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Global well-posedness theory for a class of coupled parabolic-elliptic systems

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A B S T R A C T

We consider a fully coupled system consisting of a parabolic equation, with boundary and initial conditions, and an abstract elliptic equation in a variational form with time as a parameter. Such systems appear in applications related to the modeling of coupled diffusion and elastic deformation processes in inhomogeneous porous media within a quasi-static assumption. We establish the global existence, uniqueness, and continuous dependence on initial and boundary data of a weak solution to the system. The proof of this result involves the proposed pseudo-decoupling method which reduces the coupled system to an initial-boundary value problem for a single implicit equation and a refined approach to deriving a priori energy estimates based on component-wise contributions of system parameters to energy norms.

1. Introduction

We first introduce some notation that will be used throughout the paper. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a sufficiently smooth boundary Γ, $\bar{\Omega} = \Omega \cup \Gamma$, and $\Gamma_0 \subset \Gamma$ have a nonempty interior relative to Γ. The symbol $t_f \in (0, \infty)$ stands for a final time and the dot notation $(\dot{\ })$ is used for the time derivative. Boldface letters represent vectors, both variables and functions, e.g. $\mathbf{x} \in \bar{\Omega}$, $\mathbf{u}$, or $\mathbf{\Phi}$. The ith component of a vector $\mathbf{u}$ is denoted by u_i and $\mathbf{u}^T$ means the transpose of $\mathbf{u}$. The following notation will be used for the spaces of vector-valued functions: $\mathbb{H}^n_\Omega = L^2(\Omega)^n$, $\mathbb{H}^n_\Gamma = L^2(\Gamma)^n$, and $\mathbb{V}^n = H^1(\Omega)^n, n \in \mathbb{N}$, with the usual norms. The subspace $\mathbb{V}^n_0 = H^1_0(\Omega)^n$ is equipped with the norm

$$\|\mathbf{u}\|_{0,n} = \left[\sum_{k=1}^{n} \int_\Omega |\nabla u_k|^2 \mathrm{d}\Omega\right]^{1/2},$$

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as is the subspace $\tilde{\mathbb{V}}_0^n = \{\boldsymbol{\varphi} \in \mathbb{V}^n : \boldsymbol{\varphi}|_{\Gamma_0} = \mathbf{0}\}$. The norm on $L^\infty(\Omega)^n$ will be denoted by $\|\cdot\|_{\infty,n}$. The superscript and subscript $n = 1$ will be suppressed for notational simplicity, e.g. $\mathbb{H}_\Omega = L^2(\Omega)$, $\|\cdot\|_\infty = \|\cdot\|_{\infty,1}$. We denote by $L^2(a, b; \mathrm{X})$ the space of L^2-integrable functions from $[a, b] \subset \mathbb{R}$ into a Hilbert space X with the norm

$$\|u\|_{L^2(a,b;\mathrm{X})} = \left[\int_a^b \|u(t)\|_{\mathrm{X}}^2 \mathrm{d}t\right]^{1/2}$$

Finally, the space $L^\infty(a, b; \mathrm{X})$ is the space of essentially bounded functions from $[a, b]$ into X equipped with the norm

$$\|u\|_{L^\infty(a,b;\mathrm{X})} = \operatorname*{ess\,sup}_{[a,b]} \|u(t)\|_X$$

Let $a(\boldsymbol{\lambda}) : \mathbb{V}^n \times \mathbb{V}^n \to \mathbb{R}$ be a continuous and $\tilde{\mathbb{V}}_0^n$-coercive bilinear form depending on a parameter $\boldsymbol{\lambda} \in L^\infty(\Omega)^k$. Let $B(\mathbf{b}) : \tilde{\mathbb{V}}_0^n :\to \mathbb{H}_\Omega^m$ be a linear continuous operator depending on a parameter $\mathbf{b} \in L^\infty(\Omega)^m$ such that, for every $\boldsymbol{\psi} \in \mathbb{H}_\Omega^m$ and every $\boldsymbol{\Phi} \in \tilde{\mathbb{V}}_0^n$,

$$\big(\boldsymbol{\psi}, B(\mathbf{b})\boldsymbol{\Phi}\big)_{\mathbb{H}_\Omega^m} = \big(\mathbf{e}(\mathbf{b}^T\boldsymbol{\psi}), B(\mathbf{e})\boldsymbol{\Phi}\big)_{\mathbb{H}_\Omega^m} \tag{1}$$

where $\mathbf{e}$ is an m-dimensional unit vector such that all but one of its components are zero. Condition (1) immediately implies the linearity of the operator B with respect to its parameter $\mathbf{b}$. Let $\boldsymbol{\sigma} \in L^2\big(0, t_f; \mathbb{H}_\Gamma^n \otimes \mathbb{H}_\Gamma^n\big)$ be an $n \times n$ second-order tensor such that $\dot{\boldsymbol{\sigma}} \in L^2\big(0, t_f; \mathbb{H}_\Gamma^n \otimes \mathbb{H}_\Gamma^n\big)$.

We consider the following class of coupled systems consisting of a parabolic equation, with boundary and initial conditions, and an abstract elliptic equation in a variational form with time $t \in (0, t_f)$ as a parameter.

$$M(\mathbf{x})\dot{\bar{\mathbf{V}}} - \nabla \cdot \big[A(\mathbf{x})\nabla\bar{\mathbf{V}}\big] = -B\big(\mathbf{b}_0(\mathbf{x})\big)\dot{\mathbf{u}}, \quad \text{in } \Omega \times (0, t_f) \tag{2}$$

$$a(\boldsymbol{\lambda}(\mathbf{x}); \mathbf{u}, \boldsymbol{\Phi}) = \big(\bar{\mathbf{V}}, B(\mathbf{b}_1(\mathbf{x}))\boldsymbol{\Phi}\big)_{\mathbb{H}_\Omega^m} + \big(\boldsymbol{\sigma}(\mathbf{x}, t)\mathbf{n}, \boldsymbol{\Phi}\big)_{\mathbb{H}_\Gamma^n}, \quad \forall \boldsymbol{\Phi} \in \tilde{\mathbb{V}}_0^n \tag{3}$$

$$\bar{\mathbf{V}}(\mathbf{x}, t) = \bar{\mathbf{V}}_B(\mathbf{x}, t), \quad \text{on } \Gamma \times (0, t_f) \tag{4}$$

$$\bar{\mathbf{V}}(\mathbf{x}, 0) = \bar{\mathbf{V}}_I(\mathbf{x}), \quad \text{in } \bar{\Omega} \tag{5}$$

Here M and A are $m \times m$ non-symmetric coefficient matrices, $\bar{\mathbf{V}}$ is an m-vector, $\mathbf{b}_0$ and $\mathbf{b}_1$ are coupling vectors such that $\mathbf{b}_0 \neq S\mathbf{b}_1$ for any scaling transformation S, and $\mathbf{n}$ is the outward unit normal vector on the boundary.

Systems like (2)-(5) are motivated by the modeling of coupled diffusion and elastic deformation processes in inhomogeneous porous media within the quasi-static assumption.

The objective of this paper is to develop existence-uniqueness-continuous dependence theory for a weak solution of the coupled parabolic-elliptic system (2)-(5).

Many physical phenomena are modeled by various coupled parabolic-elliptic systems, and the problems of existence, uniqueness, and continuous dependence, as well as regularity of solutions to such systems, have attracted a great deal of attention in the literature. Focusing on the class of coupled parabolic-elliptic systems under consideration, we mention the works of Showalter [12,13] and Malysheva and White [7,8]. In [12,13], based on the theory of linear degenerate evolution equations in Hilbert spaces, the existence-uniqueness-regularity theory was developed for strong and weak solutions to the mixed parabolic-elliptic system describing the classical quasi-static Biot consolidation models of poroelasticity and thermoelasticity. It should be noted that, with slight modifications, these results can be extended to the special case of the system (2)-(5) with symmetric coefficient matrices M and A and equal coupling vectors $\mathbf{b}_0$ and $\mathbf{b}_1$.

However, the arguments employed in [12,13] are not applicable to the general case of the system (2)-(5). Well-posedness for a coupled parabolic-elliptic system constituting the general model of chemical thermo-poroelasticity [3] that satisfies (2)-(5) with constant coefficients and $B(\mathbf{b})\mathbf{\Phi} = \mathbf{b}(\nabla \cdot \mathbf{\Phi})$, $\mathbf{\Phi} \in \tilde{\mathbb{V}}_0^n$, has been treated in [7,8]. Sufficient conditions for Hadamard well-posedness in a weak sense for this system were obtained in [7], and in [8], using the proposed pseudo-decoupling method, the full well-posedness theory was derived for a weak solution of the system. Existence, uniqueness, and regularity results for other classes of linear and nonlinear coupled parabolic-elliptic systems can be found, for example, in [1,2,5,9–11].

In this paper, we establish the global existence, uniqueness, and continuous dependence on initial and boundary data of weak solutions to the class of coupled parabolic-elliptic systems of the form (2)-(5). We show that far more general well-posedness results can be obtained by extending the idea of the pseudo-decoupling technique introduced in [8].

The remainder of the paper is structured as follows. In Section 2 we describe the pseudo-decoupling method that transforms the coupled parabolic-elliptic system (2)-(5) into an initial-boundary value problem for a single implicit equation and discuss the decomposition of the obtained problem using the principle of superposition. In Section 3 we prove the global existence, uniqueness, and continuous dependence on initial and boundary data of weak solutions to subproblems resulting from the decomposition carried out in Section 2. In Section 4 we develop global existence-uniqueness-continuous dependence theory for weak solutions to the class of coupled parabolic-elliptic systems (2)-(5).

2. The pseudo-decoupling method

In this section we present the pseudo-decoupling method that allows one to transform the coupled parabolic-elliptic system (2)-(5) into an initial-boundary value problem for a single implicit equation. Using the superposition principle, the obtained initial-boundary value problem is then decomposed into two subproblems: first, for an autonomous parabolic equation and second, for an implicit equation with homogeneous boundary and initial conditions.

We shall be using the following assumption on the coefficient matrices M and A and the coupling vector $\mathbf{b}_0$.

Assumption 1. We assume that the matrix $A(\mathbf{x})$ can be expressed as the product of a symmetric matrix $A_0(\mathbf{x})$ and a constant diagonal matrix D,

$$A(\mathbf{x}) = A_0(\mathbf{x})D,$$

where D is such that

$$D\mathbf{b}_0 = \mathbf{b}_0 \tag{6}$$

Considering Assumption 1, we define the vector

$$\bar{\mathbf{V}}_D = D\bar{\mathbf{V}} \tag{7}$$

The transformation (7) and the linearity of the operator $B(\cdot)$ lead to the following coupled system equivalent to (2)-(5).

$$M_D(\mathbf{x})\dot{\bar{\mathbf{V}}}_D - \nabla \cdot \left[A_0(\mathbf{x})\nabla\bar{\mathbf{V}}_D\right] + B\big(\mathbf{b}_0(\mathbf{x})\big)\dot{\mathbf{u}} = \mathbf{0}, \quad \text{in } \Omega \times (0, t_f) \tag{8}$$

$$a(\boldsymbol{\lambda}(\mathbf{x}); \mathbf{u}, \mathbf{\Phi}) = \big(\bar{\mathbf{V}}_D, B\big(D^{-1}\mathbf{b}_1(\mathbf{x})\big)\mathbf{\Phi}\big)_{\mathbb{H}^m_\Omega} + \big(\boldsymbol{\sigma}(\mathbf{x}, t)\mathbf{n}, \mathbf{\Phi}\big)_{\mathbb{H}^n_\Gamma}, \quad \forall \mathbf{\Phi} \in \tilde{\mathbb{V}}_0^n \tag{9}$$

$$\bar{\mathbf{V}}_D(\mathbf{x}, t) = D\mathbf{V}_B(\mathbf{x}, t), \quad \text{on } \Gamma \times (0, t_f) \tag{10}$$

$$\bar{\mathbf{V}}_D(\mathbf{x},0) = D\mathbf{V}_I(\mathbf{x}), \quad \text{in } \bar{\Omega} \tag{11}$$

where $M_D(\mathbf{x}) = M(\mathbf{x})D^{-1}$ is a non-symmetric matrix.

Our next step is to incorporate the abstract elliptic equation (9) into the equation (8). According to the Lax-Milgram theorem [14], the properties of continuity and $\tilde{\mathbb{V}}_0^n$-coercivity of the bilinear form $a(\boldsymbol{\lambda};\cdot,\cdot)$ allow us to define the linear operator $E_B(\mathbf{b}) : \mathbb{H}_\Omega^m \to \tilde{\mathbb{V}}_0^n$ depending on the operator $B(\mathbf{b}) : \tilde{\mathbb{V}}_0^n \to \mathbb{H}_\Omega^m$ and its parameter $\mathbf{b} \in L^\infty(\Omega)^m$ by

$$a(\boldsymbol{\lambda}; E_B(\mathbf{b})\boldsymbol{\psi}, \boldsymbol{\Phi}) = \big(\boldsymbol{\psi}, B(\mathbf{b})\boldsymbol{\Phi}\big)_{\mathbb{H}_\Omega^m}, \quad \forall \boldsymbol{\Phi} \in \tilde{\mathbb{V}}_0^n \tag{12}$$

and the function $\mathbf{u}_\Gamma = \mathbf{u}_\Gamma(\boldsymbol{\sigma}) \in \tilde{\mathbb{V}}_0^n$ by

$$a(\boldsymbol{\lambda}; \mathbf{u}_\Gamma, \boldsymbol{\Phi}) = \big(\boldsymbol{\sigma}\mathbf{n}, \boldsymbol{\Phi}\big)_{\mathbb{H}_\Gamma^n}, \quad \forall \boldsymbol{\Phi} \in \tilde{\mathbb{V}}_0^n \tag{13}$$

Remark 1. The linearity of the operator $B(\mathbf{b})$ with respect to its parameter $\mathbf{b}$ yields the linearity of $E_B(\mathbf{b})$ with respect to $\mathbf{b}$. Moreover, the operator E_B is continuous. Indeed, since $a(\boldsymbol{\lambda};\cdot,\cdot)$ is $\tilde{\mathbb{V}}_0^n$-coercive, for every $\boldsymbol{\psi} \in \mathbb{H}_\Omega^m$,

$$a(\boldsymbol{\lambda}; E_B(\mathbf{b})\boldsymbol{\psi}, E_B(\mathbf{b})\boldsymbol{\psi}) \geq \gamma(\boldsymbol{\lambda})\|E_B(\mathbf{b})\boldsymbol{\psi}\|_{0,n}^2 \tag{14}$$

where $\gamma(\boldsymbol{\lambda}) > 0$ is the coercivity constant of $a(\boldsymbol{\lambda};\cdot,\cdot)$. From the above inequality, (1), (12), and the continuity of the operator $B(\mathbf{b})$ we have

$$\begin{aligned}
\gamma(\boldsymbol{\lambda})\|E_B(\mathbf{b})\boldsymbol{\psi}\|_{0,n}^2 &\leq \big|a(\boldsymbol{\lambda}; E_B(\mathbf{b})\boldsymbol{\psi}, E_B(\mathbf{b})\boldsymbol{\psi})\big| \\
&= \Big|\big(\boldsymbol{\psi}, B(\mathbf{b})E_B(\mathbf{b})\boldsymbol{\psi}\big)_{\mathbb{H}_\Omega^m}\Big| = \Big|\big(\mathbf{e}(\mathbf{b}^T\boldsymbol{\psi}), B(\mathbf{e})E_B(\mathbf{b})\boldsymbol{\psi}\big)_{\mathbb{H}_\Omega^m}\Big| \\
&\leq \|\mathbf{b}^T\boldsymbol{\psi}\|_{\mathbb{H}_\Omega}\|B(\mathbf{e})E_B(\mathbf{b})\boldsymbol{\psi}\|_{\mathbb{H}_\Omega^m} \\
&\leq \|\mathbf{b}^T\boldsymbol{\psi}\|_{\mathbb{H}_\Omega}\|B(\mathbf{e})\|\|E_B(\mathbf{b})\boldsymbol{\psi}\|_{0,n}
\end{aligned}$$

where $\|B(\mathbf{b})\| = \sup\{\|B(\mathbf{b})\mathbf{v}\|_{\mathbb{H}_\Omega^m} : \mathbf{v} \in \tilde{\mathbb{V}}_0^n, \|\mathbf{v}\|_{0,n} = 1\}$. Therefore,

$$\|E_B(\mathbf{b})\boldsymbol{\psi}\|_{0,n} \leq \frac{1}{\gamma(\boldsymbol{\lambda})}\|B(\mathbf{e})\|\|\mathbf{b}^T\boldsymbol{\psi}\|_{\mathbb{H}_\Omega} \tag{15}$$

and hence,

$$\|E_B(\mathbf{b})\boldsymbol{\psi}\|_{0,n} \leq \frac{\sqrt{m}}{\gamma(\boldsymbol{\lambda})}\|\mathbf{b}\|_{\infty,m}\|B(\mathbf{e})\|\|\boldsymbol{\psi}\|_{\mathbb{H}_\Omega^m}$$

Applying (12) and (13) to (9) gives, for all $\boldsymbol{\Phi} \in \tilde{\mathbb{V}}_0^n$,

$$a(\boldsymbol{\lambda}(\mathbf{x}); \mathbf{u}, \boldsymbol{\Phi}) = a\big(\boldsymbol{\lambda}(\mathbf{x}); E_B\big(D^{-1}\mathbf{b}_1(\mathbf{x})\big)\bar{\mathbf{V}}_D, \boldsymbol{\Phi}\big) + a(\boldsymbol{\lambda}(\mathbf{x}); \mathbf{u}_\Gamma, \boldsymbol{\Phi})$$

It follows that

$$\mathbf{u} = E_B\big(D^{-1}\mathbf{b}_1(\mathbf{x})\big)\bar{\mathbf{V}}_D + \mathbf{u}_\Gamma \tag{16}$$

and hence,

$$B\big(\mathbf{b}_0(\mathbf{x})\big)\dot{\mathbf{u}} = \big[B\big(\mathbf{b}_0(\mathbf{x})\big)E_B\big(D^{-1}\mathbf{b}_1(\mathbf{x})\big)\big]\dot{\bar{\mathbf{V}}}_D + B\big(\mathbf{b}_0(\mathbf{x})\big)\dot{\mathbf{u}}_\Gamma \tag{17}$$

Substituting (17) into the parabolic equation (8), we obtain the implicit equation

$$\Big[M_D(\mathbf{x}) + B\big(\mathbf{b}_0(\mathbf{x})\big)E_B\big(D^{-1}\mathbf{b}_1(\mathbf{x})\big)\Big]\dot{\bar{\mathbf{V}}}_D - \nabla\cdot\big[A_0(\mathbf{x})\nabla\bar{\mathbf{V}}_D\big] + B\big(\mathbf{b}_0(\mathbf{x})\big)\dot{\mathbf{u}}_\Gamma = \mathbf{0}$$

This equation supplemented with the boundary and initial conditions (10) and (11) yields the initial-boundary value problem for a single implicit equation,

$$\begin{aligned}
\Big[M_D(\mathbf{x}) + B\big(\mathbf{b}_0(\mathbf{x})\big)E_B\big(D^{-1}\mathbf{b}_1(\mathbf{x})\big)\Big]\dot{\bar{\mathbf{V}}}_D - \nabla\cdot\big[A_0(\mathbf{x})\nabla\bar{\mathbf{V}}_D\big] + B\big(\mathbf{b}_0(\mathbf{x})\big)\dot{\mathbf{u}}_\Gamma = \mathbf{0}, &\quad \text{in } \Omega\times(0,t_f)\\
\bar{\mathbf{V}}_D(\mathbf{x},t) = D\mathbf{V}_B(\mathbf{x},t), &\quad \text{on } \Gamma\times(0,t_f)\\
\bar{\mathbf{V}}_D(\mathbf{x},0) = D\mathbf{V}_I(\mathbf{x}), &\quad \text{in } \bar{\Omega}
\end{aligned}$$

which is equivalent to the coupled parabolic-elliptic system (2)-(5). In what follows we shall denote this initial-boundary value problem by P0.

Using the superposition principle, the problem P0 can be decomposed into the following two subproblems: the subproblem P1 for an autonomous parabolic equation,

$$\begin{aligned}
M_D(\mathbf{x})\dot{\bar{\mathbf{W}}} - \nabla\cdot\big[A_0(\mathbf{x})\nabla\bar{\mathbf{W}}\big] = \mathbf{0}, &\quad \text{in } \Omega\times(0,t_f)\\
\bar{\mathbf{W}}(\mathbf{x},t) = D\mathbf{V}_B(\mathbf{x},t), &\quad \text{on } \Gamma\times(0,t_f)\\
\bar{\mathbf{W}}(\mathbf{x},0) = D\mathbf{V}_I(\mathbf{x}), &\quad \text{in } \bar{\Omega}
\end{aligned}$$

and the subproblem P2 for an implicit equation with homogeneous boundary and initial conditions,

$$\Big[M_D(\mathbf{x}) + B\big(\mathbf{b}_0(\mathbf{x})\big)E_B\big(D^{-1}\mathbf{b}_1(\mathbf{x})\big)\Big]\dot{\mathbf{V}} - \nabla\cdot\big[A_0(\mathbf{x})\nabla\mathbf{V}\big] = \mathbf{F}(\mathbf{x},t), \quad \text{in } \Omega\times(0,t_f), \tag{18}$$

$$\mathbf{V}(\mathbf{x},t) = \mathbf{0}, \quad \text{on } \Gamma\times(0,t_f), \tag{19}$$

$$\mathbf{V}(\mathbf{x},0) = \mathbf{0}, \quad \text{in } \bar{\Omega}, \tag{20}$$

where the source term is given by

$$\mathbf{F}(\mathbf{x},t) = -B\big(\mathbf{b}_0(\mathbf{x})\big)E_B\big(D^{-1}\mathbf{b}_1(\mathbf{x})\big)\dot{\bar{\mathbf{W}}} - B\big(\mathbf{b}_0(\mathbf{x})\big)\dot{\mathbf{u}}_\Gamma, \tag{21}$$

with $\dot{\bar{\mathbf{W}}}$ being the time derivative of the solution to the problem P1 and $\mathbf{u}_\Gamma$ defined by (13). Then the solution of the problem P0 is the sum of the solutions to the subproblems P1 and P2:

$$\bar{\mathbf{V}}_D = \bar{\mathbf{W}} + \mathbf{V} \tag{22}$$

and from (7), (16), and (22), the solution $(\bar{\mathbf{V}},\mathbf{u})$ of the coupled parabolic-elliptic system (2)-(5) is given by

$$\bar{\mathbf{V}} = D^{-1}(\bar{\mathbf{W}} + \mathbf{V}) \tag{23}$$

$$\mathbf{u} = E_B\big(D^{-1}\mathbf{b}_1(\mathbf{x})\big)(\bar{\mathbf{W}} + \mathbf{V}) + \mathbf{u}_\Gamma \tag{24}$$

Remark 2. Equation (19) yields the auxiliary boundary condition for problem P2:

$$\dot{\mathbf{V}}(\mathbf{x},t) = \mathbf{0}, \quad \text{on } \Gamma\times(0,t_f) \tag{25}$$

3. Weak solutions to problem P1 and problem P2 with an arbitrary source term

We shall address the questions of existence, uniqueness, and continuous dependence on initial and boundary data of weak solutions to the problem P1 and the problem P2 with an arbitrary nonhomogeneous source term $\mathbf{F}(\mathbf{x}, t)$. The following assumption will be made throughout the remainder of this paper.

Assumption 2. We assume the matrices $M_D(\mathbf{x}) = [m_{ij}(\mathbf{x})]_{i,j=1}^m$ and $A_0(\mathbf{x}) = [a_{ij}(\mathbf{x})]_{i,j=1}^m$ satisfy the following conditions:

(i) $m_{ij}, a_{ij} \in L^\infty(\Omega)$, $1 \le i, j \le m$.
(ii) There exist constants $m_i > 0$ such that $m_{ii}(\mathbf{x}) \ge m_i$, $\forall \mathbf{x} \in \bar{\Omega}$, $1 \le i \le m$.
(iii) There exists a constant $c \in (0, 1)$ such that

$$0 < \big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big)^2 < c m_i m_j$$

for all $1 \le i \le m-1$, $i+1 \le j \le m$.
(iv) The matrices $\frac{1}{2}\big(M_D + M_D^T\big)$ and A_0 are positive definite.

3.1. Problem P1

Using the superposition principle, problem P1 can further be decomposed into the elliptic boundary value problem, denoted by P1.1, with time $t \in [0, t_f]$ as a parameter:

$$\begin{aligned} -\nabla \cdot \big[A_0(\mathbf{x})\nabla \mathbf{W}_0\big] &= \mathbf{0}, \quad \text{in } \Omega \\ \mathbf{W}_0(\mathbf{x}, t) &= D\mathbf{V}_B(\mathbf{x}, t), \quad \text{on } \Gamma \end{aligned}$$

where $\dot{\mathbf{W}}_0$ satisfies

$$-\nabla \cdot \big[A_0(\mathbf{x})\nabla \dot{\mathbf{W}}_0\big] = \mathbf{0}, \quad \text{in } \Omega \tag{26}$$

$$\dot{\mathbf{W}}_0(\mathbf{x}, t) = D\dot{\mathbf{V}}_B(\mathbf{x}, t), \quad \text{on } \Gamma, \tag{27}$$

and the parabolic initial-boundary value problem, denoted by P1.2:

$$\begin{aligned} M_D(\mathbf{x})\dot{\mathbf{W}} - \nabla \cdot \big[A_0(\mathbf{x})\nabla \mathbf{W}\big] &= -M_D(\mathbf{x})\dot{\mathbf{W}}_0, \quad \text{in } \Omega \times (0, t_f) \\ \mathbf{W}(\mathbf{x}, t) &= \mathbf{0}, \quad \text{on } \Gamma \times (0, t_f) \\ \mathbf{W}(\mathbf{x}, 0) &= D\mathbf{V}_I(\mathbf{x}) - \mathbf{W}_0(\mathbf{x}, 0), \quad \text{in } \bar{\Omega} \end{aligned}$$

where $\dot{\mathbf{W}}_0$ is the solution to the boundary value problem (26) and (27). Then the solution to the problem P1 is given by

$$\bar{\mathbf{W}} = \mathbf{W}_0 + \mathbf{W} \tag{28}$$

The next two lemmas establish the global existence and uniqueness of weak solutions to the subproblems P1.1 and P1.2 and the continuous dependence of the solutions on the initial and boundary data. We omit their proofs because they follow the standard Galerkin methods for elliptic and parabolic problems, respectively. Henceforth, $\hat{C} > 0$ denotes a generic constant independent of functions to be estimated.

Lemma 1. *Given $\mathbf{V}_B \in L^2\big(0, t_f; H^{1/2}(\Gamma)^m\big)$ with $\dot{\mathbf{V}}_B \in L^2\big(0, t_f; H^{1/2}(\Gamma)^m\big)$, under Assumption 2 (i) and (iv) on the matrix A_0, there exists a global unique weak solution $\mathbf{W}_0 \in L^2\big(0, t_f; \mathbb{V}^m\big)$ to the problem P1.1 with $\dot{\mathbf{W}}_0 \in L^2\big(0, t_f; \mathbb{V}^m\big)$ and the following estimates hold*

$$\|\mathbf{W}_0\|_{L^2(0,t_f;\mathbb{V}^m)} \le \hat{C}\|\mathbf{V}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^m)}$$
$$\|\dot{\mathbf{W}}_0\|_{L^2(0,t_f;\mathbb{V}^m)} \le \hat{C}\|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^m)}$$

Moreover, for each $t \in [0, t_f]$,

$$\|\mathbf{W}_0(t)\|_{\mathbb{V}^m} \le \hat{C}\|\mathbf{V}_B(t)\|_{H^{1/2}(\Gamma)^m}$$
$$\|\dot{\mathbf{W}}_0(t)\|_{\mathbb{V}^m} \le \hat{C}\|\dot{\mathbf{V}}_B(t)\|_{H^{1/2}(\Gamma)^m}$$

Lemma 2. *Given $\dot{\mathbf{W}}_0 \in L^2(0, t_f; \mathbb{V}^m)$, $\mathbf{V}_I \in \mathbb{V}^m$, and $\mathbf{W}_0(0) \in \mathbb{V}^m$, under Assumption 2 (i) and (iv), the problem P1.2 admits a global unique weak solution $\mathbf{W} \in L^\infty\big(0, t_f; \mathbb{V}_0^m\big)$, with $\dot{\mathbf{W}} \in L^2\big(0, t_f; \mathbb{H}_\Omega^m\big)$, and the solution depends continuously on the data; that is,*

$$\|\mathbf{W}\|_{L^\infty(0,t_f;\mathbb{V}_0^m)} \le \hat{C}\big(\|\mathbf{V}_I\|_{\mathbb{V}^m} + \|\mathbf{W}_0(0)\|_{\mathbb{V}^m} + \|\dot{\mathbf{W}}_0\|_{L^2(0,t_f;\mathbb{V}^m)}\big)$$
$$\|\dot{\mathbf{W}}\|_{L^2(0,t_f;\mathbb{H}_\Omega^m)} \le \hat{C}\big(\|\mathbf{V}_I\|_{\mathbb{V}^m} + \|\mathbf{W}_0(0)\|_{\mathbb{V}^m} + \|\dot{\mathbf{W}}_0\|_{L^2(0,t_f;\mathbb{V}^m)}\big)$$

As a direct consequence of Lemmas 1 and 2, and (28), we have the following global existence-uniqueness-continuous dependence result for a weak solution to the problem P1.

Corollary 1. *Given the boundary and initial data $\mathbf{V}_B \in L^2\big(0, t_f; H^{1/2}(\Gamma)^m\big)$ with $\dot{\mathbf{V}}_B \in L^2\big(0, t_f; H^{1/2}(\Gamma)^m\big)$, and $\mathbf{V}_I \in \mathbb{V}^m$, under Assumption 2 (i) and (iv), there exists a global unique weak solution*

$$\bar{\mathbf{W}} \in L^2(0, t_f; \mathbb{V}^m) \quad \textit{with} \quad \dot{\bar{\mathbf{W}}} \in L^2(0, t_f; \mathbb{H}_\Omega^m) \tag{29}$$

to the problem P1, and the solution depends continuously on the data in the sense that the following estimates hold

$$\begin{aligned}\|\bar{\mathbf{W}}\|_{L^2(0,t_f;\mathbb{V}^m)} \le \hat{C}\big(&\|\mathbf{V}_I\|_{\mathbb{V}^m} + \|\mathbf{V}_B(0)\|_{H^{1/2}(\Gamma)^m} \\ &+ \|\mathbf{V}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^m)} + \|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^m)}\big)\end{aligned} \tag{30}$$

$$\|\dot{\bar{\mathbf{W}}}\|_{L^2(0,t_f;\mathbb{H}_\Omega^m)} \le \hat{C}\big(\|\mathbf{V}_I\|_{\mathbb{V}^m} + \|\mathbf{V}_B(0)\|_{H^{1/2}(\Gamma)^m} + \|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^m)}\big) \tag{31}$$

3.2. Problem P2

We begin by obtaining formal a priori energy estimates for the problem P2. In real-world applications elements of the matrices M and A and the coupling vectors $\mathbf{b}_0$ and $\mathbf{b}_1$ of the system (2)-(5) may differ by more than 20 orders of magnitude (see [3,4]). The same obviously holds true for the coefficient matrices and vectors of the problem P2. For this reason, we propose a refined approach to deriving a priori energy estimates that is based on element-wise contributions of system parameters to energy norms.

We first formally multiply (18) by $\dot{\mathbf{V}}^T$, integrate over Ω, and split the first integral on the left-hand side into the sum of two integrals:

$$\int_\Omega \dot{\mathbf{V}}^T M_D(\mathbf{x})\dot{\mathbf{V}}\mathrm{d}\Omega + \int_\Omega \dot{\mathbf{V}}^T B(\mathbf{b}_0(\mathbf{x}))E_B(D^{-1}\mathbf{b}_1(\mathbf{x}))\dot{\mathbf{V}}\mathrm{d}\Omega - \int_\Omega \dot{\mathbf{V}}^T\nabla\cdot\big[A_0(\mathbf{x})\nabla\mathbf{V}\big]\mathrm{d}\Omega = \int_\Omega \dot{\mathbf{V}}^T\mathbf{F}\mathrm{d}\Omega \tag{32}$$

Table 1
Coefficients $\frac{1}{\varepsilon_k}$ for $\|\dot{V}_i\|^2$.

$i \backslash j$	1	2	3	$\cdots$	$m-1$	m
1	–	$\frac{1}{\varepsilon_1}$	$\frac{1}{\varepsilon_2}$	$\cdots$	$\frac{1}{\varepsilon_{m-2}}$	$\frac{1}{\varepsilon_{m-1}}$
2	–	–	$\frac{1}{\varepsilon_m}$	$\cdots$	$\frac{1}{\varepsilon_{2m-4}}$	$\frac{1}{\varepsilon_{2m-3}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$m-1$	–	–	–	$\cdots$	–	$\frac{1}{\varepsilon_{m(m-1)/2}}$

Table 2
Coefficients ε_k for $\|\dot{V}_j\|^2$.

$i \backslash j$	1	2	3	$\cdots$	$m-1$	m
1	–	ε_1	ε_2	$\cdots$	ε_{m-2}	ε_{m-1}
2	–	–	ε_m	$\cdots$	ε_{2m-4}	ε_{2m-3}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$m-1$	–	–	–	$\cdots$	–	$\varepsilon_{m(m-1)/2}$

We shall estimate each term of (32) separately. Under Assumption 2 (i), for the first term on the left-hand side of (32) we have

$$\begin{aligned}\left|\int_\Omega \dot{\mathbf{V}}^T M_D(\mathbf{x})\dot{\mathbf{V}} - \sum_{i=1}^m m_{ii}(\mathbf{x})\dot{V}_i^2 \mathrm{d}\Omega\right| &\le \sum_{i=1}^{m-1}\sum_{j=i+1}^m \frac{1}{2}\big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big)\int_\Omega 2|\dot{V}_i||\dot{V}_j|\mathrm{d}\Omega \\ &\le \sum_{i=1}^{m-1}\sum_{j=i+1}^m \frac{1}{2}\big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big)\left[\frac{1}{\varepsilon_{ij}}\|\dot{V}_i\|^2_{\mathbb{H}_\Omega} + \varepsilon_{ij}\|\dot{V}_j\|^2_{\mathbb{H}_\Omega}\right] \quad (33)\end{aligned}$$

for every $\varepsilon_{ij} > 0$, $1 \le i \le m-1$, $i+1 \le j \le m$.

We observe that there are $\frac{1}{2}m(m-1)$ constants ε_{ij} in (33), and for the sake of convenience, we change notation from double-indexed ε_{ij}, $1 \le i \le m-1$, $i+1 \le j \le m$, to single-indexed ε_k, $1 \le k \le \frac{1}{2}m(m-1)$, as follows. The constants ε_{ij}, $1 \le i \le m-1$, $i+1 \le j \le m$, can be considered as the elements of a symmetric matrix $\left[\varepsilon_{ij}\right]_{i,j=1}^m$ located above the main diagonal. Using the row-major ordering method, the elements $\varepsilon_{12}, \dots, \varepsilon_{1m}$ are indexed as $\varepsilon_1, \dots, \varepsilon_{m-1}$, the elements $\varepsilon_{23}, \dots, \varepsilon_{2m}$ are indexed as $\varepsilon_m, \dots, \varepsilon_{2m-3}$, and so on. In general, the elements $\varepsilon_{i,i+1}$, $1 \le i \le m-1$, become ε_k, with $k = (m-1)+(m-2)+\cdots+(m-(i-1))+1 = m(i-1) - \frac{1}{2}i(i-1) + 1$, and the elements ε_{ij}, $1 \le i \le m-1$, $i+1 \le j \le m$, are indexed as ε_k, with $k = m(i-1) - \frac{1}{2}i(i-1) + (j-i) = j + m(i-1) - \frac{1}{2}i(i+1)$. Thus, (33) can be written as

$$\left|\int_\Omega \dot{\mathbf{V}}^T M_D(\mathbf{x})\dot{\mathbf{V}} - \sum_{i=1}^m m_{ii}(\mathbf{x})\dot{V}_i^2 \mathrm{d}\Omega\right| \le \sum_{i=1}^{m-1}\sum_{j=i+1}^m \frac{1}{2}\big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big)\left[\frac{1}{\varepsilon_k}\|\dot{V}_i\|^2_{\mathbb{H}_\Omega} + \varepsilon_k\|\dot{V}_j\|^2_{\mathbb{H}_\Omega}\right] \quad (34)$$

where $1 \le k \le \frac{1}{2}m(m-1)$ are given by $k = j+m(i-1)-\frac{1}{2}i(i+1)$. Tables 1 and 2 summarize the coefficients $\frac{1}{\varepsilon_k}$ for $\|\dot{V}_i\|^2$ and ε_k for $\|\dot{V}_j\|^2$ in (34), respectively.

Under Assumption 2 (ii), from (34) we obtain

$$\begin{aligned}\int_\Omega \dot{\mathbf{V}}^T M_D(\mathbf{x})\dot{\mathbf{V}}\mathrm{d}\Omega \ge \sum_{i=1}^m m_i\|\dot{V}_i\|^2_{\mathbb{H}_\Omega} &- \sum_{i=1}^{m-1}\sum_{j=i+1}^m \frac{1}{2}\big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big)\frac{1}{\varepsilon_k}\|\dot{V}_i\|^2_{\mathbb{H}_\Omega} \\ &- \sum_{i=1}^{m-1}\sum_{j=i+1}^m \frac{1}{2}\big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big)\varepsilon_k\|\dot{V}_j\|^2_{\mathbb{H}_\Omega} \quad (35)\end{aligned}$$

Table 3
Coefficients $\frac{1}{\varepsilon_k}$ and ε_l for $\|\dot{V}_i\|^2$.

$i \backslash j$	1	2	3	$\cdots$	$m-1$	m
1	–	$\frac{1}{\varepsilon_1}$	$\frac{1}{\varepsilon_2}$	$\cdots$	$\frac{1}{\varepsilon_{m-2}}$	$\frac{1}{\varepsilon_{m-1}}$
2	ε_1	–	$\frac{1}{\varepsilon_m}$	$\cdots$	$\frac{1}{\varepsilon_{2m-4}}$	$\frac{1}{\varepsilon_{2m-3}}$
3	ε_2	ε_m	–	$\cdots$	$\frac{1}{\varepsilon_{3m-7}}$	$\frac{1}{\varepsilon_{3m-6}}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$m-1$	ε_{m-2}	ε_{2m-4}	ε_{3m-7}	$\cdots$	–	$\frac{1}{\varepsilon_{m(m-1)/2}}$
m	ε_{m-1}	ε_{2m-3}	ε_{3m-6}	$\cdots$	$\varepsilon_{m(m-1)/2}$	–

with $1 \le k \le \frac{1}{2}m(m-1)$ given by $k = j + m(i-1) - \frac{1}{2}i(i+1)$. It is convenient to rewrite the last double sum in (35) in terms of $\|\dot{V}_i\|^2_{\mathbb{H}_\Omega}$. To this end, we first observe that indices i and j can be interchanged as follows.

$$\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\frac{1}{2}\big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big)\varepsilon_k\|\dot{V}_j\|^2_{\mathbb{H}_\Omega} = \sum_{j=1}^{m-1}\sum_{i=j+1}^{m}\frac{1}{2}\big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big)\varepsilon_l\|\dot{V}_i\|^2_{\mathbb{H}_\Omega} \tag{36}$$

where $1 \le l \le \frac{1}{2}m(m-1)$ is given by $l = i + m(j-1) - \frac{1}{2}j(j+1)$. Changing the order of summation in the double sum further implies

$$\sum_{j=1}^{m-1}\sum_{i=j+1}^{m}\frac{1}{2}\big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big)\varepsilon_l\|\dot{V}_i\|^2_{\mathbb{H}_\Omega} = \sum_{i=2}^{m}\sum_{j=1}^{i-1}\frac{1}{2}\big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big)\varepsilon_l\|\dot{V}_i\|^2_{\mathbb{H}_\Omega} \tag{37}$$

with $1 \le l \le \frac{1}{2}m(m-1)$, $l = i + m(j-1) - \frac{1}{2}j(j+1)$.
Applying (37) to (35) gives

$$\begin{aligned}\int_\Omega \dot{\mathbf{V}}^T M_D(\mathbf{x})\dot{\mathbf{V}}\mathrm{d}\Omega &\ge \sum_{i=1}^{m} m_i\|\dot{V}_i\|^2_{\mathbb{H}_\Omega} - \sum_{i=1}^{m-1}\sum_{j=i+1}^{m}\frac{1}{2}\big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big)\frac{1}{\varepsilon_k}\|\dot{V}_i\|^2_{\mathbb{H}_\Omega} \\ &\quad - \sum_{i=2}^{m}\sum_{j=1}^{i-1}\frac{1}{2}\big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big)\varepsilon_l\|\dot{V}_i\|^2_{\mathbb{H}_\Omega} \\ &= \sum_{i=1}^{m}\Bigg[m_i - \sum_{j=i+1}^{m}\frac{1}{2}\big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big)\frac{1}{\varepsilon_k} \\ &\quad - \sum_{j=1}^{i-1}\frac{1}{2}\big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big)\varepsilon_l\Bigg]\|\dot{V}_i\|^2_{\mathbb{H}_\Omega}\end{aligned} \tag{38}$$

where $k = j + m(i-1) - \frac{1}{2}i(i+1)$, for $i+1 \le j \le m$, and $l = i + m(j-1) - \frac{1}{2}j(j+1)$, for $1 \le j \le i-1$. Table 3 shows the coefficients $\frac{1}{\varepsilon_k}$ and ε_l for $\|\dot{V}_i\|^2$ depending on the index j in (38).

Before going further, for convenience, we introduce the following notation.

$$\mathbf{b}_D(\mathbf{x}) = D^{-1}(\mathbf{b}_1(\mathbf{x}) - \mathbf{b}_0(\mathbf{x}))$$

Using (6), (12), and linearity of the operator $E_B(\mathbf{b})$ with respect to its parameter $\mathbf{b}$, the second term on the left-hand side of (32) can be written as

$$\begin{aligned}\int_\Omega \dot{\mathbf{V}}^T B(\mathbf{b}_0(\mathbf{x}))E_B(D^{-1}\mathbf{b}_1(\mathbf{x}))\dot{\mathbf{V}}\mathrm{d}\Omega &= \Big(\dot{\mathbf{V}}, B(\mathbf{b}_0(\mathbf{x}))\big[E_B(D^{-1}\mathbf{b}_1(\mathbf{x}))\dot{\mathbf{V}}\big]\Big)_{\mathbb{H}_\Omega^m}\\ &= a\big(\boldsymbol{\lambda}; E_B(\mathbf{b}_0)\dot{\mathbf{V}}, E_B(D^{-1}\mathbf{b}_1)\dot{\mathbf{V}}\big)\\ &= a\big(\boldsymbol{\lambda}; E_B(\mathbf{b}_0)\dot{\mathbf{V}}, E_B(D^{-1}[\mathbf{b}_0+(\mathbf{b}_1-\mathbf{b}_0)])\dot{\mathbf{V}}\big)\\ &= a\big(\boldsymbol{\lambda}; E_B(\mathbf{b}_0)\dot{\mathbf{V}}, E_B(\mathbf{b}_0)\dot{\mathbf{V}}\big) + a\big(\boldsymbol{\lambda}; E_B(\mathbf{b}_0)\dot{\mathbf{V}}, E_B(\mathbf{b}_D)\dot{\mathbf{V}}\big) \end{aligned} \tag{39}$$

The coercivity property (14) of the bilinear form $a(\boldsymbol{\lambda};\cdot,\cdot)$ immediately yields

$$a\big(\boldsymbol{\lambda}; E_B(\mathbf{b}_0)\dot{\mathbf{V}}, E_B(\mathbf{b}_0)\dot{\mathbf{V}}\big) \geq \gamma(\boldsymbol{\lambda})\|E_B(\mathbf{b}_0)\dot{\mathbf{V}}\|_{0,n}^2 \tag{40}$$

On the other hand, the continuity of $a(\boldsymbol{\lambda};\cdot,\cdot)$ and the Poincaré inequality imply

$$\big|a\big(\boldsymbol{\lambda}; E_B(\mathbf{b}_0)\dot{\mathbf{V}}, E_B(\mathbf{b}_D)\dot{\mathbf{V}}\big)\big| \leq \alpha(\boldsymbol{\lambda})\|E_B(\mathbf{b}_0)\dot{\mathbf{V}}\|_{0,n}\|E_B(\mathbf{b}_D)\dot{\mathbf{V}}\|_{0,n}$$

where $\alpha(\boldsymbol{\lambda}) > 0$ is the continuity constant of $a(\boldsymbol{\lambda};\cdot,\cdot)$. From the last inequality and (15), we deduce that

$$\begin{aligned}\big|a\big(\boldsymbol{\lambda}; E_B(\mathbf{b}_0)\dot{\mathbf{V}}, E_B(\mathbf{b}_D)\dot{\mathbf{V}}\big)\big| &\leq \alpha(\boldsymbol{\lambda})\left[\frac{1}{2\varepsilon}\|E_B(\mathbf{b}_0)\dot{\mathbf{V}}\|_{0,n}^2 + \frac{\varepsilon}{2}\|E_B(\mathbf{b}_D)\dot{\mathbf{V}}\|_{0,n}^2\right]\\ &\leq \alpha(\boldsymbol{\lambda})\left[\frac{1}{2\varepsilon}\|E_B(\mathbf{b}_0)\dot{\mathbf{V}}\|_{0,n}^2 + \frac{\varepsilon}{2}\frac{1}{\gamma^2(\boldsymbol{\lambda})}\|B(\mathbf{e})\|^2\|\mathbf{b}_D^T\dot{\mathbf{V}}\|_{\mathbb{H}_\Omega}^2\right]\\ &\leq \alpha(\boldsymbol{\lambda})\left[\frac{1}{2\varepsilon}\|E_B(\mathbf{b}_0)\dot{\mathbf{V}}\|_{0,n}^2 + \frac{\varepsilon}{2}\frac{1}{\gamma^2(\boldsymbol{\lambda})}\|B(\mathbf{e})\|^2\sum_{i=1}^m\|b_{D_i}\|_\infty^2\|\dot{V}_i\|_{\mathbb{H}_\Omega}^2\right]\end{aligned}$$

where $\varepsilon > 0$, and therefore,

$$a\big(\boldsymbol{\lambda}; E_B(\mathbf{b}_0)\dot{\mathbf{V}}, E_B(\mathbf{b}_D)\dot{\mathbf{V}}\big) \geq -\frac{1}{2\varepsilon}\alpha(\boldsymbol{\lambda})\|E_B(\mathbf{b}_0)\dot{\mathbf{V}}\|_{0,n}^2 - \frac{\varepsilon}{2}\frac{\alpha(\boldsymbol{\lambda})}{\gamma^2(\boldsymbol{\lambda})}\|B(\mathbf{e})\|^2\sum_{i=1}^m\|b_{D_i}\|_\infty^2\|\dot{V}_i\|_{\mathbb{H}_\Omega}^2 \tag{41}$$

For the last term on the left-hand side of (32), using the divergence theorem and the auxiliary boundary condition (25), we obtain

$$\begin{aligned}-\int_\Omega \dot{\mathbf{V}}^T\nabla\cdot\big[A_0(\mathbf{x})\nabla\mathbf{V}\big]\mathrm{d}\Omega &= \int_\Omega\sum_{i=1}^m\nabla\dot{V}_i\cdot\Big(\sum_{j=1}^m a_{ij}(\mathbf{x})\nabla V_j\Big)\mathrm{d}\Omega\\ &= \int_\Omega\sum_{k=1}^n\partial_k\dot{\mathbf{V}}^T A_0(\mathbf{x})\partial_k\mathbf{V}\mathrm{d}\Omega\end{aligned} \tag{42}$$

where ∂_k is the partial derivative with respect to x_k.

Let us define the bilinear form $\bar{a}:\mathbb{V}_0^m\times\mathbb{V}_0^m\to\mathbb{R}$ by

$$\bar{a}(\boldsymbol{\psi},\boldsymbol{\phi}) = \int_\Omega\sum_{i=1}^n\partial_i\boldsymbol{\psi}^T A_0(\mathbf{x})\partial_i\boldsymbol{\phi}\mathrm{d}\Omega \tag{43}$$

The symmetry of the matrix $A_0(\mathbf{x})$, (42) and (43) imply

$$-\int_\Omega\dot{\mathbf{V}}^T\nabla\cdot\big[A_0(\mathbf{x})\nabla\mathbf{V}\big]\mathrm{d}\Omega = \frac{1}{2}\frac{d}{dt}\bar{a}(\mathbf{V},\mathbf{V}) \tag{44}$$

Remark 3. The symmetric bilinear form $\bar{a} : \mathbb{V}_0^m \times \mathbb{V}_0^m \to \mathbb{R}$ is coercive and continuous. Indeed, Assumption 2 (iv) on the matrix A_0 guarantees that there exists $\hat{\alpha} > 0$ such that, for every $\boldsymbol{\psi} \in \mathbb{V}_0^m$,

$$\bar{a}(\boldsymbol{\psi}, \boldsymbol{\psi}) \geq \hat{\alpha} \|\boldsymbol{\psi}\|_{0,m}^2 \tag{45}$$

On the other hand, for every $\boldsymbol{\psi}, \boldsymbol{\phi} \in \mathbb{V}_0^m$,

$$\begin{aligned} |\bar{a}(\boldsymbol{\psi}, \boldsymbol{\phi})| &= \Big| \int_\Omega \sum_{k=1}^n \partial_k \boldsymbol{\psi}^T A_0(\mathbf{x}) \partial_k \boldsymbol{\phi} \mathrm{d}\Omega \Big| \\ &\leq \max_{1 \leq i,j \leq m} \{\|a_{ij}\|_\infty\} \sum_{k=1}^n \sum_{i,j=1}^m \|\partial_k \psi_i\|_{\mathbb{H}_\Omega} \|\partial_k \phi_j\|_{\mathbb{H}_\Omega} \\ &\leq \max_{1 \leq i,j \leq m} \{\|a_{ij}\|_\infty\} \sum_{i=1}^m \sum_{k=1}^n \|\partial_k \psi_i\|_{\mathbb{H}_\Omega} \sum_{j=1}^m \sum_{k=1}^n \|\partial_k \phi_j\|_{\mathbb{H}_\Omega} \\ &\leq mn \max_{1 \leq i,j \leq m} \{\|a_{ij}\|_\infty\} \|\boldsymbol{\psi}\|_{0,m} \|\boldsymbol{\phi}\|_{0,m} \end{aligned}$$

and the result follows.

The right-hand side of (32) is estimated as

$$\int_\Omega \dot{\mathbf{V}}^T \mathbf{F} \mathrm{d}\Omega \leq \frac{1}{2} \sum_{i=1}^m \left[m_i \|\dot{V}_i\|_{\mathbb{H}_\Omega}^2 + \frac{1}{m_i} \|F_i\|_{\mathbb{H}_\Omega}^2 \right] \tag{46}$$

where m_i, $1 \leq i \leq m$, are specified in Assumption 2 (ii).

Applying (38)-(41), (44), and (46) to (32), we have

$$\begin{aligned} &2 \sum_{i=1}^m \Bigg[m_i - \sum_{j=i+1}^m \frac{(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty)}{2} \frac{1}{\varepsilon_k} - \sum_{j=1}^{i-1} \frac{(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty)}{2} \varepsilon_l \\ &\qquad - \frac{\varepsilon}{2} \frac{\alpha(\boldsymbol{\lambda})}{\gamma^2(\boldsymbol{\lambda})} \|B(\mathbf{e})\|^2 \|b_{D_i}\|_\infty^2 \Bigg] \|\dot{V}_i\|_{\mathbb{H}_\Omega}^2 \\ &+ 2 \left[\gamma(\boldsymbol{\lambda}) - \frac{1}{2\varepsilon} \alpha(\boldsymbol{\lambda}) \right] \|E_B(\mathbf{b}_0) \dot{\mathbf{V}}\|_{0,n}^2 + \frac{d}{dt} \bar{a}(\mathbf{V}, \mathbf{V}) \\ &\leq \sum_{i=1}^m \left[m_i \|\dot{V}_i\|_{\mathbb{H}_\Omega}^2 + \frac{1}{m_i} \|F_i\|_{\mathbb{H}_\Omega}^2 \right] \end{aligned}$$

which yields

$$\begin{aligned} &\sum_{i=1}^m \Bigg[m_i - \sum_{j=i+1}^m (\|m_{ij}\|_\infty + \|m_{ji}\|_\infty) \frac{1}{\varepsilon_k} - \sum_{j=1}^{i-1} (\|m_{ij}\|_\infty + \|m_{ji}\|_\infty) \varepsilon_l \\ &\qquad - \varepsilon \frac{\alpha(\boldsymbol{\lambda})}{\gamma^2(\boldsymbol{\lambda})} \|B(\mathbf{e})\|^2 \|b_{D_i}\|_\infty^2 \Bigg] \|\dot{V}_i\|_{\mathbb{H}_\Omega}^2 \\ &+ \left[2\gamma(\boldsymbol{\lambda}) - \frac{1}{\varepsilon} \alpha(\boldsymbol{\lambda}) \right] \|E_B(\mathbf{b}_0) \dot{\mathbf{V}}\|_{0,n}^2 + \frac{d}{dt} \bar{a}(\mathbf{V}, \mathbf{V}) \leq \sum_{i=1}^m \frac{1}{m_i} \|F_i\|_{\mathbb{H}_\Omega}^2 \end{aligned}$$

where $k = j + m(i-1) - \frac{1}{2} i(i+1)$, for $i+1 \leq j \leq m$, and $l = i + m(j-1) - \frac{1}{2} j(j+1)$, for $1 \leq j \leq i-1$.

Integrating the last inequality with respect to time from 0 to t and applying (20) and (45) give, for every $t \in [0, t_f]$,

$$\hat{\beta} \int_0^t \|\dot{\mathbf{V}}\|^2_{\mathbb{H}^m_\Omega} \mathrm{d}s + \hat{\gamma} \int_0^t \|E_B(\mathbf{b}_0)\dot{\mathbf{V}}\|^2_{0,n} \mathrm{d}s + \hat{\alpha}\|\mathbf{V}(t)\|^2_{0,m} \le \frac{1}{\hat{\mu}} \int_0^t \|\mathbf{F}\|^2_{\mathbb{H}^m_\Omega} \mathrm{d}s \tag{47}$$

Here $\hat{\beta} = \min\limits_{1 \le i \le m} \beta_i$ with

$$\beta_i = m_i - \sum_{j=i+1}^{m} \big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big) \frac{1}{\varepsilon_k} - \sum_{j=1}^{i-1} \big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big) \varepsilon_l - \varepsilon \frac{\alpha(\boldsymbol{\lambda})}{\gamma^2(\boldsymbol{\lambda})} \|B(\mathbf{e})\|^2 \|b_{D_i}\|^2_\infty, \tag{48}$$

$1 \le i \le m$, $k = j + m(i-1) - \frac{1}{2}i(i+1)$, for $i+1 \le j \le m$, and $l = i + m(j-1) - \frac{1}{2}j(j+1)$, for $1 \le j \le i-1$;

$$\hat{\gamma} = 2\gamma(\boldsymbol{\lambda}) - \frac{1}{\varepsilon}\alpha(\boldsymbol{\lambda}) \tag{49}$$

$$0 < \hat{\mu} = \min_{1 \le i \le m} m_i \tag{50}$$

The values $\varepsilon_k > 0$, $\varepsilon_l > 0$, and $\varepsilon > 0$ must be chosen to guarantee that $\hat{\beta} > 0$ and $\hat{\gamma} \ge 0$.

The condition $\hat{\beta} > 0$ is equivalent to $\beta_i > 0$ for all $1 \le i \le m$. From (48) we observe that the latter condition holds true if, for $1 \le i \le m$, there exist constants $p_{ij} > 1$, $i+1 \le j \le m$, $q_{ij} > 1$, $1 \le j \le i-1$, and $r_i > 1$ such that

$$\sum_{j=i+1}^{m} \frac{1}{p_{ij}} + \sum_{j=1}^{i-1} \frac{1}{q_{ij}} + \frac{1}{r_i} < 1 \tag{51}$$

and

$$\big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big) \frac{1}{\varepsilon_k} \le \frac{1}{p_{ij}} m_i, \tag{52}$$

for $i+1 \le j \le m$ and $k = j + m(i-1) - \frac{1}{2}i(i+1)$;

$$\big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big) \varepsilon_l \le \frac{1}{q_{ij}} m_i, \tag{53}$$

for $1 \le j \le i-1$ and $l = i + m(j-1) - \frac{1}{2}j(j+1)$; and

$$\varepsilon \frac{\alpha(\boldsymbol{\lambda})}{\gamma^2(\boldsymbol{\lambda})} \|B(\mathbf{e})\|^2 \|b_{D_i}\|^2_\infty \le \frac{1}{r_i} m_i \tag{54}$$

Indeed, applying (51)-(54) to (48) we have, for every $1 \le i \le m$,

$$\begin{aligned} \beta_i &\ge m_i - \sum_{j=i+1}^{m} \frac{1}{p_{ij}} m_i - \sum_{j=1}^{i-1} \frac{1}{q_{ij}} m_i - \frac{1}{r_i} m_i \\ &= \left[1 - \sum_{j=i+1}^{m} \frac{1}{p_{ij}} - \sum_{j=1}^{i-1} \frac{1}{q_{ij}} - \frac{1}{r_i}\right] m_i > 0 \end{aligned}$$

Taking into account (36) and (37), condition (53) can be written as

$$\big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big)\varepsilon_k \le \frac{1}{q_{ji}} m_j \tag{55}$$

where $1 \le i \le m-1$, $i+1 \le j \le m$, and $k = j + m(i-1) - \frac{1}{2}i(i+1)$. Putting together (52) and (55) yields

$$\frac{p_{ij}\big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big)}{m_i} \le \varepsilon_k \le \frac{m_j}{q_{ji}\big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big)}, \tag{56}$$

where $1 \le i \le m-1$, $i+1 \le j \le m$, and $1 \le k \le \frac{1}{2}m(m-1)$ is given by $k = j + m(i-1) - \frac{1}{2}i(i+1)$. The existence of positive intervals (56) for ε_k, $1 \le k \le \frac{1}{2}m(m-1)$, is guaranteed by Assumption 2 (iii).

Next we find index values i and j that correspond to a specific value k, $1 \le k \le \frac{1}{2}m(m-1)$, in (56). To this end, using the equation $k = j + m(i-1) - \frac{1}{2}i(i+1)$ we express j as a function of i,

$$j(i) = k - m(i-1) + \frac{1}{2}i(i+1) \tag{57}$$

It can be shown that $j(i)$ is strictly decreasing for $1 \le i \le m-1$. Therefore, we seek the smallest integer $1 \le i \le m-1$ such that $j \le m$. Setting

$$k - m(i-1) + \frac{1}{2}i(i+1) \le m$$

and solving it with respect to i, with $1 \le i \le m-1$, ultimately leads to

$$i = \left\lceil \frac{1}{2}\Big(2m - 1 - \sqrt{(2m-1)^2 - 8k}\Big) \right\rceil$$

where $\lceil\cdot\rceil$ denotes the ceiling function. The corresponding value of j then follows from (57).

Now we want to determine $\varepsilon > 0$ that guarantees $\hat{\gamma} > 0$. From (49) and (54), we have

$$\frac{\alpha(\boldsymbol{\lambda})}{2\gamma(\boldsymbol{\lambda})} \le \varepsilon \le \frac{\gamma^2(\boldsymbol{\lambda})}{\alpha(\boldsymbol{\lambda})\|B(\mathbf{e})\|^2} \min_{1\le i\le m} \frac{m_i}{r_i\|b_{D_i}\|_\infty^2} \tag{58}$$

To ensure the existence of a positive interval (58) for ε, we enforce the following assumption.

Assumption 3. We assume that the bilinear form $a(\boldsymbol{\lambda};\cdot,\cdot)$, the operator $B(\mathbf{e})$, the matrix $M_D(\mathbf{x}) = [m_{ij}(\mathbf{x})]_{i,j=1}^m$, and the vector $\mathbf{b}_D(\mathbf{x})$ satisfy the following condition. There exists a constant $\tilde{c} \in (0,1)$ such that

$$\frac{\alpha^2(\boldsymbol{\lambda})}{2\gamma^3(\boldsymbol{\lambda})} < \tilde{c}\frac{\hat{\mu}}{\|B(\mathbf{e})\|^2\|\mathbf{b}_D\|_{\infty,m}^2}$$

where $\alpha(\boldsymbol{\lambda})$ and $\gamma(\boldsymbol{\lambda})$ are the continuity and coercivity constants of $a(\boldsymbol{\lambda};\cdot,\cdot)$, respectively, and $\hat{\mu}$ is given by (50).

We have proved the following result.

Lemma 3. *Suppose, for every $1 \le i \le m$, constants $p_{ij} > 1$, $i+1 \le j \le m$, $q_{ij} > 1$, $1 \le j \le i-1$, and $r_i > 1$ are such that*

$$\sum_{j=i+1}^{m} \frac{1}{p_{ij}} + \sum_{j=1}^{i-1} \frac{1}{q_{ij}} + \frac{1}{r_i} < 1 \tag{51}$$

Then under Assumptions 2 and 3, the a priori energy estimate (47) holds with $\hat{\beta} > 0$ and $\hat{\gamma} \geq 0$ if ε_k, $1 \leq k \leq \frac{1}{2}m(m-1)$, and ε satisfy the following conditions.

$$\frac{p_{ij}\big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big)}{m_i} \leq \varepsilon_k \leq \frac{m_j}{q_{ji}\big(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty\big)} \tag{56}$$

where $i = \left\lceil \frac{1}{2}\Big(2m - 1 - \sqrt{(2m-1)^2 - 8k}\Big)\right\rceil$, $\lceil\cdot\rceil$ denotes the ceiling function, and $j = k - m(i-1) + \frac{1}{2}i(i+1)$; and

$$\frac{\alpha(\boldsymbol{\lambda})}{2\gamma(\boldsymbol{\lambda})} \leq \varepsilon \leq \frac{\gamma^2(\boldsymbol{\lambda})}{\alpha(\boldsymbol{\lambda})\|B(\mathbf{e})\|^2} \min_{1\leq i\leq m} \frac{m_i}{r_i \|b_{D_i}\|_\infty^2} \tag{58}$$

Remark 4. From the formal a priori energy estimate (47) we conclude that the weak solution to problem P2 is expected to be

$$\mathbf{V} \in L^\infty(0, t_f; \mathbb{V}_0^m), \quad \dot{\mathbf{V}} \in L^2(0, t_f; \mathbb{H}_\Omega^m)$$

provided $\mathbf{F} \in L^2(0, t_f; \mathbb{H}_\Omega^m)$.

The preceding remark suggests the weak formulation of the problem P2 as follows.

Given $\mathbf{F} \in L^2(0, t_f; \mathbb{H}_\Omega^m)$, find $\mathbf{V} \in L^\infty(0, t_f; \mathbb{V}_0^m)$ with $\dot{\mathbf{V}} \in L^2(0, t_f; \mathbb{H}_\Omega^m)$ such that, for all $\boldsymbol{\psi} \in \mathbb{V}_0^m$,

$$\begin{aligned} \int_\Omega \boldsymbol{\psi}^T M_D(\mathbf{x})\dot{\mathbf{V}}\mathrm{d}\Omega + \int_\Omega \boldsymbol{\psi}^T B(\mathbf{b}_0(\mathbf{x}))E_B(D^{-1}\mathbf{b}_1(\mathbf{x}))\dot{\mathbf{V}}\mathrm{d}\Omega \\ - \int_\Omega \boldsymbol{\psi}^T \nabla\cdot\big[A_0(\mathbf{x})\nabla\mathbf{V}\big]\mathrm{d}\Omega = \int_\Omega \boldsymbol{\psi}^T\mathbf{F}\mathrm{d}\Omega \end{aligned} \tag{59}$$

$$\mathbf{V}(\mathbf{x}, 0) = \mathbf{0} \tag{60}$$

By the same arguments as those used in (39), the second term on the left-hand side of (59) can be written as

$$\int_\Omega \boldsymbol{\psi}^T B(\mathbf{b}_0(\mathbf{x}))E_B(D^{-1}\mathbf{b}_1(\mathbf{x}))\dot{\mathbf{V}}\mathrm{d}\Omega = a\big(\boldsymbol{\lambda}; E_B(\mathbf{b}_0)\boldsymbol{\psi}, E_B(\mathbf{b}_0)\dot{\mathbf{V}}\big) + a\big(\boldsymbol{\lambda}; E_B(\mathbf{b}_0)\boldsymbol{\psi}, E_B(\mathbf{b}_D)\dot{\mathbf{V}}\big) \tag{61}$$

From the continuity of the bilinear form $a(\boldsymbol{\lambda}; \cdot, \cdot)$ and the continuity the linear operator E_B, we observe that

$$\boldsymbol{\psi}, \boldsymbol{\Phi} \mapsto a\big(\boldsymbol{\lambda}; E_B(\mathbf{b}_0)\boldsymbol{\psi}, E_B(\mathbf{b}_0)\boldsymbol{\Phi}\big) + a\big(\boldsymbol{\lambda}; E_B(\mathbf{b}_0)\boldsymbol{\psi}, E_B(\mathbf{b}_D)\boldsymbol{\Phi}\big)$$

is a bilinear continuous map from $\mathbb{H}_\Omega^m \times \mathbb{H}_\Omega^m$ to $\mathbb{R}$. Therefore, we can define a continuous bilinear form $\bar{l} : \mathbb{H}_\Omega^m \times \mathbb{H}_\Omega^m \to \mathbb{R}$ by

$$\bar{l}(\boldsymbol{\psi}, \boldsymbol{\Phi}) = \int_\Omega \boldsymbol{\psi}^T M_D(\mathbf{x})\boldsymbol{\Phi}\mathrm{d}\Omega + a\big(\boldsymbol{\lambda}; E_B(\mathbf{b}_0)\boldsymbol{\psi}, E_B(\mathbf{b}_0)\boldsymbol{\Phi}\big) + a\big(\boldsymbol{\lambda}; E_B(\mathbf{b}_0)\boldsymbol{\psi}, E_B(\mathbf{b}_D)\boldsymbol{\Phi}\big) \tag{62}$$

Applying (61) and (62) to (59), the first two terms on the left-hand side of (59) take the form

$$\int_\Omega \boldsymbol{\psi}^T M_D(\mathbf{x})\dot{\mathbf{V}}\mathrm{d}\Omega + \int_\Omega \boldsymbol{\psi}^T B(\mathbf{b}_0(\mathbf{x}))E_B(D^{-1}\mathbf{b}_1(\mathbf{x}))\dot{\mathbf{V}}\mathrm{d}\Omega = \bar{l}(\boldsymbol{\psi}, \dot{\mathbf{V}}) \tag{63}$$

Remark 5. The bilinear form $\bar{l} : \mathbb{H}_\Omega^m \times \mathbb{H}_\Omega^m \to \mathbb{R}$ is coercive and hence nondegenerate.

Indeed, from (38), (40), and (41), we have, for every $\dot{\mathbf{V}} \in \mathbb{H}_\Omega^m$,

$$\begin{aligned}\bar{l}(\dot{\mathbf{V}}, \dot{\mathbf{V}}) \geq \sum_{i=1}^m \Bigg[m_i &- \sum_{j=i+1}^m \frac{(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty)}{2}\frac{1}{\varepsilon_k} \\ &- \sum_{j=1}^{i-1} \frac{(\|m_{ij}\|_\infty + \|m_{ji}\|_\infty)}{2}\varepsilon_l - \frac{\varepsilon}{2}\frac{\alpha(\boldsymbol{\lambda})}{\gamma^2(\boldsymbol{\lambda})}\|B(\mathbf{e})\|^2\|b_{D_i}\|_\infty^2 \Bigg] \|\dot{V}_i\|_{\mathbb{H}_\Omega}^2 \\ &+ \left[\gamma(\boldsymbol{\lambda}) - \frac{1}{2\varepsilon}\alpha(\boldsymbol{\lambda})\right]\|E_B(\mathbf{b}_0)\dot{\mathbf{V}}\|_{0,n}^2 \\ &= \frac{1}{2}\sum_{i=1}^m (m_i + \beta_i)\|\dot{V}_i\|_{\mathbb{H}_\Omega}^2 + \frac{1}{2}\hat{\gamma}\|E_B(\mathbf{b}_0)\dot{\mathbf{V}}\|_{0,n}^2 \geq \frac{1}{2}\hat{\mu}\|\dot{\mathbf{V}}\|_{\mathbb{H}_\Omega^m}^2\end{aligned}$$

where $\beta_i > 0$, $1 \leq i \leq m$, $\hat{\gamma} \geq 0$, and $\hat{\mu} > 0$ are given by (48)-(50), respectively.

For the last term on the left-hand side of (59), using the same arguments as in the derivation of (42) together with (43), we have

$$-\int_\Omega \boldsymbol{\psi}^T \nabla \cdot \left[A_0(\mathbf{x})\nabla\mathbf{V}\right]\mathrm{d}\Omega = \bar{a}(\boldsymbol{\psi}, \mathbf{V}) \tag{64}$$

Substituting (63) and (64) into (59) yields the following abstract formulation of the problem P2 equivalent to (59) and (60). Given $\mathbf{F} \in L^2(0, t_f; \mathbb{H}_\Omega^m)$, find $\mathbf{V} \in L^\infty(0, t_f; \mathbb{V}_0^m)$ such that $\dot{\mathbf{V}} \in L^2(0, t_f; \mathbb{H}_\Omega^m)$ and, for all $\boldsymbol{\psi} \in \mathbb{V}_0^m$,

$$\bar{l}(\boldsymbol{\psi}, \dot{\mathbf{V}}) + \bar{a}(\boldsymbol{\psi}, \mathbf{V}) = (\boldsymbol{\psi}, \mathbf{F})_{\mathbb{H}_\Omega^m} \tag{65}$$

$$\mathbf{V}(\mathbf{x}, 0) = \mathbf{0} \tag{60}$$

The next theorem summarizes results on a global weak solution to the problem P2.

Theorem 1. *Given $\mathbf{F} \in L^2(0, t_f; \mathbb{H}_\Omega^m)$, under Assumptions 2 and 3, there exists a global unique weak solution*

$$\mathbf{V} \in L^\infty(0, t_f; \mathbb{V}_0^m) \quad \textit{with} \quad \dot{\mathbf{V}} \in L^2(0, t_f; \mathbb{H}_\Omega^m) \tag{66}$$

of the problem P2 in the sense of (65) and (60), and the solution depends continuously on the data $\mathbf{F}$; that is, the mapping

$$\mathbf{F} \mapsto \mathbf{V}, \dot{\mathbf{V}}$$

from $L^2(0, t_f; \mathbb{H}_\Omega^m)$ to $L^\infty(0, t_f; \mathbb{V}_0^m) \times L^2(0, t_f; \mathbb{H}_\Omega^m)$ is continuous.

Proof. The continuity and coercivity of the bilinear forms $\bar{l}$ and $\bar{a}$, the symmetry of $\bar{a}$, as well as energy estimates arising from (47) ensure the existence and uniqueness of the weak solution to the problem P2. The proof follows routinely from the standard Galerkin method, and we omit the details.

To show that the weak solution depends continuously on $\mathbf{F}$, we use the a priori energy estimate (47), which yields

$$\|\mathbf{V}(t)\|_{0,m}^2 \leq \frac{1}{\hat{\alpha}\hat{\mu}} \int_0^{t_f} \|\mathbf{F}\|_{\mathbb{H}_\Omega^m}^2 \mathrm{d}s$$

for every $t \in [0, t_f]$, and hence,

$$\|\mathbf{V}\|_{L^\infty(0,t_f;\mathbb{V}_0^m)} \leq \hat{C}\|\mathbf{F}\|_{L^2(0,t_f;\mathbb{H}_\Omega^m)} \tag{67}$$

On the other hand, taking $t = t_f$ in (47), we obtain

$$\|\dot{\mathbf{V}}\|_{L^2(0,t_f;\mathbb{H}_\Omega^m)} \leq \hat{C}\|\mathbf{F}\|_{L^2(0,t_f;\mathbb{H}_\Omega^m)} \tag{68}$$

This completes the proof.

4. A weak solution to the coupled parabolic-elliptic system: main result

In this section, we establish global existence, uniqueness, and continuous dependence on initial and boundary data for a weak solution to the coupled parabolic-elliptic system (2)-(5). Our ultimate result is given in the following theorem.

Theorem 2. *Given the initial data $\mathbf{V}_I \in \mathbb{V}^m$, the boundary data $\mathbf{V}_B \in L^2\big(0, t_f; H^{\frac{1}{2}}(\Gamma)^m\big)$ with $\dot{\mathbf{V}}_B \in L^2\big(0, t_f; H^{\frac{1}{2}}(\Gamma)^m\big)$, and $\boldsymbol{\sigma} \in L^2\big(0, t_f; \mathbb{H}_\Gamma^n \otimes \mathbb{H}_\Gamma^n\big)$ with $\dot{\boldsymbol{\sigma}} \in L^2\big(0, t_f; \mathbb{H}_\Gamma^n \otimes \mathbb{H}_\Gamma^n\big)$, under Assumptions 1, 2, and 3, the coupled system (2)-(5) admits a global unique weak solution*

$$(\bar{\mathbf{V}}, \mathbf{u}) \in L^2(0, t_f; \mathbb{V}^m) \times L^2(0, t_f; \tilde{\mathbb{V}}_0^n) \tag{69}$$

with

$$(\dot{\bar{\mathbf{V}}}, \dot{\mathbf{u}}) \in L^2(0, t_f; \mathbb{H}_\Omega^m) \times L^2(0, t_f; \tilde{\mathbb{V}}_0^n) \tag{70}$$

and this solution depends continuously on the data $\mathbf{V}_I$, $\mathbf{V}_B(0)$, $\mathbf{V}_B$, $\dot{\mathbf{V}}_B$, $\boldsymbol{\sigma}$, and $\dot{\boldsymbol{\sigma}}$.

Proof. *Existence and Uniqueness.* The solution $(\bar{\mathbf{V}}, \mathbf{u})$ of the coupled parabolic-elliptic system (2)-(5) in terms of the solutions $\bar{\mathbf{W}}$ and $\mathbf{V}$ to the problems P1 and P2 is given by (23) and (24). Corollary 1 and Theorem 1, respectively, guarantee the global existence and uniqueness of weak solutions to the problems P1, and P2 with an arbitrary source term $\mathbf{F}$, provided $\mathbf{F} \in L^2(0, t_f; \mathbb{H}_\Omega^m)$.

Next, we show that the source term (21) corresponding to the system (2)-(5) satisfies the above condition. The definition (13) of the function $\mathbf{u}_\Gamma(\boldsymbol{\sigma}) \in \tilde{\mathbb{V}}_0^n$, $\boldsymbol{\sigma} \in L^2\big(0, t_f; \mathbb{H}_\Gamma^n \otimes \mathbb{H}_\Gamma^n\big)$, and $\dot{\boldsymbol{\sigma}} \in L^2\big(0, t_f; \mathbb{H}_\Gamma^n \otimes \mathbb{H}_\Gamma^n\big)$ give

$$\mathbf{u}_\Gamma \in L^2(0, t_f; \tilde{\mathbb{V}}_0^n) \tag{71}$$

$$\dot{\mathbf{u}}_\Gamma \in L^2(0, t_f; \tilde{\mathbb{V}}_0^n) \tag{72}$$

From Corollary 1, the definitions of the operators $B(\cdot) : \tilde{\mathbb{V}}_0^n :\to \mathbb{H}_\Omega^m$ and $E_B(\cdot) : \mathbb{H}_\Omega^m \to \tilde{\mathbb{V}}_0^n$, and (72) we infer that the source term $\mathbf{F}$ of the form (21) belongs to $L^2(0, t_f; \mathbb{H}_\Omega^m)$. Thus, Theorem 1 holds for the problem P2 associated with the system (2)-(5). The global existence and uniqueness of the weak solution $(\bar{\mathbf{V}}, \mathbf{u})$ satisfying (69) and (70) now follow immediately from (29), (66), (71), (72), and the definition of the operator E_B applied to (23), (24), and their time derivatives.

Continuous dependence on data. From (23), (30), (67), and (69) we have

$$\begin{aligned}\|\bar{\mathbf{V}}\|_{L^2(0,t_f;\mathbb{V}^m)} &\le \hat{C}\big(\|\bar{\mathbf{W}}\|_{L^2(0,t_f;\mathbb{V}^m)} + \|\mathbf{V}\|_{L^\infty(0,t_f;\mathbb{V}_0^m)}\big) \\ &\le \hat{C}\big(\|\mathbf{V}_I\|_{\mathbb{V}^m} + \|\mathbf{V}_B(0)\|_{H^{1/2}(\Gamma)^m} + \|\mathbf{V}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^m)} \\ &\quad + \|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^m)} + \|\mathbf{F}\|_{L^2(0,t_f;\mathbb{H}_\Omega^m)}\big)\end{aligned} \tag{73}$$

The time derivative of (23) together with (31), (68), and (70) give

$$\begin{aligned}\|\dot{\bar{\mathbf{V}}}\|_{L^2(0,t_f;\mathbb{H}_\Omega^m)} &\le \hat{C}\big(\|\dot{\bar{\mathbf{W}}}\|_{L^2(0,t_f;\mathbb{H}_\Omega^m)} + \|\dot{\mathbf{V}}\|_{L^2(0,t_f;\mathbb{H}_\Omega^m)}\big) \\ &\le \hat{C}\big(\|\mathbf{V}_I\|_{\mathbb{V}^m} + \|\mathbf{V}_B(0)\|_{H^{1/2}(\Gamma)^m} + \|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^m)} + \|\mathbf{F}\|_{L^2(0,t_f;\mathbb{H}_\Omega^m)}\big)\end{aligned} \tag{74}$$

To estimate the source term $\mathbf{F}$ on the right-hand side of (73) and (74), we apply the continuity of the operators B and E_B and (31) to (21) and obtain

$$\begin{aligned}\|\mathbf{F}\|_{L^2(0,t_f;\mathbb{H}_\Omega^m)} &\le \hat{C}\big(\|\dot{\bar{\mathbf{W}}}\|_{L^2(0,t_f;\mathbb{H}_\Omega^m)} + \|\dot{\mathbf{u}}_\Gamma\|_{L^2(0,t_f;\tilde{\mathbb{V}}_0^n)}\big) \\ &\le \hat{C}\big(\|\mathbf{V}_I\|_{\mathbb{V}^m} + \|\mathbf{V}_B(0)\|_{H^{1/2}(\Gamma)^m} + \|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^m)} + \|\dot{\mathbf{u}}_\Gamma\|_{L^2(0,t_f;\tilde{\mathbb{V}}_0^n)}\big)\end{aligned} \tag{75}$$

Next, we estimate $\mathbf{u}_\Gamma$ and its time derivative. To that end, we take $\mathbf{\Phi} = \mathbf{u}_\Gamma$ in (13) and use the $\tilde{\mathbb{V}}_0^n$-coercivity of the form $a(\boldsymbol{\lambda};\cdot,\cdot)$, as well as the trace theorem [6] and Poincaré-Friedrichs' inequality [14]. Then, for every $t \in [0,t_f]$,

$$\begin{aligned}\gamma(\boldsymbol{\lambda})\|\mathbf{u}_\Gamma(t)\|_{0,n}^2 &\le \big|a(\boldsymbol{\lambda};\mathbf{u}_\Gamma(t),\mathbf{u}_\Gamma(t))\big| = \big|\big(\boldsymbol{\sigma}(t)\mathbf{n},\mathbf{u}_\Gamma(t)\big)_{\mathbb{H}_\Gamma^n}\big| \\ &\le \hat{C}\|\boldsymbol{\sigma}(t)\|_{\mathbb{H}_\Gamma^n\otimes\mathbb{H}_\Gamma^n}\|\mathbf{u}_\Gamma(t)\|_{0,n}\end{aligned}$$

From the above inequality, for every $t \in [0,t_f]$,

$$\|\mathbf{u}_\Gamma(t)\|_{0,n} \le \hat{C}\|\boldsymbol{\sigma}(t)\|_{\mathbb{H}_\Gamma^n\otimes\mathbb{H}_\Gamma^n}$$

and the integration over $[0,t_f]$ yields

$$\|\mathbf{u}_\Gamma\|_{L^2(0,t_f;\tilde{\mathbb{V}}_0^n)} \le \hat{C}\|\boldsymbol{\sigma}\|_{L^2(0,t_f;\mathbb{H}_\Gamma^n\otimes\mathbb{H}_\Gamma^n)} \tag{76}$$

By the similar argument, differentiating (13) with respect to time and taking $\mathbf{\Phi} = \dot{\mathbf{u}}_\Gamma$, we eventually obtain

$$\|\dot{\mathbf{u}}_\Gamma\|_{L^2(0,t_f;\tilde{\mathbb{V}}_0^n)} \le \hat{C}\|\dot{\boldsymbol{\sigma}}\|_{L^2(0,t_f;\mathbb{H}_\Gamma^n\otimes\mathbb{H}_\Gamma^n)} \tag{77}$$

Substituting (77) into (75) and applying the result to (73) and (74) we respectively get

$$\begin{aligned}\|\bar{\mathbf{V}}\|_{L^2(0,t_f;\mathbb{V}^m)} &\le \hat{C}\big(\|\mathbf{V}_I\|_{\mathbb{V}^m} + \|\mathbf{V}_B(0)\|_{H^{1/2}(\Gamma)^m} + \|\mathbf{V}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^m)} \\ &\quad + \|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^m)} + \|\dot{\boldsymbol{\sigma}}\|_{L^2(0,t_f;\mathbb{H}_\Gamma^n\otimes\mathbb{H}_\Gamma^n)}\big)\end{aligned} \tag{78}$$

and

$$\begin{aligned}\|\dot{\bar{\mathbf{V}}}\|_{L^2(0,t_f;\mathbb{V}^m)} &\le \hat{C}\big(\|\mathbf{V}_I\|_{\mathbb{V}^m} + \|\mathbf{V}_B(0)\|_{H^{1/2}(\Gamma)^m} \\ &\quad + \|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^m)} + \|\dot{\boldsymbol{\sigma}}\|_{L^2(0,t_f;\mathbb{H}_\Gamma^n\otimes\mathbb{H}_\Gamma^n)}\big)\end{aligned} \tag{79}$$

Our next step is to show that the **u**-component of the solution depends continuously on the initial and boundary data. Applying (7) and the continuity of the operator E_B to (16) and combining the result with (76) and (78) give

$$\begin{aligned}\|\mathbf{u}\|_{L^2(0,t_f;\tilde{\mathbb{V}}_0^n)} \leq \hat{C}\big(&\|\mathbf{V}_I\|_{\mathbb{V}^m} + \|\mathbf{V}_B(0)\|_{H^{1/2}(\Gamma)^m} \\ &+ \|\mathbf{V}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^m)} + \|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^m)} \\ &+ \|\boldsymbol{\sigma}\|_{L^2(0,t_f;\mathbb{H}_\Gamma^n\otimes\mathbb{H}_\Gamma^n)} + \|\dot{\boldsymbol{\sigma}}\|_{L^2(0,t_f;\mathbb{H}_\Gamma^n\otimes\mathbb{H}_\Gamma^n)}\big)\end{aligned}$$

Finally, differentiating (7) and (16) with respect to time and applying the continuity of the operator E_B together with (77) and (79), we have

$$\|\dot{\mathbf{u}}\|_{L^2(0,t_f;\tilde{\mathbb{V}}_0^n)} \leq \hat{C}\big(\|\mathbf{V}_I\|_{\mathbb{V}^m} + \|\mathbf{V}_B(0)\|_{H^{1/2}(\Gamma)^m} + \|\dot{\mathbf{V}}_B\|_{L^2(0,t_f;H^{1/2}(\Gamma)^m)} + \|\dot{\boldsymbol{\sigma}}\|_{L^2(0,t_f;\mathbb{H}_\Gamma^n\otimes\mathbb{H}_\Gamma^n)}\big)$$

The proof of Theorem 2 is thus completed.

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