



Symmetries of the C^* -algebra of a vector bundle

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ABSTRACT

We consider C^* -algebras constructed from compact group actions on complex vector bundles $E \rightarrow X$ endowed with a Hermitian metric. An action of G by isometries on $E \rightarrow X$ induces an action on the C^* -correspondence $\Gamma(E)$ over $C(X)$ consisting of continuous sections, and on the associated Cuntz–Pimsner algebra $\mathcal{O}_E$, so we can study the crossed product $\mathcal{O}_E \rtimes G$. If the action is free and $\text{rank } E = n$, then we prove that $\mathcal{O}_E \rtimes G$ is Morita–Rieffel equivalent to a field of Cuntz algebras $\mathcal{O}_n$ over the orbit space X/G . If the action is fiberwise, then $\mathcal{O}_E \rtimes G$ becomes a continuous field of crossed products $\mathcal{O}_n \rtimes G$. For transitive actions, we show that $\mathcal{O}_E \rtimes G$ is Morita–Rieffel equivalent to a graph C^* -algebra.

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1. Introduction

In this paper, in the spirit of duality between topology and operator algebras, we study some C^* -algebras associated to compact group actions on complex vector bundles $E \rightarrow X$ endowed with a Hermitian metric. By taking continuous sections of the bundle, we get a finitely generated projective Hilbert module $\Gamma(E)$ over $A = C(X)$, viewed as a C^* -correspondence with the same left and right multiplications and the usual inner product. The associated Cuntz–Pimsner algebra $\mathcal{O}_{C(X)}(\Gamma(E))$, also denoted $\mathcal{O}_E$ for short, is a continuous field of Cuntz algebras, see Proposition 2 in [23].

An action of G on E by isometries induces an action of G on $\Gamma(E)$ and on $\mathcal{O}_E$, so one can study the crossed product $\mathcal{O}_E \rtimes G$. Using a theorem of Hao–Ng from [10], we have the isomorphism $\mathcal{O}_E \rtimes G \cong \mathcal{O}_{C(X) \rtimes G}(\Gamma(E) \rtimes G)$, so it is useful to understand the crossed product correspondence $\Gamma(E) \rtimes G$ as a kind of noncommutative bundle over the crossed product $C(X) \rtimes G$, which in some cases is Morita–Rieffel equivalent to an abelian C^* -algebra.

In particular, if the action of G is free and $\text{rank } E = n$, then we show that $\mathcal{O}_E \rtimes G$ is Morita–Rieffel equivalent to a field of Cuntz algebras $\mathcal{O}_n$ over the orbit space X/G . If the action of G is fiberwise, then we prove that $\mathcal{O}_E \rtimes G$ becomes a continuous field of crossed products $\mathcal{O}_n \rtimes G$. For transitive actions, the vector bundle E has a special form, and $\mathcal{O}_E \rtimes G$ is Morita–Rieffel equivalent to a graph C^* -algebra.

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2. Group actions on vector bundles

We collect here some facts about G -vector bundles over G -spaces. For simplicity, in section 1.6 of [2], the author considers only the finite group case, but assuming continuity of the maps, all the results generalize to compact groups, see [22].

Let us fix a compact group G with identity e . A G -space is a topological space X together with a continuous action $G \times X \rightarrow X$ written $(g, x) \mapsto g \cdot x$ satisfying the usual conditions $g \cdot (g' \cdot x) = (gg') \cdot x$ and $e \cdot x = x$.

Definition 2.1. If X is a G -space, then a G -vector bundle on X is a (locally trivial) complex vector bundle $p : E \rightarrow X$ such that E is a G -space, p is a G -map i.e. $p(g \cdot v) = g \cdot p(v)$, and the maps $E_x \rightarrow E_{g \cdot x}$ between fibers are linear.

Remark 2.2. Given a vector bundle $p : E \rightarrow X$ over a compact Hausdorff space X , the elements of the automorphism group $\text{Aut}(E)$ are pairs (φ, Φ) where $\varphi : X \rightarrow X$ is a homeomorphism, $\Phi : E \rightarrow E$ is a map such that $p \circ \Phi = \varphi \circ p$ and if $E_x = p^{-1}(x)$ is the fiber over x , then all restrictions $\Phi_x : E_x \rightarrow E_{\varphi(x)}$ are linear isomorphisms. An action of a group G on E is understood as a homomorphism $G \rightarrow \text{Aut}(E)$. If G acts trivially on X , then we say that the action on E is fiberwise. The set of fiberwise automorphisms is a subgroup of $\text{Aut}(E)$, denoted $\text{Aut}_X(E)$.

Example 2.3. If X is a differentiable manifold and G is a group acting smoothly on X , then the complexified tangent bundle $E = TX \otimes \mathbb{C}$ and all the associated tensor bundles $E^{\otimes n}$ become G -vector bundles.

Example 2.4. If E is a complex vector bundle on X , then the k -fold tensor product $E^{\otimes k}$ becomes an S_k -vector bundle on X , where the symmetric group S_k fixes the points in X and permutes the tensor factors.

Remark 2.5. If X reduces to a point, then a G -vector bundle is just a finite dimensional representation of G . If X is a trivial G -space, i.e. $g \cdot x = x$ for all $g \in G$ and $x \in X$, then a G -vector bundle is a family of representations of G varying continuously with x .

Recall that X is a free G -space if $g \neq e$ implies $g \cdot x \neq x$. We denote by X/G the orbit space, and let $\pi : X \rightarrow X/G$ be the canonical map.

Remark 2.6. Given a G -vector bundle E over a free G -space X , then E is necessarily a free G -space, and E/G has a natural vector bundle structure over X/G . In fact, $E/G \rightarrow X/G$ is locally isomorphic to $E \rightarrow X$ and the action of G on E/G is trivial.

If X is a free G -space and $p : F \rightarrow X/G$ is any vector bundle over X/G , then the pull back

$$\pi^*F = \{(x, v) \in X \times F \mid \pi(x) = p(v)\}$$

is a G -vector bundle over X with action $g \cdot (x, v) = (g \cdot x, v)$.

In particular, we have

Proposition 2.7. *There is a bijective correspondence between G -vector bundles over a free G -space X and vector bundles over X/G .*

Assume now that the action on X is trivial, hence G acts on $E \rightarrow X$ fiberwise. By averaging over G , we get a projection

$$P : E \rightarrow E^G, \quad P(v) = \int_G g \cdot v \, dg,$$

where E^G is the subbundle of fixed vectors and dg is the normalized Haar measure on G . In particular, if E, F are G -bundles over X and the action on X is trivial, we have

$$\text{Hom}_G(E, F) \cong \text{Hom}(E, F)^G,$$

where $\text{Hom}(E, F)$ is the vector bundle over X with fibers $\text{Hom}(E_x, F_x)$ and $\text{Hom}_G(E, F)$ is the set of G -equivariant morphisms.

Denote by $\{V_i\}_{i \in I}$ the set of irreducible representations of G , so that any representation V of G is isomorphic to a direct sum $\bigoplus_{i \in I} n_i V_i$, where n_i are multiplicities. Take $W_i = X \times V_i$ to be the product bundle with the natural G -action $g \cdot (x, v) = (x, g \cdot v)$. If E is a G -bundle over X , then the bundle map

$$\bigoplus_i W_i \otimes \text{Hom}_G(W_i, E) \rightarrow E$$

is an isomorphism of G -bundles and the action on $\text{Hom}_G(W_i, E)$ is trivial. We conclude that

Proposition 2.8. *For X a trivial G -space, every G -bundle E over X is isomorphic to a direct sum $\bigoplus_i W_i \otimes E_i$, where $W_i = X \times V_i$ as above and $E_i = \text{Hom}_G(W_i, E)$ are vector bundles with trivial G -action.*

Suppose now that X is a transitive G -space and fix $x \in X$ with stabilizer group G_x . Then there is a natural continuous bijection of G -spaces

$$f : G/G_x \rightarrow X, \quad f(tG_x) = t \cdot x.$$

If $H \subseteq G$ is a subgroup and $p : E \rightarrow G/H$ is a G -vector bundle over the homogeneous space $G/H = X$, then the stabilizer group H of the coset $x = eH$ acts on the fiber E_x as a group representation and E_x becomes an H -module. Let $G \times_H E_x$ denote the orbit space of $G \times E_x$ under the action of H given by $h(g, v) = (gh^{-1}, h \cdot v)$. Then $G \times_H E_x$ becomes a G -vector bundle over G/H via the map

$$q : G \times_H E_x \rightarrow G/H, \quad q(g, v) = gH$$

and the action $g' \cdot (g, v) = (g'g, v)$. Moreover, the map

$$\Phi : G \times_H E_x \rightarrow E, \quad \Phi(g, v) = g \cdot v$$

is an isomorphism of G -bundles over G/H .

Conversely, to each representation V of H , we can associate a vector bundle $G \times_H V$ over G/H . Hence we obtain

Proposition 2.9. *Any G -vector bundle over the homogeneous space G/H is of the form $G \times_H V$ for some H -module V .*

3. Group actions on C^* -correspondences and crossed products

The dual notion of a G -space is a C^* -dynamical system. This is a triple (A, G, α) consisting of a C^* -algebra A , a locally compact group G and a continuous homomorphism $\alpha : G \rightarrow \text{Aut}(A)$. A covariant representation of (A, G, α) is a pair (π, u) consisting of a representation $\pi : A \rightarrow \mathcal{L}(H)$ and a unitary representation $u : G \rightarrow U(H)$ on the same Hilbert space H such that

$$\pi(\alpha_g(a)) = u_g \pi(a) u_g^*$$

for all $a \in A$ and $g \in G$. The crossed product C^* -algebra $A \rtimes_\alpha G$ is a universal object for covariant representations.

Recall that given C^* -algebras A and B , a C^* -correspondence $\mathcal{H} = {}_A\mathcal{H}_B$ from A to B is a right Hilbert B -module with a $*$ -homomorphism $\phi : A \rightarrow \mathcal{L}(\mathcal{H})$ which gives the left multiplication. Here $\mathcal{L}(\mathcal{H}) = \mathcal{L}_B(\mathcal{H})$ denotes the set of adjointable operators, which are automatically bounded and B -linear. We often write $a\xi$ for $\phi(a)\xi$.

If the map ϕ is injective, then $\mathcal{H}$ is called faithful. If the span of $\phi(A)\mathcal{H}$ is dense in $\mathcal{H}$, then $\mathcal{H}$ is called essential or nondegenerate. If the inner products $\langle \xi, \eta \rangle$ generate B , then $\mathcal{H}$ is called full.

Definition 3.1. A representation of a C^* -correspondence ${}_A\mathcal{H}_B$ is a triple $(\pi_A, \tau_{\mathcal{H}}, \pi_B)$ consisting of nondegenerate representations π_A and π_B of A and B on Hilbert spaces H_A and H_B respectively, together with a linear map $\tau_{\mathcal{H}} : \mathcal{H} \rightarrow \mathcal{L}(H_B, H_A)$ such that

$$\tau_{\mathcal{H}}(a\xi) = \pi_A(a)\tau_{\mathcal{H}}(\xi), \quad \tau_{\mathcal{H}}(\xi b) = \tau_{\mathcal{H}}(\xi)\pi_B(b), \quad \pi_B(\langle \xi, \eta \rangle) = \tau_{\mathcal{H}}(\xi)^* \tau_{\mathcal{H}}(\eta)$$

for all $a \in A, \xi, \eta \in \mathcal{H}$ and $b \in B$. Here $\mathcal{L}(H_B, H_A)$ is the set of linear bounded operators between the Hilbert spaces H_B and H_A .

Given a locally compact group G , an action on ${}_A\mathcal{H}_B$ is determined by a homomorphism $\rho : G \rightarrow \mathcal{L}_{\mathbb{C}}(\mathcal{H})$ such that ρ_g is a $\mathbb{C}$ -linear isomorphism and $g \mapsto \rho_g \xi$ is continuous for all $\xi \in \mathcal{H}$, and by continuous actions of G on A and B given by $\alpha : G \rightarrow \text{Aut}(A)$, $\beta : G \rightarrow \text{Aut}(B)$ with compatibility relations

$$\langle \rho_g(\xi), \rho_g(\eta) \rangle = \beta_g(\langle \xi, \eta \rangle), \quad \rho_g(\xi b) = \rho_g(\xi) \beta_g(b), \quad \rho_g(a\xi) = \alpha_g(a) \rho_g(\xi),$$

where $\xi, \eta \in \mathcal{H}, a \in A, b \in B$.

Definition 3.2. Suppose the locally compact group G acts on the C^* -correspondence $\mathcal{H}$ from A to B via maps ρ, α, β as above. The crossed product $\mathcal{H} \rtimes_{\rho} G$ is a C^* -correspondence from $A \rtimes_{\alpha} G$ to $B \rtimes_{\beta} G$, and it can be obtained as the completion of $C_c(G, \mathcal{H})$ using the operations

$$\begin{aligned} \langle \xi, \eta \rangle(t) &= \int_G \beta_{s^{-1}}(\langle \xi(s), \eta(st) \rangle) ds, \\ (\xi \cdot f)(t) &= \int_G \xi(s) \beta_s(f(s^{-1}t)) ds, \\ (h \cdot \xi)(t) &= \int_G h(s) \rho_s(\xi(s^{-1}t)) ds, \end{aligned}$$

where $\xi, \eta \in C_c(G, \mathcal{H}), f \in C_c(G, B), h \in C_c(G, A)$.

Remark 3.3. The crossed product correspondence $\mathcal{H} \rtimes_{\rho} G$ can also be obtained as $\mathcal{H} \otimes_{\varphi} (B \rtimes_{\beta} G)$, where $\varphi : B \rightarrow \mathcal{L}(B \rtimes_{\beta} G)$ is the embedding of B in the multiplier algebra of $B \rtimes_{\beta} G$, regarded as a Hilbert module over itself.

The group G acts on $\mathcal{H} \otimes_{\varphi} (B \rtimes_{\beta} G)$ by

$$g(\xi \otimes f) = \rho_g(\xi) \otimes u_g f u_g^{-1},$$

where $f \in B \rtimes_{\beta} G$ and $u : G \rightarrow \mathcal{L}(B \rtimes_{\beta} G)$ is the unitary representation such that $\beta_g(b) = u_g b u_g^{-1}$. If we define

$$U_g(\xi \otimes f) = \rho_g(\xi) \otimes u_g f,$$

then $U_g \in \mathcal{L}(\mathcal{H} \rtimes_{\rho} G)$ with $U_g^* = U_{g^{-1}}$ and

$$U_g(\xi \otimes f) U_{g^{-1}} = \rho_g(\xi) \otimes u_g f u_g^{-1}.$$

The left and right multiplications on $\mathcal{H} \otimes_{\varphi} (B \rtimes_{\beta} G)$ are given by convolution,

$$(h \cdot (\xi \otimes f))(t) = \int_G h(s) \rho_s(\xi) \beta_s(f(s^{-1}t)) ds, \quad (\xi \otimes f) \cdot f' = \xi \otimes f f'$$

for $h \in C_c(G, A)$ and $f, f' \in C_c(G, B)$. The inner product formula could be expressed as

$$\langle \xi \otimes f, \eta \otimes f' \rangle = f^* \langle \xi, \eta \rangle f',$$

where $\xi, \eta \in \mathcal{H}$ and $f^*(t) = \Delta(t)^{-1} \alpha_t(f(t^{-1})^*)$, with Δ the modular function on G . The isomorphism with the crossed product defined using a completion of $C_c(G, \mathcal{H})$ is induced by the function

$$\Phi : \mathcal{H} \otimes C_c(G, B) \rightarrow C_c(G, \mathcal{H}), \quad \Phi(\xi \otimes f)(g) = \xi f(g).$$

Definition 3.4. Suppose the locally compact group G acts on the C^* -correspondence $\mathcal{H}$ from A to B via maps ρ, α, β as above. If (π_A, u) and (π_B, v) are covariant representations of (A, G, α) and (B, G, β) on Hilbert spaces H_A and H_B respectively, then $(\pi_A, \tau_{\mathcal{H}}, \pi_B, u, v)$ is called a covariant representation of $(\mathcal{H}, G, \rho)$ in $\mathcal{L}(H_B, H_A)$ if

$$\tau_{\mathcal{H}}(\rho_g(\xi)) = u_g \tau_{\mathcal{H}}(\xi) v_g^*$$

for all $\xi \in \mathcal{H}$ and $g \in G$.

In this case, we can define the representation $\tau_{\mathcal{H}} \times v$ of $\mathcal{H} \rtimes_{\rho} G$ into $\mathcal{L}(H_B, H_A)$ by

$$(\tau_{\mathcal{H}} \times v)(\eta) = \int_G \tau_{\mathcal{H}}(\eta(s)) v_s ds$$

for $\eta \in C_c(G, \mathcal{H})$. Then $(\pi_A \times u, \tau_{\mathcal{H}} \times v, \pi_B \times v)$ becomes a representation of the C^* -correspondence $\mathcal{H} \rtimes_{\rho} G$ from $A \rtimes_{\alpha} G$ to $B \rtimes_{\beta} G$.

Example 3.5. (The reduced crossed product) Consider $(\pi_A, \tau_{\mathcal{H}}, \pi_B)$ a representation of ${}_A \mathcal{H}_B$ such that $\pi_A : A \rightarrow \mathcal{L}(H_A)$ and $\pi_B : B \rightarrow \mathcal{L}(H_B)$ are faithful. If λ is the left regular representation of G given by $(\lambda_s f)(t) = f(s^{-1}t)$ for $f \in L^2(G, H_A) \cup L^2(G, H_B)$, then we define

$$(\tilde{\pi}_A(a)f)(t) = \pi_A(\alpha_{t^{-1}}(a))f(t), \quad (\tilde{\pi}_B(b)f)(t) = \pi_B(\beta_{t^{-1}}(b))f(t).$$

We get a covariant representation $(\tilde{\pi}_A, \tilde{\tau}_\mathcal{H}, \tilde{\pi}_B, \lambda, \lambda)$ of $\mathcal{H}$ by taking

$$(\tilde{\tau}_\mathcal{H}(\xi)f)(t) = \tau_\mathcal{H}(\rho_{t^{-1}}(\xi))f(t)$$

for $f \in L^2(G, H_B)$ and $(\tilde{\pi}_A \times \lambda, \tilde{\tau}_\mathcal{H} \times \lambda, \tilde{\pi}_B \times \lambda)$ is a representation of $\mathcal{H} \rtimes_\rho G$ into $\mathcal{L}(L^2(G, H_B), L^2(G, H_A))$. Its image $(\tilde{\tau}_\mathcal{H} \times \lambda)(\mathcal{H} \rtimes_\rho G)$ is called the reduced crossed product, denoted $\mathcal{H} \rtimes_{\rho, r} G$, which is a C^* -correspondence from $A \rtimes_{\alpha, r} G$ to $B \rtimes_{\beta, r} G$. If G is amenable, then $\mathcal{H} \rtimes_\rho G \cong \mathcal{H} \rtimes_{\rho, r} G$.

Remark 3.6. The crossed product $\mathcal{H} \rtimes_\rho G$ can be characterized as universal for covariant representations.

Consider the category $\mathcal{C}$ in which the objects are C^* -algebras and in which the morphisms from A to B are the isomorphism classes of full C^* -correspondences from A to B . The composition of $[\mathcal{H}] : A \rightarrow B$ with $[\mathcal{M}] : B \rightarrow C$ is the isomorphism class of $\mathcal{H} \otimes_B \mathcal{M}$ as a correspondence from A to C . The isomorphisms in $\mathcal{C}$ are given by imprimitivity bimodules, see [8].

Denote by $\mathcal{A}(G)$ the category in which the objects are C^* -algebras with actions of G and in which the morphisms from (A, G, α) to (B, G, β) are equivariant isomorphism classes of full C^* -correspondences from A to B with compatible actions of G . The composition of $[\mathcal{H}, G, \rho] : (A, G, \alpha) \rightarrow (B, G, \beta)$ with $[\mathcal{M}, G, \sigma] : (B, G, \beta) \rightarrow (C, G, \gamma)$ is the isomorphism class of the tensor product action $(\mathcal{H} \otimes_B \mathcal{M}, G, \rho \otimes_B \sigma)$.

Remark 3.7. There is a functor from the category $\mathcal{A}(G)$ to $\mathcal{C}$ taking a dynamical system (A, G, α) into $A \rtimes_\alpha G$ and a C^* -correspondence $(\mathcal{H}, G, \rho)$ into $\mathcal{H} \rtimes_\rho G$ (see section 3 in [8]).

In particular, for G a compact group, $A \rtimes_\alpha G$ can be identified with a subalgebra of $A \otimes \mathcal{K}(L^2(G))$, see Proposition 4.3 in [21] and $\mathcal{H} \rtimes_\rho G$ with a subspace of $\mathcal{H} \otimes \mathcal{K}(L^2(G))$.

4. Group actions on Cuntz-Pimsner algebras and Morita-Rieffel equivalence

If $A = B$, a C^* -correspondence $\mathcal{H}$ from A to A is called a C^* -correspondence over A . An action ρ of G on the C^* -correspondence $\mathcal{H}$ over A determines an action on $\mathcal{K}(\mathcal{H}) = \mathcal{K}_A(\mathcal{H})$, the C^* -algebra generated by the finite rank operators $\theta_{\xi, \eta}$, where $\theta_{\xi, \eta}(\zeta) = \xi \langle \eta, \zeta \rangle$. The action is given by $g \cdot \theta_{\xi, \eta} = \theta_{\rho_g(\xi), \rho_g(\eta)}$ and it follows that

$$(g \cdot T)(\xi) = \rho_g(T(\rho_{g^{-1}}\xi))$$

for $T \in \mathcal{K}(\mathcal{H})$. Moreover, there is an isomorphism

$$\mathcal{K}(\mathcal{H}) \rtimes G \cong \mathcal{K}(\mathcal{H} \rtimes_\rho G),$$

see [10]. Using the universal property, we get an action of G on the Cuntz-Pimsner algebra $\mathcal{O}_A(\mathcal{H})$ defined in [11]. It is known that $\mathcal{O}_A(\mathcal{H})$ has a gauge action defined by $z \cdot a = a$, $z \cdot \xi = z\xi$ where $z \in \mathbb{T}$. The action of G commutes with the gauge action, therefore we get an action on the core algebra $\mathcal{O}_A(\mathcal{H})^\mathbb{T}$, the fixed point algebra under the gauge action. We recall the following result:

Theorem 4.1. (*G. Hao, C.-K. Ng, [10]*). *Let $\mathcal{H}$ be a C^* -correspondence over A and let the locally compact amenable group G act on $(\mathcal{H}, A)$ via (ρ, α) . Then G acts on $\mathcal{O}_A(\mathcal{H})$ via γ and*

$$\mathcal{O}_{A \rtimes_\alpha G}(\mathcal{H} \rtimes_\rho G) \cong \mathcal{O}_A(\mathcal{H}) \rtimes_\gamma G.$$

Corollary 4.2. For G locally compact amenable acting on $\mathcal{H}$ we have

$$\mathcal{O}_{A \rtimes_{\alpha} G}(\mathcal{H} \rtimes_{\rho} G)^{\mathbb{T}} \cong \mathcal{O}_A(\mathcal{H})^{\mathbb{T}} \rtimes_{\gamma} G.$$

Example 4.3. Let G be the symmetric group

$$S_3 = \langle t, s \rangle = \{1, t, ts, ts^2, s, s^2\},$$

which acts on $\mathcal{H} = \mathbb{C}^2$ by

$$t \cdot e_1 = e_2, t \cdot e_2 = e_1, s \cdot e_1 = we_1, s \cdot e_2 = w^2 e_2.$$

Here $t = (12), s = (123)$, $\{e_1, e_2\}$ is the canonical basis in $\mathbb{C}^2$, and $w^2 + w + 1 = 0$.

The Hilbert space $\mathcal{H} = \mathbb{C}^2$ is viewed as a C^* -correspondence over $A = \mathbb{C}$ with the usual operations. The group acts trivially on A , hence

$$A \rtimes S_3 \cong C^*(S_3) \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{M}_2.$$

The second isomorphism is obtained by using the minimal central projections

$$p_1 = \frac{1}{6}\chi_{\iota}, \quad p_2 = \frac{1}{6}\chi_{\varepsilon}, \quad p_3 = \frac{1}{3}\chi_{\sigma},$$

where

$$\begin{aligned} \chi_{\iota}(1) = \chi_{\iota}(t) = \chi_{\iota}(s) = 1; \quad \chi_{\varepsilon}(t) = -1, \quad \chi_{\varepsilon}(1) = \chi_{\varepsilon}(s) = 1, \\ \chi_{\sigma}(1) = 2, \quad \chi_{\sigma}(t) = 0, \quad \chi_{\sigma}(s) = -1 \end{aligned}$$

are the characters of the trivial representation ι , the signature representation ε and the 2-dimensional irreducible representation σ defined by

$$\sigma(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma(s) = \begin{bmatrix} w & 0 \\ 0 & w^2 \end{bmatrix}.$$

The crossed product C^* -correspondence $\mathcal{H} \rtimes S_3$ over $C^*(S_3)$ is a 12-dimensional vector space with basis $\{\xi_x^j : x \in S_3, j = 1, 2\}$ where $\xi_x^j(y) = e_j$ for $y = x$ and $\xi_x^j(y) = 0$ for $y \neq x$.

To decompose $\mathcal{H} \rtimes S_3$ into pieces according to the decomposition of $C^*(S_3) \cong \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{M}_2$, we need to compute the products $p_i \xi_x^j p_k$ for $i, k \in \{1, 2, 3\}, x \in S_3$ and $j \in \{1, 2\}$. Now

$$\begin{aligned} (p_i \xi_x^j p_k)(y) &= \sum_{u \in S_3} \left(\sum_{v \in S_3} p_i(v) v \cdot \xi_x^j(v^{-1}u) \right) u \cdot p_k(u^{-1}y) = \\ &= \sum_{u \in S_3} p_i(ux^{-1}) p_k(u^{-1}y)(ux^{-1}) \cdot e_j. \end{aligned}$$

A long computation shows that

$$\begin{aligned} p_i(\mathcal{H} \rtimes S_3) p_k &= 0 \text{ for } i, k \in \{1, 2\}, \\ \dim p_i(\mathcal{H} \rtimes S_3) p_3 &= \dim p_3(\mathcal{H} \rtimes S_3) p_k = 2 \text{ for } i, k \in \{1, 2\} \end{aligned}$$

and

$$\dim p_3(\mathcal{H} \rtimes S_3)p_3 = 4.$$

Since $\mathcal{O}_A(\mathcal{H}) \cong \mathcal{O}_2$, using Theorem 4.1 and results in [7], we get that $\mathcal{O}_2 \rtimes S_3 \cong \mathcal{O}_{C^*(S_3)}(\mathcal{H} \rtimes S_3)$ is Morita–Rieffel equivalent to the graph algebra with incidence matrix

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

In particular, it follows that $K_0(\mathcal{O}_2 \rtimes S_3) \cong \mathbb{Z}_2$ and $K_1(\mathcal{O}_2 \rtimes S_3) \cong 0$. Since the core algebra $\mathcal{O}_2^{\mathbb{T}}$ is the UHF-algebra M_{2^∞} , we get an action of S_3 on M_{2^∞} and $K_0(M_{2^\infty} \rtimes S_3) \cong \varinjlim (\mathbb{Z}^3, B)$, the direct limit of the stationary system $\mathbb{Z}^3 \xrightarrow{B} \mathbb{Z}^3 \xrightarrow{B} \dots$.

Definition 4.4. Given C^* -correspondences $\mathcal{H}$ and $\mathcal{M}$ over A and B respectively, we say that $\mathcal{H}$ and $\mathcal{M}$ are Morita–Rieffel equivalent in case A and B are Morita–Rieffel equivalent via an imprimitivity bimodule $\mathcal{Z}$ such that $\mathcal{Z} \otimes_B \mathcal{M}$ and $\mathcal{H} \otimes_A \mathcal{Z}$ are isomorphic as C^* -correspondences from A to B .

Using linking algebras, Muhly and Solel proved in [16] that for faithful and essential Morita–Rieffel equivalent C^* -correspondences $\mathcal{H}$ and $\mathcal{M}$, the Cuntz–Pimsner algebras $\mathcal{O}_A(\mathcal{H})$ and $\mathcal{O}_B(\mathcal{M})$ are Morita–Rieffel equivalent. This result was generalized for possibly nonfaithful correspondences in [9].

Remark 4.5. A C^* -correspondence $\mathcal{H}$ over A determines an imprimitivity bimodule between $\mathcal{K}(\mathcal{H})$ and A . The linking algebra $C = \mathcal{K}(\mathcal{H} \oplus A)$ consists of matrices

$$\begin{bmatrix} T & \xi \\ \tilde{\eta} & a \end{bmatrix},$$

where $T \in \mathcal{K}(\mathcal{H})$, $\xi \in \mathcal{H}$, $\tilde{\eta} \in \tilde{\mathcal{H}}$, $a \in A$. Here $\tilde{\mathcal{H}} = \mathcal{K}(\mathcal{H}, A)$ denotes the dual of the Hilbert module $\mathcal{H}$.

Assume now that the group G acts on $\mathcal{H}$. Then G acts on $\mathcal{H} \oplus A$ componentwise and $(\mathcal{H} \oplus A) \rtimes G \cong \mathcal{H} \rtimes G \oplus A \rtimes G$. If we write

$$C = \begin{bmatrix} \mathcal{K}(\mathcal{H}) & \mathcal{H} \\ \tilde{\mathcal{H}} & A \end{bmatrix},$$

then G acts on $C = \mathcal{K}(\mathcal{H} \oplus A)$ and

$$C \rtimes G \cong \mathcal{K}(\mathcal{H} \rtimes G \oplus A \rtimes G) = \begin{bmatrix} \mathcal{K}(\mathcal{H} \rtimes G) & \mathcal{H} \rtimes G \\ \tilde{\mathcal{H}} \rtimes G & A \rtimes G \end{bmatrix}.$$

In particular, $\mathcal{H} \rtimes G$ determines an imprimitivity bimodule between $\mathcal{K}(\mathcal{H} \rtimes G) \cong \mathcal{K}(\mathcal{H}) \rtimes G$ and $A \rtimes G$.

In order to interpret $\mathcal{H} \rtimes G$ as a C^* -correspondence over $A \rtimes G$, we need to define the left multiplication using the map $A \rtimes G \rightarrow \mathcal{L}(\mathcal{H} \oplus A) \rtimes G \cong M(C) \rtimes G$ induced by $\phi : A \rightarrow \mathcal{L}(\mathcal{H})$.

Example 4.6. Suppose that $\mathcal{H}$ is a C^* -correspondence over A and the group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ acts via an order 2 automorphism α of A and an order 2 automorphism σ of $\mathcal{H}$. Recall that the crossed product $A \rtimes_\alpha \mathbb{Z}_2$ is isomorphic with the subalgebra of $\mathbb{M}_2(A)$ made of matrices

$$\begin{bmatrix} a & b \\ \alpha(b) & \alpha(a) \end{bmatrix}$$

with $a, b \in A$. Using the linking algebra, the crossed product $\mathcal{H} \rtimes_{\sigma} \mathbb{Z}_2$ is a C^* -correspondence over $A \rtimes_{\alpha} \mathbb{Z}_2$ and it can be understood as the set of 2×2 matrices

$$\begin{bmatrix} x & y \\ \sigma(y) & \sigma(x) \end{bmatrix}$$

with $x, y \in \mathcal{H}$. The operations are determined by the equations

$$\begin{aligned} & \left\langle \begin{bmatrix} x & y \\ \sigma(y) & \sigma(x) \end{bmatrix}, \begin{bmatrix} x' & y' \\ \sigma(y') & \sigma(x') \end{bmatrix} \right\rangle = \\ &= \begin{bmatrix} \langle x, x' \rangle + \langle y, \sigma(y') \rangle & \langle x, y' \rangle + \langle y, \sigma(x') \rangle \\ \langle \sigma(y), x' \rangle + \langle \sigma(x), \sigma(y') \rangle & \langle \sigma(y), y' \rangle + \langle \sigma(x), \sigma(x') \rangle \end{bmatrix}, \\ & \begin{bmatrix} x & y \\ \sigma(y) & \sigma(x) \end{bmatrix} \begin{bmatrix} a & b \\ \alpha(b) & \alpha(a) \end{bmatrix} = \begin{bmatrix} xa + y\alpha(b) & xb + y\alpha(a) \\ \sigma(y)a + \sigma(x)\alpha(b) & \sigma(y)b + \sigma(x)\alpha(a) \end{bmatrix}, \\ & \begin{bmatrix} a & b \\ \alpha(b) & \alpha(a) \end{bmatrix} \begin{bmatrix} x & y \\ \sigma(y) & \sigma(x) \end{bmatrix} = \begin{bmatrix} ax + b\sigma(y) & ay + b\sigma(x) \\ \alpha(b)x + \alpha(a)\sigma(y) & \alpha(b)y + \alpha(a)\sigma(x) \end{bmatrix}. \end{aligned}$$

Theorem 4.7. *Suppose that a locally compact amenable group G acts on faithful and essential Morita–Rieffel equivalent C^* -correspondences $\mathcal{H}$ and $\mathcal{M}$ over A and B respectively, via an imprimitivity bimodule $\mathcal{Z}$. Then $\mathcal{Z} \rtimes G$ becomes an imprimitivity bimodule between $A \rtimes G$ and $B \rtimes G$. Moreover, $\mathcal{O}_A(\mathcal{H}) \rtimes G$ is Morita–Rieffel equivalent to $\mathcal{O}_B(\mathcal{M}) \rtimes G$.*

Proof. We use results of Combes, see [4], to realize $\mathcal{Z} \rtimes G$ as an imprimitivity bimodule between $A \rtimes G$ and $B \rtimes G$. The isomorphism $\mathcal{Z} \otimes_B \mathcal{M} \cong \mathcal{H} \otimes_A \mathcal{Z}$ is equivariant, and

$$\begin{aligned} (\mathcal{Z} \otimes_B \mathcal{M}) \rtimes G &\cong (\mathcal{Z} \rtimes G) \otimes_{B \rtimes G} (\mathcal{M} \rtimes G) \cong \\ &\cong (\mathcal{H} \rtimes G) \otimes_{A \rtimes G} (\mathcal{Z} \rtimes G) \cong (\mathcal{H} \otimes_A \mathcal{Z}) \rtimes G, \end{aligned}$$

hence $\mathcal{H} \rtimes G$ is Morita–Rieffel equivalent to $\mathcal{M} \rtimes G$ via $\mathcal{Z} \rtimes G$. By the results in [16] combined with [10], it follows that $\mathcal{O}_A(\mathcal{H}) \rtimes G$ is Morita–Rieffel equivalent to $\mathcal{O}_B(\mathcal{M}) \rtimes G$. $\square$

Remark 4.8. Note that for G compact, the set of fixed points

$$\mathcal{H}^G = \{\xi \in \mathcal{H} \mid g\xi = \xi\}$$

becomes a C^* -correspondence over the fixed point algebra A^G . If the action on A is saturated, then A^G is Morita–Rieffel equivalent to $A \rtimes G$ and it follows that $\mathcal{H}^G$ is Morita–Rieffel equivalent to $\mathcal{H} \rtimes G$. In particular, $\mathcal{O}_A(\mathcal{H}) \rtimes G$ is Morita–Rieffel equivalent to $\mathcal{O}_{A^G}(\mathcal{H}^G)$, which in some cases might be easier to handle.

Remark 4.9. Consider $j_{\mathcal{H}} : \mathcal{H} \rightarrow \mathcal{O}_A(\mathcal{H})$ the canonical inclusion and let G be a locally compact abelian group acting on $\mathcal{H}$. Then the dual $\hat{G}$ acts on $A \rtimes G$ and $\mathcal{H} \rtimes G$ in the usual way. Using the map $j_{\mathcal{H}}$ and Takai duality, we obtain

$$\begin{aligned} (\mathcal{H} \rtimes G) \rtimes \hat{G} &\subset \mathcal{O}_{(A \rtimes G) \rtimes \hat{G}}((\mathcal{H} \rtimes G) \rtimes \hat{G}) \cong \\ &\cong (\mathcal{O}_A(\mathcal{H}) \rtimes G) \rtimes \hat{G} \cong \mathcal{O}_A(\mathcal{H}) \otimes \mathcal{K}(L^2(G)). \end{aligned}$$

A duality result for crossed products of Hilbert C^* -modules by group coactions was obtained by Kusuda, see [15].

5. Applications to Cuntz–Pimsner algebras of vector bundles

Consider $E \rightarrow X$ a Hermitian vector bundle of rank n , where X is compact, metrizable and path connected. The set of continuous sections $\Gamma(E)$ becomes a C^* -correspondence over $C(X)$, with left and right multiplications given by

$$(f\xi)(x) = (\xi f)(x) = f(x)\xi(x)$$

for $f \in C(X)$, $\xi \in \Gamma(E)$ and inner product

$$\langle \xi, \eta \rangle(x) = \langle \xi(x), \eta(x) \rangle_{E_x}.$$

We denote by $\mathcal{O}_E$ the Cuntz–Pimsner algebra of the C^* -correspondence $\Gamma(E)$ over $C(X)$, which is a (locally trivial) continuous field of Cuntz algebras $\mathcal{O}_n$, see Proposition 2 in [23]. In particular, $\mathcal{O}_E$ is a $C(X)$ -algebra.

Remark 5.1. The C^* -algebra $\mathcal{O}_E$ is generated by $C(X)$ and $S_1, \dots, S_N$ such that

$$f S_j = S_j f, \quad S_j^* S_k = P_{jk}, \quad \sum_{j=1}^N S_j S_j^* = 1$$

where $f \in C(X)$ and $P \in M_N \otimes C(X)$ is a choice of projection that gives the bundle E by the Serre–Swan Theorem. Since $\Gamma(E)$ is a direct summand of $C(X)^N$, the Hilbert module $\Gamma(E) \rtimes G$ is finitely generated and projective since it is a direct summand of $C(X)^N \rtimes G \cong (C(X) \rtimes G)^N$.

Recall that the cohomology set $H^1(X, U_n)$ classifies the Hermitian vector bundles of rank n over X , where $U_n = U_n(\mathbb{C})$ denotes the unitary group. The set of isomorphism classes of locally trivial continuous fields of C^* -algebras over X with fiber A is in bijection with the cohomology set $H^1(X, \text{Aut}(A))$. If (X_i, α_{ij}) is the cocycle corresponding to the bundle E , then the cocycle with values in $\text{Aut}(\mathcal{O}_n)$ describing $\mathcal{O}_E$ is given by

$$\alpha_{ij}^*(x)(t) = \alpha_{ij}(x)^{\otimes(r+k)} t (\alpha_{ij}(x)^{\otimes r})^*, \quad \text{where } t \in (n^r, n^{r+k}) \subset \mathcal{O}_n.$$

Here $\mathcal{O}_n$ is viewed as the Doplicher–Roberts algebra associated to the n -dimensional Hilbert space H , and (n^r, n^{r+k}) denotes the set of linear maps $H^{\otimes(r+k)} \rightarrow H^{\otimes r}$. We reproduce the following example from [24], which shows that for some vector bundles, the field of Cuntz algebras associated to $\mathcal{O}_E$ is not trivial.

Example 5.2. Let $X = S^{2k}$ be an even sphere. Recall that $H^1(S^{2k}, U_n) \cong \pi_{2k-1}(U_n)$ for $k \geq 1$ and that

$$K^0(S^{2k}) \cong \mathbb{Z}^2, \quad K^1(S^{2k}) \cong 0.$$

Moreover, as a ring, $K^0(S^{2k})$ is isomorphic to $\mathbb{Z}[\lambda]/(\lambda^2)$. Tensoring with a fixed vector bundle $E \rightarrow S^{2k}$ with class $n + m\lambda$ defines an endomorphism $[E]$ of $K^0(S^{2k})$ given by

$$[E](x + y\lambda) = nx + (ny + mx)\lambda.$$

Using the six-term exact sequence for the K -theory of $\mathcal{O}_E$, see [11] we get

$$\begin{array}{ccccccc} \mathbb{Z}^2 & \xrightarrow{id-[E]} & \mathbb{Z}^2 & \xrightarrow{i_0} & K_0(\mathcal{O}_E) & & \\ \uparrow 0 & & & & \downarrow 0 & & \\ K_1(\mathcal{O}_E) & \xleftarrow{i_1} & 0 & \xleftarrow{0} & 0 & & \end{array}$$

hence

$$K_0(\mathcal{O}_E) \cong \mathbb{Z}^2 / (id - [E])\mathbb{Z}^2, \quad K_1(\mathcal{O}_E) = 0.$$

In the case the bundle E is such that $n - 1$ and m are relatively prime and $n \geq 3$, we get

$$K_0(\mathcal{O}_E) \cong \mathbb{Z} / (n - 1)^2 \mathbb{Z},$$

hence $\mathcal{O}_E$ determines a non-trivial field of Cuntz algebras since by the Künneth formula we have

$$K_0(C(S^{2k}) \otimes \mathcal{O}_n) \cong \mathbb{Z} / (n - 1) \mathbb{Z} \oplus \mathbb{Z} / (n - 1) \mathbb{Z}.$$

Remark 5.3. For $X = S^{2k+1}$ an odd sphere and for $[E] = n \in K^0(S^{2k+1}) \cong \mathbb{Z}$, Vasselli showed in [23] that we have

$$(\mathcal{O}_E \otimes \mathcal{K}, \mathbb{Z}) \cong (C(S^{2k+1}) \otimes \mathcal{O}_n \otimes \mathcal{K}, \mathbb{Z})$$

as graded algebras.

Remark 5.4. If E is a line bundle over X , then $\mathcal{O}_E$ is commutative with spectrum homeomorphic to the circle bundle of E . The resulting K -theory exact sequence is a version of the classical Gysin sequence for circle bundles, see [1]. If E, F are line bundles over X , then $\mathcal{O}_E \cong \mathcal{O}_F$ as $C(X)$ -algebras if and only if $E \cong F$ or $E \cong \bar{F}$, where $\bar{F}$ is the conjugate of F , see [6], Proposition 2.2.

In general, for a Hermitian vector bundle $E \rightarrow X$ of rank n , where X is compact, metrizable and path connected, Dădârlat showed in [6] that the principal ideal $(1 - [E])K^0(X)$ of the K -theory ring $K^0(X)$ determines $\mathcal{O}_E$ up to isomorphism and an inclusion of principal ideals $(1 - [E])K^0(X) \subseteq (1 - [F])K^0(X)$ corresponds to unital embeddings $\mathcal{O}_E \subseteq \mathcal{O}_F$. In particular if $n \geq 2$, then $\mathcal{O}_E \cong C(X) \otimes \mathcal{O}_n$ if and only if $[E] - 1$ is divisible by $n - 1$, see [6], Theorem 1.1.

If the compact group G acts on the Hermitian vector bundle $E \rightarrow X$ by isometries, we define

$$(g \cdot f)(x) = f(g^{-1}x) \text{ for } f \in C(X), \quad (g \cdot \xi)(x) = g\xi(g^{-1}x) \text{ for } \xi \in \Gamma(E).$$

A section $\xi \in \Gamma(E)$ is invariant if $g \cdot \xi = \xi$. The set of invariant sections forms a subspace $\Gamma(E)^G$ of $\Gamma(E)$. The averaging operator defines a map $\Gamma(E) \rightarrow \Gamma(E)^G$.

Using the universal property, G acts on the Cuntz–Pimsner algebra $\mathcal{O}_E$ and from Theorem 4.1, we get

$$\mathcal{O}_E \rtimes G \cong \mathcal{O}_{C(X) \rtimes G}(\Gamma(E) \rtimes G).$$

The structure of the crossed product $C(X) \rtimes G$ is known under certain hypotheses. For example, Williams showed in [25] that if X/G is Hausdorff, then $C(X) \rtimes G$ is a section algebra over X/G with fiber over the orbit Gx isomorphic to $C^*(G_x) \otimes \mathcal{K}(L^2(G/G_x, \mu))$. Here G_x is the stabilizer group and μ is any quasi-invariant measure on G/G_x . In some cases, the study of C^* -correspondences over $C(X) \rtimes G$ can be reduced to understanding the C^* -correspondences over abelian C^* -algebras $C_0(Y)$ or over continuous trace algebras.

Remark 5.5. An imprimitivity bimodule over a commutative C^* -algebra $C_0(Y)$ can be regarded as the space of sections of a complex line bundle over Y , see Appendix (A) in [18]. A general C^* -correspondence $\mathcal{H}$ over an abelian C^* -algebra $C_0(Y)$ is given by the sections of a continuous field of Hilbert spaces $\{\mathcal{H}_y\}$ over Y and by a continuous family of measures $\{\mu_y\}$ on Y which determine the left multiplication $\phi : C_0(Y) \rightarrow \mathcal{L}(\mathcal{H})$ via a family of representations $C_0(Y) \rightarrow \mathcal{L}(\mathcal{H}_y)$. If all fibers $\mathcal{H}_y$ are nonzero, then $\mathcal{H}$ is full. The Cuntz–Pimsner algebras $\mathcal{O}_{C_0(Y)}(\mathcal{H})$ recover large classes of C^* -algebras. For more on Hilbert modules over non-unital C^* -algebras, see [20].

Examples 5.6. Particular cases of C^* -correspondences $\mathcal{H}$ over an abelian C^* -algebra arise from topological quivers $(E^0, E^1, s, r, \lambda)$, see [17], or from topological relations, see [3]. In the first case, $\mathcal{H}$ is obtained as a completion of $C_c(E^1)$ and the fibers are $\mathcal{H}_v = L^2(s^{-1}(v), \lambda_v)$ for $v \in E^0$. Here $\lambda = \{\lambda_v\}_{v \in E^0}$ is a family of Radon measures which defines the inner products with values in $C_0(E^0)$, the source map $s : E^1 \rightarrow E^0$ defines the right multiplication, $\xi \cdot f(e) = \xi(e)f(s(e))$ and the range map $r : E^1 \rightarrow E^0$ defines the left multiplication $\phi : C_0(E^0) \rightarrow \mathcal{L}(\mathcal{H})$, $\phi(f)\xi(e) = f(r(e))\xi(e)$, for $e \in E^1, \xi \in C_c(E^1), f \in C_0(E^0)$.

This C^* -correspondence $\mathcal{H}$ over $A = C_0(E^0)$ is essential. The left multiplication $\phi : C_0(E^0) \rightarrow \mathcal{L}(\mathcal{H})$ is injective when r is onto, and $\phi(f) \in \mathcal{K}(\mathcal{H})$ if and only if $f \circ r \in C_0(E^1)$ and for every $e \in E^1$ such that $(f \circ r)(e) \neq 0$ there exists a neighborhood U of e such that the restriction $s|_U : U \rightarrow s(U)$ is a homeomorphism.

If $s : E^1 \rightarrow E^0$ is a local homeomorphism and λ_v are the counting measures, we recover the situation of C^* -correspondences associated to topological graphs of Katsura, see [12].

Example 5.7. Another source of examples of C^* -correspondences over abelian C^* -algebras comes from the concepts of diagonal and of Cartan subalgebra B in a C^* -algebra A , see [14] and [19]. Recall that in particular B is a maximal abelian subalgebra in A and there is a faithful conditional expectation $P : A \rightarrow B$, so one can define a B -valued inner product on A by $\langle a_1, a_2 \rangle = P(a_1^* a_2)$. By taking a completion and using the obvious left and right multiplications, one obtains a C^* -correspondence $\mathcal{H}$ over $B = C_0(\hat{B})$. It would be interesting to explore what kind of Cuntz–Pimsner algebras $\mathcal{O}_B(\mathcal{H})$ one obtain in this way. For example, if $A = \mathbb{M}_n$ and $B \cong \mathbb{C}^n$ is the subalgebra of diagonal matrices, then we get the Cuntz algebra $\mathcal{O}_n$.

To explore the structure of the crossed product $\mathcal{O}_E \rtimes G$, we first consider the case of trivial bundles. Let $E = X \times V$, where $V \cong \mathbb{C}^n$. Then $\Gamma(E) \cong C(X) \otimes V$ and $\mathcal{O}_E \cong C(X) \otimes \mathcal{O}_n$. A group action on $X \times V$ is of the form $g(x, v) = (gx, J(g, x)v)$, where $J : G \times X \rightarrow \text{End}(V)$ is continuous. We must have $J(e, x) = I$ for all $x \in X$, and from associativity of the action, we obtain the cocycle relation $J(gh, x) = J(g, hx)J(h, x)$. The structure of $\mathcal{O}_E \rtimes G \cong (C(X) \otimes \mathcal{O}_n) \rtimes G$ can be determined in principle by understanding the C^* -correspondence $(C(X) \otimes V) \rtimes G$ over $C(X) \rtimes G$.

In particular, if the action of G on $E = X \times V$ is free, then $E/G \cong (X/G) \times V$ and $\Gamma(E/G) \cong C(X/G) \otimes V$. Since $C(X) \rtimes G$ is Morita–Rieffel equivalent to $C(X/G)$ and $(C(X) \otimes V) \rtimes G$ is Morita–Rieffel equivalent to $C(X/G) \otimes V$, it follows that $\mathcal{O}_E \rtimes G$ is Morita–Rieffel equivalent to $C(X/G) \otimes \mathcal{O}_n$.

If G acts trivially on X , then $C(X) \rtimes G \cong C(X) \otimes C^*(G)$ and $(C(X) \otimes V) \rtimes G \cong C(X) \otimes (V \rtimes G)$, so $\mathcal{O}_E \rtimes G \cong C(X) \otimes (\mathcal{O}_n \rtimes G)$.

For the more general structure of $\mathcal{O}_E \rtimes G$ we have the following results.

Theorem 5.8. (Free action). *If G is a compact group acting freely on the Hermitian vector bundle $E \rightarrow X$, then $\mathcal{O}_E \rtimes G$ is Morita–Rieffel equivalent to a continuous field of Cuntz algebras over X/G .*

Proof. Indeed, in this case $C(X) \rtimes G$ is Morita–Rieffel equivalent to $C(X/G)$ via the imprimitivity bimodule $\mathcal{Z}$ which is a completion of $C(X)$, and $\Gamma(E) \rtimes G$ is Morita–Rieffel equivalent to $\Gamma(E)^G \cong \Gamma(E/G)$. $\square$

Example 5.9. Let $X = S^2$ be the 2-sphere and let $E = TS^2 \otimes \mathbb{C}$ be the complexified tangent bundle. Even though E is not a trivial bundle, we have $\mathcal{O}_E \cong C(S^2) \otimes \mathcal{O}_2$, see Theorem 1.1 in [5].

The group $\mathbb{Z}_2 = \{e, g\}$ acts on S^2 by $g \cdot x = -x$ and on E by its differential dg . Since the action is free, $E/\mathbb{Z}_2$ is a vector bundle over the real projective space $S^2/\mathbb{Z}_2 = \mathbb{RP}^2$. Moreover, $C(S^2) \rtimes \mathbb{Z}_2$ is Morita–Rieffel equivalent to $C(\mathbb{RP}^2)$ and it follows that $\mathcal{O}_E \rtimes \mathbb{Z}_2$ is Morita–Rieffel equivalent to $C(\mathbb{RP}^2) \otimes \mathcal{O}_2$.

Theorem 5.10. (*Fiberwise action*). *If G is a compact group acting on the Hermitian vector bundle $E \rightarrow X$ of rank n and the action on X is trivial, then $\mathcal{O}_E \rtimes G$ is a continuous field over X with fibers $\mathcal{O}_n \rtimes G$.*

Proof. Indeed, in this case we have a fiberwise action on the continuous field of Cuntz algebras $\mathcal{O}_n$ and we can use Theorem 4.1 in [13]. $\square$

Remark 5.11. For $G = S_n$ the symmetric group, we know from [7] that $\mathcal{O}_n \rtimes S_n$ is simple and purely infinite. If X is finite dimensional, Dădărlat gives in [5] a complete list of the UCT Kirchberg algebras D with finitely generated K -theory for which every unital separable continuous field over X with fibers isomorphic to D is automatically locally trivial or trivial.

Theorem 5.12. (*Transitive action*). *Let G be a compact group and let H be a closed subgroup. Given a Hermitian vector bundle E over $X = G/H$ we know that $E \cong G \times_H V$ for an H -module V . Then $\mathcal{O}_E \rtimes G$ is Morita–Rieffel equivalent to a graph C^* -algebra.*

Proof. Indeed, $C(G/H) \rtimes G$ is Morita–Rieffel equivalent to $C^*(H)$ which is a direct sum of matrix algebras. This in turn is Morita–Rieffel equivalent to $C_0(Y)$ with Y at most countable. Now it is known that a C^* -correspondence over $C_0(Y)$ gives rise to a discrete graph. The result follows from Theorem 4.7 since $\mathcal{O}_E \rtimes G \cong \mathcal{O}_{C(G/H) \rtimes G}(\Gamma(E) \rtimes G)$. $\square$

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