

# Convexity and concavity constants in Lorentz and Marcinkiewicz spaces

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## Abstract

We provide here the formulas for the  $q$ -convexity and  $q$ -concavity constants for function and sequence Lorentz spaces associated to either decreasing or increasing weights. It yields also the formula for the  $q$ -convexity constants in function and sequence Marcinkiewicz spaces. In this paper we extend and enhance the results from [G.J.O. Jameson, The  $q$ -concavity constants of Lorentz sequence spaces and related inequalities, *Math. Z.* 227 (1998) 129–142] and [A. Kamińska, A.M. Parrish, The  $q$ -concavity and  $q$ -convexity constants in Lorentz spaces, in: *Banach Spaces and Their Applications in Analysis*, Conference in Honor of Nigel Kalton, May 2006, Walter de Gruyter, Berlin, 2007, pp. 357–373].

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Given  $0 < q < \infty$  and a quasi-normed lattice  $(E, \|\cdot\|_E)$ , we define the following constants:

- the  $q$ -convexity constant  $M^{(q)}(E)$  is the least constant  $M$  such that for all  $f_1, \dots, f_n \in E$ ,

$$\left\| \left( \sum_{i=1}^n |f_i|^q \right)^{1/q} \right\|_E \leq M \left( \sum_{i=1}^n \|f_i\|_E^q \right)^{1/q},$$

- the  $q$ -concavity constant  $M_{(q)}(E)$  is the least constant  $K$  such that for all  $f_1, \dots, f_n \in E$ ,

$$\left( \sum_{i=1}^n \|f_i\|_E^q \right)^{1/q} \leq K \left\| \left( \sum_{i=1}^n |f_i|^q \right)^{1/q} \right\|_E.$$

We then say that  $(E, \|\cdot\|_E)$  is  $q$ -convex, respectively  $q$ -concave, if  $M_{(q)}(E), M^{(q)}(E) < \infty$ . These notions are closely related to the notions of type and cotype, and they play an essential role in studies of the local geometry of Banach

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spaces/lattices [4,8,17]. It is of special interest to determine the exact values of the convexity and concavity constants for particular classes of lattices equipped with their original (quasi-)norms [5–7]. For instance it is well known that every  $q$ -convex Banach lattice can be given an equivalent lattice norm that is  $q$ -convex with constant one (and the same holds for  $q$ -concavity) [17]. Such a renorming is then a starting point for investigation of several geometric properties.

In [6], G.J.O. Jameson found the  $q$ -concavity and  $q$ -convexity constants in Lorentz sequence spaces  $d(w, p)$  for  $1 \leq p < \infty$  and  $w$  decreasing. In that paper he stated the problem of finding a direct approach in proving his result [6, Theorem 3], where the formula for  $q$ -concavity constant in  $d(w, p)$  is established. In his proof he applied general relationships between convexity and concavity constants in Banach spaces and their duals, as well as the well-known representation of the dual space to the Lorentz space. In this paper we present a direct and simpler way in proving that theorem (Theorem 6). The method we use here has also an additional advantage, namely, it can be applied after some modifications for calculation of the convexity and concavity constants in  $d(w, p)$  for increasing weights as well. Notice that if  $w$  is increasing, then Lorentz spaces are not Banach spaces, and for a large class of weights they are not even normable [11]. Thus the duality method is not applicable in this class of spaces.

Marcinkiewicz and Lorentz spaces play an important role in the theory of Banach spaces, in particular they are key objects in the interpolation theory of linear operators. The origins of the Marcinkiewicz spaces go back to the theorem on weak type operators [17, th. 2.b.15], which was originally due to K. Marcinkiewicz in the 1930's. The Lorentz spaces introduced by G.G. Lorentz in 1950 have appeared in a natural way as interpolation spaces between suitable Lebesgue spaces by classical result of Lions and Peetre [17, th. 2.g.18]. This theory has been extensively developed and along with these investigations the theory of Lorentz and Marcinkiewicz spaces—including the studies of their local geometric structure—has evolved independently (e.g. [2,3,9,11,12,16,17,19]).

In our paper we provide the formulas for  $q$ -concavity and  $q$ -convexity constants in Lorentz function and sequence spaces,  $\Lambda_{p,w}$  and  $d(w, p)$ , for all  $0 < p < \infty$ , and any decreasing or increasing weight  $w$ . Consequently, we obtain the appropriate formulas for these constants in Marcinkiewicz—function or sequence spaces that are duals to Lorentz spaces  $\Lambda_{1,w}$  or  $d(w, 1)$ . Here we complement and improve the Jameson's results in [6] as well as the results of our earlier paper [13].

The first section is devoted to function spaces, the second one to sequence spaces, and the third one to specific sequence spaces associated to the power weight sequences  $u_n = n^\alpha$  with  $\alpha > 0$ . As a consequence of the established formulas, we get among others that these constants are equal to one if and only if the spaces are isometric to the Lebesgue spaces  $L^p$  or  $\ell^p$ .

Let  $\mathbb{R}$ ,  $\mathbb{R}_+$  and  $\mathbb{N}$  stand for the sets of real numbers, positive real numbers and natural numbers, respectively. Let  $L^0$  be the set of all real-valued  $|\cdot|$ -measurable functions defined on  $\mathbb{R}_+$  or  $\mathbb{N}$ , where  $|\cdot|$  is the Lebesgue measure on  $\mathbb{R}$  or the counting (discrete) measure on  $\mathbb{N}$ . The *distribution function*  $d_f$  of a function  $f \in L^0$  is given by  $d_f(\lambda) = |\{t > 0: |f(t)| > \lambda\}|$ , for all  $\lambda \geq 0$ . We say that two functions  $f, g \in L^0$  are *equimeasurable* and we denote it by  $f \sim g$  if  $d_f(\lambda) = d_g(\lambda)$ , for all  $\lambda \geq 0$ . For  $f \in L^0$  we define its *decreasing rearrangement* as  $f^*(t) = \inf\{s > 0: d_f(s) \leq t\}$ ,  $t > 0$ . In the case of discrete measure, the elements of  $L^0$  coincide with real-valued sequences  $x = (x(n))$ , and then  $x^* = (x^*(n))$  is a decreasing rearrangement of  $x$  defined equivalently as  $x^*(n) = \inf\{s > 0: d_x(s) \leq n - 1\}$ , for  $n \in \mathbb{N}$ .

By  $w: \mathbb{R}_+ \rightarrow \mathbb{R}_+$  or  $w: \mathbb{N} \rightarrow \mathbb{R}_+$  we denote the *weight function*  $w(t)$ ,  $t > 0$ , or the *weight sequence*  $(w_n)$ . Letting  $W(t) := \int_0^t w$ ,  $t > 0$ , or  $W(j) = W_j = \sum_{i=1}^j w_i$ ,  $j \in \mathbb{N}$ , we shall assume that  $W(t) < \infty$  for all  $t > 0$ , and  $W(\infty) = \int_0^\infty w = \sum_{i=1}^\infty w_i = \infty$ . We also assume that  $W$  satisfies the  $\Delta_2$ -condition, that is  $W(2s) \leq K W(s)$  for some  $K > 0$  and all  $s > 0$  or  $s \in \mathbb{N}$ .

Given  $0 < p < \infty$  and a weight function  $w$ , the *Lorentz space*  $\Lambda_{p,w}$  is a subset of  $L^0$  such that

$$\|f\| = \|f\|_{p,w} := \left( \int_0^\infty f^{*p} w \right)^{1/p} = \left( \int_0^\infty f^{*p}(t) w(t) dt \right)^{1/p} < \infty.$$

Recall also that the *Marcinkiewicz space*  $M_W$  is the space of all functions  $f \in L^0$  satisfying

$$\|f\|_{M_W} = \sup_{t>0} \frac{\int_0^t f^*(s) ds}{W(t)} < \infty.$$

In the case of discrete measure, we denote by  $d(w, p)$  the Lorentz sequence space associated to a weight sequence  $w = (w_n)$ . The space  $d(w, p)$  consists of all real sequences  $x = (x(n))$  such that

$$\|x\| = \|x\|_{p,w} := \left( \sum_{j=1}^{\infty} x^*(j)^p w_j \right)^{1/p} < \infty.$$

Similarly, the Marcinkiewicz sequence space  $m_W$  is the space of all sequences  $x = (x(n))$  satisfying

$$\|x\|_{m_W} = \sup_{k \geq 1} \frac{\sum_{j=1}^k x^*(j)}{W_k} < \infty.$$

Under the assumption of  $\Delta_2$ -condition on  $W$ ,  $\|\cdot\|_{p,w}$  is a quasi-norm, and  $(\Lambda_{p,w}, \|\cdot\|_{p,w})$  or  $(d(w, p), \|\cdot\|_{p,w})$  is a quasi-Banach space [11,14]. If  $w$  is decreasing, then  $\|\cdot\|_{p,w}$  is a norm [1,15] and

$$\|f\|_{p,w} = \left( \sup_{v \sim w} \int_0^{\infty} |f|^p v \right)^{1/p} \quad \text{or} \quad \|x\|_{p,w} = \left( \sup_{v \sim w} \sum_{j=1}^{\infty} |x(j)|^p v_j \right)^{1/p}. \tag{1}$$

In this case,  $\Lambda_{1,w}$  or  $d(w, 1)$  is a separable Banach space and its dual is the Marcinkiewicz space  $M_W$  [15, Theorem 5.2], or  $m_W$  [10, Theorem 4.4], respectively.

In the following, we will consider both function and sequence spaces in two cases, when  $w$  is decreasing or increasing. To prove the results for increasing weight, we need to recall the definition of the increasing rearrangement and some of its properties [13]. Similarly to the distribution function  $d_f$ , define for  $f \in L^0$  and for all  $\lambda > 0$  the function  $\gamma_f(\lambda) = |\{s \in \text{supp } f : |f(s)| < \lambda\}|$ . We say that two functions  $f$  and  $g$  are *equivalent* and denote it by  $f \sim_{\gamma} g$ , if  $\gamma_f(\lambda) = \gamma_g(\lambda)$  for all  $\lambda > 0$ . Then the increasing rearrangement of  $f$  is the function  $f_*$  defined as

$$f_*(t) = \begin{cases} \sup\{\lambda \geq 0 : \gamma_f(\lambda) \leq t\}, & \text{if } t \in [0, |\text{supp } f|); \\ 0, & \text{if } t \geq |\text{supp } f|. \end{cases}$$

Analogously, for a sequence  $x = (x(n))$ , define for  $s > 0$ ,

$$\gamma_x(s) = |\{i \in \text{supp } x : |x(i)| < s\}|,$$

and the increasing rearrangement  $x_*$  of  $x$  by

$$x_*(j) = \begin{cases} \sup\{s \geq 0 : \gamma_x(s) \leq j - 1\}, & \text{if } j \in (0, |\text{supp } x|]; \\ 0, & \text{if } j > |\text{supp } x|. \end{cases}$$

Recall [13, Theorem 2.5] that for any  $f, g \in L^0$  such that  $g > 0$  a.e. and  $\gamma_f(\lambda) < \infty$  for every  $\lambda > 0$ , we have that

$$\int_0^{\infty} fg \geq \int_0^{\infty} f_* g_*. \tag{2}$$

By Theorem 2.7 in [13], if  $w$  is increasing and  $\lim_{t \rightarrow \infty} w(t) = \infty$ , then for any bounded function  $f$  with  $|\text{supp } f| < \infty$ ,

$$\|f\|_{p,w} = \inf \left\{ \left( \int_0^{\infty} |f|^p v \right)^{1/p} : v \sim_{\gamma} w, v > 0 \text{ a.e.} \right\}. \tag{3}$$

The similar fact can be proved analogously in sequence spaces, and thus for any increasing weight  $w = (w_n)$  with  $\lim_{n \rightarrow \infty} w_n = \infty$ , and for any  $x$  with finite support, we have

$$\|x\|_{p,w} = \inf \left\{ \left( \sum_{j=1}^{\infty} |x(j)|^p v_j \right)^{1/p} : v \sim_{\gamma} w, v > 0 \right\}. \tag{4}$$

Recall that given  $0 < p < \infty$ , the  $p$ -convexification of a quasi-Banach lattice  $(E, \|\cdot\|_E)$  is the space  $E^{(p)} = \{f: |f|^p \in E\}$  with the quasi-norm  $\|f\|_{E^{(p)}} = \| |f|^p \|_E^{1/p}$ . It is well known and easy to show that

$$M_{(q)}(E^{(p)}) = M_{(q/p)}(E) \quad \text{and} \quad M^{(q)}(E^{(p)}) = M^{(q/p)}(E). \quad (5)$$

The spaces  $\Lambda_{p,w}$  or  $d(w, p)$  are  $p$ -convexifications of  $\Lambda_{1,w}$  or  $d(w, 1)$ , respectively. This allows us to assume that  $p = 1$  in the process of computing the convexity and concavity constants for these spaces.

## 1. Function spaces

The first result presented here was proved in [13, Theorem 3.2], where we have used Jameson's duality method [6, Theorem 3] accommodated to function spaces. Below we provide a direct proof.

**Theorem 1.** *Let  $q > p$  and  $w$  be a decreasing weight function. Then*

$$M_{(q)}(\Lambda_{p,w}) = \sup_{t>0} \frac{(\frac{1}{t} \int_0^t w^r)^{1/r}}{\frac{1}{t} \int_0^t w},$$

where  $\frac{p}{q} + \frac{1}{r} = 1$ .

**Proof.** Assuming  $p = 1$ , we shall prove that

$$M_{(q)}(\Lambda_{1,w}) \leq \sup_{t>0} \frac{(\frac{1}{t} \int_0^t w^r)^{1/r}}{\frac{1}{t} \int_0^t w} := B.$$

Letting  $(f_i)_{i=1}^n \subset \Lambda_{1,w}$ , there exists a sequence  $(\lambda_i)_{i=1}^n$  of non-negative numbers such that

$$\sum_{i=1}^n \lambda_i^r = 1 \quad \text{and} \quad \sum_{i=1}^n \lambda_i \|f_i\| = \left( \sum_{i=1}^n \|f_i\|^q \right)^{1/q}.$$

By (1), for  $\varepsilon > 0$  and for  $i = 1, \dots, n$  there exist  $h_i \geq 0$ ,  $h_i \sim w$  such that

$$\|f_i\| = \sup_{v \sim w} \int_0^\infty |f_i| v \leq \int_0^\infty |f_i| h_i + \frac{\varepsilon}{n\lambda_i}.$$

Applying then the Hölder inequality we obtain

$$\left( \sum_{i=1}^n \|f_i\|^q \right)^{1/q} \leq \sum_{i=1}^n \lambda_i \left( \int_0^\infty |f_i| h_i + \frac{\varepsilon}{n\lambda_i} \right) = \int_0^\infty \sum_{i=1}^n \lambda_i |f_i| h_i + \varepsilon \leq \int_0^\infty \left( \sum_{i=1}^n |f_i|^q \right)^{1/q} \left( \sum_{i=1}^n \lambda_i^r h_i^r \right)^{1/r} + \varepsilon. \quad (6)$$

Let now  $g = (\sum_{i=1}^n \lambda_i^r h_i^r)^{1/r}$ . Thus for all  $t > 0$ ,

$$\int_0^t g^{*r} = \int_0^t \left( \sum_{i=1}^n \lambda_i^r h_i^r \right)^* \leq \sum_{i=1}^n \int_0^t (\lambda_i^r h_i^r)^* = \sum_{i=1}^n \lambda_i^r \int_0^t h_i^{*r} = \int_0^t w^r.$$

Hence

$$\int_0^t g^* = \int_0^t g^* \cdot 1 \leq \left( \int_0^t w^r \right)^{1/r} \cdot t^{1/q} = \left( \frac{1}{t} \right)^{\frac{1}{r}-1} \left( \int_0^t w^r \right)^{1/r} = \frac{(\frac{1}{t} \int_0^t w^r)^{1/r}}{\frac{1}{t} \int_0^t w} \int_0^t w \leq B \cdot \int_0^t w. \quad (7)$$

By Hardy's lemma and by (7), we have

$$\int_0^\infty \left\{ \left( \sum_{i=1}^n |f_i|^q \right)^{1/q} \right\}^* g^* \leq B \int_0^\infty \left\{ \left( \sum_{i=1}^n |f_i|^q \right)^{1/q} \right\}^* w.$$

Thus by (6) and by the Hardy–Littlewood inequality,

$$\begin{aligned} \left(\sum_{i=1}^n \|f_i\|^q\right)^{1/q} &\leq \int_0^\infty \left(\sum_{i=1}^n |f_i|^q\right)^{1/q} g + \varepsilon \leq \int_0^\infty \left\{ \left(\sum_{i=1}^n |f_i|^q\right)^{1/q} \right\}^* g^* + \varepsilon \\ &\leq B \int_0^\infty \left\{ \left(\sum_{i=1}^n |f_i|^q\right)^{1/q} \right\}^* w + \varepsilon = B \left\| \left(\sum_{i=1}^n |f_i|^q\right)^{1/q} \right\| + \varepsilon, \end{aligned}$$

and the proof is completed.  $\square$

We obtain the formula for the constant in Marcinkiewicz space  $M_W$  by duality to  $\Lambda_{1,w}$ .

**Corollary 2.** For  $1 < p < \infty$  and decreasing weight function  $w$ ,

$$M^{(p)}(M_W) = \sup_{t>0} \frac{(\frac{1}{t} \int_0^t w^p)^{1/p}}{\frac{1}{t} \int_0^t w}. \tag{8}$$

The space  $M_W$  is not  $p$ -concave for any  $0 < p < \infty$ .

**Proof.** It is well known [17, Proposition 1.d.4] that if  $E$  is a Banach lattice, then for  $1 \leq q \leq \infty$ ,  $M^{(q)}(E) = M_{(q^*)}(E^*)$  and  $M_{(q)}(E) = M^{(q^*)}(E^*)$ , where  $1/q + 1/q^* = 1$ . Given that for  $w$  decreasing,  $M_W$  is a dual space to  $\Lambda_{1,w}$ , we obtain (8).

Using Theorem 117.3 in [18], if the norm of a Banach lattice is not order continuous, then the Banach lattice contains an order isomorphic copy of  $l_\infty$ , and since  $l_\infty$  is not  $p$ -concave for any  $0 < p < \infty$ , neither is the Banach lattice. So all we need is to show that  $\|\cdot\|_{M_W}$  is not order continuous. For this, consider the functions  $f_n = w\chi_{(0,1/n)}$ . So  $0 \leq f_n \downarrow 0$ ,  $f_n \leq w$ , and since for all  $t > 0$ ,

$$\frac{\int_0^t w\chi_{(0,1/n)}}{W(t)} = \begin{cases} 1, & \text{if } t < 1/n; \\ \frac{W(\frac{1}{n})}{W(t)}, & \text{if } t \geq 1/n, \end{cases}$$

and  $t/W(t)$  is increasing, we obtain that  $\|f_n\|_{M_W} = 1$  for all  $n \in \mathbb{N}$ . Thus the norm is not order continuous.  $\square$

In the next theorem we state the formula for the  $q$ -convexity constant in Lorentz space  $\Lambda_{p,w}$  for the weight  $w$  increasing. It is an improvement of the previous result given in [13, Theorem 3.5, Corollary 3.6], where we have found only some estimates of the constant.

**Theorem 3.** If  $0 < q < p$  and  $w$  is an increasing weight satisfying  $\lim_{t \rightarrow \infty} w(t) = \infty$ , then

$$M^{(q)}(\Lambda_{p,w}) = \sup_{t>0} \frac{\frac{1}{t} \int_0^t w}{(\frac{1}{t} \int_0^t w^r)^{1/r}},$$

where  $\frac{p}{q} + \frac{1}{r} = 1$ .

**Proof.** In view of (5) and the obvious fact that the  $p$ -convexification of  $\Lambda_{1,w}$  is  $\Lambda_{p,w}$ , we assume that  $p = 1$ . Since the inequality

$$M^{(q)}(\Lambda_{1,w}) \geq \sup_{t>0} \frac{\frac{1}{t} \int_0^t w}{(\frac{1}{t} \int_0^t w^r)^{1/r}}$$

was proved in [13, Theorem 3.5], we shall show only the reverse one.

Let  $(f_i)_{i=1}^n \subset \Lambda_{1,w}$ . We can assume that each  $f_i$  is a simple function with bounded support, since such functions are dense in  $\Lambda_{1,w}$ . By the reverse Hölder inequality for  $0 < q < 1$ , there exists a sequence  $(a_i)_{i=1}^n$  of positive numbers such that

$$\sum_{i=1}^n a_i^r = 1 \quad \text{and} \quad \sum_{i=1}^n a_i \|f_i\| = \left(\sum_{i=1}^n \|f_i\|^q\right)^{1/q}.$$

Let  $\varepsilon > 0$ . Since by (3),

$$\|f_i\| = \inf \left\{ \int_0^\infty |f_i|v : v \sim_\gamma w, v > 0 \text{ a.e.} \right\}$$

for all  $i = 1, 2, \dots, n$ , there exist  $h_i \sim_\gamma w, h_i > 0$ , a.e. such that

$$\|f_i\| \geq \int_0^\infty |f_i|h_i - \frac{\varepsilon}{na_i}.$$

By Hölder’s inequality for  $0 < q < 1$ ,

$$\left( \sum_{i=1}^n \|f_i\|^q \right)^{1/q} \geq \sum_{i=1}^n a_i \left( \int_0^\infty |f_i|h_i - \frac{\varepsilon}{na_i} \right) = \int_0^\infty \sum_{i=1}^n a_i |f_i|h_i - \varepsilon \geq \int_0^\infty \left( \sum_{i=1}^n |f_i|^q \right)^{1/q} \left( \sum_{i=1}^n a_i^r h_i^r \right)^{1/r} - \varepsilon. \tag{9}$$

Let  $g = (\sum_{i=1}^n a_i^r h_i^r)^{1/r} > 0$  and let  $t > 0$ . Notice that  $r < 0$  and that  $(f^r)^* = (f_*)^r$  a.e. [13, Proposition 2.3(4)]. Hence by  $\sum_{i=1}^n a_i^r = 1$  and the subadditivity of the operator  $f \mapsto \int_0^t f^*$ ,

$$\begin{aligned} \int_0^t (g_*)^r &= \int_0^t \left[ \left( \sum_{i=1}^n a_i^r h_i^r \right)^{1/r} \right]^r = \int_0^t \left( \sum_{i=1}^n a_i^r h_i^r \right)^* \leq \int_0^t \sum_{i=1}^n (a_i^r h_i^r)^* = \sum_{i=1}^n \int_0^t a_i^r (h_i^r)^* \\ &= \sum_{i=1}^n a_i^r \int_0^t [(h_i)_*]^r = \sum_{i=1}^n a_i^r \int_0^t w^r = \int_0^t w^r, \end{aligned}$$

since  $[(h_i)_*]^r$  are equimeasurable to  $w^r$ . By  $r < 0$  we get  $(\int_0^t (g_*)^r)^{1/r} \geq (\int_0^t w^r)^{1/r}$ . Thus if we denote by

$$B := \sup_{t>0} \frac{\frac{1}{t} \int_0^t w}{\left(\frac{1}{t} \int_0^t w^r\right)^{1/r}},$$

we get that for all  $t > 0$ ,

$$\int_0^t g_* = \int_0^t g_* \cdot 1 \geq \left( \int_0^t (g_*)^r \right)^{1/r} \cdot t^{1/q} \geq \left( \int_0^t w^r \right)^{1/r} \left( \frac{1}{t} \right)^{\frac{1}{r}-1} = \frac{(\int_0^t w^r)^{1/r} (\frac{1}{t})^{1/r}}{\frac{1}{t} \int_0^t w} \int_0^t w \geq B^{-1} \int_0^t w.$$

So by Hardy’s lemma [1, Proposition 3.6] we get

$$\int_0^\infty \left\{ \left( \sum_{i=1}^n |f_i|^q \right)^{1/q} \right\}^* g_* \geq B^{-1} \int_0^\infty \left\{ \left( \sum_{i=1}^n |f_i|^q \right)^{1/q} \right\}^* w.$$

Now by (9) and (2),

$$\begin{aligned} \left( \sum_{i=1}^n \|f_i\|^q \right)^{1/q} &\geq \int_0^\infty \left( \sum_{i=1}^n |f_i|^q \right)^{1/q} g - \varepsilon \geq \int_0^\infty \left\{ \left( \sum_{i=1}^n |f_i|^q \right)^{1/q} \right\}^* g_* - \varepsilon \\ &\geq B^{-1} \int_0^\infty \left\{ \left( \sum_{i=1}^n |f_i|^q \right)^{1/q} \right\}^* w - \varepsilon = B^{-1} \left\| \left( \sum_{i=1}^n |f_i|^q \right)^{1/q} \right\| - \varepsilon. \end{aligned}$$

Therefore

$$\left\| \left( \sum_{i=1}^n |f_i|^q \right)^{1/q} \right\| \leq B \left( \sum_{i=1}^n \|f_i\|^q \right)^{1/q}$$

and so  $M^{(q)}(\Lambda_{1,w}) \leq B$ .  $\square$

**Corollary 4.** *If  $w$  is decreasing (respectively increasing), then for all  $q > p$ ,  $M_{(q)}(\Lambda_{p,w}) = 1$  (respectively for all  $0 < q < p$ ,  $M^{(q)}(\Lambda_{p,w}) = 1$ ) if and only if  $\Lambda_{p,w}$  is isometric to  $L^p$ .*

**Proof.** Assume that  $p = 1$  and  $w$  is decreasing. Then  $q > 1$  and by Theorem 1, for every  $t > 0$ ,

$$\left(\frac{1}{t} \int_0^t w^r\right)^{1/q} \leq \frac{1}{t} \int_0^t w.$$

On the other hand by Hölder’s inequality

$$\frac{1}{t} \int_0^t w \leq \frac{1}{t} \left(\int_0^t w^r\right)^{1/r} t^{1/r} = \left(\frac{1}{t} \int_0^t w^r\right)^{1/r}.$$

Then for every  $t > 0$ ,  $(\frac{1}{t} \int_0^t w^r)^{1/q} = \frac{1}{t} \int_0^t w$ , and by the equality condition in Hölder’s inequality,  $w(t) = C$ , for all  $t > 0$ , and some  $C > 0$ . Hence  $\|f\|_{p,w} = C^{1/p} \|f\|_{L^p}$ . For increasing  $w$ , the proof is analogous.  $\square$

Recall that for  $0 < p, q < \infty$ , the classical Lorentz spaces  $L_{q,p}$  are obtained from  $\Lambda_{p,w}$  by setting  $w(t) = t^{p/q} - 1$  (see [1,17]). The following result is an improvement and complement of Corollary 3.7 in [13].

**Corollary 5.**

(1) *If  $p \leq q$ , then*

- (a)  $M^{(s)}(L_{q,p}) = 1$  for  $s \leq p$  and the space is not  $s$ -convex for  $s > p$ ;
- (b) for  $s > q$ ,

$$M_{(s)}(L_{q,p}) = \frac{p}{q[(\frac{p}{q} - 1)r + 1]^{1/r}},$$

where  $\frac{1}{r} + \frac{p}{s} = 1$ . For  $s \leq q$ , the space is not  $s$ -concave.

(2) *If  $p > q$ , then*

- (a)  $M_{(s)}(L_{q,p}) = 1$  for  $s \geq p$  and the space is not  $s$ -concave for  $s < p$ ;
- (b) for  $0 < s < q$ ,

$$\frac{q[(\frac{p}{q} - 1)r + 1]^{1/r}}{p} \leq M^{(s)}(L_{q,p}) \leq \left[\left(\frac{p}{q} - 1\right)r + 1\right]^{1/r},$$

where  $\frac{1}{r} + \frac{p}{s} = 1$ . For  $s \geq q$  the space is not  $s$ -convex.

## 2. Sequence spaces

We start with presenting a direct proof of the formula for  $q$ -concavity constant of  $d(w, p)$  in case when  $w$  is decreasing, originally proved by G.J.O. Jameson in [6, Theorem 3] and by this answering his question posed there.

**Theorem 6.** *Let  $q > p$  and  $w$  be a decreasing weight sequence. Then*

$$M_{(q)}(d(w, p)) = \sup_{k \geq 1} \frac{(\frac{1}{k} \sum_{j=1}^k w_j^r)^{1/r}}{\frac{1}{k} \sum_{j=1}^k w_j},$$

where  $\frac{p}{q} + \frac{1}{r} = 1$ .

**Proof.** Assume  $p = 1$ . In view of Theorem 3 in [6], we wish to prove only the inequality

$$M_{(q)}(d(w, 1)) \leq \sup_{k \geq 1} \frac{(\frac{1}{k} \sum_{j=1}^k w_j^r)^{1/r}}{\frac{1}{k} \sum_{j=1}^k w_j}.$$

We notice first that for all  $k \geq 1$ ,

$$B := \sup_{k \geq 1} \frac{(\frac{1}{k} \sum_{j=1}^k w_j^r)^{1/r}}{\frac{1}{k} \sum_{j=1}^k w_j} \geq \frac{k^{1/q}}{W_k} \left( \sum_{j=1}^k w_j^r \right)^{1/r}. \tag{10}$$

Let  $(x_i)_{i=1}^n \subset d(w, 1)$ . Then there exists a non-negative sequence  $(a_i)_{i=1}^n$  such that

$$\sum_{i=1}^n a_i^r = 1 \quad \text{and} \quad \sum_{i=1}^n a_i \|x_i\| = \left( \sum_{i=1}^n \|x_i\|^q \right)^{1/q}.$$

Let  $\varepsilon > 0$ . Proceeding in the same way as in Theorem 1, there exist  $h_i \sim w, h_i \geq 0$  and such that for all  $i = 1, \dots, n$ ,

$$\|x_i\| \leq \sum_{j=1}^{\infty} |x_i(j)| h_i(j) + \frac{\varepsilon}{na_i}.$$

Then by Hölder’s inequality

$$\left( \sum_{i=1}^n \|x_i\|^q \right)^{1/q} \leq \sum_{j=1}^{\infty} \left[ \left( \sum_{i=1}^n |x_i(j)|^q \right)^{1/q} \left( \sum_{i=1}^n a_i^r h_i(j)^r \right)^{1/r} \right] + \varepsilon. \tag{11}$$

Letting  $g$  be a sequence defined by  $g(j) = (\sum_{i=1}^n a_i^r h_i(j)^r)^{1/r}$ , we have

$$\sum_{j=1}^k (g(j)^*)^r \leq \sum_{j=1}^k w_j^r,$$

and from (10), we also have that

$$\sum_{j=1}^k g^*(j) \leq B \sum_{j=1}^k w_j$$

for all  $k \geq 1$ . So by Hardy’s lemma,

$$\sum_{j=1}^{\infty} \left\{ \left( \sum_{i=1}^n |x_i(j)|^q \right)^{1/q} \right\}^* g^*(j) \leq B \sum_{j=1}^{\infty} \left\{ \left( \sum_{i=1}^n |x_i(j)|^q \right)^{1/q} \right\}^* w_j.$$

Then from (11) and the Hardy–Littlewood inequality

$$\left( \sum_{i=1}^n \|x_i\|^q \right)^{1/q} \leq B \sum_{j=1}^{\infty} \left\{ \left( \sum_{i=1}^n |x_i(j)|^q \right)^{1/q} \right\}^* w_j + \varepsilon = B \left\| \left( \sum_{i=1}^n |x_i(j)|^q \right)^{1/q} \right\| + \varepsilon,$$

which shows the theorem.  $\square$

**Corollary 7.** For  $1 < p < \infty$  and  $w = (w_n)$  a decreasing weight sequence,

$$M^{(p)}(m_W) = \sup_{k \geq 1} \frac{(\frac{1}{k} \sum_{j=1}^k w_j^p)^{1/p}}{\frac{1}{k} \sum_{j=1}^k w_j}. \tag{12}$$

The space  $m_W$  is not  $p$ -concave for any  $0 < p < \infty$ .

**Proof.** Using the same reasoning as in Corollary 2 and that  $m_W$  is the dual space of  $d(w, 1)$ , we obtain the  $p$ -convexity constant for  $m_W$ .

Similarly, we show that the space  $m_W$  is not  $p$ -concave for any  $0 < p < \infty$  by using that  $m_W$  contains an order isomorphic copy of  $l_\infty$  if the norm is not order continuous. To show that the norm is not order continuous, consider

the sequence  $(x_n)_{n=1}^\infty$  defined by

$$x_n = w_n \chi_{E_n} + w \chi_{\mathbb{N} \setminus E_n},$$

where  $E_n = \{1, \dots, n\}$ . Then clearly  $0 < x_n \leq w$ ,  $x_n \downarrow 0$ , and elementary calculations show that  $\lim_{n \rightarrow \infty} \|x_n\|_{m_W} > 0$ .  $\square$

In the next result we present the convexity and concavity constants for the weight  $w$  increasing.

**Theorem 8.** *Let  $w$  be an increasing weight sequence with  $\lim_{n \rightarrow \infty} w_n = \infty$ .*

*If  $0 < q < p$ , then  $d(w, p)$  is not  $q$ -concave and*

$$M^{(q)}(d(w, p)) = \sup_{k \geq 1} \frac{\frac{1}{k} \sum_{j=1}^k w_j}{\left(\frac{1}{k} \sum_{j=1}^k w_j^r\right)^{1/r}},$$

where  $\frac{p}{q} + \frac{1}{r} = 1$ .

*If  $q \geq p$ , then  $d(w, p)$  is not  $q$ -convex and  $M_{(q)}(d(w, p)) = 1$ .*

**Proof.** Since  $d(w, p)$  contains an order isomorphic copy of  $\ell_p$  (see [16, Proposition 4.e.3] for  $w$  decreasing, and [14, Theorem 3.11] for arbitrary  $w$ ), it is not  $q$ -concave for  $q < p$  and not  $q$ -convex for  $q > p$ .

Let now  $q = p$  and assume that  $p = 1$ . Since  $w$  is increasing,  $W(n)/n$  is also increasing. Moreover, by the assumption  $\lim_{n \rightarrow \infty} w_n = \infty$ , we have  $\lim_{n \rightarrow \infty} W(n)/n = \infty$ . Indeed, denoting by  $\lceil s \rceil$  the least integer bigger than or equal to  $s$ , for any  $n \in \mathbb{N}$ ,

$$\frac{W(n)}{n} \geq \frac{1}{n} \sum_{i=\lceil n/2 \rceil}^n w_i \geq \left(\frac{1}{2} - \frac{1}{n}\right) w_{\lceil n/2 \rceil} \rightarrow \infty.$$

Assume for a contrary that  $d(w, 1)$  is 1-convex. Let  $1 < k < l$ ,  $n = \lceil l/k \rceil$ , and  $E_i = \{(i-1)k + 1, \dots, ik\}$  for  $i = 1, \dots, n$ . Define  $x_i = \chi_{E_i}$ . Then  $\|x_i\| = W(k)$  and  $\|\sum_{i=1}^n x_i\| = \|\chi_{\{1, \dots, nk\}}\| = W(nk)$ . By  $l \leq nk$  and  $n \leq l/k + 1$ , and by 1-convexity of the space, for some  $C > 0$ ,

$$W(l) \leq W(nk) = \left\| \sum_{i=1}^n x_i \right\| \leq C \sum_{i=1}^n \|x_i\| = CnW(k) \leq C(l/k + 1)W(k) \leq 2C(l/k)W(k).$$

Hence  $W(l)/l \leq 2CW(k)/k$  for every  $1 < k < l$ . But  $W(n)/n \uparrow \infty$ , a contradiction. So  $d(w, 1)$  is not 1-convex.

If  $q \geq p$  we get that  $M_{(q)}(d(w, p)) = 1$  by applying formula (4).

Let now  $0 < q < p$ . In view of (5) we also suppose that  $p = 1$ . In order to show that

$$M^{(q)}(d(w, 1)) \geq \sup_{k \geq 1} \frac{\frac{1}{k} \sum_{j=1}^k w_j}{\left(\frac{1}{k} \sum_{j=1}^k w_j^r\right)^{1/r}}, \tag{13}$$

we will follow the method of the proof of Theorem 3 in [6]. Indeed, let  $k \geq 1$  and let  $x_1 = (w_1^\alpha, w_2^\alpha, \dots, w_k^\alpha, 0, 0, \dots)$ , where we define  $\alpha$  by  $\alpha q = \alpha + 1 = r$  (notice that  $\alpha < 0$ ). Define  $x_2, \dots, x_k$  by all different cyclic permutations of the first  $k$  coordinates of  $x_1$ . Then for all  $i = 1, \dots, k$ ,

$$\|x_i\| = \sum_{j=1}^k w_j^{\alpha+1} \quad \text{and} \quad \left( \sum_{i=1}^k \|x_i\|^q \right)^{1/q} = k^{1/q} \left( \sum_{j=1}^k w_j^{\alpha+1} \right).$$

On the other hand,

$$\left\| \left( \sum_{i=1}^k x_i^q \right)^{1/q} \right\| = W_k \left( \sum_{j=1}^k w_j^{\alpha q} \right)^{1/q}.$$

Hence, denoting  $M = M^{(q)}(d(w, 1))$ , we get

$$W_k \left( \sum_{j=1}^k w_j^r \right)^{1/q} \leq M k^{1/q} \left( \sum_{j=1}^k w_j^r \right),$$

and so (13) is proved.

To show the reverse inequality to (13), let

$$B := \sup_{k \geq 1} \frac{\frac{1}{k} \sum_{j=1}^k w_j}{\left( \frac{1}{k} \sum_{j=1}^k w_j^r \right)^{1/r}} = \sup_{k \geq 1} \frac{W_k}{k^{1/q}} \frac{1}{\left( \sum_{j=1}^k w_j^r \right)^{1/r}}.$$

Let  $(x_i)_{i=1}^n \subset d(w, 1)$  be elements with finite supports. Then analogously to the proof of Theorem 3, there exists a sequence of positive numbers  $(a_i)_{i=1}^n$  such that

$$\sum_{i=1}^n a_i^r = 1 \quad \text{and} \quad \sum_{i=1}^n a_i \|x_i\| = \left( \sum_{i=1}^n \|x_i\|^q \right)^{1/q}.$$

Let  $\varepsilon > 0$ . Following the similar steps as in Theorem 3, in view of (4) there exists  $h_i \sim_\gamma w, h_i > 0$ , such that

$$\|x_i\| \geq \sum_{j=1}^{\infty} |x_i(j)| h_i(j) - \frac{\varepsilon}{n a_i}.$$

By Hölder’s inequality for  $0 < q < 1$ ,

$$\left( \sum_{i=1}^n \|x_i\|^q \right)^{1/q} \geq \sum_{j=1}^{\infty} \left[ \left( \sum_{i=1}^n |x_i(j)|^q \right)^{1/q} \left( \sum_{i=1}^n a_i^r h_i(j)^r \right)^{1/r} \right] - \varepsilon. \tag{14}$$

Let  $g$  be a sequence defined by  $g(j) = \left( \sum_{i=1}^n a_i^r h_i(j)^r \right)^{1/r}$ . Then for all  $k \geq 1$ ,

$$\sum_{j=1}^k (g(j))^r \leq \sum_{j=1}^k w_j^r,$$

and therefore

$$\sum_{j=1}^k g_*(j) \geq B^{-1} \sum_{j=1}^k w_j.$$

By Hardy’s lemma,

$$\sum_{j=1}^{\infty} \left\{ \left( \sum_{i=1}^n |x_i(j)|^q \right)^{1/q} \right\}^* g_*(j) \geq B^{-1} \sum_{j=1}^{\infty} \left\{ \left( \sum_{i=1}^n |x_i(j)|^q \right)^{1/q} \right\}^* w_j,$$

and so from (14), we have

$$\left( \sum_{i=1}^n \|x_i\|^q \right)^{1/q} \geq B^{-1} \left\| \left( \sum_{i=1}^n |x_i(j)|^q \right)^{1/q} \right\| - \varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, we finish the proof.  $\square$

The proof of the next result is analogical to the one of Corollary 4.

**Corollary 9.** *If  $w$  is decreasing (respectively increasing), then for all  $q > p, M_{(q)}(d(w, p)) = 1$  (respectively for all  $0 < q < p, M^{(q)}(d(w, p)) = 1$ ) if and only if  $d(w, p)$  is isometric to  $L^p$ .*

### 3. Power weight sequences

In this chapter we shall consider the class of weight sequences given by power functions, that is  $u = (u_n)$ , where  $u_n = n^\alpha$  and  $\alpha > 0$ . The case of  $u_n = n^{-\alpha}$ ,  $0 < \alpha \leq 1$ , was studied in [6]. We are interested in finding similar results when the weights are increasing. For this, define also a weight sequence  $v = (v_n)$  given by

$$v_n = \int_{n-1}^n (1 + \alpha)t^\alpha dt.$$

It is clear that  $v_n = n^{1+\alpha} - (n - 1)^{1+\alpha}$  and  $V_k = \sum_{j=1}^k v_j = k^{1+\alpha}$ . By standard comparison with the integral of  $f(t) = t^\alpha$  on  $[0, k]$  and  $[1, k + 1]$  we obtain that

$$\frac{k^{1+\alpha}}{1 + \alpha} \leq U_k \leq \frac{(k + 1)^{1+\alpha} - 1}{1 + \alpha}. \tag{15}$$

In the following, for simplicity, we will denote by

$$A_k = \frac{\frac{1}{k} \sum_{j=1}^k u_j}{\left(\frac{1}{k} \sum_{j=1}^k u_j^r\right)^{1/r}} \quad \text{and} \quad B_k = \frac{\frac{1}{k} \sum_{j=1}^k v_j}{\left(\frac{1}{k} \sum_{j=1}^k v_j^r\right)^{1/r}}.$$

By (15) we immediately get the following result:

**Proposition 10.** *Let  $0 < q < p$  and  $\frac{1}{r} = 1 - \frac{p}{q}$ . Then, for  $0 < \alpha < -\frac{1}{r}$ ,*

$$\lim_{k \geq 1} \frac{\frac{1}{k} \sum_{j=1}^k u_j}{\left(\frac{1}{k} \sum_{j=1}^k u_j^r\right)^{1/r}} = \frac{(1 + \alpha r)^{1/r}}{1 + \alpha}.$$

**Theorem 11.** *Let  $r < 0$ ,  $v$  defined as above and  $0 < \alpha < -\frac{1}{r}$ . Then*

$$\sup_{k \geq 1} \frac{\frac{1}{k} \sum_{j=1}^k v_j}{\left(\frac{1}{k} \sum_{j=1}^k v_j^r\right)^{1/r}} \leq \frac{(1 + \alpha r)^{1/r}}{1 + \alpha}.$$

**Proof.** Notice first that since  $V_k = k^{1+\alpha}$ ,

$$B_k^r = \frac{k^{(1+\alpha)r}}{k^{r-1}} \frac{1}{\sum_{j=1}^k v_j^r}.$$

For  $r < 0$ , the function  $\varphi(u) = u^r$  is convex, so by Jensen’s inequality we have

$$\left( \int_{j-1}^j (1 + \alpha)t^\alpha dt \right)^r \leq \int_{j-1}^j (1 + \alpha)^r t^{\alpha r} dt.$$

Thus by the definition of  $v_j$ ,

$$\sum_{j=1}^k v_j^r \leq (1 + \alpha)^r \int_0^k t^{\alpha r} dt = \frac{(1 + \alpha)^r}{1 + \alpha r} k^{1+\alpha r}.$$

It follows that for all  $k \geq 1$ ,

$$B_k^r \geq \frac{k^{(1+\alpha)r}}{k^{r-1}} \frac{1 + \alpha r}{(1 + \alpha)^r k^{1+\alpha r}} = \frac{1 + \alpha r}{(1 + \alpha)^r},$$

and since  $r < 0$ ,

$$\sup_{k \geq 1} B_k \leq \frac{(1 + \alpha r)^{1/r}}{1 + \alpha}. \quad \square \tag{16}$$

The next lemma can be proved following the same reasoning as in [6, Lemma 7].

**Lemma 12.** *With  $u, v$  defined as above,  $(\frac{v_j}{u_j})$  is increasing.*

**Lemma 13.** *Let  $r < 0$ ,  $(x_j), (y_j)$  be increasing and such that  $(\frac{x_j}{y_j})$  is also increasing. Then*

$$\frac{\sum_{j=1}^n x_j}{(\sum_{j=1}^n x_j^r)^{1/r}} \geq \frac{\sum_{j=1}^n y_j}{(\sum_{j=1}^n y_j^r)^{1/r}}.$$

**Proof.** For  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ , we will denote by  $X_k = x_1 + \dots + x_n$ . By [6, Lemma 6], if  $(x_j), (y_j)$  are (finite or infinite) sequences of positive numbers, and  $(\frac{x_j}{y_j})$  is decreasing (or increasing), then so is  $(\frac{X_j}{Y_j})$ .

Let  $\frac{\sum_{j=1}^n x_j}{\sum_{j=1}^n y_j} = C$ , that is  $X_n = CY_n$ . Since  $(\frac{x_j}{y_j})$  is increasing,  $(\frac{X_j}{Y_j})$  is also increasing, so

$$\frac{X_j}{Y_j} \leq \frac{X_n}{Y_n} = C \quad \text{for all } 1 \leq j \leq n - 1.$$

Since for  $r < 0$ , the function  $f(t) = t^r$  is convex, and  $(x_j), (y_j)$  are increasing non-negative elements of  $\mathbb{R}^n$  such that  $X_k \leq Y_k$ , for all  $1 \leq k \leq n - 1$ , and  $X_n = Y_n$ , using the discrete version of Karamata’s inequality for decreasing sequences [6, Lemma 8] and rearranging the terms, it can be shown that

$$\sum_{j=1}^n f(x_j) \geq \sum_{j=1}^n f(y_j).$$

Now by  $r < 0$ ,

$$\left(\sum_{j=1}^n x_j^r\right)^{1/r} \leq C \left(\sum_{j=1}^n y_j^r\right)^{1/r},$$

which proves our claim.  $\square$

**Theorem 14.** *Let  $r < 0$  and  $v$  be defined as above. Then for  $0 < \alpha < -\frac{1}{r}$ ,*

$$\sup_{k \geq 1} \frac{\frac{1}{k} \sum_{j=1}^k v_j}{(\frac{1}{k} \sum_{j=1}^k v_j^r)^{1/r}} = \frac{(1 + \alpha r)^{1/r}}{1 + \alpha}.$$

*If  $r = q^*$ , then this is the exact value of  $M^{(q)}(d(v, 1))$ .*

**Proof.** By Lemma 12,  $(\frac{v_j}{u_j})$  is increasing, and then by Lemma 13,

$$\frac{\sum_{j=1}^k u_j}{(\sum_{j=1}^k u_j^r)^{1/r}} \leq \frac{\sum_{j=1}^k v_j}{(\sum_{j=1}^k v_j^r)^{1/r}},$$

that is  $A_k \leq B_k$ . Thus from Theorem 11,

$$A_k \leq B_k \leq \sup_{k \geq 1} B_k \leq \frac{(1 + \alpha r)^{1/r}}{1 + \alpha}.$$

Therefore using Proposition 10,

$$\lim_{k \rightarrow \infty} B_k = \frac{(1 + \alpha r)^{1/r}}{1 + \alpha},$$

so the claim is proved.  $\square$

**Theorem 15.** Let  $r < 0$  and  $u$  be defined as above. Then for  $0 < \alpha < -\frac{1}{r}$ ,

$$\sup_{k \geq 1} \frac{\frac{1}{k} \sum_{j=1}^k u_j}{\left(\frac{1}{k} \sum_{j=1}^k u_j^r\right)^{1/r}} = \frac{(1 + \alpha r)^{1/r}}{1 + \alpha}.$$

If  $r = q^*$ , then this is the exact value of  $M^{(q)}(d(u, 1))$ .

**Proof.** For all  $k \geq 1$ , as noted in the previous proof,  $A_k \leq B_k \leq \frac{(1+\alpha r)^{1/r}}{1+\alpha}$ , and so by Proposition 10,

$$\sup_{k \geq 1} A_k \leq \frac{(1 + \alpha r)^{1/r}}{1 + \alpha} = \lim_{k \rightarrow \infty} A_k,$$

and we are done.  $\square$

**Theorem 16.** Let  $0 < q < p$  and  $\frac{1}{r} = 1 - \frac{p}{q}$ . Then  $d(u, p)$  and  $d(v, p)$  are  $q$ -convex for  $0 < \alpha < -\frac{1}{r}$  and not  $q$ -convex for  $\alpha \geq -\frac{1}{r}$ .

**Proof.** By Theorems 8, 14 and 15, if  $0 < \alpha < -\frac{1}{r}$ ,

$$M^{(q)}(d(u, p)) = M^{(q)}(d(v, p)) = \frac{(1 + \alpha r)^{1/r}}{1 + \alpha},$$

so the  $q$ -convexity constants are bounded and the spaces are  $q$ -convex.

For the space  $d(u, p)$ , in the case when  $\alpha = -\frac{1}{r} = \frac{1}{|r|}$ ,

$$\begin{aligned} A_k &= \frac{1}{k^{1-1/r}} \frac{\sum_{j=1}^k j^{1/|r|}}{\left(\sum_{j=1}^k \frac{1}{j}\right)^{-1/|r|}} = \frac{1}{k^{1-1/r}} \left(\sum_{j=1}^k j^{1/|r|}\right) \left(\sum_{j=1}^k \frac{1}{j}\right)^{1/|r|} \geq \frac{1}{k^{1-1/r}} \left(\int_0^k t^{1/|r|} dt\right) \left(\int_1^{k+1} \frac{1}{t} dt\right)^{1/|r|} \\ &= \frac{1}{k^{1+1/|r|}} \left(\frac{k^{\frac{1}{|r|}+1}}{\frac{1}{|r|}+1}\right) (\ln(k+1))^{\frac{1}{|r|}} = \frac{|r|}{|r|+1} (\ln(k+1))^{\frac{1}{|r|}}. \end{aligned}$$

The right side is unbounded as  $k \rightarrow \infty$ , so in view of Theorem 8, the space is not  $q$ -convex.

Consider now the space  $d(v, p)$ . Since by definition  $v_n = \int_{n-1}^n (1 + \alpha)t^\alpha dt$ ,

$$(1 + \alpha)(n - 1)^\alpha \leq v_n \leq (1 + \alpha)n^\alpha. \tag{17}$$

Then by  $r < 0$ ,

$$(1 + \alpha)^r \sum_{j=1}^k j^{\alpha r} \leq \sum_{j=1}^k v_j^r \leq (1 + \alpha)^r \sum_{j=0}^{k-1} j^{\alpha r}.$$

Since  $\alpha = -\frac{1}{r}$ , we have that  $V_k = k^{1+\alpha}$  and

$$(1 + \alpha)^r \sum_{j=1}^k j^{-1} \leq \sum_{j=1}^k v_j^r \leq (1 + \alpha)^r \sum_{j=0}^{k-1} j^{-1}.$$

Therefore

$$\begin{aligned} B_k &= \frac{1}{k^{1-1/r}} \frac{k^{1-1/r}}{\left(\sum_{j=1}^k v_j^r\right)^{1/r}} \geq \frac{1}{(1 + \alpha)\left(\sum_{j=1}^k \frac{1}{j}\right)^{1/r}} = \frac{1}{(1 + \alpha)\left(\sum_{j=1}^k \frac{1}{j}\right)^{-1/|r|}} \\ &= \frac{1}{1 + \alpha} \left(\sum_{j=1}^k \frac{1}{j}\right)^{1/|r|} \geq \frac{1}{1 + \alpha} \left(\int_1^{k+1} \frac{1}{t} dt\right)^{1/|r|} = \frac{1}{1 + \alpha} (\ln(k+1))^{1/|r|}. \end{aligned}$$

Again, the right side is unbounded as  $k \rightarrow \infty$ , so the space is not  $q$ -convex. If  $\alpha > -\frac{1}{r}$ , then by Eq. (15),

$$\frac{U_k}{k^{1-1/r}} \geq \frac{1}{k^{1-1/r}} \frac{k^{1+\alpha}}{1+\alpha} = \frac{k^{\alpha+1/r}}{1+\alpha}.$$

Since  $\alpha + 1/r > 0$ , the left side is unbounded as  $k \rightarrow \infty$ , so  $A_k$  is unbounded. Similar calculations apply to  $d(v, p)$ .  $\square$

For  $0 < p, q < \infty$ , we define the classical Lorentz sequence spaces  $l_{q,p}$  analogously to  $L_{q,p}$ , that is as the space  $d(w, p)$  with  $w = (w_n)$  such that  $w_n = n^{p/q-1}$ . In the sequence case we cannot expect to obtain the constants as easy as for function spaces (compare Corollary 5 here and Corollary 3.7 in [13]). However applying results from [6] and the previous theorem, we get the concavity and convexity constants for the space  $l_{q,1}$ .

### Corollary 17.

(1) If  $q \geq 1$ , then for  $s > q$ ,

$$M^{(s)}(l_{q,1}) = \frac{1}{q[(\frac{1}{q} - 1)r + 1]^{1/r}},$$

where  $\frac{1}{r} + \frac{1}{s} = 1$ . For  $s \leq q$ , the space is not  $s$ -concave.

(2) If  $0 < q < 1$ , then for  $s < q$ ,

$$M^{(s)}(l_{q,1}) = q \left[ \left( \frac{1}{q} - 1 \right) r + 1 \right]^{1/r},$$

where  $\frac{1}{r} + \frac{1}{s} = 1$ . For  $s \geq q$ , the space is not  $s$ -convex.

**Proof.** (1) Applying Theorem 6 in [6] for  $\alpha = 1 - \frac{1}{q}$  and  $r = s^*$ , we obtain the  $s$ -concavity constant. By Proposition 4 in [6], the space is not  $s$ -concave for  $s \leq q$ .

(2) Applying Theorem 16 for  $\alpha = \frac{1}{q} - 1$  and  $r = s^*$ , the claim is true.  $\square$

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