respect to the left uniqueness of Birkhoff–James orthogonality is useful for proving our next main result.

Theorem 2.12. [17, Theorem 4] *Let X be a real normed linear space. Then, the following statements are equivalent.*

- (i) X is strictly convex.
- (ii) *The Birkhoff–James orthogonality is unique at left, i.e., for every $x, y \in X$ with $x \neq 0$, a unique $r \in \mathbb{R}$ exists such that $(rx + y) \perp_B x$.*

Theorem 2.13. *Let $(X, \|\cdot\|)$ be a real normed linear space. Then, the following statements are equivalent.*

- (i) $\perp_{HH-C} \subseteq \perp_B$.
- (ii) X is an inner product space.

Proof. The implication of (ii) \Rightarrow (i) is clear. To prove (i) \Rightarrow (ii), it is sufficient to show that every two-dimensional subspace of X is an inner product space. Let Y be an arbitrary two-dimensional subspace of X . First, we prove that Y is strictly convex. By contrast, we assume that Y is not strictly convex. Then, $u, v \in S_Y$ with $u \neq v$ exists such that u is an extreme point of S_X and $\alpha u + (1 - \alpha)v \in S_Y$ for all $\alpha \in (0, 1)$. Now, let $w_i = \alpha_i u + (1 - \alpha_i)v$ for some $0 < \alpha_i < 1$ ($i = 1, 2$) with $\alpha_1 \neq \alpha_2$. The existence property of HH-C-orthogonality (Theorem 2.1) implies that for all $\lambda \in \mathbb{R}$, $s \in \mathbb{R}$ exist such that: $\lambda w_1 \perp_{HH-C} (sw_1 + w_2)$, and thus (i) means that $\lambda w_1 \perp_B (sw_1 + w_2)$ for all $\lambda \in \mathbb{R}$. Moreover, Birkhoff–James orthogonality is homogeneous, so we have $w_1 \perp_B (sw_1 + w_2)$. Hence, from Theorem 2.11, it follows that $\tau_{-}(w_1, w_2) \leq -s \leq \tau_{+}(w_1, w_2)$. In addition, we have:

$$\tau_{\pm}(w_1, w_2) = \lim_{t \rightarrow 0^{\pm}} \frac{\|w_1 + tw_2\| - 1}{t}$$

$$\begin{aligned}
&= \lim_{t \rightarrow 0^\pm} \frac{\left\| (\alpha_1 + t\alpha_2)u + ((1 - \alpha_1) + t(1 - \alpha_2))v \right\| - 1}{t} \\
&= \lim_{t \rightarrow 0^\pm} \frac{(\alpha_1 + t\alpha_2) + (1 - \alpha_1) + t(1 - \alpha_2) - 1}{t} \\
&= \lim_{t \rightarrow 0^\pm} \frac{1 + t - 1}{t} = 1
\end{aligned}$$

(the third equality is given because if $t \rightarrow 0^\pm$, then $\beta := \alpha_1 + t\alpha_2 > 0$ and $\gamma := (1 - \alpha_1) + t(1 - \alpha_2) > 0$, and thus $\|\frac{\beta}{\beta+\gamma}u + \frac{\gamma}{\beta+\gamma}v\| = 1$ since every convex combination of u and v is in S_Y , which implies that $\|\beta u + \gamma v\| = \beta + \gamma$). Then, it follows that $s = -1$. Therefore, $\lambda w_1 \perp_{HH-C} w_2 - w_1$ for all $\lambda \in \mathbb{R}$. Let $w = \frac{u-v}{\|u-v\|}$. Then $w_2 - w_1 = (\alpha_2 - \alpha_1)(u - v)$. Hence,

$$(\alpha_2 - \alpha_1)\|u - v\|w_1 \perp_{HH-C} (\alpha_2 - \alpha_1)\|u - v\|(\lambda w) \quad (\forall \lambda \in \mathbb{R}).$$

HH-C-orthogonality has a simplification property, so we conclude that $w_1 \perp_{HH-C} \lambda w$ for all $\lambda \in \mathbb{R}$. Therefore, the convex function:

$$\varphi(\lambda) = 3 \int_0^1 \|(1-t)w_1 + t\lambda w\|^2 dt$$

is a continuously differentiable solution of the functional equation:

$$\sum_{k=1}^r p_k \varphi(q_k \lambda) = C_1 + C_2 \lambda^2 \quad (\lambda \in \mathbb{R}), \quad (4)$$

where $r \in \mathbb{N}$, $q_k \neq 0$ ($k = 1, \dots, r$) and $\sum_{k=1}^r p_k = C_1$, $\sum_{k=1}^r p_k q_k^2 = C_2$, $\sum_{k=1}^r p_k q_k = 1$. In addition, Lemma 2.8 implies that φ satisfies:

$$\varphi(\lambda) = \lambda^2 + O(\lambda), \quad |\lambda| \rightarrow \infty.$$

Moreover, for fixed $t, \alpha_1 \in (0, 1)$, a $\delta > 0$ exists such that:

$$(1-t)\alpha_1 + \frac{\lambda t}{\|u-v\|} > 0 \quad \text{and} \quad (1-t)(1-\alpha_1) - \frac{\lambda t}{\|u-v\|} > 0$$

for all $|\lambda| < \delta$. Hence, we have:

$$\begin{aligned}
\varphi(\lambda) &= 3 \int_0^1 \|(1-t)w_1 + t\lambda w\|^2 dt \\
&= 3 \int_0^1 \left\| \left((1-t)\alpha_1 + \frac{\lambda t}{\|u-v\|} \right) u + \left((1-t)(1-\alpha_1) - \frac{\lambda t}{\|u-v\|} \right) v \right\|^2 dt \\
&= 3 \int_0^1 \left((1-t)\alpha_1 + \frac{\lambda t}{\|u-v\|} + (1-t)(1-\alpha_1) - \frac{\lambda t}{\|u-v\|} \right)^2 dt
\end{aligned}$$

$$= 3 \int_0^1 (1-t)^2 dt = 1.$$

It follows that $\varphi(\lambda) = 1 + O(\lambda^2)$ when $\lambda \rightarrow 0$. Now, we consider two cases.

Case 1: If the functional equation (4) is non-symmetrical, then from Theorem 2.7, it follows that $\varphi(\lambda) = 1 + \lambda^2$ for all $\lambda \in \mathbb{R}$. Now, if we choose w_1 and w_2 as the unit vectors of a coordinate system in Y and write $z = \alpha w_1 + \beta w_2$, then $\|z\|^2 = \|\alpha w_1 + \beta w_2\|^2 = \alpha^2 + (\frac{1-t}{t})^2 \beta^2$ for some $t \in \mathbb{R}$. Hence, $\|z\| = 1$ if and only if $\alpha^2 + (\frac{1-t}{t})^2 \beta^2 = 1$, which means that the unit sphere of Y is an ellipse. Therefore, Y is an inner product space. However, it is known that every inner product space is strictly convex, which is a contradiction.

Case 2: If functional equation (4) is symmetrical, then from Theorem 2.7, it follows that $\varphi(-\lambda) = \varphi(\lambda)$ for all $\lambda \in \mathbb{R}$. λ is an arbitrary real number, so after replacing λ with $\|u - v\|\lambda$, we have:

$$\begin{aligned} & \int_0^1 \|(1-t)(\alpha_1 u + (1-\alpha_1)v) - t\lambda(u-v)\|^2 dt \\ &= \int_0^1 \|(1-t)(\alpha_1 u + (1-\alpha_1)v) + t\lambda(u-v)\|^2 dt. \end{aligned} \quad (5)$$

In addition, after replacing λ with $\frac{1-t}{t}\lambda$ and $t \in (0, 1)$ in (5), we conclude that:

$$\begin{aligned} & \int_0^1 (1-t)^2 \|\alpha_1 u + (1-\alpha_1)v - \lambda(u-v)\|^2 dt \\ &= \int_0^1 (1-t)^2 \|\alpha_1 u + (1-\alpha_1)v + \lambda(u-v)\|^2 dt. \end{aligned}$$

Then, it follows that $\|\alpha_1 u + (1-\alpha_1)v - \lambda(u-v)\| = \|\alpha_1 u + (1-\alpha_1)v + \lambda(u-v)\|$ for all $\lambda \in \mathbb{R}$. If $\lambda = 1$ and $\alpha_1 \rightarrow 1$, then $\|2u - v\| = \|v\| = 1$. In addition, $u = \frac{2u-v+v}{2}$, which is impossible because u is an extreme point of S_Y . Therefore, Y is strictly convex.

Now, we prove that Y is an inner product space. It is sufficient to show that $\perp_B \subseteq \perp_{HH-C}$ in Y , and the conclusion then follows by Theorem 2.10. Assume that $x, y \in Y$ and $x \perp_B y$. From Theorem 2.1, it follows that a real number r exists such that $(ry + x) \perp_{HH-C} y$. Then, (i) implies that $(ry + x) \perp_B y$. In addition, Y is strictly convex. Hence, the left uniqueness of Birkhoff–James orthogonality (Theorem 2.12) shows that $r = 0$. Therefore, $x \perp_{HH-C} y$. This completes the proof. \square

The main properties of HH-P- and HH-I-orthogonality relationships were determined by Dragomir and Kikianty [15], who proved that HH-P- and HH-I-orthogonalities satisfy the nondegeneracy, continuity, and symmetry properties. They also investigated the existence and uniqueness of these orthogonalities. Moreover, some characterizations of inner product spaces were obtained by [15] using the homogeneity of HH-P- and HH-I-orthogonality relationships, as follows.

Theorem 2.14. [15, Theorems 3.5 and 3.6] *Let $(X, \|\cdot\|)$ be a normed linear space. Then, HH-I-orthogonality (or HH-P-orthogonality) is homogeneous in X if and only if X is an inner product space.*

By utilizing HH-P- and HH-I-orthogonalities as well as considering their relationships with Birkhoff–James orthogonality, we provide some new characterizations of real inner product spaces. The hypothesis regarding the homogeneity of HH-I-orthogonality in Theorem 2.14 can be weakened in the following manner.

Theorem 2.15. *Let $(X, \|\cdot\|)$ be a real normed linear space. Then, the following are equivalent.*

- (i) X is an inner product space.
- (ii) $r \in (0, 1)$ exists such that:

$$x \perp_{HH-I} y \Rightarrow x \perp_{HH-I} ry \quad (x, y \in X).$$

Proof. Obviously, (i) \Rightarrow (ii).

Suppose that (ii) holds. It is sufficient to show that $\perp_{HH-I} \subseteq \perp_B$ by Theorem 2.13. Assume that $x, y \in X$ and $x \perp_{HH-I} y$. Put $A := \{\lambda \in \mathbb{R} : x \perp_{HH-I} \lambda y\}$. Clearly, A is a nonempty closed and symmetric (i.e., if $\lambda \in A$, then $-\lambda \in A$) set. From (ii), it follows that $x \perp_{HH-I} r^n y$ for all positive integer n . $r \in (0, 1)$, so we conclude that for all $\varepsilon > 0$, a positive integer n_0 exists such that $x \perp_{HH-I} r^{n_0} y$ and $r^{n_0} < \varepsilon$. Therefore, $\inf\{\lambda > 0 : \lambda \in A\} = 0$. Then, a sequence $(\lambda_n) \subseteq A$ exists such that $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$. Consider the convex function:

$$\varphi(\lambda) = \int_0^1 \|(1-t)x + \lambda ty\|^2 dt \quad (\lambda \in A).$$

Then, we have $\varphi(\lambda_n) = \varphi(-\lambda_n)$ for all positive integers n since A is symmetric. Thus, the line $y = \frac{1}{3}\|x\|^2$ supports $\varphi(\lambda)$ at the point $(0, \frac{1}{3}\|x\|^2)$ and:

$$\varphi(\lambda) \geq \varphi(0) = \int_0^1 \|(1-t)x\|^2 dt,$$

which shows that $\|(1-t)x + \lambda ty\| \geq \|(1-t)x\|$ for almost all $t \in [0, 1]$ and all $\lambda \in \mathbb{R}$. Then, it follows that $(1-t)x \perp_B ty$ for all $t \in [0, 1]$ by the continuity property of Birkhoff–James orthogonality. Therefore, $x \perp_B y$. \square

The following lemma is useful for our further results.

Lemma 2.16. [15, Theorem 3.2] *HH-P-orthogonality is unique in any real normed linear space X , i.e., for every $x, y \in X$ with $x \neq 0$, a unique $r \in \mathbb{R}$ exists such that $x \perp_{HH-P} (rx + y)$.*

The following result suggests a condition that is weaker than the homogeneity of HH-P-orthogonality in Theorem 2.14.

Theorem 2.17. *Let $(X, \|\cdot\|)$ be a real normed linear space. Then, the following are equivalent.*

- (i) X is an inner product space.
- (ii) $r \in (0, 1)$ exists such that:

$$x \perp_{HH-P} y \Rightarrow x \perp_{HH-P} \pm ry \quad (x, y \in X).$$

Proof. It is obvious that (i) \Rightarrow (ii).

To prove the implication of (ii) \Rightarrow (i), suppose that (ii) holds and $x, y \in X$ such that $x \perp_{HH-P} y$. Then, we have:

$$\int_0^1 \|(1-t)x + try\|^2 dt = \frac{1}{3}(\|x\|^2 + \|ry\|^2)$$

and

$$\int_0^1 \|(1-t)x - try\|^2 dt = \frac{1}{3}(\|x\|^2 + \|ry\|^2).$$

It follows that $\int_0^1 \|(1-t)x + try\|^2 dt = \int_0^1 \|(1-t)x - try\|^2 dt$, and thus $x \perp_{HH-I} ry$. Hence, we have proved that $r \in (0, 1)$ exists such that:

$$x \perp_{HH-P} y \Rightarrow x \perp_{HH-I} ry \quad (x, y \in X).$$

Therefore, by using the same technique employed in the proof of Theorem 2.15, we can prove that $\perp_{HH-P} \subseteq \perp_B$, and thus from Theorem 2.13, it follows that X is an inner product space. \square

Let us quote two results given by [15] regarding HH-I-orthogonality.

Lemma 2.18. [15, Theorem 3.3] *Let X be a real normed linear space. Then, HH-I-orthogonality exists, i.e., for every $x, y \in X$ with $x \neq 0$, $r \in \mathbb{R}$ exists such that $x \perp_{HH-I} (rx + y)$.*

Lemma 2.19. [15, Theorem 3.4] *For a real normed linear space X , the following conditions are equivalent.*

- (i) X is strictly convex.
- (ii) HH-I-orthogonality is unique, i.e., for every $x, y \in X$ with $x \neq 0$, a unique $r \in \mathbb{R}$ exists such that $x \perp_{HH-I} (rx + y)$.

The following are further properties of HH-P- and HH-I-orthogonality relationships.

Theorem 2.20. *Let X be a real normed linear space. Then, the following are equivalent.*

- (i) $\perp_{HH-I} \subseteq \perp_{HH-P}$.
- (ii) $\perp_{HH-P} \subseteq \perp_{HH-I}$.

Moreover, X is strictly convex in these two cases.

Proof. (i) \Rightarrow (ii) Assume that (i) holds. First, we prove that X is strictly convex. By contrast, assume that X is not strictly convex. HH-I-orthogonality exists, so by Lemma 2.18 and from Lemma 2.19, it follows that $x, y \in X$ with $x \neq 0$ and distinct real numbers r_1 and r_2 exist such that $x \perp_{HH-I} (r_1x + y)$ and $x \perp_{HH-I} (r_2x + y)$. Hence, (i) implies that $x \perp_{HH-P} (r_1x + y)$ and $x \perp_{HH-P} (r_2x + y)$ while $r_1 \neq r_2$, which is impossible since HH-P-orthogonality is unique by Lemma 2.16.

Now, assume that $x, y \in X$ exist such that $x \perp_{HH-P} y$ but that $x \not\perp_{HH-I} y$. From Lemma 2.19, it follows that a unique $r \in \mathbb{R}$ exists such that $x \perp_{HH-I} (rx + y)$ since X is strictly convex. Then, (i) implies that $x \perp_{HH-P} (rx + y)$, and thus $r = 0$. Therefore, $x \perp_{HH-I} y$, which is impossible.

(ii) \Rightarrow (i) First, we note that the uniqueness of HH-P-orthogonality and (ii) imply that HH-I-orthogonality is unique, and thus X is strictly convex. Suppose that (ii) holds but (i) is not true. Then, $x, y \in X$ exist such that $x \perp_{HH-I} y$ and $x \not\perp_{HH-P} y$. By Lemma 2.16, a unique real number r exists such that $x \perp_{HH-P} (rx + y)$. From (ii), it follows that $x \perp_{HH-I} (rx + y)$, and thus $r = 0$. Therefore, $x \perp_{HH-P} y$, which is impossible. \square

3. Linear mappings that preserve HH type orthogonalities

In this section, we consider linear mappings that preserve some HH type orthogonalities. An orthogonality preserving property can be introduced in the most natural manner for linear mappings between inner product spaces. If X and Y are inner product spaces with the standard orthogonality relationship, then a linear mapping $T : X \rightarrow Y$ that satisfies the condition:

$$x \perp y \implies Tx \perp Ty, \quad (x, y \in X),$$

is called orthogonality preserving. It is well known that an orthogonality preserving linear mapping between two inner product spaces is necessarily a similarity, i.e., a positive constant γ exists such that $\|Tx\| = \gamma\|x\|$ for all $x \in X$ (e.g., see [10,12,25–27]). Now, let X and Y be two real normed linear spaces, and let $\diamond \in \{B, I\}$. Let us consider the linear mappings $T : X \rightarrow Y$ that preserve the \diamond -orthogonality in the following sense:

$$x \perp_{\diamond} y \implies Tx \perp_{\diamond} Ty \quad (x, y \in X).$$

Koldobsky [20] proved that a linear mapping $T : X \rightarrow Y$ that preserves Birkhoff–James orthogonality has to be a similarity (also see [11]). The respective results for both the real and complex cases were given by Blanco and Turnšek [6, Theorem 3.1]. Martini and Wu [22] proved the same result for mappings that preserve isosceles orthogonality. In the following theorem, we show that every linear mapping between real normed linear spaces that preserve HH-I-orthogonality is also necessarily a similarity.

Theorem 3.1. *Let X and Y be real normed linear spaces and let $T : X \rightarrow Y$ be a nonzero linear mapping. Then, the following conditions are equivalent.*

- (i) $x \perp_{HH-I} y \implies Tx \perp_{HH-I} Ty$ for all $x, y \in X$.
- (ii) A positive constant γ exists such that $\|Tx\| = \gamma\|x\|$ for all $x \in X$.

Proof. To prove (i) \implies (ii), we fix an arbitrary vector $u \in S_X$ and let $\gamma := \|Tu\|$. Assume that $v \in S_X$, and set: $x = \frac{v+u}{1-t}$ and $y = \frac{v-u}{t}$ for any $t \in (0, 1)$. Then, we have:

$$\int_0^1 \|(1-t)x + ty\|^2 dt = 4\|v\|^2$$

and

$$\int_0^1 \|(1-t)x - ty\|^2 dt = 4\|u\|^2.$$

Hence,

$$\int_0^1 \|(1-t)x + ty\|^2 dt = \int_0^1 \|(1-t)x - ty\|^2 dt.$$

It follows that $x \perp_{HH-I} y$. Therefore, $Tx \perp_{HH-I} Ty$ by (i), or equivalently:

$$\int_0^1 \|(1-t)Tx + tTy\|^2 dt = \int_0^1 \|(1-t)Tx - tTy\|^2 dt. \quad (6)$$

Since

$$\begin{aligned}\int_0^1 \|(1-t)Tx + tTy\|^2 dt &= \int_0^1 \left\| (1-t)T \left(\frac{v+u}{1-t} \right) + tT \left(\frac{v-u}{t} \right) \right\|^2 dt \\ &= 4\|Tv\|^2\end{aligned}$$

and

$$\begin{aligned}\int_0^1 \|(1-t)Tx - tTy\|^2 dt &= \int_0^1 \left\| (1-t)T \left(\frac{v+u}{1-t} \right) - tT \left(\frac{v-u}{t} \right) \right\|^2 dt \\ &= 4\|Tu\|^2,\end{aligned}$$

then from (6), we obtain: $4\|Tv\|^2 = 4\|Tu\|^2 = 4\gamma^2$. Thus, $\|Tv\| = \gamma$ for all $v \in \mathbb{S}_X$, so $\|Tx\| = \gamma\|x\|$ for all $x \in X$.

Now, assume that (ii) holds, $x, y \in X$, and $x \perp_{HH-I} y$. Then, we have:

$$\begin{aligned}\int_0^1 \|(1-t)Tx - tTy\|^2 dt &= \int_0^1 \|T((1-t)x - ty)\|^2 dt \\ &= \int_0^1 (\gamma\|(1-t)x - ty\|)^2 dt \\ &= \int_0^1 (\gamma\|(1-t)x + ty\|)^2 dt \\ &= \int_0^1 \|T((1-t)x + ty)\|^2 dt \\ &= \int_0^1 \|(1-t)Tx + tTy\|^2 dt.\end{aligned}$$

Therefore, $Tx \perp_{HH-I} Ty$. \square

Now let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms in a vector space X and let $\perp_{HH-I,1}$, $\perp_{HH-I,2}$ denote the HH-I-orthogonality relationships with respect to the first or second norm. After applying Theorem 3.1 for the identity mapping from $(X, \|\cdot\|_1)$ to $(X, \|\cdot\|_2)$, we obtain the following result.

Corollary 3.2. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms in a real vector space X . Then, the following conditions are equivalent:*

- (i) $\perp_{HH-I,1} \subseteq \perp_{HH-I,2}$.
- (ii) $\perp_{HH-I,1} = \perp_{HH-I,2}$.
- (iii) A positive constant γ exists such that $\|x\|_2 = \gamma\|x\|_1$ for all $x \in X$.

We can obtain the following theorem for linear mappings that preserve HH type orthogonality in the same manner.

Theorem 3.3. *Let X and Y be real normed linear spaces and let $T : X \longrightarrow Y$ be a nonzero linear mapping. Then, the following conditions are equivalent.*

- (i) $x \perp_{HH-I} y \Rightarrow Tx \perp_{HH-P} Ty$ for all $x, y \in X$.
- (ii) A positive constant γ exists such that $\|Tx\| = \gamma\|x\|$ for all $x \in X$.

Proof. (i) \Rightarrow (ii) Fix $u \in \mathbb{S}_X$. Suppose that $x = \frac{u+v}{1-t}$ and $y = \frac{u-v}{t}$ for all $v \in \mathbb{S}_X$ and all $t \in (0, 1)$. Similar to the proof of Theorem 3.1, we obtain $x \perp_{HH-I} y$, and hence $x \perp_{HH-I} (-y)$. Thus, (i) implies that $Tx \perp_{HH-P} Ty$ and $Tx \perp_{HH-P} (-Ty)$, or equivalently:

$$\int_0^1 \|(1-t)Tx + tTy\|^2 dt = \frac{1}{3} (\|Tx\|^2 + \|Ty\|^2)$$

and

$$\int_0^1 \|(1-t)Tx - tTy\|^2 dt = \frac{1}{3} (\|Tx\|^2 + \|Ty\|^2).$$

Thus,

$$\int_0^1 \|(1-t)Tx + tTy\|^2 dt = \int_0^1 \|(1-t)Tx - tTy\|^2 dt. \quad (7)$$

Since

$$\int_0^1 \|(1-t)Tx + tTy\|^2 dt = 4\|Tu\|^2$$

and

$$\int_0^1 \|(1-t)Tx - tTy\|^2 dt = 4\|Tv\|^2,$$

then from (7), we obtain $\|Tu\| = \|Tv\|$. Now, let $\|Tu\| = \gamma$. We have proved that if $\|v\| = 1$, then $\|Tv\| = \gamma$. Thus, $\|Tx\| = \gamma\|x\|$ for all $x \in X$.

To prove (ii) \Rightarrow (i), assume that (ii) holds, $x, y \in X$, and $x \perp_{HH-I} y$. Then, we have:

$$\begin{aligned} \int_0^1 \|(1-t)Tx + tTy\|^2 dt &= \int_0^1 \|T((1-t)x + ty)\|^2 dt \\ &= \int_0^1 (\gamma\|(1-t)x + ty\|)^2 dt \\ &= \frac{1}{3} ((\gamma\|x\|)^2 + (\gamma\|y\|)^2) \end{aligned}$$

$$= \frac{1}{3}(\|Tx\|^2 + \|Ty\|^2).$$

Therefore, $Tx \perp_{HH-P} Ty$. \square

We complete this section by presenting some other characterizations of inner product spaces in terms of HH-P- and HH-I-orthogonality preserving mappings.

Theorem 3.4. *Let X and Y be real normed linear spaces. Any of the following assertions implies that T is injective and $T(X)$ is an inner product space, and thus X is an inner product space.*

(i) *A nonzero linear mapping $T : X \rightarrow Y$ exists such that:*

$$x \perp_B y \implies Tx \perp_{HH-I} Ty \quad (x, y \in X).$$

(ii) *A nonzero linear mapping $T : X \rightarrow Y$ exists such that:*

$$x \perp_B y \implies Tx \perp_{HH-P} Ty \quad (x, y \in X).$$

Proof. Suppose that (i) holds. $T \neq 0$, so $u \in S_X$ exists such that $Tu \neq 0$. Let us assume that T is not injective. Then, $v \in S_X$ exists such that $Tv = 0$. It is known that $\alpha \in \mathbb{R}$ exists such that $(\alpha u + v) \perp_B \alpha u$. Then, $T(\alpha u + v) \perp_{HH-I} T(\alpha u)$. Hence, $\alpha Tu \perp_{HH-I} \alpha Tu$, so $Tu \perp_{HH-I} Tu$, and it follows that $Tu = 0$, which is a contradiction. Thus, T is injective.

Now, let $x \in X$ and V be an arbitrary two-dimensional subspace of $T(X)$ containing Tx . Let $Ty \in V$ be a vector that is linearly independent from Tx . $\alpha \in \mathbb{R}$ exists such that $x \perp_B (\alpha x + y)$, so we conclude that $x \perp_B k(\alpha x + y)$ for all $k \in \mathbb{R}$, by the homogeneity of Birkhoff–James orthogonality. Hence, $Tx \perp_{HH-I} k(\alpha Tx + Ty)$ for all $k \in \mathbb{R}$. Let $k = \frac{1-t}{t}\lambda$, where $t \in (0, 1)$ and λ is an arbitrary real number. Thus,

$$\int_0^1 \left\| (1-t)(Tx + \lambda(\alpha Tx + Ty)) \right\|^2 dt = \int_0^1 \left\| (1-t)(Tx - \lambda(\alpha Tx + Ty)) \right\|^2 dt.$$

It follows that

$$\|Tx + \lambda(\alpha Tx + Ty)\|^2 \int_0^1 (1-t)^2 dt = \|Tx - \lambda(\alpha Tx + Ty)\|^2 \int_0^1 (1-t)^2 dt,$$

which implies that $\|Tx + \lambda(\alpha Tx + Ty)\| = \|Tx - \lambda(\alpha Tx + Ty)\|$ for all $\lambda \in \mathbb{R}$, and thus $\alpha Tx + Ty$ is an element of V such that $Tx \perp_R (\alpha Tx + Ty)$. Therefore, from Lemma 1.2, it follows that $T(X)$ is an inner product space.

Next, we show that X is an inner product space. Since (i) holds and $T(X)$ is an inner product space, then we have the following implication:

$$x \perp_B y \implies Tx \perp_{HH-I} Ty \Leftrightarrow Tx \perp_B Ty \quad (\forall x, y \in X),$$

which implies that T preserves Birkhoff–James orthogonality. Therefore, T is a similarity by [6, Theorem 3.1], so X is an inner product space because $T(X)$ is an inner product space.

Let us suppose that (ii) holds. First, we prove that T is injective. By contrast, assume that $v \in S_X$ exists such that $Tv = 0$. Let $u \in S_X$. It is known that a real number $\alpha \neq 0$ exists such that $(\alpha u + v) \perp_B u$. Then, $(\alpha u + v) \perp_B \alpha u$, and thus $\alpha Tu \perp_{HH-P} \alpha Tu$. It follows that $Tu = 0$ for all $u \in X$, which is impossible.

Now, let $x, y \in X$ be nonzero vectors. Then, $\alpha \in \mathbb{R}$ exists such that $(\alpha x + y) \perp_B kx$ for all real numbers k . Hence, $T(\alpha x + y) \perp_{HH-P} T(kx)$, and so $(\alpha Tx + Ty) \perp_{HH-P} kTx$ for all $k \in \mathbb{R}$. Therefore, by using similar arguments to those used in the proof of Theorem 3.5 of [15], we can conclude that $T(X)$ is an inner product space. In addition, from (ii), we conclude that:

$$x \perp_B y \Rightarrow Tx \perp_{HH-P} Ty \Leftrightarrow Tx \perp_B Ty \quad (\forall x, y \in X),$$

and it follows that T is a similarity. Thus, X is an inner product space. \square

Theorem 3.5. *Let X and Y be real normed linear spaces. If a nonzero linear mapping $T : X \longrightarrow Y$ exists such that the dimension of $T(X)$ is greater than two and:*

$$x \perp_{HH-P} y \implies Tx \perp_B Ty \quad (x, y \in X),$$

then T is injective and $T(X)$ is an inner product space.

Proof. Let $u \in \mathbb{S}_X$ exist such that $Tu \neq 0$. If T is not injective, then $v \in \mathbb{S}_X$ exists such that $Tv = 0$. It is known that $\alpha \in \mathbb{R}$ exists such that $(\alpha u + v) \perp_{HH-P} u$, and thus $T(\alpha u + v) \perp_B Tu$. Then, $\alpha Tu \perp_B Tu$ and so $Tu \perp_B Tu$. Therefore $Tu = 0$, which is a contradiction. Thus, T is injective.

Now, suppose that $z, w \in T(X)$ and $z \perp_B w$. Then, $x, y \in X$ exist such that $z = Tx$ and $w = Ty$. In addition, a unique $\alpha \in \mathbb{R}$ exists such that $(\alpha y + x) \perp_{HH-P} y$, and thus $T(\alpha y + x) \perp_B Ty$. It follows that $(\alpha Ty + Tx) \perp_B Ty$, so $\alpha = 0$. Thus, $x \perp_{HH-P} y$. HH-P-orthogonality is symmetric, so we conclude that $y \perp_{HH-P} x$, and thus $Ty \perp_B Tx$, or equivalently that $w \perp_B z$. Therefore, Birkhoff-James orthogonality is symmetric on $T(X)$, which implies that $T(X)$ is an inner product space. \square

Theorem 3.6. *Let X and Y be real normed linear spaces. If a nonzero linear mapping $T : X \longrightarrow Y$ exists such that $T(X)$ is strictly convex with a dimension greater than two and:*

$$x \perp_{HH-I} y \implies Tx \perp_B Ty \quad (x, y \in X),$$

then T is injective and $T(X)$ is an inner product space, so X is an inner product space.

Proof. First, we prove that T is injective. By contrast, we assume that T is not injective. Then, $v \in \mathbb{S}_X$ exist such that $Tv = 0$. For each $u \in \mathbb{S}_X$ and each $t \in (0, 1)$, let $x = \frac{u+v}{1-t}$ and $y = \frac{u-v}{t}$. Therefore, $x \perp_{HH-I} y$. It follows that $Tx \perp_B Ty$, so $T(u+v) \perp_B T(u-v)$ by the homogeneity of Birkhoff-James orthogonality. Hence, $Tu \perp_B Tu$, so $Tu = 0$. Thus, $Tu = 0$ for all $u \in X$, which is impossible.

Now, suppose that $z, w \in T(X)$ and $z \perp_B w$. Then, $x, y \in X$ exist such that $z = Tx$ and $w = Ty$, so $\alpha \in \mathbb{R}$ exist such that $(\alpha y + x) \perp_{HH-I} y$. Thus, $(\alpha Ty + Tx) \perp_B Ty$. $T(X)$ is strictly convex, so it follows that Birkhoff-James orthogonality is unique at left, and thus $\alpha = 0$. Hence, $x \perp_{HH-I} y$. HH-I-orthogonality is symmetric, so we have $y \perp_{HH-I} x$. Thus, $w = Ty \perp_B z = Tx$. Therefore, Birkhoff-James orthogonality is symmetric on $T(X)$, which implies that $T(X)$ is an inner product space. In addition, we have:

$$x \perp_{HH-I} y \Rightarrow Tx \perp_B Ty \Leftrightarrow Tx \perp_{HH-I} Ty \quad (\forall x, y \in X),$$

and thus T is a similarity by Theorem 3.1. Consequently, X is an inner product space. \square

As an immediate consequence of Theorem 3.5, we obtain the following characterization of the inner product spaces in terms of HH-P-orthogonal preserving mappings.

Corollary 3.7. *A real normed linear space X with a dimension greater than two is an inner product space if and only if a nonzero surjective linear mapping $T : X \rightarrow X$ exists such that:*

$$x \perp_{HH-P} y \implies Tx \perp_B Ty \quad (x, y \in X).$$

Finally, we have the following result as a consequence of Theorem 3.6.

Corollary 3.8. *A real strictly convex normed linear space X with a dimension greater than two is an inner product space if and only if a nonzero surjective linear mapping $T : X \rightarrow X$ exists such that:*

$$x \perp_{HH-I} y \implies Tx \perp_B Ty$$

for all $x, y \in X$.

4. Concluding remarks

In this study, we characterized real inner product spaces using some known HH type orthogonalities introduced by [15] and [19].

In Section 2, we first showed that a real normed linear space X is an inner product space if and only if the Birkhoff–James orthogonality of two elements implies their HH-C-orthogonality and if and only if HH-C-orthogonality of two elements implies their Birkhoff–James orthogonality. Next, we presented a condition that is weaker than the homogeneity of HH-I-orthogonality for a real normed linear space being an inner product space. Similar results were provided for HH-P-orthogonality.

In Section 3, we considered the linear mappings that preserve HH-type orthogonalities. For real normed linear spaces X and Y , we proved that a nonzero linear mapping $T : X \rightarrow Y$ preserves HH-I-orthogonality if and only if T is a similarity with a real constant. Moreover, we obtained some other characterizations of real inner product spaces in terms of HH-P- and HH-I-orthogonal preserving mappings. For example, we proved that a real normed linear space X is an inner product space if and only if a nonzero linear mapping $T : X \rightarrow X$ exists such that the Birkhoff–James orthogonality changes to HH-P-orthogonality (or HH-I-orthogonality).

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