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Backward Φ-shifts and universality ☆

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Abstract

In this paper we consider spaces of sequences which are valued in a topological space E and study generalized backward shifts associated to certain selfmappings of E. We characterize their universality in terms of dynamical properties of the underlying selfmappings. Applications to hypercyclicity theory are given. In particular, Rolewicz's theorem on hypercyclicity of scalar multiples of the classical backward shift is extended.

Keywords: Sequence space; Universal mapping; Backward Φ-shift; Φ-product map; l_p spaces; Hypercyclic operator; Rolewicz's theorem

1. Introduction

In 1969 Rolewicz [26] was able to prove that for any scalar c with $|c| > 1$ (and only for these scalars) the multiple cB of the backward shift B on the sequence spaces l_p $(1 \leqslant p < \infty)$ and on c_0 is universal, that is, there exists some vector with dense orbit. The operator B is defined as

$$B(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots).$$

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Classical and weighted backward shift operators have been extensively studied during the last two decades in connection with hypercyclicity and chaos, see for instance [3,7,12, 14,15,18–22,27], and the references contained in them. Interest in shift operators comes, among other reasons, from the fact that many classical operators can be regarded as such operators. For instance, the differentiation operator $Df = f'$ on the space $H(\mathbb{C})$ of entire functions on the complex plane $\mathbb{C}$ may be viewed as the weighted backward shift

$$D:(a_0, a_1, a_2, \ldots) \mapsto (a_1, 2a_2, 3a_3, \ldots),$$

as soon as $H(\mathbb{C})$ is considered as the space of complex sequences $(a_0, a_1, \ldots)$ with $|a_n|^{1/n} \to 0$ $(n \to \infty)$. Here the sequence of weights is $1, 2, 3, \ldots$. In [15,18,21,27], among others, the universality of weighted backward shift operators acting on certain sequence spaces has been completely characterized. In particular, Grosse-Erdmann considers rather general sequence spaces in [15].

A. Peris [25] studied universality and chaos (in the sense of Devaney [9]; in usual settings, chaos is equivalent to universality plus existence of a dense set of periodic points, see [2]) of polynomials $P: l_q \to l_q$ given by $P(x_1, x_2, \ldots) = (p(x_2), p(x_3), \ldots)$, where $p: \mathbb{C} \to \mathbb{C}$ is a complex polynomial. This study was extended to Köthe sequence spaces in [20, Capítulo 4] for the cases $p(z) = z^m$ and $p(z) = (z+1)^m - 1$.

In this paper we are concerned with the dynamics of a class of (unweighted, this time) backward shift operators, namely, the Φ-shifts, which contains the previously cited cases. A Φ-shift is a map $B_f: S \to S$ given by

$$B_f(x_1, x_2, \ldots) := \big(f(x_2), f(x_3), \ldots\big),$$

where S is a certain subspace of $E^{\mathbb{N}}$, E is a topological space, and $f: E \to E$ is a continuous selfmap, see Section 3. We also consider the related notion of Φ-product map Π_f. Both kinds of maps will be completely characterized on l_p spaces. Our main goal is to characterize the "wild behavior" of a Φ-shift B_f in terms of the dynamics of f. This will be done in Section 4. Finally, we provide in Section 5 some applications to the theory of hypercyclic operators.

2. Universality, discrete dynamical systems and sequence spaces

The current section is devoted to fix some notation and to collect some definitions and known results coming from Topological Dynamics and from elementary theory of spaces of sequences. We refer the interested reader to the excellent surveys [8,14,16] for a summary of concepts, history and statements dealing with universality and hypercyclicity.

Assume that X and Y are topological spaces and that $T_n: X \to Y$ $(n \in \mathbb{N} := \{1, 2, 3, \ldots\})$ is a sequence of continuous mappings. Then (T_n) is said to be *universal* whenever there exists some element $x \in X$ whose orbit

$$\{T_n x:\ n \in \mathbb{N}\}$$

under (T_n) is dense in Y. In this case x is called a universal element for (T_n). Observe that the universality of (T_n) forces Y to be separable. The sequence (T_n) is called *densely universal* when the set $U((T_n))$ of universal elements for (T_n) is dense in X. Finally, (T_n) is said to be *topologically transitive* (in the sense of Birkhoff) provided that to every pair of nonempty open subsets U of X and V of Y there exists some $n \in \mathbb{N}$ with $T_n(U) \cap V \neq \emptyset$.

Assume now that $X = Y$ and that $T : X \to X$ is a continuous selfmapping. From the point of view of the behaviour of the sequence $(T^{\circ n})$ of its iterates—that is $T^{\circ 1} = T$, $T^{\circ 2} = T \circ T, \ldots$—$T$ can be considered as a "discrete dynamical system." Then T is called *universal* whenever the sequence $(T^{\circ n})$ is universal; in this case the set $U(T) := U((T^{\circ n}))$ of universal elements for T is dense in X; indeed, if x_0 is universal for T then T (and so each $T^{\circ m}$) has dense range, hence each point $T^{\circ m}x_0$ is universal. A continuous selfmapping T is said to be *topologically transitive* whenever $(T^{\circ n})$ is topologically transitive. Finally, T is called *weakly mixing* provided that the mapping

$$T \times T : (x, y) \in X \times X \mapsto (Tx, Ty) \in X \times X$$

is topologically transitive, where $X \times X$ is assumed to carry the product topology. Furstenberg [11, Proposition II.3] proved that in this case the J-product map $T \times T \times \cdots \times T$ (J times): $(x_1, \ldots, x_J) \in X^J \mapsto (Tx_1, \ldots, Tx_J) \in X^J$ is also transitive for every J. And Banks [1, Lemma 5] has shown that T is weakly mixing if for given nonempty open subsets $U_1, U_2, V \subset X$ there is an $N \in \mathbb{N}$ such that $T^{\circ N}(V) \cap U_j \neq \emptyset$ for $j = 1, 2$. Therefore we have that T is weakly mixing if and only if for given finitely many nonempty open subsets $U_1, \ldots, U_J, V \subset X$ there is an $N \in \mathbb{N}$ such that $T^{\circ N}(V) \cap U_j \neq \emptyset$ for $j = 1, \ldots, J$. This motivates the following definition, which will be used in Section 4.

Definition 2.1. Assume that $f_n : E \to E$ ($n \in \mathbb{N}$) is a sequence of continuous selfmappings on a topological space E and that a is a point in E. We say that (f_n) is *weakly mixing at a* if and only if for given finitely many nonempty open sets $U_1, \ldots, U_m, V$ such that $a \in V$ there exists $N \in \mathbb{N}$ satisfying

$$f_N(V) \cap U_j \neq \emptyset \quad \text{for all } j = 1, \ldots, m.$$

And we say that a continuous selfmapping $f : E \to E$ is *weakly mixing at a* whenever its sequence $(f^{\circ n})$ of iterates is weakly mixing at a.

The following universality criterion will reveal useful in Section 4. It can be found in, for instance, [14, Section 1a] (see also [13, Kapitel 1]).

Theorem 2.1. *Suppose that X, Y are topological spaces, in such a way that X is a Baire space and Y is second-countable. Let (T_n) be a sequence of continuous mappings from X to Y. Then the following assertions are equivalent*:

(a) *The sequence (T_n) is densely universal.*
(b) *The sequence (T_n) is topologically transitive.*

In a linear setting, that is, when X, Y are topological vector spaces on $\mathbb{K}$ ($:= \mathbb{C}$ or the real line $\mathbb{R}$) and T_n ($n \in \mathbb{N}$) (or T) are linear and continuous, the words *universal* and *hypercyclic* are synonymous. By an *operator* we mean a continuous linear selfmapping on a topological vector space.

By ω we denote, as usual, the space of all scalar sequences $\omega = \mathbb{K}^{\mathbb{N}}$. It becomes a Fréchet space (= complete metrizable locally convex space) when it is endowed with the metric

$$d(x, y) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|x_j - y_j|}{1 + |x_j - y_j|},$$

where $x = (x_j)$ and $y = (y_j)$. For $0 < p < \infty$ we consider the l_p spaces

$$l_p = \left\{ x = (x_j) \in \omega\colon \sum_{j=1}^{\infty} |x_j|^p < \infty \right\}.$$

If $\mu(p) = p$ $(p \geqslant 1)$, $\mu(p) = 1$ $(p < 1)$ and $\|x\|_p := (\sum_{j=1}^{\infty} |x_j|^p)^{1/\mu(p)}$, then for $p \geqslant 1$ the space l_p becomes a Banach space under the norm $\|\cdot\|_p$, while for $p < 1$ the space l_p is an F-space (= complete metrizable linear space) under the metric $d(x, y) = \|x - y\|_p$. Recall also that the space $c_0 = \{x = (x_j) \in \omega\colon \lim_{j\to\infty} x_j = 0\}$ is a Banach space when it is endowed with the norm $\|x\|_0 := \sup_{j\in\mathbb{N}} |x_j|$. Generalizations of this kind of sequences spaces will be considered in Section 3.

3. Φ-product maps and backward Φ-shifts

In this section and in the next one E will denote a Hausdorff topological space, and S will stand for a subset of the space $E^{\mathbb{N}}$ of E-valued sequences $x = (x_j)$, so $x_j \in E$ for all $j \in \mathbb{N}$. For every $a \in E$ we denote by $\sigma(a)$ the set of sequences ending with a, that is, $\sigma(a) = \bigcup_{J=1}^{\infty} \sigma_J(a)$, where $\sigma_J(a) = \{x = (x_j) \in E^{\mathbb{N}}\colon\ x_j = a$ for all $j > J\}$. From now on, we will assume that S is a standard sequence space in the sense established by the next new concept.

Definition 3.1. We define a *standard sequence space* (SSS) on E as a subset $S \subset E^{\mathbb{N}}$ endowed with a topology such that there exists a point $a \in E$ satisfying the following four properties:

(S1) The space S is Baire and second-countable.
(S2) The topology on S is stronger than that inherited from the product topology on $E^{\mathbb{N}}$.
(S3) The set $\sigma(a)$ is a dense subset of S.
(S4) For each $J \in \mathbb{N}$, the topology of each $\sigma_J(a)$ inherited from S is the product topology.

If S is a SSS and $a \in E$ is a point satisfying (S3)–(S4), then we will say a is a *distinguished* point for S. If E is a Hausdorff topological vector space, then a topological vector space $S \subset E^{\mathbb{N}}$ is called a *linear standard sequence space* on E whenever it satisfies (S1)–(S4) for the point $a = 0$.

Sometimes we will also consider the following property:

(S4*) Given $J \in \mathbb{N}$, an open set $U \subset S$ and a point $\alpha \in \sigma_J(a) \cap U$, there exist open sets $U_1, \dots, U_J, A$ in E such that $\alpha \in \prod_{j=1}^{\infty} A_j^{(N)} \subset U$ for all $N > J$, where

$$A_j^{(N)} = \begin{cases} U_j & (1 \leqslant j \leqslant J), \\ A & (N+1 \leqslant j \leqslant N+J), \\ \{a\} & (J < j \leqslant N \text{ or } j > N+J). \end{cases} \tag{1}$$

Remark 3.1. Due to the presence of $U_1, \dots, U_J$, (S4*) implies (S4). In (S4*), the existence of "sliding J-wagon trains" $A \times \cdots \times A$ in the projections of every neighborhood of each point of $\sigma_J(a)$ reveals some "indifference" among the coordinates of the elements of S when they are close to a. On the other hand, (S4) implies that for every $j \in \mathbb{N}$ the immersion $i_j : t \in E \mapsto (a, a, \dots, a, t, a, a, \dots) \in S$ (where t occurs at the jth place) is continuous. Of course, (S2) tells us that convergence in S implies coordinatewise convergence.

Examples 3.2.

(1) The spaces ω, l_p $(0 < p < \infty)$ and c_0 are linear SSSs: suffice it to take $E = \mathbb{K}$ with the usual topology. Property (S1) for these spaces is easily checked just by taking into account that a completely metrizable separable space is Baire and second-countable. The remaining conditions are straightforward. A different example is the space $S = \{x = (x_j) \in \mathbb{K}^{\mathbb{N}}\colon \lim_{j\to\infty} x_{2j-1} = 0 \text{ and } \sum_{j=1}^{\infty} |x_{2j}| < \infty\}$, which becomes a Banach space under the norm $\|x\| = \sup_{j \geqslant 1} |x_{2j-1}| + \sum_{j=1}^{\infty} |x_{2j}|$. All these spaces also satisfy (S4*).

(2) The direct sum $S = \bigoplus_{n \in \mathbb{N}} \mathbb{K}$ of countably many lines endowed with the inductive limit locally convex topology is a second-countable topological vector space satisfying (S2) to (S4) for $E = \mathbb{K}$ and $a = 0$, but it is not a linear SSS since S is not a Baire space.

(3) Let $S = \{x = (x_j) \in \mathbb{R}^{\mathbb{N}}\colon \lim_{j\to\infty} j x_j = 0\}$. Then S becomes a separable Banach space when it is endowed with the norm $\|x\| = \sup_{j \in \mathbb{N}} |j x_j|$. Then S satisfies (S1) to (S4) (for $a = 0$; no other point $a \in \mathbb{R}$ is possible), so it is a linear SSS. But (S4*) fails because given a ball $U = \{\|x\| < \varepsilon\}$ then we have $\operatorname{diam}(\pi_j(U)) \leqslant 2\varepsilon/j \to 0$ $(j \to \infty)$, see Remark 3.1. Here, as usual, π_j denotes the j-projection $\pi_j : x = (x_n) \in S \mapsto x_j \in E$ $(j \in \mathbb{N})$.

Another family of examples of linear SSSs which are extensions of ω, c_0, l_p is described as follows. Assume that E is a separable Banach space over $\mathbb{R}$ or $\mathbb{C}$ with norm $\|\cdot\|$. Consider the spaces of E-valued sequences $\omega(E)$, $c_0(E)$, $l_p(E)$ $(0 < p < \infty)$. They are defined as the former spaces just by replacing $\mathbb{K}$ with E, and the absolute value with $\|\cdot\|$. As a matter of fact, in the case $\omega(E)$ it is enough to assume that E is a completely metrizable separable topological space.

Now we are going to motivate the new concepts provided in Definition 3.2, see below. As seen in Section 1, Rolewicz [26] proved the universality of the backward shift $cB : (x_j) \mapsto (c x_{j+1})$ $(|c| > 1)$ on c_0 and l_p $(1 \leqslant p < \infty)$, while Grosse-Erdmann [15, Corollary 2] noted that any weighted backward shift (in particular, B itself) is universal, even chaotic, on ω. Martínez and Peris [21] have recently studied backward shifts on

Köthe echelon spaces $c_0(A)$, $\lambda_p(A)$ $(1 \leqslant p < \infty)$ (see [17] and [23] for definitions and properties; they are separable Fréchet spaces and include c_0, l_p $(1 \leqslant p < \infty)$ for adequate matrices A), and in particular they characterize the universality of B in terms of the matrix A [21, Proposition 3.1]. On the other hand, Bernardes [6] showed that for given $m > 1$ there is no universal m-homogeneous continuous polynomial on any Banach space; in particular, the shift $(x_j) \mapsto (x_{j+1}^m)$ is not universal on l_p $(1 \leqslant p < \infty)$ or c_0. However, the last mapping is universal, even chaotic, on the Fréchet space $\omega = \mathbb{C}^{\mathbb{N}}$ [24] and in fact on some (non-Banach, of course) Köthe spaces $\lambda_p(A)$ [20]. Furthermore, Peris [25] proves that the (nonhomogeneous) polynomial $(z_j) \mapsto ((z_j + 1)^m - 1)$ is universal (and chaotic) on the complex Banach spaces l_p $(1 \leqslant p < \infty)$ and c_0. This is again true on certain spaces $\lambda_p(A)$, see [20].

Definition 3.2. Suppose that S is a SSS and that $T : S \to S$ is a continuous selfmapping on S.

(a) We say that T is a *Φ-product map* on S if there exists a selfmapping $f : E \to E$ such that $T = \Pi_f$ on S, where $\Pi_f x = (f(x_j))$ for every $x = (x_j) \in E^{\mathbb{N}}$.
(b) We say that T is a *backward Φ-shift* on S if there exists a selfmapping $f : E \to E$ such that $T = B_f$ on S, where $B_f x = (f(x_{j+1}))$ for every $x = (x_j) \in E^{\mathbb{N}}$.

Remarks 3.3.

(1) Observe that $B_f = \Pi_f \circ B = B \circ \Pi_f$, where B is the ordinary backward shift, i.e., $Bx = (x_{j+1})$ for $x = (x_j)$. Of course, if g is the identity on E then $\Pi_g =$ the identity on $E^{\mathbb{N}}$ and $B_g = B$. Note also that if either Π_f is a Φ-product map or B_f is a backward Φ-shift on S then f is *continuous*. Indeed, $f = \pi_1 \circ \Pi_f \circ i_1 = \pi_1 \circ B_f \circ i_2$, where i_1, i_2 are the 1- and 2-immersions (see Remark 3.1)—which are well-defined by (S3) and continuous due to (S4)—and π_1 is the 1-projection, which is continuous by (S2).
(2) Note that even in the case of a linear SSS S on E the mapping $f : E \to E$ may be nonlinear, so both Π_f and B_f may well be nonlinear.

It is interesting to obtain necessary and sufficient conditions for Π_f (B_f, respectively) to be well-defined on a SSS S (that is, for S to be Π_f-invariant: $\Pi_f(S) \subset S$, or respectively, B_f-invariant: $B_f(S) \subset S$) and to be a Φ-product map or a Φ-shift (that is, continuous). This will be carried out at least for the most usual spaces $\omega(E)$, $l_p(E)$ $(0 < p < \infty)$, $c_0(E)$. Recall that the continuity of f is a general necessary condition for the continuity of B_f. In connection with this, we remark that Wildenberg [28] discovered in 1988 the absence of nontrivial functions $f : \mathbb{R} \to \mathbb{R}$ for which a sequence (x_j) in $\mathbb{R}^{\mathbb{N}}$ is summable only if $(f(x_j))$ $(= \Pi_f(x_j)$, in our terminology) is summable. Specifically, he stated that the last property holds for any sequence (x_j) if and only if there exists a constant k with $f(t) = kt$ in a neighborhood of the origin. Of course, this is not necessary for $\Pi_f(l_1) \subset l_1$: take, for instance, $f(t) = t^2$. In fact, we will use an approach similar to [28, Lemma 1] in order to obtain the Π_f-invariance of the spaces $l_p(E)$.

Lemma 3.4. *Let X be a topological space. Assume that a is a point in X and that $\varphi : X \to [0,\infty)$ is a function such that*

(i) *There is a countable basis of neighborhoods for a, and a is not an isolated point in X.*
(ii) $\lim_{x\to a}\varphi(x) = 0$ *and* $\varphi(x) > 0$ *for all* $x \in X \setminus \{a\}$.
(iii) *If* $(x_j) \in X^{\mathbb{N}}$ *and* $\lim_{j\to\infty}\varphi(x_j) = 0$, *then* $\lim_{j\to\infty} x_j = a$.

Let us suppose that $f : X \to X$. Then the series $\sum_{j=1}^{\infty}\varphi(f(x_j))$ is convergent for every convergent series $\sum_{j=1}^{\infty}\varphi(x_j)$ if and only if there exists a constant $M \in (0,\infty)$ such that

$$\varphi\big(f(x)\big) \leqslant M\varphi(x) \tag{2}$$

on some neighborhood of a.

Proof. Assume first that (2) holds for some $M \in (0,\infty)$ and some neighborhood U of a. If $(x_j) \in X^{\mathbb{N}}$ and $\sum_{j=1}^{\infty}\varphi(x_j)$ converges then $\varphi(x_j) \to 0$ as $j \to \infty$, so $x_j \to a$ by (iii). Therefore there is $J \in \mathbb{N}$ such that $x_j \in U$ for all $j > J$. Hence $\varphi(f(x_j)) \leqslant M\varphi(x_j)$ $(j > J)$, so $\sum_{j=1}^{\infty}\varphi(f(x_j))$ converges due to the comparison criterion.

As for the converse, suppose, by way of contradiction, that there is not any constant M satisfying (2) on some neighborhood of a. Let $\{U_n:\ n \in \mathbb{N}\}$ be a decreasing basic sequence of neighborhoods of a. From (i) and (ii) there is a sequence $\{n(1) < n(2) < \cdots\} \subset \mathbb{N}$ and points $x_j \in U_{n(j)} \setminus \{a\}$ such that

$$0 < \varphi(x_j) < 1/j^2 \quad \text{and} \quad \varphi\big(f(x_j)\big) > j\varphi(x_j) \quad (j \in \mathbb{N}).$$

Define $N(j)$ to be the least integer that is $\geqslant 1/(j^2\varphi(x_j))$. Then $N(j) - 1 < 1/(j^2\varphi(x_j))$, so $N(j)\varphi(x_j) < (1/j^2) + \varphi(x_j) < 2/j^2$. Now consider the sequence

$$(y_j) = (x_1, \ldots, x_1, x_2, \ldots, x_2, x_3, \ldots, x_3, \ldots),$$

where each x_j occurs $N(j)$ times. We have that

$$\sum_{j=1}^{\infty}\varphi(y_j) = \sum_{j=1}^{\infty}N(j)\varphi(x_j) < \sum_{j=1}^{\infty}\frac{2}{j^2} < \infty.$$

But

$$\sum_{j=1}^{\infty}\varphi\big(f(y_j)\big) = \sum_{j=1}^{\infty}N(j)\varphi\big(f(x_j)\big) \geqslant \sum_{j=1}^{\infty}\frac{1}{j^2\varphi(x_j)}j\varphi(x_j) > \sum_{j=1}^{\infty}\frac{1}{j},$$

and the last series diverges. This contradiction proves the lemma. □

We are now ready to specify exactly what Φ-shifts and what Φ-product maps are well-defined operators on $\omega(E)$, $c_0(E)$ and $l_p(E)$. In the following result we are assuming that E is a completely metrizable separable topological space in the case $\omega(E)$, while E is a separable Banach space in the cases $c_0(E)$, $l_p(E)$.

Theorem 3.5. *Let $f : E \to E$ be a selfmapping on E, and $p \in (0,\infty)$.*

(1) *The following properties are equivalent*:
   (i) Π_f *is* Φ*-product map on* $\omega(E)$.
   (ii) B_f *is a backward* Φ*-shift on* $\omega(E)$.
   (iii) f *is continuous.*
(2) *The following properties are equivalent*:
   (i) *The space* $c_0(E)$ *is* Π_f*-invariant.*
   (ii) *The space* $c_0(E)$ *is* B_f*-invariant.*
   (iii) $f(0) = 0$ *and* f *is continuous at the origin.*
(3) *The following properties are equivalent*:
   (i) Π_f *is a* Φ*-product map on* $c_0(E)$.
   (ii) B_f *is a backward* Φ*-shift on* $c_0(E)$.
   (iii) f *is continuous and* $f(0) = 0$.
(4) *The following properties are equivalent*:
   (i) *The space* $l_p(E)$ *is* Π_f*-invariant.*
   (ii) *The space* $l_p(E)$ *is* B_f*-invariant.*
   (iii) $\limsup_{t\to 0} \frac{\|f(t)\|}{\|t\|} < \infty$.
(5) *The following properties are equivalent*:
   (i) Π_f *is a* Φ*-product map on* $l_p(E)$.
   (ii) B_f *is a backward* Φ*-shift on* $l_p(E)$.
   (iii) f *is continuous and* $\limsup_{t\to 0} \frac{\|f(t)\|}{\|t\|} < \infty$.

In particular, B_f *is a backward* Φ*-shift on* $l_p(E)$ *if* f *is an operator on* E.

Proof. The ordinary backward shift B is, trivially, a well-defined continuous selfmapping on S for each of the spaces $S = \omega(E)$, $c_0(E)$, $l_p(E)$. Then the implication (i) $\Rightarrow$ (ii) in all parts (1)–(5) is evident due to the fact $B_f = B \circ \Pi_f$. On the other hand, the sentence "f is continuous" appearing in each part (iii) of (1), (3) and (5) follows from the corresponding part (ii) together with Remark 3.3(1). The additional properties "$f(0) = 0$" and "$\limsup_{t\to 0} \frac{\|f(t)\|}{\|t\|} < \infty$" respectively in (3)(iii) and (5)(iii) follow from parts (2), (4). Thus, we will be done as soon as we prove the following implications.

(iii) $\Rightarrow$ (i) of (1): It is evident because Π_f is continuous if and only if $\pi_j \circ \Pi_f$ is continuous for each projection $\pi_j : \omega(E) \to E$; but $\pi_j \circ \Pi_f = f$ for every $j \in \mathbb{N}$.

(ii) $\Rightarrow$ (iii) of (2): Given a sequence $(x_j) \in c_0(E)$ we must have $f(x_{j+1}) \in c_0(E)$, that is, $f(x_{j+1}) \to 0$ $(j \to \infty)$ or, that is the same, $f(x_j) \to 0$ $(j \to \infty)$. Since E is first-countable this tells us that f is continuous at the origin and $f(0) = 0$.

(iii) $\Rightarrow$ (i) of (2): Use again that (iii) is equivalent to "$(f(x_j))$ tends to zero for every sequence (x_j) tending to zero."

(iii) $\Rightarrow$ (i) of (3): Fix $\alpha = (a_j) \in c_0(E)$ and $\varepsilon > 0$. Since $f(0) = 0$ and f is continuous at the origin, there is $\delta_0 > 0$ such that $\|f(t)\| < \varepsilon/2$ if $\|t\| < \delta_0$. On the other hand, from the fact $a_j \to 0$ $(j \to \infty)$ we deduce the existence of some $J \in \mathbb{N}$ with $\|a_j\| < \delta_0/2$ for all $j > J$, hence $\|f(a_j)\| < \varepsilon/2$ $(j > J)$. Now, from the continuity of f at $a_1, \ldots, a_J$ one gets $\delta_j > 0$ $(j \in \{1, \ldots, J\})$ such that $\|f(t) - f(a_j)\| < \varepsilon$ whenever $t \in E$ and $\|t - a_j\| < \delta_j$. Let us choose $\delta := \min\{\delta_0/2, \delta_1, \ldots, \delta_J\}$. Then $\delta > 0$. Suppose that $x = (x_j)$ in $c_0(E)$ and $\|x - \alpha\|_0 < \delta$. If $j \in \{1, \ldots, J\}$, then $\|x_j - a_j\| < \delta_j$, therefore $\|f(x_j) - f(a_j)\| < \varepsilon$.

Finally, for $j > J$, we have $\|x_j - a_j\| < \delta_0/2$, so $\|x_j\| \leqslant \|x_j - a_j\| + \|a_j\| < \delta_0$. Hence $\|f(x_j)\| < \varepsilon/2$ and

$$\|f(x_j) - f(a_j)\| \leqslant \|f(x_j)\| + \|f(a_j)\| < \varepsilon \quad (j > J).$$

Thus,

$$\|\Pi_f(x) - \Pi_f(\alpha)\|_0 = \sup_{j\in\mathbb{N}} \|f(x_j) - f(a_j)\| \leqslant \varepsilon,$$

which proves the continuity of f at α. But α was arbitrary, so the implication is proved.

(ii) $\Rightarrow$ (iii) of (4): Apply Lemma 3.4 on $X = E$, $a = 0$, $\varphi(x) = \|x\|^p$. Take into account that, trivially, $\sum_{j=1}^{\infty} \varphi(f(x_j))$ converges if and only if $\sum_{j=1}^{\infty} \varphi(f(x_{j+1}))$ does.

(iii) $\Rightarrow$ (i) of (4): Apply again Lemma 3.4.

(iii) $\Rightarrow$ (i) of (5): By the "limsup" condition and (4), Π_f is a selfmapping on $l_p(E)$. As for the continuity, fix $\alpha = (a_j) \in l_p(E)$ and $\varepsilon > 0$. By hypothesis, there are $M, \delta_0 \in (0,\infty)$ such that $\|f(t)\| \leqslant M\|t\|$ whenever $\|t\| < \delta_0$. Since $a_j \to 0$ as $j \to \infty$, we can find $J \in \mathbb{N}$ with $\|a_j\| < \delta_0/2$ for all $j > J$, so $\|f(a_j)\| \leqslant M\|a_j\|$ $(j > J)$. The number J can be chosen such that, in addition,

$$\sum_{j=J+1}^{\infty} \|a_j\|^p < \frac{\varepsilon^{\mu(p)}}{3M^p 2^p (1+2^p)}.$$

Due to the continuity of f at each a_j, it is possible to find $\delta_j > 0$ satisfying that

$$\|f(t) - f(a_j)\| < \left(\frac{\varepsilon^{\mu(p)}}{3J}\right)^{1/p} \quad \text{whenever } \|t - a_j\| < \delta_j.$$

Now, let us choose

$$\delta := \min\left\{\left(\frac{\delta_0}{2}\right)^{p/\mu(p)}, \delta_1^{p/\mu(p)}, \delta_2^{p/\mu(p)}, \dots, \delta_J^{p/\mu(p)}, \frac{\varepsilon}{(3M^p 4^p)^{1/\mu(p)}}\right\} > 0.$$

Assume that $x = (x_j) \in l_p(E)$ and $\|x - \alpha\|_p < \delta$. Then $\|x_j - a_j\| < \delta_0/2$, so $\|x_j\| < \|a_j\| + (\delta_0/2)$, whence $\|x_j\| < \delta_0$ for all $j > J$ and, consequently, $\|f(x_j)\| \leqslant M\|x_j\|$ $(j > J)$. In addition, $\|x_j - a_j\| < \delta_j$ $(j \in \{1,\dots,J\})$, therefore $\|f(x_j) - f(a_j)\|^p < \varepsilon^{\mu(p)}/3J$ $(1 \leqslant j \leqslant J)$. We recall that $(b+c)^p \leqslant 2^p(b^p + c^p)$ for all $p \in (0,\infty)$ and all $b, c \geqslant 0$, see for instance [10, p. 57]. Finally, we estimate:

$$\begin{aligned}
&\|\Pi_f(x) - \Pi_f(\alpha)\|_p^{\mu(p)} \\
&\quad = \sum_{j=1}^{\infty} \|f(x_j) - f(a_j)\|^p \\
&\quad \leqslant \sum_{j=1}^{J} \|f(x_j) - f(a_j)\|^p + \sum_{j=J+1}^{\infty} \big(\|f(x_j)\| + \|f(a_j)\|\big)^p \\
&\quad \leqslant \sum_{j=1}^{J} \|f(x_j) - f(a_j)\|^p + 2^p \sum_{j=J+1}^{\infty} \|f(x_j)\|^p + 2^p \sum_{j=J+1}^{\infty} \|f(a_j)\|^p
\end{aligned}$$

$$\begin{aligned}
&\leqslant \sum_{j=1}^{J} \bigl\| f(x_j) - f(a_j) \bigr\|^p + 2^p M^p \sum_{j=J+1}^{\infty} \|x_j\|^p + 2^p M^p \sum_{j=J+1}^{\infty} \|a_j\|^p \\
&\leqslant \sum_{j=1}^{J} \bigl\| f(x_j) - f(a_j) \bigr\|^p + 2^p M^p \sum_{j=J+1}^{\infty} \bigl(2^p \|x_j - a_j\|^p + 2^p \|a_j\|^p\bigr) \\
&\quad + 2^p M^p \sum_{j=J+1}^{\infty} \|a_j\|^p \\
&\leqslant \sum_{j=1}^{J} \bigl\| f(x_j) - f(a_j) \bigr\|^p + 4^p M^p \|x - \alpha\|_p^{\mu(p)} + 2^p \bigl(1 + 2^p\bigr) M^p \sum_{j=J+1}^{\infty} \|a_j\|^p \\
&< J \cdot \frac{\varepsilon^{\mu(p)}}{3J} + 4^p M^p \cdot \frac{\varepsilon^{\mu(p)}}{3M^p 4^p} + 2^p \bigl(1 + 2^p\bigr) M^p \cdot \frac{\varepsilon^{\mu(p)}}{3M^p 2^p (1 + 2^p)} = \varepsilon^{\mu(p)}.
\end{aligned}$$

Thus, $\|\Pi_f(x) - \Pi_f(\alpha)\|_p < \varepsilon$ whenever $\|x - \alpha\|_p < \delta$, which establishes the continuity of Π_f. □

4. Universality of Φ-product maps and of Φ-shifts

This section is devoted to providing necessary conditions and sufficient conditions for a Φ-product map Π_f or a backward Φ-shift B_f to be universal on a given SSS. Such conditions will be expressed in terms of the dynamical properties of the underlying function f, that is, in terms of the behavior of its sequence $(f^{\circ n})$ of iterates.

To start with, we establish a necessary condition for universality on $l_p(E)$ and $c_0(E)$. Recall that "$\limsup_{t \to 0} \frac{\|f(t)\|}{\|t\|} < \infty$" is necessary for Π_f and B_f to be well-defined operators on $l_p(E)$. The point is that such limsup must not be too small.

Proposition 4.1. *Assume that E is a separable Banach space, that $p \in (0, \infty)$, and that $f : E \to E$ is continuous and satisfies $\limsup_{t \to 0} \frac{\|f(t)\|}{\|t\|} < \infty$ (and satisfies $f(0) = 0$, respectively). If either Π_f or B_f is universal on $l_p(E)$ (on $c_0(E)$, respectively), then $\limsup_{t \to 0} \frac{\|f(t)\|}{\|t\|} \geqslant 1$.*

Proof. If we follow a way of contradiction and assume that $\limsup_{t \to 0} \|f(t)\|/\|t\| < 1$, then we obtain easily that for every vector $x = (x_j)$ in some ball centered at the origin of S, the orbit of x under Π_f or B_f is bounded, so nondense. This is a contradiction because the set $U(T)$ of universal elements of a continuous selfmapping T is dense if T is universal. The details are left to the interested reader. □

Observe that as a consequence of Proposition 4.1 if $f : E \to E$ is an operator on a separable Banach space E and B_f is hypercyclic on $l_p(E)$ $(1 \leqslant p < \infty)$ or on $c_0(E)$ then $\|f\| \geqslant 1$. In fact, we can say a little more: it must be $\|f\| > 1$ because $\|\Pi_f\| = \|B_f\| = \|f\|$ and a hypercyclic operator on a normed space cannot be nonexpansive.

We saw in Section 3 that the continuity of f is a necessary condition for B_f to be a backward Φ-shift on a SSS. We establish in the next result that the denseness of the range of f is necessary for the universality of B_f. This condition becomes sharp for the largest space $\omega(E)$. As for Π_f, the universality of f itself is necessary.

Theorem 4.2. *Suppose that $f_n : E \to E$ $(n \in \mathbb{N})$ and $f : E \to E$ are continuous and that S is a* SSS *on E. We have*:

(a) *If $B_f : S \to S$ is a universal backward Φ-shift, then f has dense range.*
(b) *If E is a completely metrizable separable topological space and f has dense range, then $B_f : \omega(E) \to \omega(E)$ is a universal backward Φ-shift.*
(c) *If (Π_{f_n}) is a universal sequence of Φ-product maps on S, then the sequence (f_n) is universal on E. In particular, if $\Pi_f : S \to S$ is a universal Φ-product map, then f is universal on E.*

Proof. (a) Assume that B_f is universal on S, and fix a point $y \in E$. From (S3), the point $y = (y, a, a, a, \ldots)$ is in S for some $a \in E$. By universality, there is a point $(x_j) \in S$ and a sequence (n_k) of positive integers such that the sequence $((f^{\circ n_k}(x_{j+n_k}))_{j \in \mathbb{N}})$ converges to $(y, a, a, \ldots)$ in S as $k \to \infty$. From (S2), $f^{\circ n_k}(x_{1+n_k}) \to y$, hence $f(f^{\circ n_k - 1}(x_{1+n_k})) \to y$, which proves that y is in the closure of $f(E)$. But y was arbitrary, so $f(E)$ is dense.

(b) From Theorem 3.5, B_f is in fact a backward Φ-shift on $\omega(E)$. Recall that $\omega(E)$ is a second-countable Baire space. In order to apply Theorem 2.1, take $X := \omega(E) =: Y$, $T_n := B_f^n$. Let us try to check the Birkhoff transitivity property. Fix nonempty open subsets U, V of $\omega(E)$. Then there exist $J \in \mathbb{N}$ and nonempty open subsets $U_1, \ldots, U_J, V_1, \ldots, V_J$ in E such that

$$U_1 \times \cdots \times U_J \times E \times E \times \cdots \subset U \quad \text{and} \quad V_1 \times \cdots \times V_J \times E \times E \times \cdots \subset V.$$

Since f has dense range and is continuous, we have that $f^{\circ J}$ has also dense range. From this, we derive the existence of points $t_1, \ldots, t_J \in E$ such that $f^{\circ J}(t_j) \in V_j$ $(j = 1, \ldots, J)$. Choose any points $y_j \in U_j$ $(j = 1, \ldots, J)$ and any point $t \in E$. Consider the sequence $x = (x_j) \in \omega(E)$ defined as

$$x_j = \begin{cases} y_j & (1 \leqslant j \leqslant J), \\ t_{j-J} & (J < j \leqslant 2J), \\ t & (j > 2J). \end{cases}$$

It is clear that $x \in U$. Finally, $T_J x = B_f^J x = (f^{\circ J}(x_{j+J}))$, but $f^{\circ J}(x_{j+J}) = f^{\circ J}(t_j) \in V_j$ for $j = 1, \ldots, J$, so

$$T_J x \in V_1 \times \cdots \times V_J \times E \times E \times \cdots \subset V.$$

Consequently, $T_J(U) \cap V \neq \emptyset$, as required.

(c) This is due to the following facts: the projection π_1 is continuous and surjective (by (S3)), $\pi_1 \circ \Pi_{f_n} = f_n \circ \pi_1$ and $(\Pi_f)^{\circ n} = \Pi_{f^{\circ n}}$ for all $n \in \mathbb{N}$. $\square$

Now, we focus our attention on the searching of conditions on the function f that guarantee the universality of B_f and Π_f on general SSSs. A local weakly mixing condition will be imposed in the following theorem on a sequence of selfmappings fixing the distinguished point.

Theorem 4.3. *Assume that $f_n : E \to E$ $(n \in \mathbb{N})$ is a sequence of selfmappings for which the mappings $T_n : x = (x_j) \in S \mapsto (f_n(x_{j+n})) \in S$ are well-defined and continuous, where S is a* SSS *on E satisfying* (S4*) *for some distinguished point $a \in E$. Suppose that $f_n(a) = a$ for all $n \in \mathbb{N}$ and that (f_n) is weakly mixing at a. Then (T_n) is densely universal.*

Proof. Observe first that, in a similar way to the case of B_f, every f_n must be continuous, see Remark 3.3(1).

Our aim is to apply Theorem 2.1. Choose $X := S =: Y$. Observe that X is Baire and that Y is second-countable by (S1). Consequently, our goal is, given a pair of nonempty open sets U, V of S, to find a sequence $x = (x_j) \in U$ and a positive integer N such that $T_N x \in V$. From (S3), $U \cap \sigma(a) \neq \emptyset \neq V \cap \sigma(a)$. Therefore there exist $J \in \mathbb{N}$ and points $a_1, \dots, a_J, b_1, \dots, b_J \in E$ with $\alpha \in U$ and $\beta \in V$, where

$$\alpha := (a_1, \dots, a_J, a, a, a, \dots) \quad \text{and} \quad \beta := (b_1, \dots, b_J, a, a, a, \dots).$$

First of all, let us prove the following claim: There are in fact *infinitely many* $N \in \mathbb{N}$ with $f_N(A) \cap U_j \neq \emptyset$ for all $j = 1, \dots, m$, where $A, U_1, \dots, U_m$ are prescribed nonempty open sets with $a \in A$. Indeed, choose $N_1 \in \mathbb{N}$ such that each $f_{N_1}(A) \cap U_j$ $(j \in \{1, \dots, m\})$ is not empty. If E has only one point, namely a, then the claim is trivial. If E has at least two points then, since E is Hausdorff, there are $b \in E$ and open subsets A_0, $B \subset E$ with $a \in A_0$, $b \in B$ and $A_0 \cap B = \emptyset$. Recall that each f_n is continuous. Hence there exist open subsets A_n $(n = 1, \dots, N_1)$ in E with $a \in A_n$ and $A_n \subset A$ such that $f_n(A_n) \subset A_0$; we have used that $f_n(a) = a$ for all n. Define $\tilde{A} := A_1 \cap \cdots \cap A_{N_1}$. Then $\tilde{A}$ is an open subset containing the point a and $f_n(\tilde{A}) \subset A_0$ for all $n \in \{1, \dots, N_1\}$. In addition, $\tilde{A} \subset A$. Hence $f_n(\tilde{A}) \cap B = \emptyset$ for $n = 1, \dots, N_1$. By hypothesis, there exists $N_2 \in \mathbb{N}$ (necessarily, $N_2 > N_1$) with $f_{N_2}(\tilde{A}) \cap U_j \neq \emptyset$ for all $j \in \{0, 1, \dots, m\}$, where $U_0 := B$. Therefore

$$f_{N_2}(A) \cap U_j \neq \emptyset \quad \bigl(j \in \{1, \dots, m\}\bigr),$$

which proves the claim because in the same way we would obtain $N_1 < N_2 < N_3 < \cdots$ such that $f_{N_k}(A) \cap U_j \neq \emptyset$ for all $j \in \{1, \dots, m\}$ and all $k \in \mathbb{N}$.

Now we recover our first goal and fix U, V, α, β as before. Since $\alpha \in \sigma_J(a) \cap U$, from (S4*) it can be extracted the existence of finitely many open sets $U_1, \dots, U_J, A$ in E for which $\alpha \in \prod_{j=1}^{\infty} A_j^{(N)} \subset U$ $(N > J)$, where $A_j^{(N)}$ is defined by (1). In addition, there are open sets $V_1, \dots, V_J$ in E such that $\beta \in V_1 \times \cdots \times V_J \times \{a\} \times \{a\} \times \cdots \subset V$. By the just-proved claim, a positive integer N can be chosen in such a way that $N > J$ and

$$f_N(A) \cap V_j \neq \emptyset \quad (j = 1, \dots, J).$$

Hence there exist J points $t_1, \dots, t_J$ in A satisfying $f_N(t_j) \in V_j$ $(j = 1, \dots, J)$. Let us define $x = (x_j) \in E^{\mathbb{N}}$ as

$$x_j = \begin{cases} \alpha_j & (1 \leqslant j \leqslant J), \\ t_{j-N} & (N + 1 \leqslant j \leqslant N + J), \\ a & (J < j \leqslant N \text{ or } j > N + J). \end{cases}$$

Then $x \in \prod_{j=1}^{\infty} A_j^{(N)}$, so $x \in U$. Finally,

$$T_N x = \bigl(f_N(x_{N+j})\bigr) = \bigl(t_1, \ldots, t_J, f_N(a), f_N(a), f_N(a), \ldots\bigr)$$
$$= (t_1, \ldots, t_J, a, a, a, \ldots) \in V_1 \times \cdots \times V_J \times \{a\} \times \{a\} \times \{a\} \times \cdots \subset V,$$

which concludes the proof. □

Remark 4.4. The sufficient condition for universality furnished in the last theorem may not be necessary at all. Indeed, if for instance $E = \mathbb{R}$ and $S = \omega$, then the identity $f : x \in \mathbb{R} \mapsto x \in \mathbb{R}$ has dense range and is continuous, so $B = B_f$ is universal by Theorem 4.2, but the sequence $(f_n) = (f^{\circ n})$ is clearly not weakly mixing at $a = 0$. Nevertheless, the converse holds for c_0 and for the l_p spaces. In fact, we will be able to obtain a more general result for SSSs (see Theorem 4.5 below) under the further condition that the "center" of S has neighborhoods with projections which are as "uniformly small" as desired, that is, under the condition

(S5) Given an open subset $V \subset E$ containing a, there exists an open subset $U \subset S$ with $(a, a, a, \ldots) \in U$ such that $\pi_j(U) \subset V$ for all $j \in \mathbb{N}$.

For instance, $c_0(E)$ and $l_p(E)$ $(0 < p < \infty)$ satisfy (S5) (with $a = 0$) for any Banach space E, while $\omega(E)$ does not satisfy it for any metrizable space E.

Theorem 4.5. *If S is a* SSS *on E satisfying* (S5) *for the distinguished point a and $f_n : E \to E$ $(n \in \mathbb{N})$ is a sequence of selfmappings such that $T_n : x = (x_j) \in S \mapsto (f_n(x_{j+n})) \in S$ $(n \in \mathbb{N})$ is a densely universal sequence of continuous selfmappings on S, then (f_n) is weakly mixing at a.*

Proof. Let us fix an open set $V \subset E$ containing the distinguished point a. Fix also finitely many nonempty open sets $U_j \subset E$ $(j = 1, \ldots, J)$. Since (S5) holds for S, we get the existence of an open subset $U \subset S$ containing $(a, a, a, \ldots)$ such that $\pi_j(U) \subset V$ for all $j \in \mathbb{N}$. Since (T_n) is densely universal, there must be at least one element $x = (x_j) \in U$ which is universal for (T_n). The set $W := U_1 \times \cdots \times U_J \times E \times E \times \cdots$ is open in S by (S2), therefore there exists a positive integer N such that $T_N x \in W$, that is, $(f_N(x_{j+N})) \in U_1 \times \cdots \times U_J \times E \times E \times \cdots$. In other words,

$$f_N(x_{j+N}) \in U_j \quad (j = 1, \ldots, J).$$

But every x_{j+N} belongs to V, hence $U_j \cap f_N(V) \neq \emptyset$ for $1 \leqslant j \leqslant N$, as required. □

Theorem 4.6. *Let S be a* SSS *and $\Pi_{f_n} : S \to S$ $(n \in \mathbb{N})$ be a sequence of Φ-product maps. We have*:

(a) *If (Π_{f_n}) is densely universal then, for all $J \in \mathbb{N}$, the sequence $\{f_n \times \cdots \times f_n : E^J \to E^J\}_{n \geqslant 1}$ is transitive.*
(b) *If $f_n(a) = a$ for all $n \in \mathbb{N}$ and the sequence $\{f_n \times \cdots \times f_n : E^J \to E^J\}_{n \geqslant 1}$ is transitive for all $J \in \mathbb{N}$, then (Π_{f_n}) is densely universal.*

Proof. (a) Fix $J \in \mathbb{N}$ and nonempty open subsets A, B of E^J. We must show that $(f_N \times \cdots \times f_N)(A) \cap B \neq \emptyset$ for some N. There exist nonempty open subsets $U_1, \ldots, U_J$, $V_1, \ldots, V_J$ of E such that $U_1 \times \cdots \times U_J \subset A$ and $V_1 \times \cdots \times V_J \subset B$. Hence it is enough to find an N with $f_N(U_j) \cap V_j \neq \emptyset$ for all $j = 1, \ldots, J$. From (S2) the sets $U := (U_1 \times \cdots \times U_J \times E \times E \times \cdots) \cap S$ and $V := (V_1 \times \cdots \times V_J \times E \times E \times \cdots) \cap S$ are open in S. By hypothesis and Theorem 2.1 together with (S1), the sequence (Π_{f_n}) is transitive, so there exists $N \in \mathbb{N}$ such that $\Pi_{f_N}(U) \cap V \neq \emptyset$. Pick an element $y = (y_1, y_2, \ldots)$ in such intersection. Then $y \in V$ and there is $x = (x_1, x_2, \ldots) \in U$ with $f_N(x_j) = y_j$ for all $j \in \mathbb{N}$. Hence $x_j \in U_j$, $y_j \in V_j$ and $f_N(x_j) = y_j$ $(j = 1, \ldots, J)$, which proves (a).

(b) This time we fix nonempty open subsets U, V of S. Again by Theorem 2.1 and (S1) it should be shown the existence of an N with $\Pi_{f_N}(U) \cap V \neq \emptyset$. Due to (S3), the sets $\sigma(a) \cap U$ and $\sigma(a) \cap V$ are nonempty, so $\sigma_J(a) \cap U \neq \emptyset \neq \sigma_J(a) \cap V$ for some $J \in \mathbb{N}$. Now (S4) comes to our help, yielding the existence of nonempty open sets $U_1, \ldots, U_J, V_1, \ldots, V_J$ in E with

$$U_1 \times \cdots \times U_J \times \{a\} \times \{a\} \times \cdots \subset \sigma_J(a) \cap U \quad \text{and}$$
$$V_1 \times \cdots \times V_J \times \{a\} \times \{a\} \times \cdots \subset \sigma_J(a) \cap V.$$

By hypothesis, there exists $N \in \mathbb{N}$ such that $f_N(U_j) \cap V_j \neq \emptyset$ $(j = 1, \ldots, J)$. Pick points $x_j \in U_j$, $y_j = f_N(x_j) \in V_j$ $(j = 1, \ldots, J)$. Then $\tilde{x} := (x_1, \ldots, x_J, a, a, \ldots) \in \sigma_J(a) \cap U \subset U$ and $\tilde{y} := (y_1, \ldots, y_J, a, a, \ldots) \in \sigma_J(a) \cap V \subset V$. Finally,

$$\Pi_{f_N}\tilde{x} = \big(f_N(x_1), \ldots, f_N(x_J), f_N(a), f_N(a), \ldots\big) = (y_1, \ldots, y_J, a, a, \ldots) = \tilde{y}$$

because every f_n fixes a. $\square$

We remark that in Theorem 4.6(a) only properties (S1)–(S2) of an SSS are used; in particular, it also holds for the space $\bigoplus_{n\in\mathbb{N}} \mathbb{K}$, see Example 3.2(2).

Roughly speaking, the following corollary shows that under soft conditions on a SSS the universality of the backward Φ-shifts and of the Φ-product maps becomes completely characterized in terms of the dynamical properties of their underlying selfmappings.

Corollary 4.7. *We have the following*:

(a) *Assume that $B_f : S \to S$ is a Φ-shift on a* SSS *S satisfying* (S4*) *with distinguished point a, in such a way that $f(a) = a$ and f is weakly mixing at that point. Then B_f is universal.*
(b) *Assume that $B_f : S \to S$ is a universal Φ-shift on a* SSS *S which satisfies property* (S5). *Then f is weakly mixing at the distinguished point.*
(c) *Assume that $\Pi_f : S \to S$ is a Φ-product map on a* SSS *S with distinguished point a, in such a way that $f(a) = a$ and f is weakly mixing. Then Π_f is universal.*
(d) *Assume that $\Pi_f : S \to S$ is a universal Φ-product map on a* SSS. *Then f is weakly mixing.*
(e) *Suppose that E is a separable Banach space, that $S = l_p(E)$ or $c_0(E)$* $(0 < p < \infty)$ *and that $f : E \to E$ is continuous. In addition, we assume* $\limsup_{t\to 0} \|f(t)\|/\|t\| < \infty$ *if $S = l_p(E)$, and $f(0) = 0$ if $S = c_0(E)$. Then the Φ-product Π_f (the Φ-shift B_f,*

respectively) *is universal on* S *if and only if* f *is weakly mixing* (*weakly mixing at the origin, respectively*).

Proof. The results (a)–(e) are direct consequences of Theorems 3.5, 4.3, 4.5, 4.6 and of the fact that, for a single selfmapping T on a topological space, the universality of T implies the dense universality of the sequence $(T^{\circ n})$ of its iterates. Only part (c) needs some further explanation: Since f is weakly mixing, by [11, Proposition II.3] one gets that for every $J \in \mathbb{N}$ the mapping $f \times \cdots \times f : E^J \to E^J$ is transitive. But this is the same as the transitivity of the sequence $f^{\circ n} \times \cdots \times f^{\circ n} : E^J \to E^J$ $(n \in \mathbb{N})$, hence Theorem 4.6(b) applies. □

5. Applications to hypercyclicity theory

Here we obtain two examples of hypercyclic operators and of hypercyclic sequences of operators on linear SSSs as a consequence of some preceding results.

Firstly, we get the following rather general statement that extends Rolewicz's theorem. This is just the case $E = \mathbb{K}$, $S = l_p$ $(1 \leqslant p < \infty)$ or c_0, $f =$ the identity on $\mathbb{K}$.

Theorem 5.1. *Let be prescribed a Banach space* E, *a surjective operator* f *on* E *and a linear* SSS S *on* E *satisfying* (S4*) *for* $a = 0$. *Let* U_E *be the open unit ball of* E. *Then the scalar multiple* $\lambda B_f : S \to S$ *of the backward* Φ*-shift* B_f *is hypercyclic whenever*

$$|\lambda| > \mu := \frac{1}{\sup\{\alpha > 0\colon\ f(U_E) \supset \alpha U_E\}}.$$

Proof. The Open Mapping Theorem together with the boundedness of f guarantees that $\mu \in (0, \infty)$. If $|\lambda| > \mu$, then $f(U_E) \supset \alpha U_E$ for some $\alpha \in (0, \infty)$ with $|\lambda\alpha| > 1$. Therefore $(\lambda f)^{\circ n}(U_E) \supset (\lambda\alpha)^n U_E$ for all $n \in \mathbb{N}$. Let us fix an open set $V \subset E$ with $0 \in V$ and finitely many nonempty open sets $U_1, \ldots, U_J$ in E. Pick points $t_j \in U_j$ $(j = 1, \ldots, J)$. Then $V \supset \beta U_E$ for some $\beta > 0$. Since the set $F := \{t_1, \ldots, t_J\}$ is finite and $|\lambda\alpha| > 1$, there is $N \in \mathbb{N}$ such that $\beta(\lambda\alpha)^N U_E \supset F$. Hence, trivially, $(\lambda f)^{\circ N}(V) \cap U_j \neq \emptyset$ for every $j \in \{1, \ldots, J\}$. Then λf is weakly mixing at the origin, so Corollary 4.7(a) applies with $a = 0$ if we take into account that $B_{\lambda f} = \lambda B_f$. □

We finish with a result (Theorem 5.2) that relates the hypercyclicity of a Φ-product map to the so-called Hypercyclicity Criterion, which is the condition (b) in Theorem 5.2. Such criterion is a well-known sufficient condition for hypercyclicity, see [5,7,14]. We will assume that S is a complete linear SSS on an F-space E. Hence S is a separable F-space (due to (S1), because second-countable is equivalent to metrizable plus separable) and, from (S4), E is also separable. The following concept was introduced by the author in [4]: a sequence (f_n) of operators on E is called *almost-commuting* whenever $\lim_{n\to\infty}[f_n(f_m(t)) - f_m(f_n(t))] = 0$ for every $m \in \mathbb{N}$ and every $t \in E$.

Theorem 5.2. *Suppose that S is a complete linear* SSS *on an F-space E. Assume that (f_n) is a sequence of operators on E such that (Π_{f_n}) defines a sequence of operators on S. Consider the following properties*:

(a) *The sequence (Π_{f_n}) is densely hypercyclic.*
(b) *There exist dense subsets X_0 and Y_0 of E and an increasing sequence $(n_k) \subset \mathbb{N}$ satisfying the following two conditions*: $f_{n_k}(t) \to 0$ $(k \to \infty)$ *for all $t \in X_0$*; *for any $t \in Y_0$ there is a sequence (u_k) in E such that $u_k \to 0$ and $f_{n_k}(u_k) \to t$ $(k \to \infty)$.*
(c) *The sequence $f_n \times f_n : E^2 \to E^2$ $(n \in \mathbb{N})$ is hypercyclic.*

Then we have the following:

(A) *Properties* (a) *and* (b) *are equivalent.*
(B) *If (f_n) is almost-commuting, then* (a)–(c) *are equivalent.*

Proof. (A) We are assuming that (Π_{f_n}) is densely hypercyclic. From Theorem 4.6(a), the sequence $\{f_n \times \cdots \times f_n : E^J \to E^J\}_{n \geqslant 1}$ is transitive (so densely hypercyclic by Theorem 2.1) for all $J \in \mathbb{N}$. Then Theorem 2.2 of [5] applies and one obtains that (b) holds. Conversely, assume that (b) is satisfied. Again by [5, Theorem 2.2] the sequence $\{f_n \times \cdots \times f_n : E^J \to E^J\}_{n \geqslant 1}$ is densely hypercyclic (so transitive) for all $J \in \mathbb{N}$. Since $f_n(0) = 0$ for all n we have that (Π_{f_n}) is densely hypercyclic by Theorem 4.6(b).

(B) We have already obtained that (a) and (b) are equivalent. On the other hand, if (a) holds, then by Theorem 4.6(a) (for $J = 2$) we get again the transitivity (so the dense hypercyclicity, hence the single hypercyclicity) of the sequence $\{f_n \times f_n : E^2 \to E^2\}_{n \geqslant 1}$. Conversely, if this sequence is hypercyclic, then Theorem 3.3 of [5] guarantees that (b) is satisfied. □

Of course, part (B) applies to a single operator f on E since any two iterates $f^{\circ n}$, $f^{\circ m}$ clearly commute. Other conditions on (f_n) which are equivalent to the Hypercyclicity Criterion can be seen in [5] and [7]. Observe also that the transitivity of $\{f_n \times \cdots \times f_n : E^J \to E^J\}_{n \geqslant 1}$ for all $J \in \mathbb{N}$ is stronger than the property that (f_n) is weakly mixing at the origin. Hence, in view of Theorem 4.3 and of the proof of Theorem 5.2, we have that the sequence $\{T_n : x = (x_j) \in S \mapsto (f_n(x_{j+n})) \in S\}_{n \geqslant 1}$ is hypercyclic if (b) and (S4*) (with $a = 0$) are satisfied. In particular if f is an operator on E then under the latter two conditions (with $f_n = f^{\circ n}$ for all $n \in \mathbb{N}$) the Φ-shift B_f is hypercyclic on S.

Remark 5.3. As for a nice *nonlinear* example, we point out that some arguments similar to those presented in the proofs of Theorems 4.3 and 4.5 allowed Peris to show in [25] that a polynomial $P : l_q \to l_q$ given by $P(x_1, x_2, \ldots) = (p(x_2), p(x_3), \ldots)$—where $p : \mathbb{C} \to \mathbb{C}$ is a complex polynomial with $p(0) = 0$ of degree strictly greater that one—is universal if and only if 0 belongs to the Julia set of p.

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