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Asymptotic properties of standing waves for Maxwell-Schrödinger-Poisson system

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ABSTRACT

In this paper, we study the asymptotic properties of minimizers for a class of constraint minimization problems derived from the Maxwell-Schrödinger-Poisson system

$$-\Delta u - (|u|^2 * |x|^{-1})u - \alpha|u|^{2p}u - \mu_p u = 0, \;\; x \in \mathbb{R}^3$$

on the L^2-spheres $\mathcal{A}_\lambda = \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} |u|^2 dx = \lambda\}$, where $\alpha, p > 0$. Let $\lambda^* = \|Q_{\frac{2}{3}}\|_2^2$, and $Q_{\frac{2}{3}}$ is the unique (up to translations) positive radial solution of $-\frac{3p}{2}\Delta u + \frac{2-p}{2}u - |u|^{2p}u = 0$ in $\mathbb{R}^3$ with $p = \frac{2}{3}$. We prove that if $\lambda < \alpha^{-\frac{3}{2}}\lambda^*$, then minimizers are relatively compact in $\mathcal{A}_\lambda$ as $p \nearrow \frac{2}{3}$. On the contrary, if $\lambda > \alpha^{-\frac{3}{2}}\lambda^*$, by directly using asymptotic analysis, we prove that all minimizers must blow up and give the detailed asymptotic behavior of minimizers.

1. Introduction and main results

Due to its importance in various physical frameworks: gravitation, plasma physics, semiconductor theory, quantum chemistry and so on (see, e.g. [4,13,14] and the reference therein), the following X^α-Schrödinger-Poisson (X^α-SP) model or Maxwell-Schrödinger-Poisson system has been studied extensively in recent years, see [3–5,8,9,14,16,18] for instance. The wave function $\psi : \mathbb{R}^3 \times [0, T) \to \mathbb{C}$ satisfies

$$\begin{cases} i\dfrac{\partial \psi}{\partial t} = -\Delta_x \psi + V(x,t)\psi - \alpha|\psi(x,t)|^{2p}\psi, \\ -\Delta_x V = \epsilon 4\pi |\psi|^2, \\ \psi(x,0) = \phi(x) \end{cases} \tag{1.1}$$

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with $\phi \in L^2(\mathbb{R}^3)$, and $\alpha, p > 0$. The self-consistent Poisson potential V can be rewritten explicitly in the form of $V(x,t) = \epsilon|\psi(x,t)|^2 * |x|^{-1}$, where $*$ refers to the convolution with respect to x on $\mathbb{R}^3$ and ϵ takes the value $+1$ or -1, depending on whether the interaction between the particles is repulsive or attractive. The system (1.1) can be therefore reduced to a single nonlinear and nonlocal Schrödinger-type equation

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} = -\Delta_x\psi + \epsilon(|\psi|^2 * |x|^{-1})\psi - \alpha|\psi|^{2p}\psi, \\ \psi(x,0) = \phi(x). \end{cases} \tag{1.2}$$

Such a model appears in various frameworks, such as black holes in gravitation ($\epsilon = -1$, see [17]), one-dimensional reduction of electron density in plasma physics ($\epsilon = +1$), as well as in semiconductor theory ($\epsilon = +1$), as a correction to the Schrödinger-Poisson system (which is X^α-SP with $\alpha = 0$), see [4,13,14,16] and the reference therein. The last term $|\psi(x,t)|^{2p}\psi$ is usually considered to be a correction to the nonlocal term $V\psi$, for example, $p = \frac{1}{3}$, which is called the Slater correction, or $p = \frac{2}{3}$, which is named as Dirac correction. The interested reader is recommended to find more backgrounds in the reference, see [5,18] and the reference therein.

In the following, we will be concerned with the standing waves, that is, solutions to (1.2) of the form

$$\psi(x,t) = e^{-i\mu_p t}u(x)$$

with $\mu_p \in \mathbb{R}$ and $u \in H^1(\mathbb{R}^3)$ solving

$$-\Delta u + \epsilon(u^2 * |x|^{-1})u - \alpha|u|^{2p}u - \mu_p u = 0,$$

which is a special case of Schrödinger-Maxwell equations [8]. It is well known that, minimizers of the following minimization problem solve the above equation with μ_p being some Lagrange multiplier:

$$e(p,\lambda) := \inf_{u \in \mathcal{A}_\lambda} E_p(u), \quad \lambda > 0, \tag{1.3}$$

where the functional $E_p(\cdot)$ is given by

$$E_p(u) := \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\epsilon}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|} dxdy - \frac{\alpha}{p+1}\int_{\mathbb{R}^3} |u|^{2p+2} dx$$

and

$$\mathcal{A}_\lambda = \big\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} |u|^2 dx = \lambda\big\}.$$

In the repulsive case $\epsilon = +1$, many existence results have been known. In [11], a negative answer was given to $p = 0$. In [6], a positive answer is given to $p \in (0, \frac{1}{2})$ with $\lambda > 0$ small. In [18], as part of its results, a positive answer was obtained to the Slater correction case: $p = \frac{1}{3}$. In [2], Bellazzini and Siciliano proved that (1.3) admits at least one minimizer if $p \in (\frac{1}{2}, \frac{2}{3})$ and $\lambda > 0$ is large enough. In [12], Jeanjean and Luo showed the sharp nonexistence results for (1.3) with $p \in [\frac{1}{2}, \frac{2}{3}]$, i.e. for $p \in (\frac{1}{2}, \frac{2}{3})$, there exists $\lambda_1 > 0$ such that (1.3) has a minimizer if and only if $\lambda \geq \lambda_1$. When $p = \frac{1}{2}$ or $p = \frac{2}{3}$, no minimizer exists for all $\lambda > 0$. For $\frac{2}{3} < p < 2$, problem (1.3) does not work. It has been proved in [1] that there exists at least one critical point of $E(u)$ restricted to $\mathcal{A}$ with a minimax characterization.

For the attractive case $\epsilon = -1$, the existence of minimizer for (1.3) is quite well understood. Before stating the result, we first recall from [19] the following Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}^3} u^{2p+2}dx \le \frac{p+1}{\lambda_p^{*p}}\Big(\int_{\mathbb{R}^3} |\nabla u|^2 dx\Big)^{\frac{3p}{2}}\Big(\int_{\mathbb{R}^3} u^2 dx\Big)^{\frac{2-p}{2}}, \quad u \in H^1(\mathbb{R}^3), \tag{1.4}$$

where $\lambda_p^* = \|Q_p\|_2^2$ with $Q_p(x) = Q_p(|x|)$ optimizing the above inequality and being the unique positive radially symmetric solution of

$$-\frac{3p}{2}\Delta u + \frac{2-p}{2}u - |u|^{2p}u = 0 \ \text{ in } \ \mathbb{R}^3, \ \text{ where } \ p \in (0,2). \tag{1.5}$$

It follows directly from Lemma 8.1.2 in [7] that $Q_p(|x|)$ satisfies

$$\int_{\mathbb{R}^3} |\nabla Q_p|^2 dx = \int_{\mathbb{R}^3} Q_p^2 dx = \frac{1}{p+1}\int_{\mathbb{R}^3} Q_p^{2p+2}dx. \tag{1.6}$$

Moreover, a simple analysis shows that Q_p satisfies

$$Q_p(x) \to Q_{\frac{2}{3}}(x) \ \text{ strongly in } \ H^1(\mathbb{R}^3) \ \text{ and } \ \lambda_p^* \to \lambda^* := \|Q_{\frac{2}{3}}\|_2^2 \ \text{ as } \ p \nearrow \frac{2}{3}.$$

Similar to Theorem 1.1 in [21], the existence of the minimizer for (1.3) with $\epsilon = -1$ is established by making full use of the above $Q_p(x)$ and the Gagliardo-Nirenberg inequality (1.4), and we omit the details for simplicity.

Theorem 1.1. *Let Q_p be the unique (up to translations) positive radial solution of (1.5). Then, we have the followings:*

(I). *If $0 < p < \frac{2}{3}$, then there exists at least one minimizer of (1.3) for any $\lambda \in (0, +\infty)$.*
(II). *If $\frac{2}{3} < p < 2$, then there is no minimizer of (1.3) for any $\lambda \in (0, +\infty)$.*
(II). *If $p = \frac{2}{3}$, then we have:*
- (II)$_1$. *If $0 < \lambda < \alpha^{-\frac{3}{2}}\lambda^* := \alpha^{-\frac{3}{2}}\|Q_{\frac{2}{3}}\|_2^2$, there exists at least one minimizer for (1.3).*
- (II)$_2$. *If $\lambda \ge \alpha^{-\frac{3}{2}}\lambda^*$, there is no minimizer for (1.3).*

We remark that there exists at least one minimizer u_p for (1.3) if $p \in (0, \frac{2}{3})$. In what follows, we investigate the limit behavior of minimizers of (1.3) as $p \nearrow \frac{2}{3}$. Firstly, if $0 < \lambda < \alpha^{-\frac{3}{2}}\lambda^*$ is fixed, our result shows that the minimizers of (1.3) are relatively compact in the space $\mathcal{A}_\lambda$ as $p \nearrow \frac{2}{3}$. More precisely, we have

Theorem 1.2. *For any given $0 < \lambda < \alpha^{-\frac{3}{2}}\lambda^*$, and let u_p be a nonnegative minimizer of (1.3) for each $p \in (0, \frac{2}{3})$. Then,*

$$\lim_{p \nearrow \frac{2}{3}} e(p, \lambda) = e(\frac{2}{3}, \lambda).$$

Moreover, for any sequence $\{p_k\}$ with $p_k \nearrow \frac{2}{3}$, there exists a subsequence, still denoted by $\{p_k\}$ and a sequence $\{y_{p_k}\}$, such that

$$u_{p_k}(x + y_{p_k}) \xrightarrow{k\to\infty} u_0 \in \mathcal{A}_\lambda \ \textit{ with } u_0 \textit{ being a minimizer of } e(\frac{2}{3}, \lambda).$$

On the contrary, if $\lambda > \alpha^{-\frac{3}{2}}\lambda^*$, the result is quit different and blow-up will happen in minimizers as $p \nearrow \frac{2}{3}$. Our main results in this direction can be stated as the following theorem.

Theorem 1.3. *Suppose that* $\lambda > \alpha^{-\frac{3}{2}}\lambda^*$ *and let* u_p *be a nonnegative minimizer of (1.3) for each* $p \in (0, \frac{2}{3})$. *For any sequence* $\{p_k\}$ *with* $p_k \nearrow \frac{2}{3}$, *then up to subsequence, such that each* u_{p_k} *has a unique maximum point* x_k, *and*

$$\lim_{k\to\infty} s_{p_k}^{-\frac{3}{4}} u_{p_k}(x_k + s_{p_k}^{-\frac{1}{2}} x) = \frac{1}{\lambda^{\frac{1}{4}}\lambda^{*\frac{1}{2}}} Q_{\frac{2}{3}}(\frac{x}{\lambda^{\frac{1}{2}}}) \text{ strongly in } H^1(\mathbb{R}^3),$$

where $s_{p_k} = \left(\frac{3p_k}{2}\frac{\alpha\lambda^{\frac{2-p_k}{2}}}{\lambda^{*\,p_k}_{p_k}}\right)^{\frac{2}{2-3p_k}}$. *Moreover,*

$$\lim_{p\nearrow\frac{2}{3}} \frac{3p}{3p-2} s_p^{-1} e(p,\lambda) = 1.$$

2. Asymptotic behavior of minimizers

In this section, we shall establish Theorem 1.2 and Theorem 1.3, which is focused on the asymptotic behavior of minimizers for $e(p,\lambda)$ as $p \nearrow \frac{2}{3}$. Under the assumptions of Theorem 1.1 (I), let u_p is a nonnegative minimizers for (1.3), which satisfies the Euler-Lagrange equation

$$-\Delta u_p - (|u_p|^2 * |x|^{-1})u_p - \alpha|u_p|^{2p}u_p - \mu_p u_p = 0 \quad \text{in} \quad \mathbb{R}^3, \tag{2.1}$$

where $\mu_p \in \mathbb{R}$ is a suitable Lagrange multiplier associated to u_p.

2.1. Case of $0 < \lambda < \alpha^{-\frac{3}{2}}\lambda^*$

The aim of this subsection is to prove that when $0 < \lambda < \alpha^{-\frac{3}{2}}\lambda^*$ is fixed, all minimizers of (1.3) are relatively compact in the space $\mathcal{A}_\lambda$ as $p \nearrow \frac{2}{3}$, which gives the proof of Theorem 1.2.

Lemma 2.1. *For any given* $0 < \lambda < \alpha^{-\frac{3}{2}}\lambda^*$ *and* $p \nearrow \frac{2}{3}$, $\{u_p\}$ *is bounded in* $\mathcal{A}_\lambda$.

Proof. For any $\lambda, t > 0$, let $Q^t(x) = \frac{\lambda^{\frac{1}{2}}t^{\frac{3}{2}}}{\lambda^{*\frac{1}{2}}} Q_{\frac{2}{3}}(tx)$, where $Q_{\frac{2}{3}}$ is the unique positive radial solution of (1.5) with $p = \frac{2}{3}$. Then $Q^t(x) \in \mathcal{A}_\lambda$ and

$$E_p(Q^t) = \lambda t^2 - \frac{\lambda^2 t}{\lambda^{*2}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{Q^2(x)Q^2(y)}{|x-y|} dxdy - \frac{\alpha\lambda^{p+1}t^{3p}}{(p+1)\lambda^{*p+1}} \int_{\mathbb{R}^3} Q_{\frac{2}{3}}^{2p+2} dx,$$

which implies that

$$e(p,\lambda) \le \inf_{t>0} E_p(Q^t) < 0. \tag{2.2}$$

We combine the Hardy-Littlewood-Sobolev inequality and the Sobolev inequality to yield that there exists a positive constant C such that

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|} dxdy \le& C\lambda^{\frac{3}{2}} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^{\frac{1}{2}} \\ \le& \epsilon \int_{\mathbb{R}^3} |\nabla u|^2 dx + C\epsilon^{-1}\lambda^3. \end{aligned} \tag{2.3}$$

It follows from (1.4), (2.2) and (2.3) that

$$\int_{\mathbb{R}^3} |\nabla u_p|^2 dx < \epsilon \int_{\mathbb{R}^3} |\nabla u_p|^2 dx + C_\epsilon + \frac{\alpha\lambda^{\frac{2-p}{2}}}{\lambda_p^{*p}} (\int_{\mathbb{R}^3} |\nabla u_p|^2)^{\frac{3p}{2}}.$$

We claim that

$$\limsup_{p \nearrow \frac{2}{3}} \int_{\mathbb{R}^3} |\nabla u_p|^2 dx < +\infty. \tag{2.4}$$

On the contrary, suppose there exists a subsequence such that

$$\lim_{p \nearrow \frac{2}{3}} \int_{\mathbb{R}^3} |\nabla u_p|^2 dx = \infty,$$

which implies that

$$\int_{\mathbb{R}^3} |\nabla u_p|^2 dx \le (\epsilon + \frac{3p}{2} \frac{\alpha\lambda^{\frac{2-p}{2}}}{\lambda_p^{*p}}) \int_{\mathbb{R}^3} |\nabla u_p|^2 dx + C_\epsilon.$$

Noting that

$$\lim_{p \nearrow \frac{2}{3}} \frac{3p}{2} \frac{\alpha\lambda^{\frac{2-p}{2}}}{\lambda_p^{*p}} < 1,$$

then there exists $\epsilon > 0$, such that

$$\epsilon + \frac{3p}{2} \frac{\alpha\lambda^{\frac{2-p}{2}}}{\lambda_p^{*p}} < 1 \text{ as } p \nearrow \frac{2}{3}.$$

This leads to a contradiction, thus (2.4) is obtained. From (2.4), we see that $\{u_p\}$ is bounded in $\mathcal{A}_\lambda$. $\square$

It then follows from Lemma 2.1 that for any sequence $\{p_k\}$ with $p_k \nearrow \frac{2}{3}$, there exist a subsequence, still denoted by $\{p_k\}$ and $\tilde{u} \in \mathcal{A}_\lambda$, such that $u_{p_k} \rightharpoonup \tilde{u}$ in $\mathcal{A}_\lambda$ as $k \to \infty$. We next prove that

$$\int_{\mathbb{R}^3} u_{p_k}^{2p_k+2} dx - \int_{\mathbb{R}^3} u_{p_k}^{\frac{10}{3}} dx \xrightarrow{k} 0. \tag{2.5}$$

Choosing $s > \frac{3}{10}$, it follows from Hölder inequality that

$$\|u_{p_k}\|_{\frac{10}{3}} \le \|u_{p_k}\|_{2p_k+2}^{\alpha_k} \|u_{p_k}\|_s^{1-\alpha_k}, \quad \alpha_k = \frac{(2p_k+2)(3s-10)}{10(s-2p_k-2)},$$

and

$$\|u_{p_k}\|_{2p_k+2} \le \|u_{p_k}\|_{\frac{10}{3}}^{\beta_k} \|u_{p_k}\|_2^{1-\beta_k}, \quad \beta_k = \frac{5p_k}{2p_k+2}.$$

Then,

$$\|u_{p_k}\|_{\frac{10}{3}}^{\frac{1}{\alpha_k}} \|u_{p_k}\|_s^{\frac{\alpha_k-1}{\alpha_k}} \le \|u_{p_k}\|_{2p_k+2} \le \lambda^{\frac{1-\beta_k}{2}} \|u_{p_k}\|_{\frac{10}{3}}^{\beta_k}.$$

By Lemma 2.1, assuming that $\|u_{p_k}\|_{\frac{10}{3}} \xrightarrow{k} a$, $\|u_{p_k}\|_s \xrightarrow{k} b$ and $\|u_{p_k}\|_{2p_k+2} \xrightarrow{k} c$. Noting that $\alpha_k, \beta_k \xrightarrow{k} 1$, which implies that $a = c$. (2.5) is therefore proved. From (2.5) we see that

$$\lim_{k\to\infty} e(p_k, \lambda) = \lim_{k\to\infty} E_{p_k}(u_{p_k}) = \lim_{k\to\infty} E_{\frac{2}{3}}(u_{p_k}) \ge e(\frac{2}{3}, \lambda).$$

On the other hand, suppose that $\bar{u}$ is a minimizer of $e(\frac{2}{3}, \lambda)$, then

$$e(\frac{2}{3}, \lambda) = E_{\frac{2}{3}}(\bar{u}) = \lim_{k\to\infty} E_{p_k}(\bar{u}) \ge \lim_{k\to\infty} e(p_k, \lambda),$$

which implies that

$$\lim_{k\to\infty} e(p_k, \lambda) = e(\frac{2}{3}, \lambda).$$

Since the above argument holds for any sequence $\{p_k\}$ satisfying $\lim_{k\to\infty} p_k = \frac{2}{3}$, we thus have

$$\lim_{p\nearrow\frac{2}{3}} e(p, \lambda) = e(\frac{2}{3}, \lambda).$$

Proof of Theorem 1.2. In view of above facts, $\{u_{p_k}\} \subset \mathcal{A}_\lambda$ is a bounded minimizing sequence for $e(\frac{2}{3}, \lambda)$. By the concentration-compactness principle, there exists a sequence $\{y_k\}$ such that $u_{p_k}(\cdot + y_k)$ is relatively compact in $L^p(\mathbb{R}^3)$ for $2 \le p < 6$. Therefore, there exists a subsequence still denoted by $\{p_k\}$ and $u_0 \in \mathcal{A}_\lambda$ such that

$$\lim_{k\to\infty} u_{p_k}(x + y_k) = u_0(x) \text{ strongly in } L^p(\mathbb{R}^3) \text{ for } 2 \le p < 6.$$

Using the weak lower semicontinuity, we have

$$e(\frac{2}{3}, \lambda) \le E_{\frac{2}{3}}(u_0) \le \lim_{k\to\infty} E_{\frac{2}{3}}(u_{p_k}) = e(\frac{2}{3}, \lambda).$$

This completes the proof of Theorem 1.2. $\square$

2.2. Case of $\lambda > \alpha^{-\frac{3}{2}}\lambda^*$

In the following, we intend to prove that all minimizers must blow up in $\mathcal{A}_\lambda$ as $p \nearrow \frac{2}{3}$. Towards this purpose, we introduce the following auxiliary minimization problem as:

$$\tilde{e}(p, \lambda) = \inf \left\{ \tilde{E}_p(u) : \int_{\mathbb{R}^3} |u|^2 dx = \lambda \right\}, \tag{2.6}$$

where $\tilde{E}_p(u)$ is defined by

$$\tilde{E}_p(u) = \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{\alpha}{p+1} \int_{\mathbb{R}^3} |u|^{2p+2} dx.$$

Similar to Lemma 3.1 in [20], the exact value of the minimum energy of (2.6) as well as the precise form of its minimizers are established, and we omit the details for simplicity.

Lemma 2.2. *Let $p \in (0, \frac{2}{3})$ and Q_p is the unique positive radial solution of (1.5). Then*

$$\tilde{e}(p,\lambda) = \frac{3p-2}{3p} s_p, \ \text{where } s_p = \left(\frac{3p}{2} \frac{\alpha \lambda^{\frac{2-p}{2}}}{\lambda_p^{*p}} \right)^{\frac{2}{2-3p}}, \tag{2.7}$$

and the unique (up to translations) positive minimizer of (2.6) must be of the form

$$\tilde{Q}_p(x) = \left(\frac{\lambda}{\lambda_p^*} \right)^{\frac{1}{2}} t_p^{\frac{3}{2}} Q_p(t_p x), \ \text{where } t_p = \left(\frac{s_p}{\lambda} \right)^{\frac{1}{2}}.$$

Denote now u_p to be a nonnegative minimizer of (1.3). In the following we shall derive refined estimates on $\|\nabla u_p\|_2$.

Lemma 2.3. *There exists a positive constant K, independent of p, such that*

$$K \leq s_p^{-1} \int_{\mathbb{R}^3} |\nabla u_p|^2 \leq \frac{1}{K} \quad \text{as} \quad p \nearrow \frac{2}{3},$$

where s_p is defined in (2.7).

Proof. We first give the lower bound of $\int_{\mathbb{R}^3} |\nabla u_p|^2 dx$. Using (1.4) and (2.3), we have

$$E_p(u) \geq (1-\epsilon) \int_{\mathbb{R}^3} |\nabla u|^2 dx - \frac{\alpha \lambda^{\frac{2-p}{2}}}{\lambda_p^{*p}} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{\frac{3p}{2}} - C\epsilon^{-1}\lambda^3.$$

Set

$$f(s) = (1-\epsilon)s - \frac{\alpha \lambda^{\frac{2-p}{2}}}{\lambda_p^{*p}} s^{\frac{3p}{2}},$$

then $f(s)$ has a unique minima $s_* = (\frac{1}{1-\epsilon})^{\frac{2}{2-3p}} s_p$.

We claim that there exists $\gamma > 0$ small such that for $p \nearrow \frac{2}{3}$,

$$\frac{\gamma^{\frac{3p}{2}} - \frac{3p}{2}\gamma}{1 - \frac{3p}{2}} + \gamma < 1. \tag{2.8}$$

In fact, taking $\gamma > 0$ sufficiently small such that $-\gamma \ln \gamma + 2\gamma < \frac{1}{2}$, then the conclusion follows by taking the limit $p \nearrow \frac{2}{3}$ in (2.8).

Let $\gamma > 0$ be chosen as in the above claim. We then claim that

$$s_p^{-1} \int_{\mathbb{R}^3} |\nabla u_p|^2 \geq \gamma, \quad \text{for } p \nearrow \frac{2}{3}. \tag{2.9}$$

Otherwise, there exists a sequence $\{p_k\}$ with $p_k \nearrow \frac{2}{3}$ as $k \to \infty$ satisfying

$$\int_{\mathbb{R}^3} |\nabla u_{p_k}|^2 dx < \gamma s_{p_k}.$$

Choose $\epsilon_k = \frac{2-3p_k}{3p_k} > 0$ small such that $s_* = (\frac{1}{1-\epsilon_k})^{\frac{2}{2-3p_k}} s_{p_k} > \gamma s_{p_k}$ for $p_k \nearrow \frac{2}{3}$, it then yields that

$$\begin{aligned}\frac{3p_k-2}{3p_k}s_{p_k} =&\tilde{e}(p_k,\lambda) \geq e(p_k,\lambda) \geq f\left(\int\limits_{\mathbb{R}^3}|\nabla u_{p_k}|^2dx\right) - C\epsilon_k^{-1}\lambda^3 \\ \geq& f(\gamma s_{p_k}) - C\epsilon_k^{-1}\lambda^3 = -\left(\frac{2}{3p_k}\gamma^{\frac{3p_k}{2}} - \gamma + \frac{2-3p_k}{3p_k}\gamma\right)s_{p_k} - C\epsilon_k^{-1}\lambda^3.\end{aligned}$$

Thus,

$$-C\lambda^3 \leq -\frac{2-3p_k}{3p_k}\left(\frac{2-3p_k}{3p_k} - \frac{2}{3p_k}\gamma^{\frac{3p_k}{2}} + \gamma - \frac{2-3p_k}{3p_k}\gamma\right)s_{p_k} \xrightarrow{k} -\infty,$$

which leads to a contradiction.

To get the upper bound of $\int_{\mathbb{R}^3}|\nabla u_p|^2dx$, we first deduce from the Pohožaev identity that

$$2\int\limits_{\mathbb{R}^3}|\nabla u_p|^2 = \frac{1}{2}\int\limits_{\mathbb{R}^3}\int\limits_{\mathbb{R}^3}\frac{u_p^2(x)u_p^2(y)}{|x-y|} + \frac{3\alpha p}{p+1}\int\limits_{\mathbb{R}^3}|u_p|^{2p+2}. \tag{2.10}$$

Therefore, we combine (1.4), (2.3) and (2.10) to yield that

$$2\int\limits_{\mathbb{R}^3}|\nabla u_p|^2dx \leq \frac{3\alpha p\lambda^{\frac{2-p}{2}}}{\lambda_p^{*p}}\left(\int\limits_{\mathbb{R}^3}|\nabla u_p|^2dx\right)^{\frac{3p}{2}} + C\lambda^{\frac{3}{2}}\left(\int\limits_{\mathbb{R}^3}|\nabla u_p|^2dx\right)^{\frac{1}{2}}.$$

Then, there exists constant $\bar{\lambda} > 1$, such that

$$s_p^{-1}\int\limits_{\mathbb{R}^3}|\nabla u_p|^2dx \leq \left(1 - C\lambda^{\frac{3}{2}}\gamma^{-\frac{1}{2}}\bar{\lambda}^{-\frac{1}{2-3p}}\right)^{-\frac{2}{2-3p}} \to 1 \text{ as } p \nearrow \frac{2}{3}.$$

This completes the proof of Lemma 2.3. $\square$

In view of above facts, we next define the $L^2(\mathbb{R}^2)$-normalized function

$$w_p(x) := s_p^{-\frac{3}{4}}u_p\big(s_p^{-\frac{1}{2}}x + x_p\big), \tag{2.11}$$

where x_p is a global maximum point of u_p. It follows from Lemma 2.3 that

$$K \leq \int\limits_{\mathbb{R}^3}|\nabla w_p|^2 \leq \frac{1}{K} \text{ as } p \nearrow \frac{2}{3}. \tag{2.12}$$

Before proving Theorem 1.3, we first establish the following lemma.

Lemma 2.4. *There exists a positive constant η such that*

$$\liminf_{p\nearrow\frac{2}{3}}\int\limits_{B_2(0)}|w_p|^2dx \geq \eta > 0. \tag{2.13}$$

Proof. In view of (2.1), $w_p(x)$ defined in (2.11) satisfies the elliptic equation

$$-\Delta w_p(x) - s_p^{-\frac{1}{2}}(w_p^2 * |x|^{-1})w_p - \alpha s_p^{\frac{3p-2}{2}} w_p^{2p+1} - s_p^{-1}\mu_p w_p = 0 \quad \text{in } \mathbb{R}^3. \tag{2.14}$$

We first claim that there exists a positive constant M, independent of p, such that

$$-M \le s_p^{-1}\mu_p \le -\frac{1}{M} \quad \text{as } p \nearrow \frac{2}{3}. \tag{2.15}$$

In fact, using (2.10) we have

$$-\lambda\mu_p = 3\int_{\mathbb{R}^3} |\nabla u_p|^2 - \frac{(5p-1)\alpha}{p+1}\int_{\mathbb{R}^3} |u_p|^{2p+2} \le 3\int_{\mathbb{R}^3} |\nabla u_p|^2 \quad \text{as } p \nearrow \frac{2}{3}.$$

On the other hand, it follows from (2.2) and (2.10) that

$$-\lambda\mu_p = -e(p,\lambda) + \frac{\alpha p}{p+1}\int_{\mathbb{R}^3} |u_p|^{2p+2} + \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_p^2(x)u_p^2(y)}{|x-y|} \ge \frac{2}{3}\int_{\mathbb{R}^3} |\nabla u_p|^2.$$

Therefore, (2.15) holds by Lemma 2.3.

Denote $\phi_{w_p}(x) = \int_{\mathbb{R}^3} \frac{w_p^2(y)}{|x-y|}dy$. It follows from Hölder inequality that there exists a constant C, independent of p, such that

$$\begin{aligned}\phi_{w_p}(x) &= \int_{|x-y|<1} \frac{w_p^2(y)}{|x-y|}dy + \int_{|x-y|\ge 1} \frac{w_p^2(y)}{|x-y|}dy \\ &\le \Big(\int_{|x-y|<1} \frac{1}{|x-y|^{\frac{3}{2}}}dy\Big)^{\frac{2}{3}}\Big(\int_{|x-y|<1} w_p^6(y)dy\Big)^{\frac{1}{3}} + \int_{|x-y|\ge 1} w_p^2(y)dy \\ &\le C\|w_p\|_6^2 + \lambda \le C \quad \text{as} \quad p \nearrow \frac{2}{3}.\end{aligned} \tag{2.16}$$

Note from (2.14)-(2.16) that $-\Delta w_p - c(x)w_p \le 0$ in $\mathbb{R}^3$, where $c(x) = \alpha s_p^{\frac{3p-2}{2}} w_p^{2p}(x)$. By applying Theorem 4.1 in [10], we then have

$$\max_{B_1(0)} w_p \le C\Big(\int_{B_2(0)} |w_p|^2 dx\Big)^{\frac{1}{2}}, \tag{2.17}$$

where $C > 0$ depends only on the upper bound of $\|c(x)\|_{L^2(B_2(0))}$, i.e., the upper bound of $\|w_p\|_{L^{4p}(B_2(0))}$. Therefore, it then follows from (1.4) that the constant $C > 0$ in (2.17) is bounded uniformly as $p \nearrow \frac{2}{3}$. Since $w_p(x)$ attains its local maximum at $x = 0$, we thus obtain from (2.14)-(2.16) that

$$0 \le s_p^{-1}\mu_p w_p(0) + s_p^{-\frac{1}{2}}\phi_{w_p}(0)w_p(0) + \alpha s_p^{\frac{3p-2}{2}} w_p^{2p+1}(0) \le -Cw_p(0) + C' w_p^{2p+1}(0) \quad \text{as} \quad p \nearrow \frac{2}{3},$$

which implies that $w_p(0) \ge C > 0$ as $p \nearrow \frac{2}{3}$. Then (2.13) holds, and this completes the proof of Lemma 2.4. □

Proof of Theorem 1.3. We are now ready to complete the proof of Theorem 1.3 by the following two steps. *Step 1: The detailed asymptotic behavior.* For any given sequence $\{p_k\}$ with $p_k \nearrow \frac{2}{3}$ as $k \to \infty$, we denote $w_k(x) := w_{p_k}(x) = s_k^{-\frac{3}{4}} u_{\lambda_k}\left(x_k + s_k^{-\frac{1}{2}} x\right) \geq 0$, and $s_k := s_{p_k} > 0$, where s_k is defined by (2.7) and satisfies $s_k \to \infty$ as $k \to \infty$. In view of (2.14), $w_k(x)$ satisfies the Euler-Lagrange equation

$$-\Delta w_k(x) - s_k^{-\frac{1}{2}}(w_k^2 * |x|^{-1})w_k - \alpha s_k^{\frac{3p_k-2}{2}} w_k^{2p_k+1} - s_k^{-1}\mu_k w_k = 0 \ \text{ in } \ \mathbb{R}^3, \tag{2.18}$$

and note from (2.15), up to a subsequence, that there exists a positive constant β, such that

$$s_k^{-1}\mu_k \to -\beta \ \text{ as } \ k \to \infty.$$

Therefore, by passing to a subsequence if necessary, we deduce from (2.12) that $w_k \rightharpoonup w_0 \geq 0$ in $H^1(\mathbb{R}^3)$ for some $w_0 \in H^1(\mathbb{R}^3)$. Moreover, by passing to the weak limit of (2.18), the nonnegative function w_0 satisfies

$$-\Delta w_0(x) + \beta w_0(x) = \left(\frac{\lambda^*}{\lambda}\right)^{\frac{2}{3}} w_0^{\frac{7}{3}} \quad \text{in } \ \mathbb{R}^3. \tag{2.19}$$

Furthermore, it follows from Lemma 2.4 that $w_0 \not\equiv 0$, and thus $w_0 > 0$ by the strong maximum principle. By a simple rescaling,

$$w_0(x) = \left(\frac{\lambda}{\lambda^*}\right)^{\frac{1}{2}} \left(\frac{3}{2}\beta\right)^{\frac{3}{4}} Q_{\frac{2}{3}}\left((\frac{3}{2}\beta)^{\frac{1}{2}}|x - y_0|\right) \quad \text{for some} \quad y_0 \in \mathbb{R}^3, \tag{2.20}$$

where $Q_{\frac{2}{3}}$ is a positive radially symmetric solution of (1.5) with $p = \frac{2}{3}$. Note that $\int_{\mathbb{R}^3} |w_0(x)|^2 dx = \lambda$. By the norm preservation we further conclude that

$$w_k \xrightarrow{k} w_0 \ \text{ strongly in } \ L^2(\mathbb{R}^3).$$

Together with the boundness of $H^1(\mathbb{R}^3)$ norm for w_k, this implies that

$$w_k \xrightarrow{k} w_0 \ \text{ strongly in } \ L^2(\mathbb{R}^3) \ \text{ for any } \ 2 \leq p < 6.$$

Moreover, since w_k and w_0 satisfy (2.18) and (2.19), respectively, a simple analysis shows that

$$w_k \xrightarrow{k} w_0 = \left(\frac{\lambda}{\lambda^*}\right)^{\frac{1}{2}} \left(\frac{3}{2}\beta\right)^{\frac{3}{4}} Q_{\frac{2}{3}}\left((\frac{3}{2}\beta)^{\frac{1}{2}}|x - y_0|\right) \text{ strongly in } \ H^1(\mathbb{R}^3). \tag{2.21}$$

Using the standard elliptic regular theory, one can further obtain that

$$w_k \xrightarrow{k} w_0 \text{ in } C^2_{loc}(\mathbb{R}^3). \tag{2.22}$$

We notice that the origin is a critical (local maximum) point of w_k for all $k > 0$, in view of (2.22) it is also a critical point of w_0. We therefore conclude that w_0 is spherically symmetric about the origin, i.e. $y_0 = (0, 0, 0)$ in (2.21) and

$$w_0(x) = \left(\frac{\lambda}{\lambda^*}\right)^{\frac{1}{2}} \left(\frac{3}{2}\beta\right)^{\frac{3}{4}} Q_{\frac{2}{3}}\left((\frac{3}{2}\beta)^{\frac{1}{2}}|x|\right). \tag{2.23}$$

Since w_k decays uniformly to zero w.r.t. k as $|x| \to \infty$, all local maximum points of w_k stay in a finite ball in $\mathbb{R}^3$. It then follows from (2.22) and Lemma 4.2 in [15] that for large k, w_k has no critical points other

than the origin. This gives the uniqueness of local maximum points for $w_k(x)$, which therefore implies that x_k is the unique maximum point of u_k.
Step 2: The exact value of β defined in (2.23). Using (2.10) we have

$$e(p_k,\lambda) = \frac{3p_k-2}{3p_k} s_k \int_{\mathbb{R}^3} |\nabla w_k|^2 + \frac{1-3p_k}{6p_k} s_k^{\frac{1}{2}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{w_k^2(x)w_k^2(y)}{|x-y|}.$$

It follows from (2.21) that

$$\lim_{k\to\infty} \frac{3p_k}{3p_k-2} s_k^{-1} e(p_k,\lambda) = \frac{3}{2}\lambda\beta.$$

On the other hand, by Lemma 2.2 we see that

$$\lim_{k\to\infty} \frac{3p_k}{3p_k-2} s_k^{-1} \tilde{e}(p_k,\lambda) = 1.$$

Note that

$$e(p_k,\lambda) \le \tilde{e}(p_k,\lambda) \le e(p_k,\lambda) + \frac{s_k^{\frac{1}{2}}}{2} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{w_k^2(x)w_k^2(y)}{|x-y|},$$

which implies that

$$0 \le \frac{3p_k}{3p_k-2} s_k^{-1} \left(\tilde{e}(p_k,\lambda) - e(p_k,\lambda)\right) \le \frac{3p_k}{2(3p_k-2)} s_k^{-\frac{1}{2}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{w_k^2(x)w_k^2(y)}{|x-y|} \xrightarrow{k} 0.$$

Thus,

$$\beta = \frac{2}{3\lambda} \quad\text{and}\quad \lim_{k\to\infty} \frac{3p_k}{3p_k-2} s_k^{-1} e(p_k,\lambda) = 1.$$

Since the above argument holds for any sequence $\{p_k\}$ satisfying $\lim_{k\to\infty} p_k = \frac{2}{3}$, we thus have

$$\lim_{p\nearrow\frac{2}{3}} \frac{3p}{3p-2} s_p^{-1} e(p,\lambda) = 1,$$

and the proof is finished. □

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