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Auxiliary Principle and Iterative Algorithms for Generalized Set-Valued Strongly Nonlinear Mixed Variational-like Inequalities

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In this paper, we extend the auxiliary principle technique to study a class of generalized set-valued strongly nonlinear mixed variational-like inequalities. We prove the existence of a solution of the auxiliary problem for the generalized set-valued strongly nonlinear mixed variational-like inequalities, construct the iterative algorithm for the generalized set-valued strongly nonlinear mixed variational-like inequalities, and show the existence of a solution of the generalized set-valued strongly nonlinear mixed variational-like inequality by using the auxiliary principle technique. We also prove the convergence of iterative sequences generated by the algorithm. © 2001 Academic Press

Key Words: mixed variational-like inequality; iterative algorithm; set-valued mapping; auxiliary principle technique; existence; convergence.

1. INTRODUCTION

Variational inequality theory and complementarity problem theory are very powerful tools of the current mathematical technology. In recent years, classical variational inequality and complementarity problems have been extended and generalized to study a wide class of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics, finance, regional, structural, transportation, elasticity, and applied sciences, etc.; see [1–13, 15–25] and the references therein.

A useful and important generalization of variational inequalities is the generalized mixed variational-like inequality. The generalized mixed varia-

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tional-like inequalities have potential and significant applications in optimization theory [21, 22], structural analysis [18], and economics [3, 21]. Some special cases of the mixed variational-like inequalities have been studied by Parida and Sen [20], Yao [22], and Tian [21] by using the Berge maximum theorem in finite and infinite dimensional spaces. It is remarked that their methods are not constructive. The development of an efficient and implementable technique for solving variational-like inequalities is an important and difficult problem in this theory. There are many numerical methods including the projection method and its variant forms, linear approximation, descent, and Newton's methods for variational inequalities. However, there are very few methods for some variational-like inequalities. It is worth mentioning that the projection type technique cannot be used to suggest iterative algorithms for variational-like inequalities, since it is not possible to find the projection of the solution.

Glowinski *et al.* [6] suggested another technique, which does not depend on the projection. This technique is called the auxiliary principle technique. Recently, Noor [15] extended the auxiliary principle technique to study the existence and uniqueness of a solution for a class of generalized mixed variational-like inequalities for set-valued mappings with compact values. However, the proof of the uniqueness part in [15, Theorem 3.1] is wrong. Also, the proof of the existence part is based on the assumption that the auxiliary problem has a solution, but the author did not show the existence of the solution for this auxiliary problem.

On the other hand, in 1994, Yao [23] studied a class of generalized variational inequalities for single-valued mappings by using the auxiliary principle technique. Very recently, Huang *et al.* [12] modified and extended the auxiliary principle technique to study the existence of a solution for a class of generalized set-valued strongly nonlinear implicit variational inequalities and to suggest some general iterative algorithms.

Inspired and motivated by recent research going on in this fascinating and interesting field, in this paper, we extend the auxiliary principle technique to study a class of generalized strongly nonlinear mixed variational-like inequalities for set-valued mappings without compact-values. We prove the existence of a solution of the auxiliary problem for the generalized set-valued strongly nonlinear mixed variational-like inequality, construct the iterative algorithm for the generalized set-valued strongly nonlinear mixed variational-like inequality, and show the existence of a solution of the generalized set-valued strongly nonlinear mixed variational-like inequality by using the auxiliary principle technique. We also prove the convergence of iterative sequences generated by the algorithm. Our results improve and modify the main results of Noor [15].

2. PRELIMINARIES

Let H be a real Hilbert space endowed with a norm $\|\cdot\|$, and inner product $\langle\cdot,\cdot\rangle$, respectively. Let $CB(H)$ be the family of all nonempty bounded closed subsets of H.

Given single-valued mappings N, η: $H \times H \to H$ and set-valued mappings T, A: $H \to CB(H)$, we consider the problem of finding $u \in H$, $w \in T(u)$, $y \in A(u)$ such that

$$\langle N(w, y), \eta(v, u)\rangle + b(u, v) - b(u, u) \geq 0, \qquad \forall v \in H, \quad (2.1)$$

where $b(\cdot,\cdot)$: $H \times H \to R$, which is nondifferentiable, satisfies the following properties:

(i) $b(u, v)$ is linear with respect to u;

(ii) $b(u, v)$ is bounded, that is, there exists a constant $\gamma > 0$ such that

$$b(u, v) \leq \gamma\|u\|\,\|v\|, \qquad \forall u, v \in H;$$

(iii) $b(u, v) - b(u, w) \leq b(u, v - w)$, $\forall u, v, w \in H$;

(iv) $b(u, v)$ is convex with respect to v.

The problem (2.1) is called the generalized set-valued strongly nonlinear mixed variational-like inequality introduced and studied by Noor [15] in the assumption that T and A are the set-valued mappings with compact-values.

EXAMPLE 2.1. Suppose that f: $H \to R$ is a linear bounded functional and h: $H \to R$ is a sublinear bounded functional. Let

$$b(u, v) = f(u)h(v), \qquad \forall u, v \in H.$$

Then b satisfies the conditions (i)–(iv).

Remark 2.1. (1) For arbitrary $u, v \in H$, condition (i) implies that $-b(u, v) = b(-u, v)$ and condition (ii) implies that $b(-u, v) \leq \gamma\|u\|\,\|v\|$. Hence we have

$$|b(u, v)| \leq \gamma\|u\|\,\|v\|, \qquad \forall u, v \in H$$

and

$$b(u, 0) = 0, \qquad b(0, v) = 0, \forall\, u, v \in H.$$

(2) It follows from conditions (ii) and (iii) that

$$b(u, v) - b(u, w) \leq \gamma\|u\|\,\|v - w\|$$

and

$$b(u,w) - b(u,v) \le \gamma\|u\|\,\|v - w\|$$

for all $u, v, w \in H$. Therefore

$$|b(u,v) - b(u,w)| \le \gamma\|u\|\,\|v - w\|.$$

This implies that $b(u,v)$ is continuous with respect to the second argument.

Some Special Cases. (I) If $\eta(v,u) = g(v) - g(u)$, where $g\colon H \to H$, then the problem (2.1) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in A(u)$ such that

$$\langle N(w,y), g(v) - g(u)\rangle + b(u,v) - b(u,u) \ge 0, \qquad \forall v \in H, \quad (2.2)$$

which is called the generalized set-valued strongly nonlinear mixed implicit variational inequality.

(II) If $\eta(v,u) = v - u$, then the problem (2.1) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in A(u)$ such that

$$\langle N(w,y), v - u\rangle + b(u,v) - b(u,u) \ge 0, \qquad \forall v \in H, \quad (2.3)$$

which is called the generalized set-valued strongly nonlinear mixed variational inequality.

(III) If $\eta(v,u) = g(v) - g(u)$, where $g\colon H \to H$, and $b(u,v) = f(v)$ for all $u, v \in H$, where $f\colon H \to R$, then the problem (2.1) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in A(u)$ such that

$$\langle N(w,y), g(v) - g(u)\rangle + f(v) - f(u) \ge 0, \qquad \forall v \in H. \quad (2.4)$$

which is called the generalized set-valued strongly nonlinear implicit variational inequality considered by Huang *et al.* [12].

(IV) If $b(u,v) \equiv 0$, then the problem (2.1) is equivalent to finding $u \in H$, $w \in T(u)$, $y \in A(u)$ such that

$$\langle N(w,y), \eta(v,u)\rangle \ge 0, \qquad \forall v \in H, \quad (2.5)$$

which is called the generalized variational-like inequality considered by Noor [17]. It has been shown in [18] that the nonconvex nonmonotone and multivalued problems arising in structural analysis can be formulated in terms of the generalized variational-like inequality (2.5). Parida and Sen [20], Tian [21], Yao [22, 24], and Cubiotti [3] have shown that many problems arising in optimization and economics can be studied by the generalized variational inequalities of the type (2.5).

In brief, for suitable choice of the mappings N, η, T, A, and the function b, one can obtain a number of known variational inequalities as special cases from the problems (2.1)–(2.5).

We need the following definitions.

DEFINITION 2.1. A set-valued mapping V: $H \to CB(H)$ is said to be

(i) *Lipschitz continuous* if there exists a constant $\gamma > 0$ such that

$$(V(u), V(v)) \leq \gamma \|u - v\|, \qquad \forall u, v \in H,$$

where $(\cdot, \cdot)$ is the Hausdorff metric on $CB(H)$;

(ii) *strongly monotone* if there exists a constant $\xi > 0$ such that

$$\langle w_1 - w_2, u_1 - u_2 \rangle \geq \xi \|u_1 - u_2\|^2,$$
$$\forall u_1, u_2 \in H, \ \forall w_1 \in T(u_1), \ w_2 \in T(u_2).$$

DEFINITION 2.2. Let N: $H \times H \to H$ be a nonlinear mapping and T: $H \to CB(H)$ be a set-valued mapping.

(i) T is said to be *strongly monotone* with respect to the first argument of N if there exists a constant $\alpha > 0$ such that

$$\langle N(w_1, \cdot) - N(w_2, \cdot), u_1 - u_2 \rangle \geq \alpha \|u_1 - u_2\|^2,$$
$$\forall u_1, u_2 \in H, \ \forall w_1 \in T(u_1), \ w_2 \in T(u_2);$$

(ii) N is said to be *Lipschitz continuous* with respect to the first argument if there exists a constant $\beta > 0$ such that

$$\|N(u_1, \cdot,) - N(u_2, \cdot)\| \leq \beta \|u_1 - u_2\|, \qquad \forall u_1, u_2 \in H.$$

In a similar way, we can define the strong monotonicity of a mapping T with respect to the second argument of N and the Lipschitz continuity of a mapping $N(\cdot, \cdot)$ with respect to the second argument.

Remark 2.2. If $N(u, v) = u + v$ for all $u, v \in H$, then Definition 2.1(i) reduces to the usual definition of strong monotonicity for the set-valued mapping T.

DEFINITION 2.3. A mapping η: $H \times H \to H$ is said to be

(a) *strongly monotone* if there exists a constant $\sigma > 0$ such that

$$\langle \eta(u, v), v - u \rangle \geq \sigma \|v - u\|^2, \qquad \forall u, v \in H;$$

(b) *Lipschitz continuous* if there exists a constant $\delta > 0$ such that

$$\|\eta(v,u)\| \le \delta\|v-u\|, \qquad \forall u,v \in H.$$

DEFINITION 2.4. Let D be a nonempty convex subset of H and f: $D \to (-\infty, +\infty)$,

(1) f is said to be *convex* if, for any $u, v \in D$ and for any $\alpha \in [0,1]$,

$$f(\alpha u + (1-\alpha)v) \le \alpha f(u) + (1-\alpha)f(v);$$

(2) f is said to be *lower semicontinuous* on D if, for each $\alpha \in (-\infty, +\infty)$, the set $\{u \in D\colon f(u) \le \alpha\}$ is closed in D;

(3) f is said to be *concave* if $-f$ is convex;

(4) f is said to be *upper semicontinuous* on D if $-f$ is lower semicontinuous on D.

In order to obtain our results, we need the following assumption.

ASSUMPTION 2.1. The mappings N, η: $H \times H \to H$ satisfy the following conditions:

(1) for all $w, y \in H$, there exists a constant $\tau > 0$ such that

$$\|N(w,y)\| \le \tau(\|w\| + \|y\|);$$

(2) $\eta(v,u) = -\eta(u,v)$, $\forall\, u, v \in H$;

(3) for given $x, y, u \in H$, mapping $v \mapsto \langle N(x,y), \eta(u,v)\rangle$ is concave and upper semicontinuous.

EXAMPLE 2.2. Let g: $H \to H$ be a strongly monotone Lipschitz continuous mapping and h: $H \to H$ be a monotone Lipschitz continuous mapping such that, for any given $x \in H$, the mappings $u \mapsto \langle x, g(u)\rangle$ and $u \mapsto \langle x, h(u)\rangle$ are both convex. If

$$\eta(u,v) = g(u) - g(v) + h(u) - h(v), \qquad \forall u,v \in H,$$

then it is easy to see that η satisfies the conditions (a), (b) of Definition 2.3 and the conditions (2), (3) of Assumption 2.1.

Remark 2.3. (i) By (2), we have

$$\eta(u,u) = 0, \qquad \forall u \in H.$$

(ii) From (2) and (3), we know that, for any given $x, y, v \in H$, the mapping $u \mapsto \langle N(x,y), \eta(u,v)\rangle$ is convex and lower semicontinuous.

We also need the following lemma:

LEMMA 2.1 [1, 2]. *Let X be a nonempty closed convex subset of a Hausdorff linear topological space E, ϕ, ψ: $X \times X \to R$ be mappings satisfying the following conditions*:

(i) $\psi(x, y) \leq \phi(x, y)$, $\forall x, y \in X$, *and* $\psi(x, x) \geq 0$, $\forall x \in X$;

(ii) *for each* $x \in X$, $\phi(x, y)$ *is upper semicontinuous with respect to* y;

(iii) *for each* $y \in X$, *the set* $\{x \in X: \psi(x, y) < 0\}$ *is a convex set*;

(iv) *there exists a nonempty compact set* $K \subset X$ *and* $x_0 \in K$ *such that* $\psi(x_0, y) < 0$, $\forall y \in X \setminus K$. *Then there exists an* $\bar{y} \in K$ *such that*

$$\phi(x, \bar{y}) \geq 0, \qquad \forall x \in X.$$

3. AUXILIARY PROBLEM AND ALGORITHM

In this section, we extend the auxiliary principle technique of Glowinski *et al*. [6] to study the generalized set-valued strongly nonlinear mixed variational-like inequality (2.1). We give an existence theorem of a solution of the auxiliary problem for the generalized set-valued strongly nonlinear mixed variational-like inequality (2.1). Based on this existence theorem, we construct the iterative algorithm for the generalized set-valued strongly nonlinear mixed variational-like inequality (2.1).

Given $u \in H$, $w \in T(u)$, $y \in A(u)$, we consider the following problem $P(u, w, y)$: find $z \in H$ such that

$$\langle z, v - z \rangle \geq \langle u, v - z \rangle - \rho \langle N(w, y), \eta(v, z) \rangle + \rho b(u, z) - \rho b(u, v), \quad \forall v \in H, \tag{3.1}$$

where $\rho > 0$ is a constant. The problem $P(u, w, y)$ is called the auxiliary problem for the generalized set-valued strongly nonlinear mixed variational-like inequality (2.1).

THEOREM 3.1. *Let the mapping* η: $H \to H$ *be Lipschitz continuous with* $\delta > 0$, *the function* $b(\cdot, \cdot)$ *satisfy the conditions* (i)–(iv). *If Assumption* 2.1 *holds*, *then the auxiliary problem* $P(u, v, y)$ *has a solution*.

Proof. Define the mappings ϕ, ψ: $H \times H \to R$ by

$$\phi(v, z) = \langle v, v - z \rangle - \langle u, v - z \rangle + \rho \langle N(w, y), \eta(v, z) \rangle - \rho b(u, z) + \rho b(u, v)$$

and

$$\psi(v, z) = \langle z, v - z \rangle - \langle u, v - z \rangle + \rho \langle N(w, y), \eta(v, z) \rangle - \rho b(u, z) + \rho b(u, v),$$

respectively. We show that the mappings ϕ, ψ satisfy all the conditions of Lemma 2.1 in the weak topology.

Clearly, ϕ and ψ satisfy condition (i) of Lemma 2.1. Since $b(\cdot,\cdot)$ is convex with respect to the second argument, it follows from Assumption 2.1(3) that, for each given $v \in H$ and $\alpha \in R$, the set $\{z \in H\colon \phi(v, z) \geq \alpha\}$ is convex. It follows from Remark 2.1(2) and Assumption 2.1(3) that $\phi(v, z)$ is weakly upper semicontinuous with respect to z. It is easy to show that the set $\{v \in H | \psi(v, z) < 0\}$ is a convex set for each fixed $z \in H$ and so the conditions (ii) and (iii) of Lemma 2.1 hold.

Now let

$$\omega = \|u\| + \rho\gamma\|u\| + \rho\delta\tau(\|w\| + \|y\|), \qquad K = \{z \in H\colon \|z\| \leq \omega\}.$$

Then K is a weakly compact subset of H. For any fixed $z \in X \setminus K$, take $v_0 = 0 \in K$. From Assumption 2.1, the Lipschitz continuity of η, and Remark 2.1, we have

$$\begin{aligned}
\psi(v_0, z) &= \psi(0, z)\\
&= -\|z\|^2 + \langle u, z\rangle + \rho\langle N(w, y), \eta(0, z)\rangle - \rho b(u, z)\\
&\leq -\|z\|^2 + \|u\|\,\|z\| + \rho\delta\tau\|z\|(\|w\| + \|y)\| + \rho\gamma\|u\|\,\|z\|\\
&= -\|z\|(\|z\| - \|u\| - \rho\delta\tau(\|w\| + \|y\|) - \rho\gamma\|u\|)\\
&< 0.
\end{aligned}$$

Therefore, the condition (iv) of Lemma 2.1 holds. By Lemma 2.1, there exists an $\bar{z} \in H$ such that $\phi(v, \bar{z}) \geq 0$, for all $v \in H$, that is,

$$\begin{aligned}
&\langle v, v - \bar{z}\rangle - \langle u, v - \bar{z}\rangle + \rho\langle N(w, y), \eta(v, \bar{z})\rangle\\
&\qquad -\rho b(u, \bar{z}) + \rho b(u, v) \geq 0, \qquad \forall v \in H. \qquad (3.2)
\end{aligned}$$

For arbitrary $t \in (0, 1]$ and $v \in H$, let $x_t = tv + (1 - t)\bar{z}$. Replacing v by x_t in (3.2), we obtain

$$\begin{aligned}
0 &\leq \langle x_t, x_t - \bar{z}\rangle - \langle u, x_t - \bar{z}\rangle + \rho\langle N(w, y), \eta(x_t, \bar{z})\rangle\\
&\quad - \rho b(u, \bar{z}) + \rho b(u, x_t)\\
&= t\big(\langle x_t, v - \bar{z}\rangle - \langle u, v - \bar{z}\rangle\big) - \rho\langle N(w, y), \eta(\bar{z}, tv + (1 - t)\bar{z})\rangle\\
&\quad + \rho(b(u, tv + (1 - t)\bar{z}) - b(u, \bar{z}))\\
&\leq t\big(\langle x_t, v - \bar{z}\rangle - \langle u, v - \bar{z}\rangle\big) + \rho t\langle N(w, y), \eta(v, \bar{z})\rangle\\
&\quad + \rho t\langle b(u, v) - b(u, \bar{z})\rangle.
\end{aligned}$$

Hence

$$\langle x_t, v - \bar{z} \rangle - \langle u, v - \bar{z} \rangle + \rho \langle N(w,y), \eta(v,\bar{z}) \rangle + \rho b(u,v) - \rho b(u,\bar{z})) \geq 0$$

and so

$$\langle x_t, v - \bar{z} \rangle \geq \langle u, v - \bar{z} \rangle - \rho \langle N(w,y), \eta(v,\bar{z}) \rangle - \rho b(u,v) + \rho b(u,\bar{z}).$$

Letting $t \to 0^+$, we have

$$\langle \bar{z}, v - \bar{z} \rangle \geq \langle u, v - \bar{z} \rangle - \rho \langle N(w,y), \eta(v,\bar{z}) \rangle + \rho b(u,\bar{z}) - \rho b(u,v), \quad \forall v \in H.$$

Therefore, $\bar{z} \in H$ is a solution of the auxiliary problem $P(u,y,w)$. This completes the proof.

By using Theorem 3.1, we now construct the algorithm for solving the generalized set-valued strongly nonlinear mixed variational-like inequality (2.1).

For given $u_0 \in H$, $w_0 \in T(u_0)$, $y_0 \in A(u_0)$, from Theorem 3.1, we know that the auxiliary problem $P(u_0, w_0, y_0)$ has a solution u_1, that is,

$$\langle u_1, v - u_1 \rangle \geq \langle u_0, v - u_1 \rangle - \rho \langle N(w_0, y_0), \eta(v, u_1) \rangle + \rho b(u_0, u_1) - \rho b(u_0, v), \qquad \forall v \in H.$$

Since $w_0 \in T(u_0) \in CB(H)$, $y_0 \in A(u_0) \in CB(H)$, by Nadler [14], there exist $w_1 \in T(u_1)$ and $y_1 \in A(u_1)$ such that

$$\|w_0 - w_1\| \leq (1 + 1) H(T(u_0), T(u_1)),$$
$$\|y_0 - y_1\| \leq (1 + 1) H(A(u_0), A(u_1)).$$

By Theorem 3.1 again, the auxiliary problem $P(u_1, w_1, y_1)$ has a solution u_2, that is,

$$\langle u_2, v - u_2 \rangle \geq \langle u_1, v - u_2 \rangle - \rho \langle N(w_1, y_1), \eta(v, u_2) \rangle + \rho b(u_1, u_2) - \rho b(u_1, v), \qquad \forall v \in H.$$

For $w_1 \in T(u_1)$ and $y_1 \in A(u_1)$, there exist $w_2 \in T(u_2)$ and $y_2 \in A(u_2)$ such that

$$\|w_1 - w_2\| \leq (1 + 1/2) H(T(u_1), T(u_2)),$$
$$\|y_1 - y_2\| \leq (1 + 1/2) H(A(u_1), A(u_2)).$$

By induction, we can get the algorithm for the problem (2.1) as follows:

Algorithm 3.1. For given $u_0 \in H$, $w_0 \in T(u_0)$, $y_0 \in A(u_0)$, there exist the sequences $\{w_n\}$, $\{y_n\}$, and $\{u_n\}$ in H satisfying the conditions

$$\begin{aligned} w_n &\in T(u_n), \qquad \|w_n - w_{n+1}\| \leq (1 + 1/(n+1)) H(T(u_n), T(u_{n+1})), \\ y_n &\in A(u_n), \qquad \|y_n - y_{n+1}\| \leq (1 + 1/(n+1)) H(A(u_n), A(u_{n+1})), \\ &\langle u_{n+1}, v - u_{n+1} \rangle \geq \langle u_n, v - u_{n+1} \rangle - \rho \langle N(w_n, y_n), \eta(v, u_{n+1}) \rangle \\ &\qquad + \rho b(u_n, u_{n+1}) - \rho b(u_n, v), \\ &\qquad\qquad \forall v \in H,\ n = 0, 1, 2, \ldots, \end{aligned} \tag{3.3}$$

where $\rho > 0$ is a constant.

Remark 3.1. From Algorithm 3.1, we can get some algorithms for the problems (2.2), (2.3), (2.4), and (2.5), respectively.

Remark 3.2. Based on the existence Theorem 3.1 for a solution of the auxiliary problem P, in Algorithm 3.1, we construct some iterative sequences which converge to a solution of the generalized set-valued strongly nonlinear mixed variational-like inequality (2.1) (see Theorem 3.1 in Section 4). But in [15], the author did not construct any iterative sequence for solving the problem (2.1).

4. EXISTENCE AND CONVERGENCE THEOREM

In this section, we prove the existence of a solution of the generalized set-valued strongly nonlinear mixed variational-like inequality (2.1) and the convergence of the sequences generated by the algorithm.

THEOREM 4.1. *Let N: $H \times H \to H$ be Lipschitz continuous with respect to the first and second arguments with Lipschitz constants β and $\xi > 0$, respectively, and A, T: $H \to CB(H)$ be Lipschitz continuous with Lipschitz constants $\mu, \nu > 0$, respectively. Let T be strongly monotone with respect to the first argument of N with constant $\alpha > 0$ and η: $H \times H \to H$ be strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\delta > 0$. Suppose that the function $b(\cdot,\cdot)$ satisfies the conditions* (i)–(iv). *If Assumption* 2.1 *holds and*

$$0 < \rho < 2\frac{\alpha - k}{\beta^2 \nu^2 - k^2}, \qquad \rho k < 1,\ k < \alpha, \tag{4.1}$$

where

$$k = \beta\nu\sqrt{1 - 2\sigma + \delta^2} + \gamma + \delta\xi\mu,$$

then there exists $u \in H$, $w \in T(u)$, $y \in A(u)$ *satisfying the generalized set-valued strongly nonlinear mixed variational-like inequality* (2.1) *and*

$$u_n \to u, \qquad w_n \to w, \qquad y_n \to y \ (n \to \infty),$$

where the sequences $\{u_n\}$, $\{w_n\}$, *and* $\{y_n\}$ *are defined by Algorithm* 3.1.

Proof. By (3.3), for any $v \in H$, we have

$$\begin{aligned}\langle u_n, v - u_n\rangle \geq \langle u_{n-1}, v - u_n\rangle - \rho\langle N(w_{n-1}, y_{n-1}), \eta(v, u_n)\rangle \\ + \rho b(u_{n-1}, u_n) - \rho b(u_{n-1}, v)\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\langle u_{n+1}, v - u_{n+1}\rangle \geq \langle u_n, v - u_{n+1}\rangle - \rho\langle N(w_n, y_n), \eta(v, u_{n+1})\rangle \\ + \rho b(u_n, u_{n+1}) - \rho b(u_{n-1}, v).\end{aligned} \tag{4.3}$$

Taking $v = u_{n+1}$ in (4.2) and $v = u_n$ in (4.3), respectively, we get

$$\begin{aligned}&\langle u_n, u_{n+1} - u_n\rangle \\ &\quad \geq \langle u_{n-1}, u_{n+1} - u_n\rangle - \rho\langle N(w_{n-1}, y_{n-1}), \eta(u_{n+1}, u_n)\rangle \\ &\qquad + \rho b(u_{n-1}, u_n) - \rho b(u_{n-1}, u_{n+1}),\end{aligned} \tag{4.4}$$

$$\begin{aligned}\langle u_{n+1}, u_n - u_{n+1}\rangle \geq \langle u_n, u_n - u_{n+1}\rangle - \rho\langle N(w_n, y_n), \eta(u_n, u_{n+1})\rangle \\ + \rho b(u_n, u_{n+1}) - \rho b(u_n, u_n).\end{aligned} \tag{4.5}$$

Adding (4.4) and (4.5), we obtain

$$\begin{aligned}&\langle u_{n+1} - u_n, u_n - u_{n+1}\rangle \\ &\quad \geq \langle u_n - u_{n-1}, u_n - u_{n+1}\rangle \\ &\qquad - \rho\langle N(w_n, y_n) - N(w_{n-1}, y_{n-1}), \eta(u_n, u_{n+1})\rangle \\ &\qquad + \rho b(u_{n-1} - u_n, u_n) + \rho b(u_n - u_{n-1}, u_{n+1})\end{aligned}$$

and so

$$
\begin{aligned}
&\langle u_n - u_{n+1}, u_n - u_{n+1} \rangle \\
&\qquad \leq \langle u_{n-1} - u_n, u_n - u_{n+1} \rangle \\
&\qquad\quad - \rho \langle N(w_{n-1}, y_{n-1}) - N(w_n, y_n), \eta(u_n, u_{n+1}) \rangle \\
&\qquad\quad + \rho b(u_n - u_{n-1}, u_n) - \rho b(u_n - u_{n-1}, u_{n+1}) \\
&\qquad \leq \langle u_{n-1} - u_n, u_n - u_{n+1} \rangle \\
&\qquad\quad - \rho \langle N(w_{n-1}, y_{n-1}) - N(w_n, y_{n-1}), \eta(u_n, u_{n+1}) \rangle \\
&\qquad\quad + \rho \langle N(w_n, y_n) - N(w_n, y_{n-1}), \eta(u_n, u_{n+1}) \rangle \\
&\qquad\quad + \rho b(u_n - u_{n-1}, u_n - u_{n+1}) \\
&\qquad \leq \langle u_{n-1} - u_n - \rho\{N(w_{n-1}, y_{n-1}) - N(w_n, y_{n-1})\}, u_n - u_{n+1} \rangle \\
&\qquad\quad + \rho \langle N(w_{n-1}, y_{n-1}) - N(w_n, y_{n-1}), u_n - u_{n+1} - \eta(u_n, u_{n+1}) \rangle \\
&\qquad\quad + \rho \langle N(w_n, y_n) - N(w_n, y_{n-1}), \eta(u_n, u_{n+1}) \rangle \\
&\qquad\quad + \rho b(u_n - u_{n-1}, u_n - u_{n+1}).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\|u_n - u_{n+1}\|^2 \\
&\qquad \leq \|u_{n-1} - u_n - \rho\{N(w_{n-1}, y_{n-1}) - N(w_n, y_{n-1})\}\| \, \|u_n - u_{n+1}\| \\
&\qquad\quad + \rho \|N(w_{n-1}, y_{n-1}) - N(w_n, y_{n-1})\| \, \|u_n - u_{n+1} - \eta(u_n, u_{n+1})\| \\
&\qquad\quad + \rho \|N(w_n, y_n) - N(w_n, y_{n-1})\| \, \|\eta(u_n, u_{n+1})\| \\
&\qquad\quad + \rho\gamma \|u_n - u_{n-1}\| \, \|u_n - u_{n+1}\|.
\end{aligned}
\tag{4.6}
$$

By the strong monotonicity of T with respect to the first argument of N, the Lipschitz continuity of N with respect to the first argument, and the Lipschitz continuity of T, we have

$$
\begin{aligned}
&\|u_{n-1} - u_n - \rho\{N(w_{n-1}, y_{n-1}) - N(w_n, y_{n-1})\}\|^2 \\
&\qquad = \|u_{n-1} - u_n\|^2 - 2\rho \langle N(w_{n-1}, y_{n-1}) - N(w_n, y_{n-1}), u_{n-1} - u_n \rangle \\
&\qquad\quad + \rho^2 \|N(w_{n-1}, y_{n-1}) - N(w_n, y_{n-1})\|^2 \\
&\qquad \leq \left(1 - 2\rho\alpha + \rho^2\beta^2\nu^2\left(1 + \frac{1}{n}\right)^2\right) \|u_{n-1} - u_n\|^2
\end{aligned}
\tag{4.7}
$$

and

$$\|N(w_{n-1}, y_{n-1}) - N(w_n, y_{n-1})\|$$
$$\leq \beta\|w_{n-1} - w_n\| \leq \beta\nu\left(1 + \frac{1}{n}\right)\|u_{n-1} - u_n\|. \tag{4.8}$$

Since $\eta(\cdot,\cdot)$ is strongly monotone and Lipschitz continuous, it follows that

$$\begin{aligned}\|u_n - u_{n+1} - \eta(u_n, u_{n+1})\|^2 &= \|u_n - u_{n+1}\|^2 - 2\langle u_n - u_{n+1}, \eta(u_n, u_{n+1})\rangle + \|\eta(u_n, u_{n+1})\|^2 \\ &\leq (1 - 2\sigma + \delta^2)\|u_n - u_{n+1}\|^2. \end{aligned}\tag{4.9}$$

By the Lipschitz continuity of N with respect to the second argument and H-Lipschitz continuity of A, we obtain

$$\|N(w_n, y_n) - N(w_n, y_{n-1})\| \leq \xi\|y_{n-1} - y_n\| \leq \xi\mu\left(1 + \frac{1}{n}\right)\|u_{n-1} - u_n\|. \tag{4.10}$$

It follows from (4.6)–(4.10) that

$$\|u_n - u_{n+1}\| \leq \theta_n\|u_{n-1} - u_n\|, \tag{4.11}$$

where

$$\theta_n = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\nu^2\left(1 + \frac{1}{n}\right)^2} + \rho\beta\nu\left(1 + \frac{1}{n}\right)\sqrt{1 - 2\sigma + \delta^2} + \rho\xi\mu\delta\left(1 + \frac{1}{n}\right) + \rho\gamma.$$

Let

$$\theta = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\nu^2} + \rho\beta\nu\sqrt{1 - 2\sigma + \delta^2} + \rho\xi\mu\delta + \rho\gamma.$$

Clearly,

$$\theta_n \to \theta = g(\rho) + \rho k, \tag{4.12}$$

where

$$g(\rho) = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2\nu}.$$

Now we show that $\theta < 1$. It is clear that $g(\rho)$ assumes its minimum value for $\bar{\rho} = \alpha/\beta^2\nu^2$ with $g(\bar{\rho}) = \sqrt{1 - (\alpha/\beta\nu)^2}$. For $\rho = \bar{\rho}$, $g(\rho) + \rho k < 1$

implies that $\rho k < 1$. Thus it follows that $\theta < 1$ for all ρ with

$$0 < \rho < \frac{2(\alpha - k)}{\beta^2\nu^2 - k^2}, \qquad \rho k < 1, \qquad \text{and} \qquad k < \alpha.$$

Since $\theta < 1$, it follows from (4.11) and (4.12) that $\{u_n\}$ is a Cauchy sequence in H. Let $u_n \to u$ as $n \to \infty$. By (3.3), we know that both $\{w_n\}$ and $\{y_n\}$ are Cauchy sequences in H. Let $w_n \to w$ and $y_n \to y$ as $n \to \infty$. Since $T(u) \in CB(H)$, $A(u) \in CB(H)$, we have $w \in T(u)$, $y \in A(u)$. Thus

$$\langle u, v - u\rangle \geq \langle u, v - \mu\rangle - \rho\langle N(w, y), \eta(v, u)\rangle + \rho b(u, u) - \rho b(u, v), \quad \forall v \in H.$$

that is,

$$\langle N(w, y), \eta(v, u)\rangle + b(u, v) - b(u, u) \geq 0, \qquad \forall v \in H.$$

This completes the proof.

Remark 4.1. From Theorem 4.1, we can obtain some existence and convergence theorems for the problems (2.2), (2.3), (2.4), and (2.5), respectively.

Remark 4.2. (1) In Theorem 4.1, the existence is given of a solution for the generalized set-valued strongly nonlinear mixed variational-like inequality (2.1) and the approximation is also given of a solution for the problem (2.1) by the iterative sequences generated by Algorithm 3.1.

(2) In Theorem 3.1 of Noor [15], the author considered the existence and uniqueness of a solution for the problem (2.1). Unfortunately, its proof is wrong.

(3) Theorem 4.1 improves and modifies Theorem 3.1 of Noor [15].

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