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Multiple solutions for a class of singular quasilinear problems

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ABSTRACT

In this paper we use sub-supersolution and minimax methods to show the existence and multiplicity of solutions for the following class of singular quasilinear problems:

$$\begin{cases} -\Delta u - \Delta(u^2)u = a(x)u^{-\beta} + h(x,u) & \text{in} \quad \Omega, \\ u > 0 \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ $(N \geq 3)$, the function $a(x)$ is nonnegative, $\beta > 0$ is a constant and the nonlinearity $h(x,u)$ is continuous. In our first result, the nonlinearity h has an arbitrary polynomial growth and we obtain the existence of a solution for the problem via sub-supersolution method. For the second result, h has subcritical growth and we show the existence of a second solution by applying the Mountain Pass Theorem.

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1. Introduction

Nonlinear singular boundary value problems arise in certain physical situations, for example, in fluid mechanics, pseudoplastic flow, chemical heterogeneous catalysts, non-Newtonian fluids and biological pattern formation, see [4,5,16] and the references therein.

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There is a rich literature in nonlinear analysis related to the following problem with singular nonlinearity:

$$\begin{cases} -\Delta u = a(x)u^{-\beta} + \lambda f(x,u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

see for example [1,7,8,10–13,17,21,22,28–31] and references therein.

In [28], it was used the sub-supersolution method for proving that problems like (1.1) has a unique positive solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$.

In [22], Perera and Silva have showed the existence and multiplicity of positive solutions for (1.1), by considering the p-Laplacian operator instead of $-\Delta$ in (1.1) and moreover they have assumed $\lambda > 0$ sufficiently small and appropriate hypotheses on $a(x)$ and $f(x,u)$.

In [10,11], the authors used a theorem due to Rabinowitz (see [24]) in order to study the following generalization of (1.1):

$$\begin{cases} -\text{div}[a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u] = p(x)u^{-\beta} + h(x)u^q & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where $0 < q < 1$, $\beta \in (0,1)$ and $p(x), h(x)$ are continuous functions.

In another context, the study of quasilinear equations involving the operator $Lu := -\Delta u - \Delta(u^2)u$ arises in various branches of mathematical physics. It is well-known that nonlinear Schrödinger equations of the form

$$i\partial_t \psi = -\Delta\psi + V(x)\psi - \Delta(|\psi|^2)\psi - h(|\psi|^2)\psi, \tag{1.3}$$

where $\psi : \mathbb{R} \times \Omega \to \mathbb{C}$, $V = V(x)$ is a given potential and h is a real function, have been studied in relation with some mathematical models in physics (see for instance [23]). It was shown that a system describing the self-trapped electron on a lattice can be reduced in the continuum limit to (1.3) and numerical results on this equation are obtained in [3]. In [18], motivated by the nanotubes and fullerene related structures, it was proposed and shown that a discrete system describing the interaction of a 2-dimensional hexagonal lattice with an excitation caused by an excess electron can be reduced to (1.3) and numerical results have been done on domains of disc type, cylinder type and sphere type.

Setting $\psi(t,x) = \exp(-iFt)u(x)$, $F \in \mathbb{R}$, into the equation (1.3), we obtain the stationary equation

$$-\Delta u - \Delta(u^2)u = g(u) - V(x)u \quad \text{in} \quad \Omega, \tag{1.4}$$

where we have renamed $V(x) - F$ to be $V(x)$ and $g(u) = h(u^2)u$.

When $\Omega = \mathbb{R}^N$, the quasilinear equation (1.4) has received special attention in the past several years, see for instance [9,14,19,23] and references therein. In these papers the authors obtain the existence of solution by perform a change of variable, which transform the quasilinear equation into a new semilinear equation, and they used variational approach. When Ω is a bounded domain, we can cite the papers [6,15], where the authors study the existence, uniqueness and multiplicity of positive solutions for (1.4). They use mainly the sub-supersolution method, bifurcation and variational arguments to obtain their results. However, they do not lead with singular nonlinearities.

For problems involving the quasilinear operator $Lu := -\Delta u - \Delta(u^2)u$ and singular nonlinearities in bounded domain, as far as we know in the literature there are only the papers [13,20,30]. In [13], do Ó and Moameni studied the problem

$$\begin{cases} -\Delta u - \Delta(u^2)u = \lambda u^3 - u - u^{-\beta} & \text{in} \quad \Omega, \\ u > 0 \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a ball in $\mathbb{R}^N$ centered at the origin, $\lambda > 0$ is a positive parameter and $0 < \beta < 1$. The authors establish the existence of solutions for the above quasilinear problem when λ belongs to a certain neighborhood of the first eigenvalue λ_1 of the following nonlinear eigenvalue problem with homogeneous Dirichlet boundary condition:

$$-\Delta u - \frac{1}{2}\Delta(u^2)u = \lambda u^3 \quad \text{in} \quad \Omega.$$

In [20], the authors obtain the same results as in [13] considering a more general class of equations. More recently, in [30], by using minimax methods and some analysis techniques, the author showed the existence and uniqueness of solutions to the problem

$$\begin{cases} -\Delta u - \Delta(u^2)u = g(x)u^{-\beta} - u^{p-1} & \text{in} \quad \Omega, \\ u > 0 \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $\beta \in (0,1)$, $g \in L^{\frac{2.2^*}{2^*-1+\beta}}(\Omega)$ with $g(x) > 0$ for almost every $x \in \Omega$ and $2^* = 2N/(N-2)$ is the critical Sobolev exponent and $p \in [2, 2.2^*]$.

Motivated by these papers and by the increasing interest on problems with singular nonlinearities, our main purpose in this paper is to investigate the existence and multiplicity of solutions for the following singular quasilinear problem:

$$\begin{cases} -\Delta u - \Delta(u^2)u = a(x)u^{-\beta} + h(x,u) & \text{in} \quad \Omega, \\ u > 0 \quad \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.5}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ $(N \geq 3)$, $a(x)$ is nonnegative function, $\beta > 0$ is a constant and the nonlinearity $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies some conditions.

We say that a function $u : \Omega \to \mathbb{R}$ is a weak solution for problem (1.5) if $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $u > 0$ almost everywhere in Ω and it holds

$$\int_\Omega (1+2u^2)\nabla u \nabla\varphi + 2\int_\Omega |\nabla u|^2 u\varphi = \int_\Omega a(x)u^{-\beta}\varphi + \int_\Omega h(x,u)\varphi, \quad \forall\ \varphi \in C_0^\infty(\Omega). \tag{1.6}$$

As observed in [9], when we are leading with problems of the type

$$\begin{cases} -\Delta u - \Delta(u^2)u = g(x,u) & \text{in} \quad \Omega, \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1.7}$$

in dimension $N \geq 2$, there are some technical difficulties in order to apply directly variational methods to the formal functional associated to (1.5) given by

$$J(u) = \frac{1}{2}\int_\Omega (1+2u^2)|\nabla u|^2 - \int_\Omega G(x,u),$$

where $G(x,t) = \int_0^t g(x,s)ds$. The main point is that J is not well defined in $H_0^1(\Omega)$, because we can not guarantee that $|u|^2|\nabla u|^2 \in L^1(\Omega)$ if $u \in H_0^1(\Omega)$. To overcome this difficulty, we use the arguments developed in [9,19]. More precisely, we make the change of variables $v = f^{-1}(u)$, where f is defined by

$$\begin{aligned} f'(t) &= \frac{1}{[1+2f^2(t)]^{1/2}} \quad \text{on} \quad [0,+\infty), \\ f(t) &= -f(-t) \qquad\qquad\quad \text{on} \quad (-\infty,0]. \end{aligned} \tag{1.8}$$

Therefore, by using this change of variables, the functional J can be rewritten in the following way:

$$I(v) := J(f(v)) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega G(x,f(v)) \tag{1.9}$$

which can be well defined in $H_0^1(\Omega)$ under convenient hypotheses on $G(x,t)$. Moreover, $I(v)$ is the Euler-Lagrange functional associated to the semilinear equation

$$\begin{cases} -\Delta v = g(x,f(v))f'(v) & \text{in} \quad \Omega, \\ \quad u = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

In [9,26], it was shown that if $v \in H_0^1(\Omega)$ is a solution of this semilinear problem then $u = f(v)$ is a solution of (1.7). By the properties of the function $f(t)$ (see the next section), we have $f(t) > 0$ for $t > 0$. Thus, by using the change of variables (1.8), problem (1.5) can be transformed in the singular problem

$$\begin{cases} -\Delta v = a(x)f(v)^{-\beta}f'(v) + h(x,f(v))f'(v) \quad \text{in} \quad \Omega, \\ v > 0 \quad \text{in} \quad \Omega, \\ v = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.10}$$

A weak solution for (1.10) is a function $v \in H_0^1(\Omega)$ verifying $v > 0$ almost everywhere in Ω and it holds

$$\int_\Omega \nabla v \nabla \phi - \int_\Omega a(x)f(v)^{-\beta}f'(v)\phi - \int_\Omega h(x,f(v))f'(v)\phi = 0 \quad \forall\ \phi \in H_0^1(\Omega). \tag{1.11}$$

Thus, if v is a weak solution of (1.10) then $u = f(v)$ is a weak solution of (1.5) and therefore our goal is reduced to showing that problem (1.10) has a solution.

Now, we establish the hypothesis on the function $a(x)$:

(a_0) There exists $\varphi_0 \in C_0^1(\overline{\Omega})$, $\varphi_0 \geq 0$ and $q > N$ such that $a\varphi_0^{-\beta} \in L^q(\Omega)$.

Notice that from this integrability condition, we must have $|\{x \in \Omega : \varphi_0(x) = 0\}| = 0$, where $|A|$ denotes the Lebesgue measure of a set A in $\mathbb{R}^N$.

Remark 1.1. Note that by (a_0) we have

$$\int_\Omega |a|^q = \int_\Omega |a\varphi_0^{-\beta}\varphi_0^{\beta}|^q dx \leq \|\varphi_0\|_\infty^{\beta q} \int_\Omega |a\varphi_0^{-\beta}|^q < \infty,$$

that is, $a \in L^q(\Omega)$.

For the our first result, the hypotheses on the function h will be the following:

(h_1) There exist constants $\delta, k > 0$ such that

$$-ka(x) \leq h(x,t), \ \forall\, t \in [0,\delta] \text{ and a.e. in } x \in \Omega.$$

(h_2) There exist $C > 0$, $r \geq 1$ and $b \in L^\infty(\Omega)$, $b \geq 0$ almost everywhere in Ω, such that

$$|h(x,t)| \leq C(1 + b(x)|t|^{r-1}), \ \forall\, t \in \mathbb{R} \text{ and a.e. in } x \in \Omega.$$

The first main result of this paper is stated as follows:

Theorem 1.2. *Assume that conditions (a_0), (h_1) and (h_2) hold. Then, there exists $\varepsilon_0 > 0$ such that problem (1.5) has a weak solution provided that $\|b\|_\infty \leq \varepsilon_0$.*

In order to establish the existence of a second solution for problem (1.5), let us assume that $h : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the additional assumption

(h_3) There exist $t_0 > 0$ and $\theta > 2$ such that

$$0 < 2\theta H(x,t) \leq th(x,t), \ \forall\, t \in [t_0, \infty) \text{ and a.e. in } x \in \Omega.$$

Our multiplicity result is stated below.

Theorem 1.3. *Assume that conditions (a_0) and $(h_1)-(h_3)$ hold and $r < 2.2^* = 4N/(N-2)$. Then, there exists $\varepsilon_1 > 0$ such that problem (1.5) has two distinct weak solutions provided that $\|b\|_\infty \leq \varepsilon_1$.*

We would like to highlight that our theorems can be applied for the model nonlinearity

$$h(x,t) = b(x)|t|^{r-2}t - a(x)k + C,$$

where $k > 0$, $C \geq 0$, $r \in (4, 2.2^*)$, $a(x), b(x)$ are nonnegative functions and belong to $L^\infty(\Omega)$.

In order to prove Theorem 1.2 and Theorem 1.3, we adapt some arguments from the papers [9,22,26]. In the first result, we used the sub-supersolution method, which allowed us to consider the growth of h for all $r \geq 1$. We also observe that if $h(x,t) \leq 0$ for all $t \in \mathbb{R}$ and almost everywhere in Ω, then Theorem 1.2 is valid only under condition (h_1). To apply minimax methods in the study of (1.10), inspired by [22] we perform a truncation in the singular term, see (3.1) and (4.1) for more details. But, since we are leading with the operator $Lu = -\Delta u - \Delta(u^2)u$, some estimates are more delicate and we need to prove others properties of the change of variables $f(t)$ involving the exponent β (see Lemma 2.2).

We emphasize that our results improve the works [13,30] in the sense that we consider the singular term $u^{-\beta}$ for all $\beta > 0$ while in [13,30] the authors only treat the case $\beta \in (0,1)$. In others directions, our paper complements the works [9,22,26].

Throughout, $H_0^1(\Omega)$ denotes the Sobolev space endowed with the inner product

$$\langle u, v\rangle = \int_\Omega \nabla u \nabla v, \quad u, v \in H_0^1(\Omega),$$

and the associated norm is represented by $\|\cdot\|$. We use $\|\cdot\|_p$ to denote the norm of the Lebesgue space $L^p(\Omega)$, $1 \leq p \leq \infty$ and the symbols C_i, $i = 0, 1, 2, \cdots$ will denote various constants.

This paper is organized as follows. Some preliminary results and technical lemmas are obtained in Section 2. In order to lead with the singular term, in the Section 3, we consider an auxiliary problem and we

prove Theorem 1.2 by using a minimization argument combined with the sub-supersolution technique. In Section 4, we consider a truncated problem and we apply Mountain Pass Theorem to prove Theorem 1.3, obtaining a second solution for problem (1.5).

2. Preliminary results

In this section we will establish some preliminaries which will be important for the understanding of the work. In what follows, we present the main properties of the function $f : \mathbb{R} \to \mathbb{R}$.

Lemma 2.1 *(see [9,26]). The function f and its derivative enjoy the following properties:*

(i) f *is uniquely defined,* C^2 *and invertible;*
(ii) $|f'(t)| \leq 1$ *for all* $t \in \mathbb{R}$*;*
(iii) $|f(t)| \leq |t|$ *for all* $t \in \mathbb{R}$*;*
(iv) $f(t)/t \to 1$ *as* $t \to 0$*;*
(v) $|f(t)| \leq 2^{1/4}|t|^{1/2}$ *for all* $t \in \mathbb{R}$*;*
(vi) $f(t)/2 < tf'(t) < f(t)$ *for all* $t > 0$*, and the reverse inequalities hold for* $t < 0$*;*
(vii) $f(t)/\sqrt{t} \to a > 0$ *as* $t \to +\infty$*;*
(viii) there exists a constant $C > 0$ *such that*

$$|f(t)| \geq \begin{cases} C|t|, & |t| \leq 1 \\ C|t|^{1/2}, & |t| \geq 1; \end{cases}$$

(ix) $f^2(ts) \geq tf^2(s)$ *for all* $t \geq 1$ *and* $s \geq 0$*;*
(x) $|f(t)f'(t)| \leq 1/\sqrt{2}$ *for all* $t \in \mathbb{R}$*.*

The next result will be very important in our arguments.

Lemma 2.2. *The function* $t \to f(t)^{-\beta}f'(t),\ t > 0$ *satisfies the following properties:*

(i) $f(t)^{-\beta}f'(t)$ *is decreasing on* $(0, \infty)$*;*
(ii) $\lim_{t\to 0^+} f(t)^{-\beta}f'(t) = +\infty$*;*
(iii) There exists a constant $C_1 > 0$ *such that* $0 < f(t)^{-\beta}f'(t) \leq C_1$*, for all* $t \geq 1$*.*

Proof. Using the definition (1.8), it follows that the function $t \to f(t)$ is increasing for $t > 0$ and the function $t \to f'(t)$ is decreasing on $(0, \infty)$ Thus, $t \to f^{-\beta}(t)$ is decreasing for $t > 0$ and since $f'(t) > 0$ for $t > 0$ we have proved (i).

Since $\beta > 0$ and $f(t) \to 0^+$ as $t \to 0^+$, $f'(t) \to 1$ as $t \to 0^+$, item (ii) is proved. Item (iii) is an immediate consequence of (ii) and $(viii)$ of Lemma 2.1. □

Next, we say that $\underline{v}$ is a subsolution and $\overline{v}$ is a supersolution for problem (1.10), if $\underline{v}, \overline{v} \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and it holds

(a) $0 < \underline{v}(x) \leq \overline{v}(x)$ a.e. in $x \in \Omega$;
(b) for each $\phi \in H_0^1(\Omega)$, with $\phi \geq 0$, one has

$$\int_\Omega \nabla \underline{v} \nabla \phi \leq \int_\Omega a(x) f(\underline{v})^{-\beta} f'(\underline{v})\phi + \int_\Omega h(x, f(\underline{v}))f'(\underline{v})\phi$$

and

$$\int_\Omega \nabla\overline{v}\nabla\phi \geq \int_\Omega a(x)f(\overline{v})^{-\beta}f'(\overline{v})\phi + \int_\Omega h(x,f(\overline{v}))f'(\overline{v})\phi.$$

The following lemma establishes the existence of a subsolution and supersolution for the problem (1.10):

Lemma 2.3. *Assume that (a), (h_1) and (h_2) hold. Then, there exists $\varepsilon_0 > 0$ such that, for $\|b\|_\infty \leq \varepsilon_0$, we obtain $\underline{v}, \overline{v} \in L^\infty(\Omega)$ satisfying:*

(i) $af(\underline{v})^{-\beta}f'(\underline{v}) \in L^q(\Omega)$ *and* $\|\underline{v}\|_\infty \leq \delta/2$, *where* $\delta > 0$ *is given in* (h_1);
(ii) $0 < \underline{v}(x) \leq \overline{v}(x)$ *a.e. in* $x \in \Omega$;
(iii) $\underline{v}$ *is a subsolution and* $\overline{v}$ *is a supersolution of (1.10).*

Proof. Let $w \in H_0^1(\Omega)$ be the unique weak solution of the problem

$$\begin{cases} -\Delta w = a(x) & \text{in} \quad \Omega, \\ \quad w = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{2.1}$$

By Remark 1.1, we know that $a \in L^q(\Omega)$ for some $q > N$. By the regularity theory, $w \in W^{2,q}(\Omega)$ and since $q > N$ it follows by Sobolev embedding that $w \in C_0^{1,\gamma}(\overline{\Omega})$. Moreover, by the Strong Maximum Principle and Hopf's Lemma, we get $w > 0$ in Ω and $\partial w/\partial\nu > 0$ on $\partial\Omega$, where ν is the interior unit normal on $\partial\Omega$. Hence, considering φ_0 given by (a_0) we have $\inf_\Omega(w/\varphi_0) > 0$ and therefore $aw^{-\beta} \in L^q(\Omega)$. Now, we take $0 < \varepsilon < 1$ sufficiently small so that $\underline{v} := \varepsilon w \in C_0^1(\overline{\Omega})$ satisfies $0 < \underline{v}(x) \leq \min\{\delta/2, \delta_0, 1\}$, with $\delta > 0$ given by (h_1) and, by (ii) of Lemma 2.2, $\delta_0 > 0$ is such that

$$f(t)^{-\beta}f'(t) \geq k+1, \quad \forall\, t \in (0,\delta_0),$$

where k is the constant of (h_1). Using item (iii) of Lemma 2.1, we have $af(\underline{v})^{-\beta}f'(\underline{v}) \in L^q(\Omega)$ and $\|\underline{v}\|_\infty \leq \delta/2$, which shows item (i). Now, analogously we consider $z \in C_0^1(\overline{\Omega})$ the unique solution of the problem

$$\begin{cases} -\Delta z = a(x)f(\underline{v})^{-\beta}f'(\underline{v}) + 2C & \text{in} \quad \Omega, \\ \quad z = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2}$$

where C is the constant of (h_2). By the choice of $\delta_0 > 0$, we have

$$f(\underline{v}(x))^{-\beta}f'(\underline{v}(x)) \geq k+1 \ \text{ a.e. in } x \in \Omega. \tag{2.3}$$

Moreover, considering $\overline{v} = z$, for all $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$, we have

$$\begin{aligned} \int_\Omega \nabla\overline{v}\nabla\phi &\geq \int_\Omega a(x)f(\underline{v})^{-\beta}f'(\underline{v})\phi \\ &\geq \int_\Omega a(x)\phi \geq \varepsilon\int_\Omega a(x)\phi = \varepsilon\int_\Omega \nabla w\nabla\phi = \int_\Omega \nabla\underline{v}\nabla\phi. \end{aligned} \tag{2.4}$$

Hence, by the Weak Comparison Principle we conclude that $\underline{v} \leq \overline{v}$ almost everywhere in Ω, which proves item (ii).

Now, we are going to prove item (iii). First, note that by construction $\underline{v}, \overline{v} \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and using (iii) of Lemma 2.1 we have $f(\underline{v}) \in [0, \delta]$. In view of (2.3), (h_1) and (ii) of Lemma 2.1, we obtain

$$\int_\Omega \nabla \underline{v} \nabla \phi - \int_\Omega a(x) f(\underline{v})^{-\beta} f'(\underline{v}) \phi - \int_\Omega h(x, f(\underline{v})) f'(\underline{v}) \phi \leq \int_\Omega a(x)[\varepsilon + k - f(\underline{v})^{-\beta} f'(\underline{v})]\phi \leq 0,$$

which shows that $\underline{v}$ is a subsolution of (1.10). Next, by the definition of $\overline{v}$ and since $a(x) f(\underline{v})^{-\beta} f'(\underline{v}) \in L^q(\Omega)$ for some $q > N$, by the elliptic regularity theory it follows that $\overline{v} \in W^{2,q}(\Omega)$ and moreover there exists $C_1 > 0$ (independent of $\overline{v}$) such that

$$\|\overline{v}\|_{W^{2,q}} \leq C_1 \|a(x) f(\underline{v})^{-\beta} f'(\underline{v}) + 2C\|_q$$

As $q > N$ by Sobolev embedding we have $\overline{v} \in C_0^{1,\gamma}(\overline{\Omega})$ and therefore

$$\|\overline{v}\|_\infty \leq C_2 \|a(x) f(\underline{v})^{-\beta} f'(\underline{v}) + 2C\|_q =: C_0 \tag{2.5}$$

and C_0 is independent of $\overline{v}$ and $b(x)$, because $\underline{v}$ depends only on $a(x)$. Since $f'(\overline{v}) \in (0,1)$ and $\underline{v} \leq \overline{v}$ almost everywhere in Ω, using item (i) of Lemma 2.2 and (h_2) we reach

$$\begin{aligned}
&\int_\Omega \nabla \overline{v} \nabla \phi - \int_\Omega a(x) f(\overline{v})^{-\beta} f'(\overline{v}) \phi - \int_\Omega h(x, f(\overline{v})) f'(\overline{v}) \phi \\
&\geq \int_\Omega a(x) \big[f(\underline{v})^{-\beta} f'(\underline{v}) - f(\overline{v})^{-\beta} f'(\overline{v}) \big] \phi + \int_\Omega \big[2C - h(x, f(\overline{v})) \big] f'(\overline{v}) \phi \\
&\geq \int_\Omega \big[2C - h(x, f(\overline{v})) \big] f'(\overline{v}) \phi \\
&\geq C \int_\Omega \big(1 - b(x) f(\overline{v})^{r-1} \big) f'(\overline{v}) \phi.
\end{aligned} \tag{2.6}$$

At this point, taking $\varepsilon_0 = 1/C_0^{r-1} > 0$, where C_0 is the constant in (2.5), we have that if $\|b\|_\infty \leq \varepsilon_0$ then

$$b(x) \leq \frac{1}{C_0^{r-1}} \leq \frac{1}{\|\overline{v}\|_\infty^{r-1}} \leq \frac{1}{f(\|\overline{v}\|_\infty)^{r-1}} \leq \frac{1}{f(\overline{v}(x))^{r-1}}, \quad \forall\, x \in \Omega,$$

that is, $1 - b(x) f(\overline{v})^{r-1} \geq 0$ in Ω. Thus, according to (2.6), $\overline{v}$ is a supersolution of (1.10) and the proof is finished. □

3. Proof of Theorem 1.2

In this section, we present the proof of our first main result. More precisely, we show that Problem (1.10) has a weak solution $v \in H_0^1(\Omega)$ satisfying

$$0 < \underline{v}(x) \leq v(x) \leq \overline{v}(x) \text{ a.e. in } x \in \Omega.$$

For this, we consider the function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$g(x,t) = \begin{cases} a(x) f(\underline{v}(x))^{-\beta} f'(\underline{v}(x)) + h(x, f(\underline{v}(x))) f'(\underline{v}(x)), & t < \underline{v}(x), \\ a(x) f(t)^{-\beta} f'(t) + h(x, f(t)) f'(t), & \underline{v}(x) \leq t \leq \overline{v}(x), \\ a(x) f(\overline{v}(x))^{-\beta} f'(\overline{v}(x)) + h(x, f(\overline{v}(x))) f'(\overline{v}(x)), & t > \overline{v}(x) \end{cases} \tag{3.1}$$

and the following auxiliary problem:

$$\begin{cases} -\Delta v = g(x,v) & \text{in} \quad \Omega, \\ v = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.2}$$

The energy functional $\Phi : H_0^1(\Omega) \to \mathbb{R}$ associated to (3.2) is given by

$$\Phi(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega G(x,v), \tag{3.3}$$

where $G(x,t) = \int_0^t g(x,s)ds$.

Finalizing the Proof of Theorem 1.2: By the definition of $g(x,t)$, (i) of Lemma 2.2 and $|f'(t)| \leq 1$, we obtain

$$|g(x,t)| \leq a(x)f(\underline{v})^{-\beta}f'(\underline{v}) + M_1, \quad \forall\, t \in \mathbb{R} \ \text{ and a.e in } x \in \Omega, \tag{3.4}$$

where $M_1 = \sup_{(x,t)\in\Omega\times(0,\|\overline{v}\|_\infty)} |h(x,t)|$. Next, using (3.3), (3.4), the fact that $a(x)f(\underline{v})^{-\beta}f'(\underline{v}) \in L^q(\Omega)$ for $q > N$, Hölder inequality and Sobolev embedding, we reach

$$\Phi(v) \geq \frac{1}{2}\|v\|^2 - C_1\|v\|_{\frac{q}{q-1}} - C_2\|v\|_1 \geq \frac{1}{2}\|v\|^2 - C_3\|v\|,$$

for all $v \in H_0^1(\Omega)$ and for some constants $C_1, C_2, C_3 > 0$. Notice that the set

$$\mathcal{M} := \left\{v \in H_0^1(\Omega) : \underline{v}(x) \leq v(x) \leq \overline{v}(x) \ \text{ a.e. in } \ x \in \Omega\right\}$$

is closed and convex in $H_0^1(\Omega)$ and therefore it is weakly closed.

Now, let $v_n \rightharpoonup v$ in $H_0^1(\Omega)$. By using (3.4), Sobolev embedding and Lebesgue Dominated Convergence Theorem, it follows that

$$\int_\Omega G(x,v_n) \to \int_\Omega G(x,v).$$

Thus, Φ is (sequentially) weakly lower semi-continuous on $\mathcal{M}$ with respect to $H_0^1(\Omega)$. Moreover, since $H_0^1(\Omega)$ is a reflexive Banach space and Φ is coercive, by [27, Theorem 1.2], the restriction $\Phi\big|_{\mathcal{M}}$ attains its infimum at a point $v_1 \in \mathcal{M}$. Using the same arguments as in the proof of [27, Theorem 2.4], we infer that $\Phi'(v_1)=0$. Therefore, v_1 is a weak solution of (3.2). Since $g(x,t) = a(x)f(t)^{-\beta}f'(t) + h(x,f(t))f'(t)$ for $t \in [\underline{v}(x)), \overline{v}(x)]$, then v_1 is a weak solution of (1.10) and the proof is finished. □

Remark 3.1. The arguments used in the proof of Lemma 2.3 show that if $h(x,t) \leq 0$ almost everywhere in $x \in \Omega$ and for all $t \in \mathbb{R}$, then only the hypothesis (h_1) is sufficient to prove Lemma 2.3.

4. Proof of Theorem 1.3

In this section, under an additional restriction on r, namely $r < 2.2^*$, we shall prove that problem (1.5) has a second solution. For this, consider $\underline{v} \in C_0^1(\overline{\Omega})$ obtained by Lemma 2.3, a Carathéodory function $\widehat{g}$ defined on $\Omega \times \mathbb{R}$ by

$$\widehat{g}(x,t)=\begin{cases}a(x)f(\underline{v}(x))^{-\beta}f'(\underline{v}(x))+h(x,f(\underline{v}(x)))f'(\underline{v}(x)), & t<\underline{v}(x)\\ a(x)f(t)^{-\beta}f'(t)+h(x,f(t))f'(t), & t\geq\underline{v}(x),\end{cases} \tag{4.1}$$

and the following singular elliptic problem:

$$\begin{cases}-\Delta v=\widehat{g}(x,v) & \text{in} \quad \Omega,\\ \quad v=0 & \text{on} \quad \partial\Omega.\end{cases} \tag{4.2}$$

The energy functional $\widehat{\Phi}:H_0^1(\Omega)\to\mathbb{R}$ associated to (4.2) is given by

$$\widehat{\Phi}(v)=\frac{1}{2}\|v\|^2-\int_\Omega \widehat{G}(x,v), \tag{4.3}$$

where $\widehat{G}(x,t)=\int_0^t\widehat{g}(x,s)ds$. By the defintion of $\widehat{g}(x,t)$, (v) of Lemma 2.1, (i) of Lemma 2.2 and (h_2), one has

$$|\widehat{g}(x,t)|\leq a(x)f(\underline{v})^{-\beta}f'(\underline{v})+M_1+C(1+b(x)|t|^{\frac{r-2}{2}}),\ \forall\ t\in\mathbb{R}\text{ and a.e in } x\in\Omega,$$

and hence

$$|\widehat{G}(x,t)|\leq l_1(x)t+C_1b(x)|t|^{\frac{r}{2}},\ \forall\ t\in\mathbb{R}\text{ and a.e in } x\in\Omega, \tag{4.4}$$

where $l_1:=a(x)f(\underline{v})^{-\beta}f'(\underline{v})+M_1+C$ which belongs to $L^q(\Omega)$ for some constant $C_1>0$. Therefore, it is not difficult to verify that $\widehat{\Phi}\in C^1(H_0^1(\Omega),\mathbb{R})$. Furthermore, any critical point of $\widehat{\Phi}$ is a weak solution of (4.2).

Lemma 4.1. *The functional $\widehat{\Phi}$ satisfies the Palais-Smale condition at any level $c\in\mathbb{R}$.*

Proof. Let $(v_n)\subset H_0^1(\Omega)$ be a sequence such that $\widehat{\Phi}(v_n)\to c$ and $\widehat{\Phi}'(v_n)\to 0$. By definition of $\widehat{g}(x,t)$, a straightforward computation shows that

$$\begin{aligned}\widehat{g}(x,t)t-\theta\widehat{G}(x,t)&=(1-\theta)t\left[a(x)f(\underline{v}(x))^{-\beta}f'(\underline{v}(x))+h(x,f(\underline{v}(x)))f'(\underline{v}(x))\right]\\ &\geq(1-\theta)t\left[a(x)f(\underline{v}(x))^{-\beta}f'(\underline{v}(x))+M_1\right],\end{aligned} \tag{4.5}$$

for $t<\underline{v}(x)$. Now, we are going to consider three cases: $\beta<1$, $\beta>1$ and $\beta=1$.

Case 1: $\beta<1$;

First, without lost de generality, we can suppose that $t_0>0$, given in (h_3), satisfies $t_0>f(\|\underline{v}\|_\infty)$. Thus, setting $M_0=\sup_{(x,t)\in\Omega\times(0,t_0)}|h(x,t)|$ and $M_2=\sup_{(x,t)\in\Omega\times(0,t_0)}|H(x,t)|$, we reach

$$\begin{aligned}\widehat{g}(x,t)t-\theta\widehat{G}(x,t)&\geq -M_0t_0-\theta t\left[a(x)f(\underline{v}(x))^{-\beta}f'(\underline{v}(x))+M_1\right]\\ &\quad-\frac{\theta}{1-\beta}a(x)f(t)^{1-\beta}-\theta H(x,f(t))+\theta H(x,f(\underline{v}(x)))\\ &\geq -M_0t_0-\theta M_1t-\left(\frac{3-\beta}{1-\beta}\right)\theta a(x)f(\underline{v}(x))^{-\beta}f'(\underline{v}(x))t-2M_2,\end{aligned} \tag{4.6}$$

for $\underline{v}(x)\leq t<f^{-1}(t_0)$, where we have used that $2f'(t)t\geq f(t)$ for $t>0$ and item (i) of Lemma 2.2. Now, for $t\geq f^{-1}(t_0)$, by (h_3) we obtain

$$\widehat{g}(x,t)t - \theta\widehat{G}(x,t) \geq -\theta M_1 t - \left(\frac{3-\beta}{1-\beta}\right)\theta a(x) f(\underline{v}(x))^{-\beta} f'(\underline{v}(x))t - M_2. \tag{4.7}$$

Combining the estimates (4.5), (4.6), (4.7) and since $a(x)f(\underline{v})^{-\beta}f'(\underline{v}) \in L^q(\Omega)$, we can write

$$\widehat{g}(x,t)t - \theta\widehat{G}(x,t) \geq -C_1 - C_2 w(x)t, \text{ for all } t \in \mathbb{R} \text{ and a.e. in } x \in \Omega.$$

Hence,

$$C_0 + C_3\|v_n\| \geq \theta\widehat{\Phi}(v_n) - \widehat{\Phi}'(v_n).v_n = \left(\frac{\theta}{2} - 1\right)\|v_n\|^2 + \int_\Omega [\widehat{g}(x,v_n)v_n - \theta\widehat{G}(x,v_n)]$$

which implies that

$$C_0 + C_1|\Omega| + C_4\|w\|_q\|v_n\| + C_3\|v_n\| \geq \left(\frac{\theta}{2} - 1\right)\|v_n\|^2,$$

for some constants $C_1, C_2, C_3, C_4 > 0$. This shows that (v_n) is bounded in $H_0^1(\Omega)$. Therefore, up to subsequence, we have

$$\begin{cases} v_n \rightharpoonup v \text{ in } H_0^1(\Omega), \\ v_n \to v \text{ in } L^s(\Omega),\ 1 \leq s < 2^*, \\ v_n(x) \to v(x) \text{ a.e. in } x \in \Omega. \end{cases} \tag{4.8}$$

By using (h_2) and the Lebesgue Dominated Convergence Theorem we can see that $\|v_n - v\|^2 \to 0$ and therefore the Palais-Smale condition is proved in this case.

Case 2: $\beta > 1$;

We can treat this case similarly to the Case 1 and we omit it.

Case 3: $\beta = 1$;

Assume that $\underline{v}(x) \leq t < f^{-1}(t_0)$. Using (i) of Lemma 2.2, items (i) and (ii) of Lemma 2.1, we have

$$\widehat{g}(x,t)t - \theta\widehat{G}(x,t) \geq a(x)f(s_0)^{-1}f'(s_0)\underline{v} - M_0 t - \theta\left[\int_0^{\underline{v}(x)} \widehat{g}(x,s)ds + \int_{\overline{v}(x)}^{t} \widehat{g}(x,s)ds\right],$$

where $s_0 = f^{-1}(t_0)$. Then, from definition of $\widehat{g}$ and since $a(x)f(s_0)^{-1}f'(s_0)\underline{v} > 0$, we get

$$\widehat{g}(x,t)t - \theta\widehat{G}(x,t) \geq -\theta a(x)f(\underline{v})^{-1}f'(\underline{v})t + \theta H(x, f(\underline{v})) - M_0 t - \theta M_2. \tag{4.9}$$

Now, for $t \geq f^{-1}(t_0)$, using $2f'(t)t \geq f(t)$, (h_3) and $a(x)f(t)^{-1}f'(t) > 0$, we have

$$\begin{aligned} \widehat{g}(x,t)t - \theta\widehat{G}(x,t) &\geq a(x)f(t)^{-1}f'(t)\underline{v} + \big[h(x,f(t))f'(t)t - \theta H(x,f(t))\big] \\ &\quad - \theta h(x,f(\underline{v}))f'(\underline{v})\underline{v} + \theta H(x,f(\underline{v})) - \theta a(x)f(\underline{v})^{-1}f'(\underline{v})t \\ &\geq \theta\big[H(x,f(\underline{v})) - h(x,f(\underline{v}))f'(\underline{v})\underline{v}\big] - \theta a(x)f(\underline{v})^{-1}f'(\underline{v})t. \end{aligned} \tag{4.10}$$

Combining (4.5), (4.9), (4.10), we can write

$$\widehat{g}(x,t)t - \theta\widehat{G}(x,t) \geq -C_1 w_1(x)t - C_2,\ \forall\ t \in \mathbb{R} \text{ and a.e in } x \in \Omega,$$

where $w_1 \in L^q(\Omega)$ and $C_1, C_2 > 0$ are constants. Therefore, we obtain constants $C_0, C_1, C_2, C_3, C_4 > 0$ such that

$$C_0 + C_4\|v_n\| \geq \theta\widehat{\Phi}(v_n) - \widehat{\Phi}'(v_n).v_n = \left(\frac{\theta}{2} - 1\right)\|v_n\|^2 + \int_\Omega [\widehat{g}(x, v_n)v_n - \theta\widehat{G}(x, v_n)]$$
$$\geq \left(\frac{\theta}{2} - 1\right)\|v_n\|^2 - C_3\|w_1\|_q\|v_n\| - C_2|\Omega|,$$

which implies that (v_n) is bounded in $H_0^1(\Omega)$. The same arguments used in the case $\beta < 1$ show that, up to subsequence, $v_n \to v$ in $H_0^1(\Omega)$ for some $v \in H_0^1(\Omega)$. □

Next, we show that the functional $\widehat{\Phi}$ satisfies the geometric conditions of the Mountain Pass Theorem [2].

Lemma 4.2. *Assume that $(h_2) - (h_3)$ are satisfied. The following statements are valid:*

i) *There exist $\varepsilon_1 > 0$, $R > 0$ and $\rho > 0$ such that if $\|b\|_\infty \leq \varepsilon_1$ then*

$$\inf_{v \in \partial B_R(0)} \widehat{\Phi}(v) \geq \rho;$$

ii) *There exists $e \in H_0^1(\Omega) \setminus B_R(0)$ such that $\widehat{\Phi}(e) < 0$.*

Proof. By virtue of (4.4), if $v \in H_0^1(\Omega)$ then

$$\int_\Omega \widehat{G}(x, v) \leq \|l_1\|_q\|v\|_{\frac{q}{q-1}} + C\|b\|_\infty\|v\|^{\frac{r}{2}}.$$

Hence, by the Sobolev embedding

$$\widehat{\Phi}(v) \geq \frac{1}{2}\|v\|^2 - C_1\|l_1\|_q\|v\| - C_2\|b\|_\infty\|v\|^{\frac{r}{2}}.$$

Now, we fix $\|v\| = R > 0$ so that $\sigma := R^2/2 - C_1\|l_1\|_q R > 0$. Thus, for $\|v\| = R$ we get

$$\widehat{\Phi}(v) \geq \sigma - C_2\|b\|_\infty R^{\frac{r}{2}}$$

and choosing $\varepsilon_1 > 0$ such that $\sigma - C_2\varepsilon_1 R^{\frac{r}{2}} := \rho > 0$, the proof of item (i) follows.

To prove item (ii), first we observe that by (h_2) and (h_3) there exists $K \in L^1(\Omega)$ such that

$$H(x, t) \geq \frac{H(x, t_0)}{t_0^{2\theta}}t^{2\theta} - K(x), \quad \forall\, t \geq 0 \text{ and a.e. in } x \in \Omega. \tag{4.11}$$

Moreover, it is easy to see that for $t \geq \|\underline{v}\|_\infty$ one has

$$\widehat{G}(x, t) \geq \widehat{G}(x, \underline{v}(x)) + H(x, f(t)) - H(x, f(\underline{v}(x))), \quad \forall\, t \geq 0 \text{ and a.e. in } x \in \Omega.$$

Thus, by using (ix) of Lemma 2.1, for $t \geq \max\{1, \|\underline{v}\|_\infty\}$ we obtain

$$\widehat{\Phi}(t\underline{v}) \leq \frac{t^2}{2}\|\underline{v}\|^2 - C_2 - \int_\Omega \frac{H(x, t_0)}{t_0^{2\theta}}f(t\underline{v})^{2\theta}$$
$$\leq \frac{t^2}{2}\|\underline{v}\|^2 - C_2 - t^\theta \int_\Omega \frac{H(x, t_0)}{t_0^{2\theta}}f(\underline{v})^{2\theta},$$

which implies that $\widehat{\Phi}(t\underline{v}) \to -\infty$ as $t \to \infty$. Taking $e := t\underline{v}$ with $t > 0$ large we have proved the lemma. □

Finalizing the Proof of Theorem 1.3: According to Lemmas 4.1 and 4.2 and by using Mountain Pass Theorem, see [25], the functional $\widehat{\Phi}$ has a critical point $v_2 \in H_0^1(\Omega)$ satisfying $\widehat{\Phi}(v_2) \geq \rho > 0$ and v_2 is a weak solution of problem (4.2). We claim that $\underline{v} \leq v_2$ almost everywhere in Ω. Indeed, take $\phi = (\underline{v} - v_2)^+$ as a test function in the definition of weak solution of (4.2), we get

$$\begin{aligned}\int_\Omega \nabla v_2 \nabla(\underline{v} - v_2)^+ &= \int_{\{v_2<\underline{v}\}} \widehat{g}(x, v_2)(\underline{v} - v_2)^+ \\ &= \int_{\{v_2<\underline{v}\}} [a(x)f(\underline{v})^{-\beta} f'(\underline{v}) + h(x, f(\underline{v}))f'(\underline{v})](\underline{v} - v_2)^+ \\ &\geq \int_\Omega \nabla \underline{v} \nabla(\underline{v} - v_2)^+,\end{aligned}$$

which implies that $\|(\underline{v} - v_2)^+\|=0$ and therefore $(\underline{v} - v_2)^+ = 0$, that is, $\underline{v} \leq v_2$ almost everywhere in Ω. Hence, v_2 is a weak solution of (1.10). On the other hand, as $\underline{v}$ is a subsolution of (1.10) and by definition $\widehat{G}(x, \underline{v}) = a(x)f(\underline{v})^{-\beta} f'(\underline{v}) + h(x, f(\underline{v}))f'(\underline{v})\underline{v}$, we have

$$\Phi(v_1) \leq \Phi(\underline{v}) = \widehat{\Phi}(\underline{v}) < 0 < \Phi(v_2)$$

and the proof of Theorem 1.3 is finished. □

Remark 4.3. By the same arguments used in the proof of Theorem 1.3 and replacing $\underline{v}$ by v_1 in (4.1), we can conclude that $v_1 \leq v_2$ almost everywhere in Ω.

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