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Multiple solutions for semilinear elliptic boundary value problems with double resonance

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ABSTRACT

In this paper we study the multiplicity of nontrivial solutions of semilinear elliptic boundary value problems which may be double resonance near infinity between two consecutive eigenvalues of $-\Delta$ with zero Dirichlet boundary data. The methods we use here are Morse theory, minimax methods and bifurcation theory.

1. Introduction

In this paper we study the multiplicity of solutions to the semilinear elliptic boundary value problem

$$(P)\quad \begin{cases} -\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$ and $f \in C^1(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfies the subcritical growth condition

$$\left|f(x,t)\right| \leqslant C\left(1+|t|^{p-1}\right), \quad x \in \Omega,\ 1 < p < 2^* = \begin{cases} \frac{2N}{N-2}, & N \geqslant 3, \\ \infty, & N = 1, 2. \end{cases}$$

It follows that the functional,

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(x,u)\,dx, \quad u \in H_0^1(\Omega),$$

is well-defined and of C^2, and the solutions of (P) are exactly the critical points of I, where $F(x,t) = \int_0^t f(x,s)\,ds$ and the Sobolev space $H_0^1(\Omega)$ is a Hilbert space with standard norm and inner product.

We assume that $f(x,0) \equiv 0$ which implies that (P) has a trivial solution $u = 0$ and we are interested in the existence of nontrivial solutions. The existence of nontrivial solutions of (P) depends on the local properties of f or F near the origin and infinity. In the current paper, we consider the multiplicity of nontrivial solutions of (P) in the sense that near infinity (P) may be double resonant between two consecutive eigenvalues of $-\Delta$ in $H_0^1(\Omega)$.

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Now we state the conditions and conclusions in this paper. Denoted by $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots$ the distinct eigenvalues of $-\Delta$ in $H_0^1(\Omega)$. For $j \in \mathbb{N}$, $H_0^1(\Omega)$ can be decomposed as

$$H_0^1(\Omega) = E_j^- \oplus E^j \oplus E^{j+1} \oplus E_j^+,$$

where

$$E^j = \mathrm{Ker}(-\Delta - \lambda_j), \qquad E_j^- = \bigoplus_{i<j} E^i, \qquad E_j^+ = \overline{\bigoplus_{i>j+1} E^i}.$$

Any function $u \in H_0^1(\Omega)$ can be written as

$$u = u^- + u^j + u^{j+1} + u^+, \qquad u^\pm \in E_j^\pm, \quad u^j \in E^j, \ u^{j+1} \in E^{j+1}.$$

We impose on f the following global conditions.

(f_1) There exists $k \geqslant 2$ such that

$$\lambda_k \leqslant \liminf_{|t|\to\infty} \frac{f(x,t)}{t} \leqslant \limsup_{|t|\to\infty} \frac{f(x,t)}{t} \leqslant \lambda_{k+1} \quad \text{uniformly for } x \in \Omega.$$

(f_2) If $\frac{\|u_n^k\|}{\|u_n\|} \to 1$ as $\|u_n\| \to \infty$, then there exist $\delta_1, N_1 > 0$ such that

$$\int_\Omega \big(f(x,u_n) - \lambda_k u_n\big) u_n^k dx \geqslant \delta_1, \quad n \geqslant N_1, \ x \in \Omega.$$

(f_3) If $\frac{\|u_n^{k+1}\|}{\|u_n\|} \to 1$ as $\|u_n\| \to \infty$, then there exist $\delta_2, N_2 > 0$ such that

$$\int_\Omega \big(\lambda_{k+1} u_n - f(x,u_n)\big) u_n^{k+1}\, dx \geqslant \delta_2, \quad n \geqslant N_2, \ x \in \Omega.$$

The main results in this paper are the following theorems. First we consider the case that $u = 0$ is a local minimizer.

Theorem 1.1. *Assume that (f_1)–(f_3) hold and $f_t'(x,0) < \lambda_1$. Then (P) has at least three nontrivial solutions in which one is positive and one is negative.*

Next we consider the case that $f_t'(x,0) = \lambda_m$ which means the trivial solution $u = 0$ is degenerate or (P) is resonant at the origin. We need a local sign condition.

$(F_0^\pm)$ There exists $\delta > 0$ such that

$$\pm\big(2F(x,t) - \lambda_m t^2\big) \geqslant 0, \quad |t| \leqslant \delta, \ x \in \Omega.$$

Moreover, we also need the following hypotheses.

(f_0) There exists $t_0 \neq 0$ such that $f(x,t_0) = 0$ for $x \in \Omega$.

Theorem 1.2. *Assume that (f_0)–(f_3) hold and $f_t'(x,0) = \lambda_m$. Then (P) has at least four nontrivial solutions in each of the following cases*

(i) (F_0^+) *with* $2 \leqslant m \neq k$;
(ii) (F_0^-) *with* $2 < m \neq k+1$.

As one will see below that the conclusion of Theorem 1.2 holds for $f_t'(x,0) \in (\lambda_m, \lambda_{m+1})$ with $m \neq k$ without the sign condition $(F_0^\pm)$. We can get more solutions for (P) when $f_t'(x,0) := \lambda$ is very close to λ_m.

Theorem 1.3. *Assume that (f_0)–(f_3) hold. Then there is $\epsilon > 0$ such that (P) has at least six nontrivial solutions in each of the following cases*

(i) $\lambda \in (\lambda_m - \epsilon, \lambda_m)$ *with* $m > 2, m \neq k, k+1$, *and*

$$(f_4) \quad \big(f(x,t) - \lambda t\big)t > 0, \quad |t| > 0 \text{ small}, \ x \in \Omega;$$

(ii) $\lambda \in (\lambda_m, \lambda_m + \epsilon)$ *with* $m \geqslant 2$, $m \neq k, k+1$, *and*

$$(f_5) \quad \big(f(x,t) - \lambda t\big)t < 0, \quad |t| > 0 \text{ small}, \ x \in \Omega.$$

Now we give some remarks and comments. Resonance is a very common natural phenomenon existing in the real world from macrocosm to microcosm. This physical phenomenon may take human serious disasters as well as great benefits if one develops scientific technology to abstain, control and utilize it. When an model is explained by a semilinear problem, the mathematical feature of resonance lies in the interactions between linear spectrum and the nonlinearity.

Semilinear elliptic equation with resonance has its own meanings. In population biology, for example (see [21]), this kind of problem often arise in the study of a steady-state population density u where $f(x,u)/u$ represents a population dependent growth rate, and $f'(x,0)$ represents a growth rate in the absence of certain environmental restrictions such as crowding.

Resonance problem have received much attention in the literature since the appearance of the pioneering paper by Landesman and Lazer [10]. Many authors considered the complete resonance situation in the sense that

$$\lim_{|t|\to\infty} \frac{f(x,t)}{t} = \lambda_k \quad \text{uniformly for } x \in \Omega \tag{1.1}$$

via different methods such as Minimax methods [18], Morse theory [6,16] and topological theory. The main difficulty caused by the resonance phenomena lies in verification of the global compactness for the associated energy functional I when minimax methods or Morse theory is applied. To ensure the global compactness, one needs to impose various conditions on the nonlinearity f or F near infinity. When $g(x,t) := f(x,t) - \lambda_k t$ is bounded, the global compactness of I is ensured by the condition

$$\lim_{\|v\|\to\infty} \int_\Omega G(x,v)\,dx = \pm\infty, \quad v \in E^k, \tag{1.2}$$

where $G(x,t) = \int_0^t g(x,s)\,ds$. These are related to the famous Landesman–Lazer resonance conditions and have been used in [1,7,11,27]. For g unbounded, global compactness of I is guaranteed by the conditions

$$\pm g(x,t)t \geqslant 0, \qquad C_1|t|^r \leqslant \big|g(x,t)\big| \leqslant C_2|t|^r, \quad x \in \Omega, \ |t| \geqslant R, \tag{1.3}$$

where $C_1, C_2, R > 0$ and $r \in (0,1)$ are constants; these conditions were first introduced in [25] (see [22] for a stronger version). For other conditions imposed on the nonlinearity near infinity, we refer to [2,6,7,22,24,26] and references therein. From (f_1) we regard that (P) may be double resonant near infinity between two consecutive eigenvalues. Notice that (1.1) is a special case of (f_1). Under the conditions (f_1)–(f_3), Robinson [21] obtained, by using Leray–Schauder degree approach, the solvability result for (P). In [21], the existence of two nontrivial solutions was also obtained when $k = 1$ and $f_t'(x,0) \in (\lambda_m, \lambda_{m+1})$ with $m \geqslant 2$, i.e., the trivial solution was nondegenerate with Morse index large than 2. This is necessary when degree argument is involved. The results in [21] extended the early work in [9]. Under (f_1) and some stronger conditions than (f_2) and (f_3), Su [23] proved by using Morse theory some multiplicity results for (P) including an interesting case that $f_t'(x,0) = \lambda_m$ and $m = k$, i.e., (P) may be resonant around a same eigenvalue near the origin and near infinity.

Our results Theorems 1.1 and 1.2 in the current paper extend the results in [23] and we use different methods from that used in [21]. In Theorem 1.3, we investigate the situation when $f_t'(x,0)$ is very closed to an eigenvalue and get more solutions for (P) by combining Morse theory, minimax methods and bifurcation method. This result is completely new and the idea was first used in [20]. With the three theorems, most situations have been studied when $f_t'(x,0) \in R$.

The paper is organized as follows. In Section 2 we give a simple revisit to Morse theory and in Section 3 we give some lemmas. The proofs of main results will be given in the last section.

2. Preliminaries about Morse theory

Let H be a Hilbert space and $I \in C^2(H,\mathbb{R})$ be a functional possessing the deformation properties [2] which follows from the Palais–Smale condition ((PS) in short) or Cerami condition ((C) in short) introduced in [5]. Denote by $H_q(A,B)$ the qth singular relative homology group of the topological pair (A,B) with coefficients in a field $\mathbb{F}$. Let u be an isolated critical point of I with $I(u) = c \in \mathbb{R}$. The group,

$$C_q(I,u) := H_q\big(I^c, I^c \setminus \{u\}\big), \quad q \in \mathbb{Z},$$

is called the qth critical group of I at u, where $I^c = \{u \in H \mid I(u) \leqslant c\}$. Denote

$$\mathcal{K} = \big\{u \in H \bigm| I'(u) = \theta\big\}.$$

Assume that $\mathcal{K}$ is a finite set. Take $a < \inf I(\mathcal{K})$. The critical groups of I at infinity are defined by [2]

$$C_q(I,\infty) := H_q\big(H, I^a\big), \quad q \in \mathbb{Z}.$$

The relationship between the Morse type numbers

$$M_q := \sum_{u\in\mathcal{K}} \operatorname{rank} C_q(I,u), \quad q \in \mathbb{Z}$$

and the Betti numbers $\beta_q := \operatorname{rank} C_q(I,\infty)$ is described by the following Morse inequalities [6,16].

$$\sum_{j=0}^{q}(-1)^{q-j}M_j \geqslant \sum_{j=0}^{q}(-1)^{q-j}\beta_j, \quad q \in \mathbb{Z}, \tag{2.1}$$

$$\sum_{q=0}^{\infty}(-1)^{q}M_q = \sum_{q=0}^{\infty}(-1)^{q}\beta_q. \tag{2.2}$$

Let $u \in \mathcal{K}$ be an isolated critical point of I such that $I''(u)$ is a Fredholm operator and the Morse index $\mu(u)$ and the nullity $\nu(u)$ of u are finite. The following facts are known [6,8,16].

(i) $C_q(I,u) \cong 0$, $q \notin [\mu(u), \mu(u)+\nu(u)]$.
(ii) If u is nondegenerate, i.e., $\nu(u) = 0$, then $C_q(I,0) \cong \delta_{q,\mu(u)}\mathbb{F}$.

When u is degenerate, without additional conditions, there would be nothing known about the groups for $q \in [\mu(u), \mu(u)+\nu(u)]$. Recently it was discovered in [23] that the critical groups of I at a degenerate critical point u can be described completely when I has a *local linking* structure at u, a concept introduced in [14] (see also [12]). We state this result for $u = 0$ (in applications the origin is always a trivial critical point). Recall that I is said to have a local linking structure at 0 with respect to the direct sum decomposition $H = H^- \oplus H^+$ if there exists $r > 0$ such that

$$I(u) > 0 \quad \text{for } u \in H^+,\ 0 < \|u\| \leqslant r, \qquad I(u) \leqslant 0 \quad \text{for } u \in H^-,\ \|u\| \leqslant r. \tag{2.3}$$

Proposition 2.1. *(See [23].) Let* 0 *be an isolated critical point of $I \in C^2(E,\mathbb{R})$ with Morse index μ_0 and nullity ν_0. Assume I has a local linking structure at* 0 *with respect to the decomposition $H = H^- \oplus H^+$ and $k = \dim H^- < \infty$. If $k = \mu_0$ or $k = \mu_0 + \nu_0$ then*

$$C_q(I,0) \cong \delta_{q,k}\mathbb{F}, \quad q \in \mathbb{Z}.$$

When I is C^1 and has a local linking structure as (2.3) with $k = \dim H^- < \infty$, it was discovered in [15] that $C_k(I,0) \ncong 0$ which is crucial in obtaining the above conclusion.

3. Compactness and critical group at infinity

In this section we verify the compactness for the energy functional I corresponding to (P) and then compute the critical group of I at infinity. Recall that in the current paper the functional,

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega F(x,u)\,dx, \quad u \in H_0^1(\Omega) := H,$$

is well-defined and is of C^2 with derivatives given by

$$\big\langle I'(u), v\big\rangle = \int_\Omega \nabla u\nabla v\,dx - \int_\Omega f(x,u)v\,dx, \quad u, v \in H,$$

$$\big\langle I''(u)v, w\big\rangle = \int_\Omega \nabla v\nabla w\,dx - \int_\Omega f_t'(x,u)vw\,dx, \quad u, v, w \in H.$$

We first verify that I possesses the compactness in the sense of Cerami [5], i.e., every sequence $\{u_n\} \subset H$ such that

$$\big|I(u_n)\big| \leqslant c, \qquad \big(1+\|u_n\|\big)I'(u_n) \to 0 \quad \text{as } n \to \infty \tag{3.1}$$

has a convergent subsequence. We always use c to denote various positive constants throughout the paper.

Lemma 3.1. *Assume that f satisfies (f_1)–(f_3). Then the functional I satisfies* (C)*.*

Proof. The idea of the following proof is due to Robinson in [21]. Let $\{u_n\} \subset H$ satisfy (3.1). We only need to show that $\{u_n\}$ is bounded since f has a subcritical growth [18]. Suppose, by the way of contradiction, that $\|u_n\| \to \infty$ as $n \to \infty$. Setting $v_n = u_n/\|u_n\|$, then $\|v_n\| \equiv 1$ and, up to a subsequence if necessary, we may assume that there exists $v \in H_0^1(\Omega)$ such that

$$\begin{aligned} &v_n \rightharpoonup v \quad \text{in } H_0^1(\Omega),\\ &v_n \to v \quad \text{in } L^2(\Omega),\\ &v_n(x) \to v(x) \quad \text{a.e. } x \in \Omega. \end{aligned} \tag{3.2}$$

It follows from (f_1) that there exists c such that

$$\left|f(x,t)\right| \leqslant c\big(1+|t|\big), \quad t \in \mathbb{R},\ x \in \Omega.$$

Therefore for n large enough,

$$\frac{|f(x,u_n(x))|}{\|u_n\|} \leqslant c\big(1+\left|v_n(x)\right|\big) \quad \text{a.e. } x \in \Omega, \tag{3.3}$$

and then $\{f(x,u_n)/\|u_n\|\}$ is bounded in $L^2(\Omega)$. By (3.1) we have that

$$\left|\int_\Omega \nabla v_n \nabla(v_n - v)\,dx - \int_\Omega \frac{f(x,u_n)}{\|u_n\|}(v_n - v)\,dx\right| = \left|\left\langle \frac{I'(u_n)}{\|u_n\|}, v_n - v\right\rangle\right| \leqslant \frac{\|I'(u_n)\|(1+\|v\|)}{\|u_n\|} \to 0 \quad \text{as } n \to \infty. \tag{3.4}$$

It follows from (3.2) and (3.3) that

$$\int_\Omega \frac{f(x,u_n)}{\|u_n\|}(v_n - v)\,dx \to 0 \quad \text{as } n \to \infty.$$

So from (3.5), it is easy to see that

$$(v_n, v_n - v) = \int_\Omega \nabla v_n \nabla(v_n - v)\,dx \to 0 \quad \text{as } n \to \infty.$$

Hence $\|v_n\| \to \|v\|$ as $n \to \infty$. So $v_n \to v$ in $H_0^1(\Omega)$ and $\|v\| = 1$.

By (3.2)–(3.4) and (f_1), there is some function $p \in L^2(\Omega)$ with

$$\lambda_k \leqslant p(x) \leqslant \lambda_{k+1}, \quad \text{a.e. } x \in \Omega$$

such that

$$\frac{f(x,u_n)}{\|u_n\|} \rightharpoonup pv \quad \text{in } L^2(\Omega),\ n \to \infty. \tag{3.5}$$

It follows that for all $\varphi \in H_0^1(\Omega)$,

$$\int_\Omega \nabla v \varphi\,dx - \int_\Omega pv\varphi\,dx = 0.$$

Therefore v is a nontrivial solution of the linear problem

$$\begin{cases} -\Delta v = pv & \text{in } \Omega,\\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

By a unique continuation property and the maximum principle, we see either $p \equiv \lambda_k$ or $p \equiv \lambda_{k+1}$.

If $p \equiv \lambda_k$, then $v \in E^k$ and $\frac{\|u_n^k\|}{\|u_n\|} \to 1$ as $n \to \infty$. Using (3.1), we obtain that

$$\int_\Omega \big(f(x,u_n) - \lambda_k u_n\big)u_n^k\,dx = -\big\langle I'(u_n), u_n^k\big\rangle \leqslant \|u_n\|\left\|I'(u_n)\right\| \to 0 \quad \text{as } n \to \infty,$$

which contradicts to (f_2). If $p \equiv \lambda_{k+1}$, we will get a conclusion contradicting to (f_3). The proof is completed. □

Now we compute the critical groups of the functional I at infinity. We first recall a homotopy result on critical group which is a slight generalization of that in [17].

Proposition 3.2. *Let H be Hilbert space and $\{I_t \in C^1(H,\mathbb{R}) \mid t \in [0,1]\}$ a family of functionals such that I_t' and $\partial_t I_t$ are locally Lipschitz continuous. Assume I_0 and I_1 satisfy* (C). *If there exist $a \in \mathbb{R}$ and $\delta > 0$ such that*

$$I_t(u) \leqslant a \quad \Rightarrow \quad \big(1+\|u\|\big)\big\|I_t'(u)\big\| \geqslant \delta, \quad t \in [0,1],$$

then

$$C_q(I_0,\infty) = C_q(I_1,\infty), \quad q \in \mathbb{Z}. \tag{3.6}$$

In particular, if there exists $R > 0$ such that

$$\inf_{t\in[0,1],\,\|u\|>R}\big(1+\|u\|\big)\big\|I_t'(u)\big\| > 0, \qquad \inf_{t\in[0,1],\,\|u\|\leqslant R} I_t(u) > -\infty,$$

then (3.7) *holds.*

Proof. For completeness, we sketch out the proof. Let $\eta(t,u)$ be the flow generated by the Cauchy problem

$$\begin{cases} \dot\eta = -\dfrac{\partial_t I_t(\eta(t,u))}{\|I_t'(\eta(t,u))\|^2} I_t'\big(\eta(t,u)\big), \\ \eta(0,u) = u \in I_0^a. \end{cases}$$

Then

$$\frac{d}{dt} I_t\big(\eta(t,u)\big) = \big\langle I_t'\big(\eta(t,u)\big), \dot\eta\big\rangle + \partial_t I_t\big(\eta(t,u)\big) = 0.$$

Hence,

$$I_t\big(\eta(t,u)\big) = I_0\big(\eta(0,u)\big) = I_0(u).$$

In particular, since $I_t(\eta(t,u)) \leqslant a$, this flow exists for $t \in [0,1]$. It can be reversed by replacing I_t with I_{1-t}. Thus $\eta(1,\cdot)$ is a homeomorphism of I_0^a onto I_1^a. It follows that

$$C_q(I_0,\infty) = H_q\big(H, I_0^a\big) \cong H_q\big(H, I_1^a\big) = C_q(I_1,\infty), \quad q \in \mathbb{Z}.$$

Lemma 3.3. *Assume that f satisfies (f_1)–(f_3). Then*

$$C_q(I,\infty) \cong \delta_{q,\mu_\infty}\mathbb{F}, \quad \mu_\infty = \dim E_{k+1}^-.$$

Proof. Define the family of functionals $I_t : H_0^1(\Omega) \to \mathbb{R}$, $t \in [0,1]$ as follows:

$$I_t(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1-t}{4}\lambda_k \int_\Omega u^2\,dx - \frac{1-t}{4}\lambda_{k+1}\int_\Omega u^2\,dx - t\int_\Omega F(x,u)\,dx.$$

Firstly, we want to show that there exists $R > 0$ such that

$$\inf_{t\in[0,1],\,\|u\|>R}\big(1+\|u\|\big)\big\|I_t'(u)\big\| > 0.$$

If not, then there exist $\{u_n\} \subset H_0^1(\Omega)$ and $t_n \in [0,1]$ such that

$$\|u_n\| \to \infty \quad \text{and} \quad \big(1+\|u_n\|\big)I_{t_n}'(u_n) \to 0 \quad \text{as } n \to \infty.$$

Let $v_n = u_n/\|u_n\|$. Similar to the proof of Lemma 3.1, we may assume that there exists $v \in H_0^1(\Omega)$ such that (3.2) holds. Noting that $\{t_n\}$ is bounded and

$$\left|\left\langle \frac{I_{t_n}'(u_n)}{\|u_n\|}, v_n - v\right\rangle\right| \leqslant \frac{\|I_{t_n}'(u_n)\|(1+\|v\|)}{\|u_n\|} \to 0 \quad \text{as } n \to \infty,$$

we have $v_n \to v$ in $H_0^1(\Omega)$ and $v \neq 0$. Since $\{f(x,u_n)/\|u_n\|\}$ is bounded in $L^2(\Omega)$ by (3.3), we may also assume that there exists $p \in L^2(\Omega)$ with $\lambda_k \leqslant p(x) \leqslant \lambda_{k+1}$ for a.e. $x \in \Omega$ such that

$$\frac{f(x,u_n)}{\|u_n\|} \rightharpoonup pv, \quad \text{in } L^2(\Omega).$$

Suppose that $t_n \to t_0$. With the same argument in Lemma 3.1, we know that $v \neq 0$ is a nontrivial solution of

$$\begin{cases} -\triangle v = \zeta(t_0)v, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\zeta(t_0) = \frac{1-t_0}{2}\lambda_k + \frac{1-t_0}{2}\lambda_{k+1} + t_0 p$ and satisfies $\lambda_k \leqslant \zeta(t_0) \leqslant \lambda_{k+1}$. Using a unique continuation property and the maximum principle, we have either $\zeta(t_0) \equiv \lambda_k$ or $\zeta(t_0) \equiv \lambda_{k+1}$. In both case we have that $t_0 = 1$, i.e., $t_n \to 1$ as $n \to \infty$. Therefore we have that

$$\text{either} \quad \frac{\|u_n^k\|}{\|u_n\|} \to 1 \quad \text{or} \quad \frac{\|u_n^{k+1}\|}{\|u_n\|} \to 1.$$

It follows from (f_2) or (f_3) that either

$$\int_\Omega \big(f(x,u_n) - \lambda_k u_n\big)u_n^k\,dx \geqslant \delta_1 > 0, \quad n > N_1$$

or

$$\int_\Omega \big(\lambda_{k+1}u_n - f(x,u_n)\big)u_n^{k+1}\,dx \geqslant \delta_2 > 0, \quad n > N_2.$$

However, for n large enough,

$$\int_\Omega \big(f(x,u_n) - \lambda_k u_n\big)u_n^k\,dx = \frac{1-t_n}{2t_n}(\lambda_k - \lambda_{k+1})\big\|u_n^k\big\|^2_{L^2(\Omega)} + o(1)$$

or

$$\int_\Omega \big(\lambda_{k+1}u_n - f(x,u_n)\big)u_n^{k+1}\,dx = \frac{1-t_n}{2t_n}(\lambda_k - \lambda_{k+1})\big\|u_n^{k+1}\big\|^2_{L^2(\Omega)} + o(1).$$

These are contradictions since $\lambda_k < \lambda_{k+1}$.

It is obvious that

$$\inf_{t\in[0,1],\ \|u\|\leqslant R} I_t(u) > -\infty.$$

Furthermore, I_0 obviously satisfies (C) and so does I_1 by Lemma 3.1. Therefore from Proposition 3.2, we have

$$C_q(I,\infty) = C_q(I_1,\infty) \cong C_q(I_0,\infty), \quad \forall q \in \mathbb{Z}. \tag{3.7}$$

Noticing that

$$I_0(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{4}(\lambda_k + \lambda_{k+1})\int_\Omega u^2\,dx,$$

we see that $u = 0$ is a unique critical point of I_0 and is nondegenerate with Morse index $\mu_\infty = \dim E^-_{k+1}$. Therefore, we have that

$$C_q(I_0,\infty) \cong C_q(I_0,0) \cong \delta_{q,\mu_\infty}\mathbb{F}.$$

It follows from (3.8) that

$$C_q(I,\infty) \cong \delta_{q,\mu_\infty}\mathbb{F}.$$

The proof is completed. □

4. Proofs of the main results

In this section we give the proofs of the main results in this paper and give more comments. Our approaches will be the combination of Morse theory, Minimax methods and cut-off techniques. To this purpose, we need the following compactness lemmas for the functional corresponding to cut-off functions.

Lemma 4.1. *(See [23].) Let the function $g \in C(\bar\Omega \times \mathbb{R})$ be such that $g(x,t) = 0$ for $t < 0$, and satisfy, for $k \geqslant 2$,*

$$\lambda_k \leqslant \liminf_{t\to+\infty}\frac{g(x,t)}{t} \leqslant \limsup_{t\to+\infty}\frac{g(x,t)}{t} \leqslant \lambda_{k+1} \quad \textit{uniformly for } x \in \Omega. \tag{4.1}$$

Then the functional,

$$J(u)=\frac{1}{2}\int\limits_{\Omega}|\nabla u|^2\,dx-\int\limits_{\Omega}G(x,u)\,dx,$$

satisfies the (PS) *condition, where* $G(x,t)=\int_0^t g(x,s)\,ds$.

Lemma 4.2. *(See [23].) Let the function* $g\in C(\bar{\Omega}\times\mathbb{R})$ *be such that* $g(x,t)=0$ *for* $t>0$, *and satisfy, for* $k\geqslant 2$,

$$\lambda_k\leqslant\liminf_{t\to-\infty}\frac{g(x,t)}{t}\leqslant\limsup_{t\to-\infty}\frac{g(x,t)}{t}\leqslant\lambda_{k+1}\quad\textit{uniformly for } x\in\Omega. \tag{4.2}$$

Then the functional,

$$J(u)=\frac{1}{2}\int\limits_{\Omega}|\nabla u|^2\,dx-\int\limits_{\Omega}G(x,u)\,dx,$$

satisfies the (PS) *condition, where* $G(x,t)=\int_0^t g(x,s)\,ds$.

The proofs of the above two lemmas were referred to [23]. The ideas of the proofs were from [7] for the case that (1.1) holds.

From now on we prove Theorems 1.1–1.3.

Proof of Theorem 1.1. By $f_t'(x,0)<\lambda_1$, a direct computation shows that $u=0$ is a local minimizer of I and then

$$C_q(I,0)\cong\delta_{q,0}\mathbb{F}. \tag{4.3}$$

It follows from Lemma 3.3 that

$$C_q(I,\infty)\cong\delta_{q,\mu_\infty}\mathbb{F}. \tag{4.4}$$

Therefore by Morse theory, I has a critical point u^* satisfying

$$C_{\mu_\infty}\big(I,u^*\big)\ncong 0. \tag{4.5}$$

From (4.3) and (4.5) we see that $u^*\neq 0$.

Set $f^+(x,t)=f(x,t)$ for $t\geqslant 0$, $f^+(x,t)=0$ for $t<0$ and let $F^+(x,t)=\int_0^t f^+(x,s)\,ds$. Then the critical points of

$$I^+(u)=\frac{1}{2}\int\limits_{\Omega}|\nabla u|^2\,dx-\int\limits_{\Omega}F^+(x,u)\,dx,\quad u\in H$$

are exactly solutions of the problem

$$\big(P^+\big)\quad\begin{cases}-\Delta u=f^+(x,u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega,\end{cases}$$

and the nonnegative solutions (P^+) are solutions of (P). By Lemma 4.1, we see that $I^+\in C^{2-0}(H,\mathbb{R})$ satisfies the (PS) condition. Moreover, it follows from $f_t'(x,0)<\lambda_1$ and (f_1) that I^+ has a mountain pass geometry. Indeed, a direct calculation shows that there exist $\rho>0$, $\tau>0$ such that

$$I^+(u)\geqslant\tau,\quad u\in H \text{ with } \|u\|=\rho,$$

and

$$I^+(t\varphi)\to-\infty\quad\text{as } t\to+\infty,$$

where φ is the first eigenfunction of $-\Delta$ with zero Dirichlet boundary data. By Mountain Pass Theorem [18], I^+ has a critical point u^+. By the maximum principle $u^+>0$ and hence is a critical point of I. Furthermore, by using the results in [6] and [16] and the critical group property for a mountain pass point [6], we have

$$C_q\big(I,u^+\big)\cong\delta_{q,1}\mathbb{F}. \tag{4.6}$$

A same argument shows that I has a nontrivial critical point $u^-<0$ with

$$C_q\big(I,u^-\big)\cong\delta_{q,1}\mathbb{F}. \tag{4.7}$$

As $k\geqslant 2$ which implies $\mu_\infty\geqslant 2$, by comparing the critical groups, we see that $u^\pm$ and u^* are three nontrivial critical points of I. This completes the proof. □

Remark 4.3. (i) According to Gromoll–Meyer's result [8], it always holds that $\mu(u^*) \leqslant \mu_\infty \leqslant \mu(u^*) + \nu(u^*)$. Therefore, by Shifting theorem [6,16] and Morse inequality (2.1) and (2.2), we can get the conclusion that (P) has at least four nontrivial solutions in the cases that $\mu(u^*) = \mu_\infty$ or $\mu(u^*) + \nu(u^*) = \mu_\infty$ or $\mu(u^*) > 2$.

(ii) The conclusion of Theorem 1.1 is still true if $f'_t(x,t) = \lambda_1$ and $2F(x,t) \leqslant \lambda_1 t^2$ for $|t|$ small. See [26] for details.

Now we prove Theorem 1.2. We denote $\mu_0 = \dim E_m^-$ and $\nu_0 = \dim E^m$. It follows from $f'_t(x,0) = \lambda_m$ that $u = 0$ is a degenerate critical point of I with Morse index μ_0 and nullity ν_0. In order to apply Proposition 2.1, we need the following lemma about the local linking.

Lemma 4.4. *(See [12,23].) Assume $f'_t(x,0) = \lambda_m$ and (F_0^+) (or (F_0^-)). Then I has a local linking* (2.3) *at* 0 *with respect to the decomposition $H = H^+ \oplus H^-$ where $H^- = E_{m+1}^-$ (or $H^- = E_m^-$).*

Proof of Theorem 1.2. We give the proof for the case (i). The procedure consists of three steps.

Step 1. Now $u = 0$ is an isolated critical point with Morse index μ_0 and nullity ν_0. By Lemma 4.4 and Proposition 2.1, we get that

$$C_q(I,0) \cong \delta_{q,\mu_0+\nu_0}\mathbb{F}, \quad q \in \mathbb{Z}. \tag{4.8}$$

As in the proof of Theorem 1.1, I has a critical point u^* with

$$C_{\mu_\infty}\big(I,u^*\big) \ncong 0. \tag{4.9}$$

Notice that $m \neq k$ implies $\mu_\infty \neq \mu_0 + \nu_0$, it follows that $u^* \neq 0$.

Step 2. We may assume that $t_0 > 0$ in (f_0). We will find a local minimizer for I by a cut-off technique which is motivated by [7].

Define

$$\tilde f(x,t) = \begin{cases} 0 & t < 0, \\ f(x,t) & t \in [0,t_0], \\ 0 & t > t_0 \end{cases}$$

and

$$\tilde I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega \tilde F(x,u)\,dx, \quad u \in H,$$

where $\tilde F(x,t) = \int_0^t \tilde f(x,s)\,ds$. Since $\tilde I$ is coercive and weakly lower semi-continuous [18], there is a minimizer u_0 of $\tilde I$. By the maximum principle, $u_0 = 0$ or $0 < u_0(x) < t_0$ for all $x \in \Omega$ and $\frac{\partial u_0}{\partial n}|_{\partial\Omega} < 0$. By assumption $f'_t(x,0) = \lambda_m$ and $m \geqslant 2$, 0 is not a minimizer. In addition, u_0 is a local minimizer of I in the $C_0^1(\bar\Omega)$ topology. By [4], u_0 is a local minimizer of I in $H_0^1(\Omega)$ topology, therefore

$$C_q(I,u_0) \cong \delta_{q,0}\mathbb{F}, \quad q \in \mathbb{Z}. \tag{4.10}$$

Step 3. Define $\hat f(x,t) = f(x,t+u_0(x)) - f(x,u_0(x))$, $x \in \Omega$, $t \in \mathbb{R}$ and consider the equation

$$(\hat P) \quad \begin{cases} -\Delta v = \hat f(x,v), & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases}$$

and the corresponding energy functional

$$\hat I(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega \hat F(x,v)\,dx, \quad u \in H.$$

A simple calculation shows that if v is a positive critical point of $\hat I$ then $u_0 + v$ is a critical point of I, and moreover $C_q(\hat I, v) = C_q(I, u_0 + v)$.

Furthermore define

$$\hat f^+(x,t) = \begin{cases} \hat f(x,t), & t \geqslant 0, \\ 0, & t < 0, \end{cases}$$

and consider the equation

$$\big(\hat P^+\big) \quad \begin{cases} -\Delta v = \hat f^+(x,v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

and its energy functional

$$\hat{I}^+(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega \hat{F}^+(x,v)\,dx,$$

where $\hat{F}^+(x,t) = \int_0^t \hat{f}^+(x,s)\,ds$. By (f_1), we see that $\hat{f}$ satisfies

$$\lambda_k \leqslant \liminf_{t\to+\infty} \frac{\hat{f}(x,t)}{t} \leqslant \limsup_{t\to+\infty} \frac{\hat{f}(x,t)}{t} \leqslant \lambda_{k+1} \quad \text{uniformly for } x \in \Omega.$$

It follows from Lemma 4.1 that $\hat{I}^+$ satisfies the (PS) condition. Since $u_0 > 0$ is a local minimizer of I, $v = \theta$ is a strictly local minimizer of $\hat{I}^+$. By (f_1), we can also get

$$\hat{I}^+(t\varphi) \to -\infty, \quad \text{as } t \to +\infty,$$

as that in the proof of Theorem 1.1. Now applying the mountain pass lemma, we have that $\hat{I}^+$ has a critical point v^+, and by the maximum principle, $v^+ > 0$ and then is a critical point of $\hat{I}$. By the same argument as that in [6,23], we have

$$C_q(\hat{I}, v^+) \cong \delta_{q,1}\mathbb{F}, \quad q \in \mathbb{Z}.$$

Hence $u^+ = u_0 + v^+ > u_0$ is a critical point of I satisfying

$$C_q(I, u^+) \cong \delta_{q,1}\mathbb{F}, \quad q \in \mathbb{Z}. \tag{4.11}$$

In a similar way, we have that I has a critical point $u^- < u_0$ satisfying

$$C_q(I, u^-) \cong \delta_{q,1}\mathbb{F}, \quad q \in \mathbb{Z}. \tag{4.12}$$

Finally by comparing the critical groups and by using the condition $m, k \geqslant 2$ with $m \neq k$, u^*, u_0 and $u^\pm$ are four nontrivial critical points of I in which two are positive. The proof is completed. □

Remark 4.5. (i) In Theorem 1.2 we assume that $\mu_0, \mu_\infty \geqslant 2$. The conclusion of Theorem 1.2 is valid if $u = 0$ is degenerate with Morse index μ_0 and nullity ν_0 satisfying $\mu_\infty \notin [\mu_0, \mu_0 + \nu_0]$ without other conditions near the origin.

(ii) The conclusion of Theorem 1.2 is valid provided $f_t'(x,0) := \lambda \in (\lambda_m, \lambda_{m+1})$ and $2 \leqslant m \neq k$, that is, $u = 0$ is a nondegenerate critical point of I with Morse index $\mu_0 = \dim E_{m+1}^- \geqslant 2$. We will use this result in the proof of Theorem 1.3.

(iii) Although we know the Morse index $\mu(u^*)$ and nullity $\nu(u^*)$ of I at u^* are finite, in general, we do not know them exactly. It would be interesting to give certain conditions that can be used to control these indices. Anyway, (P) will has a fifth nontrivial solution if $\mu(u^*) = \mu_\infty$ or $\mu(u^*) + \nu(u^*) = \mu_\infty$. It is the case for the spatial dimension $N = 1$. See [13].

Proposition 4.6. *(See [18, Theorem 11.35].) Let H be Hilbert space and $I \in C^2(H, \mathbb{R})$ with*

$$\nabla I(u) = Lu + T(u),$$

where $L \in \mathcal{L}(H, H)$ is symmetric and $T(u) = o(\|u\|)$ as $\|u\| \to 0$. Consider the operator equation

$$Lu + T(u) = \lambda u. \tag{4.13}$$

Let $\mu \in \sigma(L)$ be an isolated eigenvalue of finite multiplicity. Then either

(i) *$(\mu, 0)$ is not an isolated solution of (4.13) in $\{\mu\} \times E$, or*
(ii) *there is an one-sided neighborhood Λ of μ such that for all $\lambda \in \Lambda \setminus \{\mu\}$, (4.13) has at least two distinct nontrivial solutions, or*
(iii) *there is a neighborhood Λ of μ such that for all $\lambda \in \Lambda \setminus \{\mu\}$, (4.13) has at least one nontrivial solution.*

Now we apply Proposition 4.6 to get two solutions of (P) and meanwhile give the estimates of their Morse indices and nullities.

Lemma 4.7. *Let f satisfies $f_t'(x,0) = \lambda$. Then there exists $\epsilon > 0$ such that (P) has at least two nontrivial solutions u_i^λ $(i = 1, 2)$ in each of the following cases*

(i) *$\lambda \in (\lambda_\ell - \epsilon, \lambda_\ell)$ with $\ell \geqslant 1$ and*

$$(f_4) \quad \big(f(x,t) - \lambda t\big)t > 0, \quad |t| > 0 \text{ small}, \ x \in \Omega;$$

(ii) *$\lambda \in (\lambda_\ell, \lambda_\ell + \epsilon)$ with $\ell \geqslant 1$ and*

$$(f_5) \quad \big(f(x,t) - \lambda t\big)t < 0, \quad |t| > 0 \text{ small}, \ x \in \Omega.$$

Furthermore, the Morse indices $\mu(u_i^\lambda)$ and nullities $\nu(u_i^\lambda)$ of u_i^λ satisfy

$$\dim E_\ell^- \leqslant \mu(u_i^\lambda) \leqslant \mu(u_i^\lambda) + \nu(u_i^\lambda) \leqslant \dim E_{\ell+1}^-. \tag{4.14}$$

Proof. We can rewrite (P) as

$$\begin{cases} -\Delta u = \lambda u + h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.15}$$

where $h(x,t) = f(x,t) - \lambda t$. We have $h(x,0) = h'(x,0) = 0$. Hence for any λ lies in a finite interval, we have that for $v \in H$,

$$\langle T(u), v\rangle = -\int_\Omega h(x,u)v\,dx = o(\|u\|)\|v\|, \quad u \in H,\ \|u\| \to 0.$$

Moreover, every eigenvalue λ_j of $-\Delta$ in $H_0^1(\Omega)$ gives rise to a bifurcation point $(\lambda_j, 0)$ of (4.15). Using the arguments in [20] we can verify that the case (ii) of Proposition 4.6 occurs under the given conditions (f_4) or (f_5). Therefore for λ is very close to λ_ℓ, (P) has two bifurcation solutions u_i^λ $(i = 1, 2)$ which are nontrivial and satisfy

$$\|u_i^\lambda\| \to 0, \quad \lambda \to \lambda_\ell. \tag{4.16}$$

By standard elliptic regularity theory, we have that

$$\|u_i^\lambda\|_C \to 0, \quad \lambda \to \lambda_\ell. \tag{4.17}$$

Therefore from $f_t'(x,0) = \lambda$ we get that there is $\epsilon > 0$ such that

$$\lambda_{\ell-1} < f_t'(x, u_i^\lambda(x)) < \lambda_{\ell+1} \quad \text{uniformly for } x \in \Omega,\ |\lambda - \lambda_\ell| < \epsilon. \tag{4.18}$$

Notice that for $\phi \in H$,

$$\langle I''(u_i^\lambda)\phi, \phi\rangle = \int_\Omega |\nabla\phi|\,dx - \int_\Omega f_t'(x, u_i^\lambda(x))\phi^2\,dx.$$

We get by using (4.18) that

$$\langle I''(u_i^\lambda)v, v\rangle = \int_\Omega |\nabla v|^2\,dx - \int_\Omega f_t'(x, u_i)v^2\,dx < 0, \quad 0 \neq v \in E_\ell^-,\ i = 1, 2,$$

$$\langle I''(u_i^\lambda)w, w\rangle > 0, \quad 0 \neq w \in E_\ell^+,\ i = 1, 2.$$

Therefore (4.14) holds. This completes the proof. □

Now we give

Proof of Theorem 1.3. We only consider the case (i). As $m > 2$, when $f_t'(x,0) := \lambda \in (\lambda_{m-1}, \lambda_m)$, $u = 0$ is a nondegenerate critical point of I with Morse index $\mu_0 := \dim E_m^- \geqslant 2$. By Theorem 1.2 and Remark 4.5(ii), I have four nontrivial critical points with their critical groups given by (4.9)–(4.12), respectively. By (f_4) and Lemma 4.7(i), as $\lambda \in (\lambda_m - \epsilon, \lambda_m)$, I have two nontrivial critical points u_i^λ $(i = 1, 2)$ with their Morse indices satisfying (4.14) with ℓ replaced by m. Therefore,

$$C_q(I, u_i^\lambda) = 0, \quad q \notin [\dim E_m^-, \dim E_{m+1}^-],\ i = 1, 2. \tag{4.19}$$

Finally the assumptions on m and k imply that $2 \leqslant \mu_\infty \notin [\dim E_m^-, \dim E_{m+1}^-]$ and hence the six nontrivial solutions we found above are different. The proof is completed. □

Remark 4.8. In Theorem 1.3, if we do not make the assumption (f_0), then we can get the existence of three nontrivial solutions for (P). This result is also new, to our knowledge.

We conclude the paper with further remarks. In it we get the existence of multiple solutions for elliptic boundary value problem with double resonance near infinity between two eigenvalues λ_k and λ_{k+1}. The solutions are given by the variational approach combining Morse theory with Minimax methods, bifurcation method and elliptic techniques. Duo to this reason we need higher regularity for the nonlinearity f which to be of C^1 in u. One may examine further the topological property of the solutions such as Morse index or critical group in order to find one more solution as we mentioned in Remark 4.5(iii). One may also examine the qualitative property such as the nodal property. One question is that whether some of our approach could be adapted to the situation that f is continuous. Another interesting question is that whether one could allow the situation that $f(x,u)/u$ may oscillate between λ_k and $\lambda_{k+\ell}$ for $\ell > 1$ for large $|u|$.

Some of our results may be extended to semilinear problems on unbounded domains or the entire space $\mathbb{R}^N$ with a compact linear operator. The modal is the problem $-\Delta u + V(x)u = f(x,u)$, $x \in \mathbb{R}^N$ where V is coercive from that the linear operator $(-\Delta + V)$ is compact [3,19] and the same results of Theorem 1.1 hold.

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