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On the application of substochastic semigroup theory to fragmentation models with mass loss

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Abstract

A linear integro-differential equation modelling multiple fragmentation with inherent mass loss is investigated by means of substochastic semigroup theory. The existence of a semigroup is established and, under natural conditions on certain coefficients, the generator of this semigroup is identified. This yields, in particular, a validation of the formal mass-loss rate equation for the model.

Keywords: Semigroups of operators; Abstract Cauchy problem; Multiple fragmentation with mass loss

1. Introduction

Fragmentation processes arise in many physical situations such as polymer degradation, liquid droplet breakup, combustion, and the crushing and grinding of rocks. Often, when modelling such processes, it is assumed that the total mass in the system is a conserved quantity. However, as pointed out in [1,2], there are many situations where mass loss can occur in a natural manner. Motivated by this, Edwards et al. [1,2] introduced the following linear rate equation to describe fragmentation with mass loss:

$$\partial_t u(x,t) = -a(x)u(x,t) + \int_x^\infty a(y)b(x|y)u(y,t)\,dy + \partial_x\big[r(x)u(x,t)\big]. \tag{1.1}$$

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This equation involves a particle mass distribution function u, a fragmentation rate a, a continuous mass loss rate r, and a nonnegative measurable function b, where $b(x|y)$ describes the distribution of particle masses x spawned by the fragmentation of a particle of mass y. The continuous mass loss rate r is defined so that $r(m(t)) = -dm/dt$ for a particle of time-dependent mass $m(t)$, while the normalizing condition

$$\int_0^y xb(x|y)\,dx = y - \lambda(y)y, \tag{1.2}$$

where $0 \leqslant \lambda(y) \leqslant 1$, allows for so-called discrete mass loss to occur in the fragmentation process.

Note that a rate equation for the mass in the system can be obtained, as in [3], by multiplying (1.1) by x and integrating. This leads to

$$\frac{d}{dt}\int_0^\infty u(x,t)x\,dx = -\int_0^\infty a(x)\lambda(x)u(x,t)x\,dx - \int_0^\infty r(x)u(x,t)\,dx. \tag{1.3}$$

Although it is expected that the mass in the system will evolve according to (1.3), it cannot be assumed that this will be the case since only formal arguments have been used to derive it. Consequently, one of our objectives in this paper is to establish the validity of (1.3) in a mathematically rigorous manner.

Past investigations into (1.1) appear to have concentrated on finding exact and asymptotic solutions, usually for specific choices of a, b and r; see, for example, [1–3]. However, it seems that little has been done to establish general conditions for the existence and uniqueness of solutions to (1.1), despite the fact that numerous results of this type have been proved for the mass conserving version of (1.1), in which r and λ are both identically zero. Relevant work in this area includes [4] and [5], and also [6] and [7] where results are presented for a combined coagulation–fragmentation equation. Our aim here is to rectify this situation by using functional-analytic techniques to establish that an abstract formulation of (1.1) has a unique solution under fairly mild conditions on a and r.

The strategy we adopt involves the theory of semigroups of linear operators [8] and is largely based on an approach developed by Voigt [9] and Arlotti [10], and used later by one of the present authors in several cases; see, e.g., [4,11]. In particular, in [4] the method was used to analyse a class of formally mass conserving fragmentation equations. Crucial to this approach is a theorem by Voigt [9] (the origins of which go back to the fundamental work by Kato [12] on Kolmogorov equations) which establishes that, under appropriate assumptions, a perturbation $A + B$ of a generator A of a substochastic semigroup by a positive (unbounded) operator B has an extension K that also generates a substochastic semigroup. An account of this theorem and Arlotti's work is given in Section 2, where we also prove a new theorem giving conditions which guarantee that K is the closure of $A + B$.

In Section 3, we examine an abstract formulation of the transport equation that is obtained from (1.1) when the integral is omitted. When a and r are suitably restricted, the existence of a positive strongly continuous contractive (and thus substochastic) semigroup is proved by using the Hille–Yosida theorem. Moreover, we find an explicit formula for this semigroup.

Returning to the full equation (1.1) in the final section, we use Voigt's result to deduce the existence of a minimal substochastic semigroup generated by an extension of the operator formally defined by the right-hand side of (1.1). Under an additional assumption on a and r, this solution is shown to satisfy the mass rate equation. It is worthwhile to note that in the particular case when r and λ are identically zero and $a(x) = x^\alpha$ (see, e.g., [13]) the sufficient conditions derived here coincide with the necessary and sufficient conditions for mass conservation obtained in [4], which suggests that the developed technique is quite sharp.

2. The Voigt perturbation theorem

The approach we shall adopt in our analysis of (1.1) is to reformulate the associated initial-value problem as an abstract Cauchy problem (ACP) which can then be treated using the theory of semigroups of operators. For convenience, we include here an account of a perturbation theorem due to Voigt that plays a prominent rôle later. Further details on this theorem can be found in [4,9].

Let (Ω, μ) be a measure space and let X denote the Banach space $L_1(\Omega, \mu)$ endowed with the standard norm $\|\cdot\|$. For any subspace $Z \subset X$, we denote by Z_+ the cone of nonnegative (a.e.) elements of Z. Let $(G(t))_{t\geqslant 0}$ be a strongly continuous semigroup on X. We say that $(G(t))_{t\geqslant 0}$ is a *substochastic semigroup* if, for each $t \geqslant 0$, $G(t) \geqslant 0$ and $\|G(t)\| \leqslant 1$. It is called a *stochastic* semigroup if additionally $\|G(t)f\| = \|f\|$ for $f \in X_+$.

We consider two linear operators $(A, D(A))$ and $(B, D(B))$ in X, which are assumed to have the following properties:

(A.1) $(A, D(A))$ generates a substochastic semigroup denoted by $(G_A(t))_{t\geqslant 0}$,
(A.2) $D(B) \supseteq D(A)$ and $Bf \geqslant 0$ for any $f \in D(B)_+$,
(A.3) for any $f \in D(A)_+$

$$\int_\Omega (Af + Bf)\, d\mu \leqslant 0. \tag{2.1}$$

Let us observe that the above list yields that the operator $B(I - A)^{-1}$ is a bounded positive operator on X [9]. In addition, the following perturbation result can be proved [4,9,10].

Theorem 2.1. *Let A and B satisfy assumptions* (A.1)–(A.3). *Then there exists a smallest substochastic semigroup $(G_K(t))_{t\geqslant 0}$ generated by an extension K of $A+B$. The generator K is characterized by*

$$(I - K)^{-1} f = \sum_{n=0}^{\infty} (I - A)^{-1} \big[B(I - A)^{-1}\big]^n f, \quad \forall f \in X. \tag{2.2}$$

Proof. See [4, Theorem 2.1], where also some more constructive formulae for $(G_K(t))_{t\geqslant 0}$ are given. □

The main drawback of Theorem 2.1 is that it fails to provide any characterization of the domain $D(K)$ of the generator K. It turns out that this is not only of mathematical interest but determines, for example, whether the solution to (1.1) satisfies the formal rate equation for mass (1.3) and thus whether this solution is physically relevant in the sense that it obeys the physical laws used to construct the model.

To explain this in more detail let us first consider applications in which the underlying model is formally conservative, that is

$$\int_\Omega (A+B)f\,d\mu = 0, \quad \forall f \in D(A)_+. \tag{2.3}$$

In this case the desired situation is that $K = \overline{A+B}$ since this results in [4]

$$\frac{d}{dt}\|G_K(t)f\| = 0, \quad \forall f \in D(K)_+,\ t > 0.$$

As we indicated in Section 1, it is expected that the system governed by (1.1) is non-conservative since, formally, there is mass loss described by (1.3). To cater for this, we replace (2.3) by

$$\int_\Omega (A+B)f\,d\mu = -c(f), \quad \forall f \in D(A)_+, \tag{2.4}$$

where c is a positive linear functional on $D(A)$. Note that (2.4) is consistent with assumption (A.3). Ideally, (2.4) should lead to

$$\frac{d}{dt}\|G_K(t)f\| = -c\bigl(G_K(t)f\bigr), \quad \forall f \in D(K)_+,\ t > 0, \tag{2.5}$$

so that the semigroup yields solutions that decay in accordance with (2.4). Clearly, if $D(K) = D(A)$, then $(G_K(t))_{t\geqslant 0}$ satisfies (2.5), since $G_K(t)f \in D(A)_+$ for all $t \geqslant 0$ and $f \in D(A)_+$, and so

$$\begin{aligned}\frac{d}{dt}\|G_K(t)f\| &= \int_\Omega \frac{d}{dt}G_K(t)f\,d\mu = \int_\Omega \bigl(AG_K(t)f + BG_K(t)f\bigr)\,d\mu\\ &= -c\bigl(G_K(t)f\bigr).\end{aligned}$$

However, as with the conservative case, the less restrictive requirement that $K = \overline{A+B}$ can also be physically acceptable provided c has some additional properties. These include cases when c is closed on $D(K)$ or, alternatively, when c is a positive linear functional on $D(A)_+$ and $(K, D(K)_+)$ is, in a suitable sense, accessible from $D(A)_+$ through monotonic sequences of functions (see Theorem 2.2). If either of these additional constraints on c is satisfied and $K = \overline{A+B}$, then

$$\int_\Omega Kf\,d\mu = \lim_{n\to\infty}\int_\Omega (A+B)f_n\,d\mu = -\lim_{n\to\infty} c(f_n) = -c(f),$$

where $(f_n)_{n\in\mathbb{N}} \subset D(A)_+$ is such that $f_n \to f \in D(K)_+$ and $(A+B)f_n \to Kf$ (and $f_n \nearrow f$ if the second assumption is satisfied). This in turn shows that if $f \in D(K)_+$, then

$$\frac{d}{dt}\|G_K(t)f\| = \int_\Omega \frac{d}{dt}\bigl(G_K(t)f\bigr)\,d\mu = \int_\Omega KG_K(t)f\,d\mu = -c\bigl(G_K(t)f\bigr), \quad \forall t > 0,$$

as required.

Note, however, that if K is a proper extension of $\overline{A+B}$, then (2.5) may not hold and solutions may decay in a manner that is not accounted for by the model. This situation has been encountered in investigations into the formally conservative fragmentation process governed by the equation

$$\partial_t u(x,t) = -x^\alpha u(x,t) + 2\int_x^\infty y^{\alpha-1}u(y,t)\,dy, \quad x > 0,\ \alpha < 0;$$

see [4,14] for further details. For the model (1.1) with some special coefficients such "shattering" solutions were found in [1].

The problem of determining sufficient conditions under which $K = \overline{A+B}$ has obviously received some attention, with relevant results presented in [4,9,10]. Unfortunately, it is difficult to apply these results directly to (1.1) as they were developed for formally conservative models. Therefore we devote the remainder of this section to establishing sufficient conditions more suited to the problem in hand. Since the right-hand side of (1.1) can be expressed in terms of three operators, we make the further assumption that

(A.4) $A \subseteq A_0 + A_1$, $D(A) \subseteq D(A_0) \cap D(A_1)$,

where A_0 and A_1 are both linear operators in X. It should be noted that a similar scenario was examined in [10] but under the much more restrictive assumption that A_0 is the generator of a substochastic semigroup on X and $A = A_0 + A_1$.

Adopting the approach used in [10], we denote by E the set of all measurable functions defined on Ω taking values in the extended set of real numbers (that is, infinity is allowed as the value of a function). Clearly $X \subset \mathsf{E}$. We define the subset $\mathsf{F} \subset \mathsf{E}$ by the following condition: $f \in \mathsf{F}$ if and only if for every nonnegative and nondecreasing sequence of functions $(f_n)_{n\in\mathbb{N}}$ satisfying $\sup_n f_n = |f|$ we have $\sup_n (I - A)^{-1} f_n \in X$.

Before proceeding any further we adopt the following assumption on B.

(A.5) $f \in D(B)$ if and only if $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$ both belong to $D(B)$. Moreover, if $(f'_n)_{n\in\mathbb{N}}$, $(f''_n)_{n\in\mathbb{N}}$ are two nondecreasing sequences of elements of $D(A)_+$ satisfying $\sup_n f'_n = \sup_n f''_n$, almost everywhere, then $\sup_n Bf'_n = \sup_n Bf''_n$ a.e.

Through B we construct another subset of E, say G, defined as the set of all functions $f \in X$ such that for any nonnegative, nondecreasing sequence $(f_n)_{n\in\mathbb{N}}$ of elements of $D(B)$ such that $\sup_n f_n = |f|$, we have $\sup_n Bf_n < +\infty$ almost everywhere. It is easy to check that $D(A) \subseteq \mathsf{G} \subseteq X \subseteq \mathsf{F} \subseteq \mathsf{E}$. A consequence of assumptions (A.1)–(A.5) is that we can define mappings $\mathsf{B} : \mathsf{G}_+ \to \mathsf{E}_+$ and $\mathsf{L} : \mathsf{F}_+ \to X_+$ by

$$\mathsf{B}f := \sup_n Bf_n, \quad \forall f \in \mathsf{G}_+, \tag{2.6}$$

$$\mathsf{L}f := \sup_n (I - A)^{-1} f_n, \quad \forall f \in \mathsf{F}_+, \tag{2.7}$$

where $0 \leqslant f_n \leqslant f_{n+1}$ for any $n \in \mathbb{N}$, and $\sup_n f_n = f$. Precisely speaking, the correctness of the definition of L follows from the fact, proved in [11], that, as defined, L is a restriction of the so-called Sobolev tower extension of $R(1, A)$; see [8].

We extend the mappings L and B onto F and G, respectively, by linearity. The relation between L and the extension of $R(1, A)$ yields easily the result that

$$\mathsf{L}f \in D(A) \quad \text{if and only if} \quad f \in X, \tag{2.8}$$

which was established in [10, Lemma 2] by a different method.

Recalling that we denoted by K the generator of the full semigroup constructed by Voigt's method, let $h \in D(K)$. Then $g = (I - K)h \in X$ and so $\mathsf{L}g = (I - A)^{-1}g \in D(A) \subseteq D(B)$ which implies that $\mathsf{BL}g = B\mathsf{L}g$. Consequently, from (2.2), we obtain

$$h = \sum_{k=0}^{\infty} \mathsf{L}(B\mathsf{L})^k g. \tag{2.9}$$

Following [10], for any given $g \in X$ and arbitrary $n \in \mathbb{N}$ we write

$$f_n = \sum_{k=0}^{n} (B\mathsf{L})^k g \tag{2.10}$$

and

$$h_n = \mathsf{L}f_n. \tag{2.11}$$

By (2.2), $(h_n)_{n\in\mathbb{N}}$ converges to h in X. However, for positive g we can consider limits of both sequences, $(f_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$, in the sense of monotonic convergence almost everywhere, as L and B are positive operators. Denoting the respective limits by f and h, it follows that $h \in X_+$ and $\mathsf{L}f = h$. We can now prove a more general version of [10, Theorem 2].

Theorem 2.2. *Let assumptions* (A.1)–(A.5) *be satisfied and let c be the positive functional defined by* (2.4). *Moreover, for arbitrary $g \in X_+$, let h, h_n and f_n be defined by* (2.9), (2.10) *and* (2.11).

(a) *The real sequence $(c(h_n))_{n\in\mathbb{N}}$ is convergent.*
(b) *If we have*

$$-\lim_{n\to\infty} c(h_n) \leqslant \int_{\Omega} Kh \, d\mu, \tag{2.12}$$

then $K = \overline{A + B}$.
(c) *If c is of the form*

$$c(f) = \int_{\Omega} lf \, d\mu, \quad f \in D(A),$$

where the function l is positive (a.e.) on Ω and $K = \overline{A+B}$, then, for any $u \in D(K)_+$,

$$\int_\Omega Ku\,d\mu = -c(u). \tag{2.13}$$

Proof. (a) Since $h_n \in D(A) \subseteq D(A_0) \cap D(A_1)$, we can write

$$f_n = (I - A_0 - A_1)h_n, \tag{2.14}$$

and so

$$\begin{aligned} \|f_n\| &= \|h_n\| - \int_\Omega A_0 h_n\,d\mu - \int_\Omega A_1 h_n\,d\mu \\ &= \|h_n\| + \int_\Omega Bh_n\,d\mu - \int_\Omega (A_0h_n + A_1h_n + Bh_n)\,d\mu \\ &= \|h_n\| + \|Bh_n\| + c(h_n). \end{aligned} \tag{2.15}$$

Noting that

$$Bh_n = B\mathsf{L}f_n = \sum_{k=0}^{n} (B\mathsf{L})^{k+1} g = f_{n+1} - g = f_n + (B\mathsf{L})^{n+1}g - g, \tag{2.16}$$

we obtain

$$\|Bh_n\| = \|f_n\| + \left\|(B\mathsf{L})^{n+1}g\right\| - \|g\|. \tag{2.17}$$

Combining (2.17) with (2.15) produces

$$\|g\| = \|h_n\| + c(h_n) + \left\|(B\mathsf{L})^{n+1}g\right\|, \tag{2.18}$$

and therefore $(c(h_n))_{n\in\mathbb{N}}$ is bounded. From the positivity of c and the definition of h_n we deduce that $(c(h_n))_{n\in\mathbb{N}}$ is also nondecreasing and hence is convergent.

(b) It follows from (a) and (2.18) that $(\|(B\mathsf{L})^{n+1}g\|)_{n\in\mathbb{N}}$ is convergent. Moreover, as $g = h - Kh$, we have

$$\|g\| = \|h\| - \int_\Omega Kh\,d\mu. \tag{2.19}$$

Consequently, from (2.18), (2.19) and assumption (2.12), we obtain

$$\lim_{n\to\infty} \left\|(B\mathsf{L})^{n+1}g\right\| = \|g\| - \|h\| - \lim_{n\to\infty} c(h_n) = -\int_\Omega Kh\,d\mu - \lim_{n\to\infty} c(h_n) \leqslant 0.$$

This shows that

$$\lim_{n\to\infty} \left\|(B\mathsf{L})^{n+1}g\right\| = 0, \quad \forall g \in X_+, \tag{2.20}$$

and, by linearity, we conclude that (2.20) is also valid for any $g \in X$. The final statement $Kh = (\overline{A+B})h$ follows as in [10, Theorem 2] or by invoking the general result [15, Theorem 3.1].

(c) If $u \in D(K)_+$, then $u = (I - K)^{-1}g = (I - K)^{-1}(g^+ - g^-)$, where $g \in X$ and $g^+, g^- \in X_+$. Writing $h^\pm = (I - K)^{-1}g^\pm$, we obtain $u = h^+ - h^-$. Both h^+ and h^- can be achieved by monotonic sequences $(h^\pm{}_n)_{n\in\mathbb{N}}$, defined as in (2.9) with $g^\pm$, respectively. Both sequences are in $D(A)_+$ and converge in X to $h^\pm$, respectively. Moreover, by combining (2.14) and (2.16), we get $(A+B)h_n^\pm = h_n^\pm + (B\mathsf{L})^{n+1}g^\pm - g^\pm$, so that by (2.20) and $K = \overline{A+B}$, $(A+B)h_n^\pm \to Kh^\pm$ in X. Thus, passing to the limit in $\int_\Omega Kh_n^\pm \, d\mu = -c(h_n^\pm)$ we obtain, by the monotone convergence theorem, $\int_\Omega Kh^\pm \, d\mu = -c(h^\pm)$, where the right-hand limit is finite. Finally, for arbitrary $u \in D(K)_+$ we obtain

$$\int_\Omega Ku \, d\mu = \int_\Omega Kh^+ \, d\mu - \int_\Omega Kh^- \, d\mu = -c(h^+ - h^-) = -c(u). \quad \square$$

3. The transport semigroup

As a first step toward applying the theory of Section 2 to Eq. (1.1), we now establish the existence of a strongly continuous semigroup $(G_A(t))_{t\geqslant 0}$ associated with the transport equation

$$\begin{aligned} &\partial_t u(x,t) = \partial_x\big[r(x)u(x,t)\big] - a(x)u(x,t), \quad t > 0,\ x > 0, \\ &u(0,x) = g(x). \end{aligned} \tag{3.1}$$

Throughout we shall assume that the functions r and a satisfy the following conditions:

(C.1) r is strictly positive on $(0,\infty)$ and absolutely continuous on any compact subinterval of $(0,\infty)$,
(C.2) $a \in L_{1,\mathrm{loc}}(0,\infty)$ and is nonnegative almost everywhere on $(0,\infty)$.

Note that the possible singularities of r allowed here make it rather difficult to apply directly the fairly general theory of first-order equations developed in [16] and subsequent papers. Thus we have decided for a straightforward approach that is presented below.

In the sequel, any function that is absolutely continuous on all compact subintervals of $(0,\infty)$ will be said to be locally absolutely continuous (abbreviated to l.a.c.). Since (C.1) and (C.2) imply that $1/r, a/r \in L_{1,\mathrm{loc}}(0,\infty)$, their respective antiderivatives R and Q, given by

$$R(x) := \int_{x_0}^{x} \frac{1}{r(s)}\, ds, \qquad Q(x) := \int_{x_0}^{x} \frac{a(s)}{r(s)}\, ds$$

(for fixed $x_0 > 0$), are both l.a.c. Consequently, $R + Q$ is bounded on any compact subinterval of $(0,\infty)$, and, since the exponential function is uniformly Lipschitz on any (fixed) compact subinterval, it follows that $e^{\lambda R+Q}$ is also l.a.c. for any fixed constant λ. Other immediate consequences of (C.1) and (C.2) are that R is strictly increasing (and hence invertible) on $(0,\infty)$, and Q is nondecreasing on $(0,\infty)$. Define m_R, M_R, m_Q and M_Q by

$$\lim_{x\to 0} R(x) = m_R, \qquad \lim_{x\to\infty} R(x) = M_R,$$
$$\lim_{x\to 0} Q(x) = m_Q, \qquad \lim_{x\to\infty} Q(x) = M_Q.$$

We note that m_R and m_Q can be finite or $-\infty$, while M_R and M_Q can be finite or $+\infty$. Clearly, $M_R > m_R$ and $M_Q \geqslant m_Q$, and the images of R and Q are (m_R, M_R) and (m_Q, M_Q), respectively.

As the total mass in the system at time t is given by $M(t) = \int_0^\infty xu(x,t)\,dx$, we reformulate (3.1) as the ACP

$$\begin{aligned} &\frac{du}{dt}(t) = A\big[u(t)\big], \quad t > 0, \\ &u(0) = g, \end{aligned} \tag{3.2}$$

posed in the Banach space $X := L_1([0,\infty), x\,dx)$, where A is given formally by $Af = (d/dx)(rf) - af$. More precisely, we define

$$Af := A_0 f + A_1 f, \quad f \in D(A) \subseteq D(A_0) \cap D(A_1),$$

where $A_0 f := (d/dx)(rf)$, $A_1 f := -af$, and

$$D(A_0) := \left\{ f \in X\colon\ rf \text{ is l.a.c. and } \frac{d}{dx}(rf) \in X \right\},$$
$$D(A_1) := \{f \in X\colon\ af \in X\}.$$

Our main result in this section is to identify a domain, $D(A)$, for A, so that $(A, D(A))$ generates a substochastic semigroup on X.

By direct integration, we find that the general solution to the differential equation

$$\lambda f(x) + a(x) f(x) - \frac{d}{dx}\big(r(x) f(x)\big) = 0, \quad \lambda > 0,$$

is given by $f(x) = Cf_\lambda(x)$, where

$$f_\lambda(x) = \frac{e^{\lambda R(x) + Q(x)}}{r(x)} = e^{(\lambda - 1)R(x)} f_1(x). \tag{3.3}$$

For some choices of r and a (e.g., for $r(x) = x^p$ with $p > 2$ and a bounded and integrable), a routine calculation shows that $\|f_\lambda\|$ is finite. In such cases, $f_\lambda \in D(A_0) \cap D(A_1)$ and therefore $(\lambda I - A, D(A_0) \cap D(A_1))$ is not invertible for $\lambda > 0$ and consequently $(A, D(A_0) \cap D(A_1))$ cannot be the generator of a C_0-semigroup. Thus, our first aim is to determine the domain $D(A)$ of A for which $(\lambda I - A, D(A))$ is invertible for all $\lambda > 0$ and all functions r and a satisfying (C.1) and (C.2).

Lemma 3.1. *For each $\lambda > 0$, let*

$$I(\lambda) := \int_0^\infty x f_\lambda(x)\,dx = \|f_\lambda\|, \tag{3.4}$$

where f_λ is given by (3.3).

(a) *If $M_R = \infty$, then $I(\lambda) = \infty$ for all $\lambda > 0$.*
(b) *If $I(\lambda) < +\infty$ for some $\lambda > 0$, then $M_R < \infty$.*
(c) *$I(\lambda) < \infty$ for any $\lambda > 0$ if and only if $I(1) < \infty$.*
(d) *For any $g \in D(A_0) \cap D(A_1)$ and $M_R < +\infty$,*

$$\lim_{x\to\infty} \frac{g(x)}{f_\lambda(x)} = 0 \quad \textit{if and only if} \quad \lim_{x\to\infty} \frac{g(x)}{f_1(x)} = 0.$$

(e) *If $I(\lambda) = \infty$, then*

$$\lim_{x\to\infty} \frac{g(x)}{f_\lambda(x)} = 0.$$

Proof. (a) and (b). Since $x \exp(\lambda R(x) + Q(x))$ is positive and increasing, we obtain

$$I(\lambda) \geqslant \int_{x_0}^{\infty} x f_\lambda(x)\, dx \geqslant x_0 e^{\lambda R(x_0) + Q(x_0)} \int_{x_0}^{\infty} \frac{dx}{r(x)} = x_0 e^{\lambda R(x_0) + Q(x_0)} M_R,$$

from which both (a) and (b) follow immediately.

(c) and (d). If $M_R < \infty$, then

$$\lim_{x\to\infty} e^{(\lambda-1)R(x)} = e^{(\lambda-1)M_R} \in (0, \infty), \tag{3.5}$$

and therefore for any $y > 0$

$$\int_y^{\infty} x f_\lambda(x)\, dx < \infty \quad \text{if and only if} \quad \int_y^{\infty} x f_1(x)\, dx < \infty.$$

Since, for any $\lambda > 0$,

$$\int_0^y x f_\lambda(x)\, dx = \frac{1}{\lambda} \int_0^y x e^{Q(x)} \frac{d}{dx}\bigl(e^{\lambda R(x)}\bigr)\, dx \leqslant \frac{y}{\lambda} e^{Q(y)}\bigl(e^{\lambda R(y)} - e^{\lambda m_R}\bigr), \tag{3.6}$$

we obtain (c). The result stated in (d) also follows directly from (3.3) and (3.5).

(e) Let $I(\lambda) = \infty$ and let $g \in D(A_0) \cap D(A_1)$. Then, for $y > 0$,

$$\int_y^{\infty} e^{-\lambda R(x) - Q(x)} \frac{d}{dx}\bigl(r(x) g(x)\bigr)\, dx < \infty. \tag{3.7}$$

Furthermore, rg and $e^{-\lambda R - Q}$ are l.a.c. and so the left-hand side of (3.7) can be integrated by parts to produce

$$\begin{aligned}
&\bigl[e^{-\lambda R(x) - Q(x)} r(x) g(x)\bigr]_y^{\infty} - \int_y^{\infty} \frac{d}{dx}\bigl(e^{-\lambda R(x) - Q(x)}\bigr) r(x) g(x)\, dx \\
&\quad = \lim_{x\to\infty} \frac{g(x)}{f_\lambda(x)} - \frac{g(y)}{f_\lambda(y)} + \int_y^{\infty} \bigl(\lambda + a(x)\bigr) g(x)\, dx,
\end{aligned} \tag{3.8}$$

from which we deduce that

$$\lim_{x\to\infty}\frac{g(x)}{f_\lambda(x)} = L < \infty.$$

Suppose $L \neq 0$. Then there exist $C > 0$ and $y > 0$ such that $|g(x)|/f_\lambda(x) \geqslant C$ for all $x \geqslant y$, in which case

$$\int_y^\infty x f_\lambda(x)\,dx = \int_y^\infty x\bigl|g(x)\bigr|\frac{f_\lambda(x)}{|g(x)|}\,dx \leqslant \frac{1}{C}\int_0^\infty x\bigl|g(x)\bigr|\,dx < \infty.$$

Thus, it follows from (3.6) that $I(\lambda) < \infty$, contrary to the assumption on $I(\lambda)$. □

The results given in the previous lemma suggest that we define $D(A) \subseteq D(A_0) \cap D(A_1)$ by

$$D(A) := \begin{cases} D(A_0) \cap D(A_1) & \text{if } I(1) = +\infty, \\ \bigl\{g \in D(A_0) \cap D(A_1):\ \lim_{x\to\infty}\frac{g(x)}{f_1(x)} = 0\bigr\} & \text{if } I(1) < +\infty, \end{cases} \tag{3.9}$$

where f_1 and $I(1)$ are given by (3.3) and (3.4), respectively. Note that $(\lambda I - A, D(A))$ is invertible and that the condition

$$\lim_{x\to\infty}\frac{g(x)}{f_1(x)} = 0, \quad \forall g \in D(A), \tag{3.10}$$

is always satisfied, irrespective of whether M_R and $I(1)$ are finite or infinite.

Lemma 3.2. *For each $\lambda > 0$, let $R(\lambda)$ be defined by*

$$\bigl(R(\lambda)g\bigr)(x) := \int_x^\infty G_\lambda(x, y)\frac{g(y)}{r(y)}\,dy, \quad g \in X,\ x > 0, \tag{3.11}$$

where $G_\lambda(x, y) = f_\lambda(x)/f_\lambda(y)$. Then $R(\lambda)$ is the resolvent of A.

Proof. For $g \in X$ and $\lambda > 0$ we have, by Tonelli's theorem,

$$\begin{aligned}\bigl\|R(\lambda)g\bigr\| &\leqslant \int_0^\infty\int_x^\infty \frac{xG_\lambda(x, y)|g(y)|}{r(y)}\,dy\,dx = \int_0^\infty \frac{y|g(y)|}{r(y)}\left(\int_0^y \frac{xG_\lambda(x, y)}{y}\,dx\right)dy \\ &\leqslant \frac{1}{\lambda}\int_0^\infty y\bigl|g(y)\bigr|\,dy = \frac{1}{\lambda}\|g\|, \end{aligned} \tag{3.12}$$

where the last inequality follows, by (3.6), from

$$\int_0^y \frac{xG_\lambda(x, y)}{y}\,dx = \frac{1}{yf_\lambda(y)}\int_0^y xf_\lambda(x)\,dx \leqslant \frac{1}{\lambda}.$$

Hence $R(\lambda)$ is a bounded operator on X with $\|R(\lambda)\| \leqslant 1/\lambda$.

Next we note that

$$\int_0^\infty a(x)x\big|\big(R(\lambda)g\big)(x)\big|\,dx \leqslant \int_0^\infty y\big|g(y)\big|\left(\frac{1}{yr(y)f_\lambda(y)}\int_0^y xa(x)f_\lambda(x)\,dx\right)dy.$$

Since

$$\begin{aligned}\int_0^y xa(x)f_\lambda(x)\,dx &= \int_0^y xe^{\lambda R(x)}\frac{d}{dx}\big(e^{Q(x)}\big)\,dx\\ &\leqslant ye^{\lambda R(y)}\big(e^{Q(y)}-e^{m_Q}\big)\leqslant yr(y)f_\lambda(y),\end{aligned}$$

we deduce that $\|A_1R(\lambda)g\|\leqslant\|g\|$ for each $g\in X$ and $\lambda>0$, and so $R(\lambda)X\subseteq D(A_1)$. Now observe that, for $g\in X$,

$$r(x)\big(R(\lambda)g\big)(x) = e^{\lambda R(x)+Q(x)}\int_x^\infty e^{-\lambda R(y)-Q(y)}g(y)\,dy,$$

and both $e^{\lambda R+Q}$ and the integral (as a function of its lower limit) are absolutely continuous and bounded on any compact subinterval of $(0,\infty)$. Therefore $rR(\lambda)g$ is l.a.c. Moreover,

$$A_0R(\lambda)g = \frac{d}{dx}\big(rR(\lambda)g\big) = (\lambda I - A_1)R(\lambda)g - g,\quad \forall g\in X, \tag{3.13}$$

so that $R(\lambda)X\subseteq D(A_0)$ and hence $R(\lambda)X\subseteq D(A_0)\cap D(A_1)$ for all $\lambda>0$. If $I(1)=\infty$, we deduce immediately that $R(\lambda)X\subseteq D(A)$. If $I(1)<\infty$, then

$$\left|\frac{(R(\lambda)g)(x)}{f_\lambda(x)}\right| \leqslant \int_x^\infty e^{-\lambda R(y)-Q(y)}\big|g(y)\big|\,dy \leqslant \frac{e^{-\lambda R(x)-Q(x)}}{x}\int_x^\infty y\big|g(y)\big|\,dy \to 0$$

as $x\to\infty$ and again $R(\lambda)X\subseteq D(A)$ for all $\lambda>0$.

Finally, it follows from (3.13) that

$$(\lambda I - A)R(\lambda)g = (\lambda I - A_0 - A_1)R(\lambda)g = (\lambda I - A_1)R(\lambda)g - A_0R(\lambda)g = g,\quad \forall g\in X.$$

Also, for $g\in D(A)$, integration by parts yields

$$\begin{aligned}\big(R(\lambda)A_0g\big)(x) &= \int_x^\infty \frac{G_\lambda(x,y)}{r(y)}\frac{d}{dy}\big(r(y)g(y)\big)\,dy\\ &= \big[G_\lambda(x,y)g(y)\big]_x^\infty - \int_x^\infty r(y)g(y)\frac{d}{dy}\left(\frac{G_\lambda(x,y)}{r(y)}\right)dy\\ &= f_\lambda(x)\lim_{y\to\infty}\frac{g(y)}{f_\lambda(y)} - g(x)\end{aligned}$$

$$+ f_\lambda(x)\int_x^\infty \bigl(\lambda + a(x)\bigr)e^{-\lambda R(y)-Q(y)}g(y)\,dy$$
$$= \bigl(R(\lambda)(\lambda I - A_1)g\bigr)(x) - g(x).$$

Consequently,

$$R(\lambda)(\lambda I - A_0 - A_1)g = -R(\lambda)A_0 g + R(\lambda)(\lambda I - A_1)g = g, \quad \forall g \in D(A),$$

and the theorem is proved. □

Theorem 3.1. *The operator $(A, D(A))$ is the infinitesimal generator of a strongly continuous positive semigroup of contractions, say $(G_A(t))_{t\geqslant 0}$, on X.*

Proof. This follows immediately from Lemma 3.2, the positivity of $R(\lambda)$ and the Hille–Yosida theorem. □

To complete our analysis of (3.2), we now find an explicit formula for $(G_A(t))_{t\geqslant 0}$. If we define

$$Y(t,x) := R^{-1}\bigl(R(x)+t\bigr), \quad x > 0,\ 0 \leqslant t < M_R - R(x),$$

then direct integration of (3.1) leads to the solution

$$u(x,t) = e^{(\int_0^t (r'-a)(Y(s,x))\,ds)}g\bigl(Y(t,x)\bigr) = \frac{e^{Q(x)}r(Y(t,x))g(Y(t,x))}{e^{Q(Y(t,x))}r(x)}, \tag{3.14}$$

where the second equation in (3.14) is obtained by using the identities

$$\frac{d}{ds}\ln r\bigl(Y(s,x)\bigr) = \frac{r'(Y(s,x))}{r(Y(s,x))}\frac{dY}{ds} = r'\bigl(Y(s,x)\bigr)$$

and

$$\int_0^t a\bigl(Y(s,x)\bigr)\,ds = \int_x^{Y(t,x)} \frac{a(\sigma)}{r(\sigma)}\,d\sigma = Q\bigl(Y(t,x)\bigr) - Q(x). \tag{3.15}$$

If M_R is finite, then (3.14) is not defined for all $t > 0$. To enable a semigroup to be defined in such cases we must find a suitable extension beyond the stipulated limits of t. To do this, we observe that $Y(t,x)$ approaches $+\infty$ as $R(x)+t$ approaches M_R and thus, by (3.9), $u(x,t)$ converges to zero (at least for $g \in D(A)$). Thus a reasonable candidate for the semigroup is

$$\bigl[Z(t)g\bigr](x) = \begin{cases} \frac{e^{Q(x)}r(Y(t,x))g(Y(t,x))}{e^{Q(Y(t,x))}r(x)} & \text{for } R(x)+t < M_R, \\ 0 & \text{for } R(x)+t \geqslant M_R. \end{cases} \tag{3.16}$$

Theorem 3.2. *For any $g \in X$, the function $(t,x) \to [Z(t)g](x)$ is a representation of the semigroup $(G_A(t))_{t\geqslant 0}$ in the sense that, for almost any $t > 0$ and $x > 0$,*

$$\bigl[G_A(t)g\bigr](x) = \bigl[Z(t)g\bigr](x).$$

If $g \in D(A)$, then the equality holds for any $t \geqslant 0$ and $x > 0$.

Proof. Let us fix $g \in X$. For any fixed $x > 0$, the function $t \to [Z(t)g](x)$ is clearly measurable and has Laplace transform

$$\int_0^\infty e^{-\lambda t}\big[Z(t)g\big](x)\,dt = \int_0^{M_R - R(x)} \frac{e^{-\lambda t + Q(x) - Q(Y(t,x))} r(Y(t,x))g(Y(t,x))}{r(x)}\,dt$$
$$= \frac{e^{\lambda R(x)+Q(x)}}{r(x)} \int_x^\infty e^{-\lambda R(z) - Q(z)} g(z)\,dz, \tag{3.17}$$

where the change of variables $z = Y(t,x) = R^{-1}(R(x) + t)$ has been used to obtain the last formula. On the other hand, from Theorem 3.1 we have, for any $g \in X$,

$$\int_0^\infty e^{-\lambda t} G_A(t)g\,dt = (\lambda I - A)^{-1} g = R(\lambda)g \quad \text{in } X.$$

Since X is a space of type L [17, pp. 68–71], and $t \to G_A(t)g$ is continuous, there is a measurable representation $(G_A(t)g)(x)$ for which we have for almost all $x > 0$

$$\int_0^\infty e^{-\lambda t}\big(G_A(t)g\big)(x)\,dt = \left[\int_0^\infty e^{-\lambda t} G_A(t)g\,dt\right](x) = \big[R(\lambda)g\big](x)$$
$$= \frac{e^{\lambda R(x)+Q(x)}}{r(x)} \int_x^\infty e^{-\lambda R(z) - Q(z)} g(z)\,dz. \tag{3.18}$$

As both $[G_A(t)g](x)$ and $[Z(t)g](x)$ are clearly locally integrable with respect to t on $[0, \infty)$ for almost any $x > 0$, and the abscissae of convergence of the Laplace integrals are equal to 0, from [18, Theorem 1.7.3] we infer that

$$\big[G_A(t)g\big](x) = \big[Z(t)g\big](x), \quad \text{for a.a. } t > 0,\ x > 0, \tag{3.19}$$

so that $[Z(t)g](x)$ is a representative of $G_A(t)g$.

If $g \in D(A)$, then, from the definition of $D(A_0)$ and the strict positivity of r, we obtain that g is continuous on $(0, \infty)$ so that, by the discussion preceding (3.16), $[Z(t)g](x)$ is continuous in $t \in (0, \infty)$ for any $x > 0$. On the other hand, for $g \in D(A)$, $G_A(t)g$ is a differentiable X-valued function so that, by [17, Theorem 3.4.2], a representative $[G_A(t)g](x)$ can be selected to be continuous in t for any $x > 0$. Repeating the previous argument we obtain the validity of (3.19) for any $t > 0$ and $x > 0$. The extension to $t = 0$ can be done by continuity as $g(Y(t,x))$ is continuous at $t = 0$ provided $x > 0$. □

From Theorems 3.1 and 3.2 we can state immediately that the ACP (3.2) has a strong solution $u : [0, \infty) \to X_+$, given by $u(t) := G_A(t)g = Z(t)g$, for all $g \in D(A)_+$. By further restricting g to be an absolutely continuous function with support in $[0, N]$, $N < \infty$, it is possible to show by direct, but lengthy, calculations that $u(x,t) := [Z(t)g](x)$ satisfies the initial value problem (3.1) for almost all $t > 0$ and $x > 0$.

4. The semigroup for the fragmentation equation with mass loss

Having established the existence of a substochastic semigroup $(G_A(t))_{t\geqslant 0}$ associated with the reduced initial-value problem (3.1), we now turn our attention to the full mass-loss fragmentation equation (1.1) and show that this can be analysed using the theory described in Section 2. As in Section 3, we express the problem as an ACP in the space $X = L_1([0,\infty), x\,dx)$. In this case, the abstract problem takes the form

$$\begin{aligned} &\frac{du}{dt}(t) = A\big[u(t)\big] + B\big[u(t)\big], \quad t > 0, \\ &u(0) = g. \end{aligned} \tag{4.1}$$

Throughout, $A \subseteq A_0 + A_1$ is defined as in the previous section, while B is given by

$$(Bf)(x) := \int_x^\infty a(y)b(x|y)f(y)\,dy, \quad f \in D(B), \tag{4.2}$$

where b satisfies (1.2) and $D(B) = D(A) = \{f \in X\colon\ af \in X\}$.

Lemma 4.1. *For any $f \in D(A)$ we have*

$$\int_0^\infty (Af + Bf)x\,dx = -c(f), \tag{4.3}$$

where

$$c(f) = \int_0^\infty r(x)f(x)\,dx + \int_0^\infty \lambda(x)a(x)f(x)x\,dx. \tag{4.4}$$

Proof. Let $f \in D(A)$. Then $f = (I-A)^{-1}g$ for some $g \in X$ and, as in (3.13), we obtain

$$\big(A_0(I-A)^{-1}g\big)(x) = \frac{1+a(x)}{r(x)}e^{R(x)+Q(x)}\int_x^\infty e^{-R(y)-Q(y)}g(y)\,dy - g(x).$$

Now

$$\begin{aligned} &\int_0^\infty \left(\frac{1+a(x)}{r(x)}e^{R(x)+Q(x)}\int_x^\infty e^{-R(y)-Q(y)}g(y)\,dy\right)x\,dx \\ &\quad = \int_0^\infty e^{-R(y)-Q(y)}g(y)\left(\int_0^y \frac{1+a(x)}{r(x)}e^{R(x)+Q(x)}x\,dx\right)dy, \end{aligned}$$

where

$$\begin{aligned}\int_0^y \frac{1+a(x)}{r(x)} e^{R(x)+Q(x)} x\,dx &= \int_0^y x\frac{d}{dx}e^{R(x)+Q(x)}\,dx\\ &= ye^{R(y)+Q(y)} - \int_0^y e^{R(x)+Q(x)}\,dx.\end{aligned}$$

Hence

$$\begin{aligned}\int_0^\infty A_0(I-A)^{-1}gx\,dx &= \int_0^\infty g(y)y\,dy - \int_0^\infty e^{-R(y)-Q(y)}g(y)\left(\int_0^y e^{R(x)+Q(x)}\,dx\right)dy\\ &\quad - \int_0^\infty g(y)y\,dy\\ &= -\int_0^\infty e^{R(x)+Q(x)}\left(\int_x^\infty e^{-R(y)-Q(y)}g(y)\,dy\right)dx\\ &= -\int_0^\infty r(x)\big((I-A)^{-1}g\big)(x)\,dx.\end{aligned}$$

Since $D(A) \subseteq D(A_0) \cap D(A_1)$, it follows that $(I-A)^{-1}g \in D(A_0) \cap D(A_1)$ and therefore, using (1.2), we deduce that

$$\begin{aligned}\int_0^\infty (Af+Bf)x\,dx &= \int_0^\infty (A_0f + A_1f + Bf)x\,dx\\ &= -\int_0^\infty r(x)\big((I-A)^{-1}g\big)(x)\,dx - \int_0^\infty xa(x)\big((I-A)^{-1}g\big)(x)\,dx\\ &\quad + \int_0^\infty x\left(\int_x^\infty a(y)b(x|y)f(y)\,dy\right)dx\\ &= -\int_0^\infty r(x)f(x)\,dx - \int_0^\infty \lambda(x)a(x)f(x)x\,dx = -c(f). \quad \square\end{aligned}$$

Theorem 4.1. *Let r and a satisfy* (C.1) *and* (C.2). *Then there exists a smallest substochastic semigroup, say $(G_K(t))_{t\geqslant 0}$, generated by an extension K of $A+B$.*

Proof. This follows immediately from Theorem 2.1 and Lemma 4.1, as $-c(f) \leqslant 0$ for $f \in D(A)_+$. $\square$

Our final aim is to obtain a complete characterization of the generator K by using Theorem 2.2 to show that $K = \overline{A + B}$. In view of Theorem 2.2(c) and the comments made after Theorem 2.1, this will then lead to

$$\frac{d}{dt}\left\|G_K(t)f\right\| = -c(f), \quad \forall f \in D(A)_+,$$

where c is defined by (4.4), thus providing a rigorous justification of the mass-loss rate equation (1.3).

We start with an auxiliary result that specifies some results of Section 2 in the present context.

Lemma 4.2. *In the setting of this section, let $h = (I - K)^{-1}g$ with $g \in X_+$, and define f_n and h_n via* (2.10) *and* (2.11), *respectively. Then we have*

$$\lim_{n\to\infty} \int_0^\infty A_0 h_n x\,dx = -\lim_{n\to\infty} \int_0^\infty r(x)h_n(x)\,dx = -\int_0^\infty r(x)(\mathsf{L}f)(x)\,dx. \tag{4.5}$$

Consequently, $r\mathsf{L}f$ is integrable with respect to the Lebesgue measure dx. Similarly

$$\lim_{n\to\infty} \int_0^\infty \lambda(x)a(x)h_n(x)x\,dx = \int_0^\infty \lambda(x)a(x)(\mathsf{L}f)(x)x\,dx, \tag{4.6}$$

so that $\lambda a\mathsf{L}f$ is integrable with respect to the measure $x\,dx$.

Proof. Recalling the notation (2.10) and (2.11), since $f_n \in X_+$, $h_n \in D(A) \subset D(A_0)$, and, as in Lemma 4.1,

$$\int_0^\infty A_0\mathsf{L}f_n x\,dx = -\int_0^\infty r(x)(\mathsf{L}f_n)(x)\,dx.$$

Now, from the definition, $\mathsf{L}f_n$ converges monotonically almost everywhere to $h = \mathsf{L}f$ and therefore $(r\mathsf{L}f_n)_{n\in\mathbb{N}}$ is a nondecreasing sequence converging almost everywhere to $r\mathsf{L}f$. From the monotone convergence theorem

$$\lim_{n\to\infty} \int_0^\infty r(x)(\mathsf{L}f_n)(x)\,dx = \int_0^\infty r(x)(\mathsf{L}f)(x)\,dx,$$

and, by (4.4), we see that this limit does not exceed the limit of $(c(h_n))_{n\in\mathbb{N}}$ which, from Theorem 2.2(a), is known to be finite.

The second part follows in the same way. □

Theorem 2.2 is not immediately useful as we know neither K nor h. To circumvent this difficulty we could try an approach that has proved successful in previous investigations, such as [4,10,11]. However, this would require a substantially stronger condition that (2.12) holds for the maximal extension of $A + B$, that is, for all positive h for which an expression on the right-hand side of (1.1) defines an integrable function. It turns out that such a condition is too crude for our purposes. Instead, we use the result below which follows from a simple observation that, having obtained some necessary properties of functions from $D(K)$ in the previous lemma, we need only test (2.12) on functions that have these properties. This leads to the following more convenient condition.

Theorem 4.2. *Let*

$$\begin{aligned}
&-\int_0^\infty r(x)(\mathsf{L}f)(x)\,dx - \int_0^\infty \lambda(x)a(x)(\mathsf{L}f)(x)x\,dx \\
&\quad \leqslant \int_0^\infty (\mathsf{L}f)(x)x\,dx + \int_0^\infty \bigl(-f(x) + (\mathsf{BL}f)(x)\bigr)x\,dx
\end{aligned} \tag{4.7}$$

for all functions $f \in \mathsf{F}_+$ which are such that the products $r\mathsf{L}f$ and $x\lambda a\mathsf{L}f$ are both integrable with respect to the Lebesgue measure dx, and $-f + \mathsf{BL}f \in X$. Then $K = \overline{A + B}$.

Proof. Since $Kh = h - g$, where $h = \mathsf{L}f$ and $\mathsf{B}h = f - g$, we can write

$$Kh = \mathsf{L}f - f + \mathsf{BL}f. \tag{4.8}$$

Moreover, if $f \in \mathsf{F}$ is such that (4.8) holds, then the right-hand side is integrable (with respect to the measure $x\,dx$). Since $\mathsf{L}f \in X$, we see that it is enough to restrict our attention to functions f satisfying $-f + \mathsf{BL}f \in X$. Substituting (4.8) and using (4.4) in (2.12) gives (4.7). Also, from Lemma 4.2 it follows that it is enough to consider functions for which the indicated products are integrable. □

To be able to apply Theorem 4.2, we require the following lemma.

Lemma 4.3. *If $f \in \mathsf{F}_+$ and*

(C.3) $\lim_{x\to 0^+} r(x)/x < +\infty$ *and* $\lim_{x\to 0^+} a(x) < \infty$,

then $f \in L_1([0,\alpha], x\,dx)$ for any $\alpha < +\infty$.

Proof. Since $(I - A)^{-1}$ is an integral operator with a positive kernel, it follows from the monotone convergence theorem that L is the same integral operator, but now defined on those measurable functions for which the integral is finite almost everywhere and defines an integrable function. Consequently, from Lemma 3.2 and (3.11), $\mathsf{L}f \in X$ is given by

$$(\mathsf{L}f)(x) = \int_x^\infty \frac{G_1(x,y)f(y)}{r(y)}\,dy, \tag{4.9}$$

and so, applying Tonelli's theorem, we obtain

$$\int_0^\infty (\mathsf{L}f)(x)x\,dx = \int_0^\infty yf(y)\left(\frac{1}{yr(y)}\int_0^y xG_1(x,y)\,dx\right)dy.$$

The function

$$\psi(y) := \frac{1}{yr(y)} \int_0^y x G_1(x, y)\, dx$$

is continuous, strictly positive and finite for all $y \in (0, \infty)$. Moreover, by l'Hospital's theorem

$$\lim_{y\to 0^+} \psi(y) = \lim_{y\to 0^+} \frac{y}{r(y) + y(1 + a(y))} = \lim_{y\to 0^+} \frac{1}{r(y)/y + 1 + a(y)} > 0$$

provided assumption (C.3) is satisfied and so the stated result follows. □

Theorem 4.3. *Let r and a satisfy assumptions* (C.1)–(C.3)*. Then the generator K of the substochastic semigroup $(G_K(t))_{t\geqslant 0}$ of Theorem* 4.1 *satisfies $K = \overline{A + B}$.*

Proof. Let $f \in \mathsf{F}_+$ satisfy the assumptions of Theorem 4.2, so that in particular $-f + \mathsf{BL}f \in X$. By Lemma 4.3 we see that $f \in L_1([0, \alpha], x\,dx)$ for any finite $\alpha > 0$ and consequently $\mathsf{BL}f \in L_1([0, \alpha], x\,dx)$ for any $\alpha \in (0, \infty)$. Hence

$$\int_0^\infty \bigl(-f(x) + (\mathsf{BL}f)(x)\bigr) x\, dx = \lim_{\alpha\to+\infty} \int_0^\alpha \bigl(-f(x) + (\mathsf{BL}f)(x)\bigr) x\, dx, \tag{4.10}$$

where

$$\int_0^\alpha \bigl(-f(x) + (\mathsf{BL}f)(x)\bigr) x\, dx = -\int_0^\alpha f(x) x\, dx + \int_0^\alpha (\mathsf{BL}f)(x) x\, dx.$$

In an analogous manner to L, the operator B is defined by the same formula as B and therefore, interchanging the order of integration and using (1.2), we obtain

$$\begin{aligned}\int_0^\alpha (\mathsf{BL}f)(x) x\, dx &= \int_0^\alpha \left(\int_x^\infty a(y) b(x|y) (\mathsf{L}f)(y)\, dy\right) x\, dx \\ &= \int_0^\alpha a(y)(\mathsf{L}f)(y) y\, dy - \int_0^\alpha a(y)(\mathsf{L}f)(y)\lambda(y) y\, dy + R_\alpha, \end{aligned}\tag{4.11}$$

where $R_\alpha := \int_\alpha^\infty \int_0^\alpha a(y) b(|y)(\mathsf{L}f)(y) x\, dx\, dy$. By assumption, $y\lambda a\mathsf{L}f$ is integrable so that

$$\lim_{\alpha\to\infty} \int_0^\alpha a(y)(\mathsf{L}f)(y)\lambda(y) y\, dy = \int_0^\infty a(y)(\mathsf{L}f)(y)\lambda(y) y\, dy < \infty, \tag{4.12}$$

and so we focus on the first integral in (4.11). Using (4.9) we obtain

$$\int_0^\alpha a(y)(\mathsf{L}f)(y)y\,dy = -\int_0^\alpha \left(\frac{e^{R(y)+Q(y)}}{r(y)}\int_y^\infty e^{-R(z)-Q(z)}f(z)\,dz\right)y\,dy + \int_0^\alpha \bigl(1+a(y)\bigr)\left(\frac{e^{R(y)+Q(y)}}{r(y)}\int_y^\infty e^{-R(z)-Q(z)}f(z)\,dz\right)y\,dy, \tag{4.13}$$

where clearly

$$\lim_{\alpha\to+\infty}\int_0^\alpha \left(\frac{e^{R(y)+Q(y)}}{r(y)}\int_y^\infty e^{-R(z)-Q(z)}f(z)\,dz\right)y\,dy = \int_0^\infty (\mathsf{L}f)(y)y\,dy < +\infty.$$

Interchanging the order of integration in the second integral in (4.13) yields

$$\int_0^\alpha e^{-R(z)-Q(z)}f(z)\left(\int_0^z \frac{(1+a(y))y}{r(y)}e^{R(y)+Q(y)}\,dy\right)dz + S_\alpha, \tag{4.14}$$

where

$$S_\alpha := \int_\alpha^\infty \int_0^\alpha \frac{(1+a(y))y}{r(y)} e^{R(y)+Q(y)-R(z)-Q(z)} f(z)\,dy\,dz.$$

Since

$$\int_0^z \frac{1+a(y)y}{r(y)}e^{R(y)+Q(y)}\,dy = \int_0^z y\frac{d}{dy}e^{R(y)+Q(y)}\,dy = ze^{R(z)+Q(z)} - \int_0^z e^{R(y)+Q(y)}\,dy,$$

it follows that

$$\int_0^\alpha e^{-R(z)-Q(z)}f(z)\left(\int_0^z \frac{(1+a(y))y}{r(y)}e^{R(y)+Q(y)}\,dy\right)dz = \int_0^\alpha f(z)z\,dz - \int_0^\alpha e^{-R(z)-Q(z)}f(z)\left(\int_0^z e^{R(y)+Q(y)}\,dy\right)dz. \tag{4.15}$$

Now, by Lemma 4.2 and Tonelli's theorem,

$$\lim_{\alpha\to\infty}\int_0^\alpha e^{-R(z)-Q(z)}f(z)\left(\int_0^z e^{R(y)+Q(y)}\,dy\right)dz$$

$$= \int_0^\infty e^{-R(z)-Q(z)} f(z) \left(\int_0^z e^{R(y)+Q(y)}\, dy \right) dz$$

$$= \int_0^\infty e^{R(y)+Q(y)} \left(\int_y^\infty e^{-R(z)-Q(z)} f(z)\, dz \right) dy = \int_0^\infty r(x)(\mathsf{L}f)(x)\, dx < \infty.$$

Substituting into (4.11) and using (4.4), we obtain

$$\int_0^\infty \big(-f(x) + (\mathsf{BL}f)(x)\big) x\, dx = \lim_{\alpha\to+\infty} (R_\alpha + S_\alpha) - \int_0^\infty (\mathsf{L}f)(y) y\, dy - c(\mathsf{L}f)$$

and so $R_\alpha + S_\alpha$ must have a finite nonnegative limit as $\alpha \to \infty$. Therefore

$$\int_0^\infty (\mathsf{L}f)(x) x\, dx + \int_0^\infty \big(-f(x) + (\mathsf{BL}f)(x)\big)\, dx \geqslant -c(\mathsf{L}f),$$

and the result follows from Theorem 4.2. □

Corollary 4.1. *Let r and a satisfy* (C.1)–(C.3). *Then, for any $f \in D(K)_+$,*

$$\frac{d}{dt}\big\|G_K(t)f\big\| = -\int_0^\infty r(x)\big[G_K(t)f\big](x)\, dx - \int_0^\infty \lambda(x)a(x)\big[G_K(t)f\big](x)x\, dx.$$

Proof. This follows from Theorem 2.2(c) and (4.4). □

To conclude, we consider the case when a and r do not satisfy (C.3) and show, by means of a simpler argument, that the generator K coincides with $A + B$ provided λ is suitably constrained.

Theorem 4.4. *Let a and r satisfy* (C.1)–(C.2) *and suppose that for some $\lambda_0 > 0$ we have $\lambda_0 \leqslant \lambda(y) \leqslant 1$ for all $y \geqslant 0$. Then*

$$D(K) = D(A) = D(A_0) \cap D(A_1). \tag{4.16}$$

Proof. Let $\mathsf{L}f = h$, where $h \in D(K)$. From Lemma 4.2 and the assumption on λ we see that $a\mathsf{L}f \in X$. Moreover, from Tonelli's theorem we obtain, as in (4.11),

$$\int_0^\infty (\mathsf{BL}f)(x)x\, dx = \int_0^\infty a(y)(\mathsf{L}f)(y)y\, dy - \int_0^\infty a(y)(\mathsf{L}f)(y)\lambda(y)y\, dy.$$

Therefore, $\mathsf{BL}f \in X$ which leads, via (4.8), to $f \in X$. If we now apply (2.8), then we obtain $h = \mathsf{L}f \in D(A)$ which yields the stated result. □

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