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A local central limit theorem on the Laguerre hypergroup

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ABSTRACT

We consider here the Laguerre hypergroup $(\mathbb{K}, *_\alpha)$, where $K = [0, +\infty[\times \mathbb{R}$ and $*_\alpha$ a convolution product on $\mathbb{K}$ coming from the product formula satisfied by the Laguerre functions $\mathcal{L}_m^{(\alpha)}$ ($m \in \mathbb{N}$, $\alpha \geqslant 0$). We set on this hypergroup a local central limit theorem which consists to give a weakly estimate of the asymptotic behavior of the convolution powers $\mu^{*_\alpha k} = \mu *_\alpha \cdots *_\alpha \mu$ (k times), μ being a given probability measure satisfying some regularity conditions on this hypergroup. It is also given a central local limit theorem for some particular radial probability measures on the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$.

1. Introduction

Our purpose is to establish on the Laguerre hypergroup (see [14], [20, pp. 243–263]) a local limit theorem similar to the classical one (see [4,18]). The aim of such a theorem is to give a weakly asymptotic behavior for the convolution powers $\mu^{*k} = \mu * \cdots * \mu$ (k times), μ being a suitable given probability measure.

Note that many authors have been interested in this kind of result in different situations. One can cite for instance [1,3,5,9–11].

The local limit theorem that we establish here states precisely that given a probability measure μ on the Laguerre hypergroup $(\mathbb{K}, *_\alpha)$, satisfying the conditions

$$\mu\big(\{0\} \times \mathbb{R}\big) = 0, \qquad \int_{\mathbb{K}} t\, d\mu(x,t) = 0 \quad \text{and} \quad \int_{\mathbb{K}} \big(1 + \big(x^2 + t^2\big)\big(x^2 + t^2 + |t|\big)\big)\, d\mu(x,t) < \infty,$$

then there is a positive constant $C_\mu^{(\alpha)}$ such that for every compactly supported continuous function $f : \mathbb{K} \to \mathbb{C}$, we have

$$\lim_{k \to \infty} \left[k^{\alpha+2} \int_{\mathbb{K}} f(x,t)\, d\mu^{*_\alpha k}(x,t) \right] = \frac{C_\mu^{(\alpha)}}{\pi \Gamma(\alpha+1)} \int_{\mathbb{K}} f(x,t) x^{2\alpha+1}\, dx\, dt.$$

This yields naturally, as it is detailed below, to a local limit theorem for some suitable radial probability measures on the $(2n+1)$-dimensional Heisenberg group.

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The technique we adopt for the proof of our local limit theorem on the Laguerre hypergroup is similar as that used in the classical case (see [4,18]). It is based on the properties of the Fourier transform on this hypergroup and especially on the asymptotic behaviors with respect both the direct and the dual variables of the characters belonging to the support of the Plancherel measure γ_α (given below) associated to this hypergroup.

Let us recall that $(\mathbb{K}, *_\alpha)$ is a commutative hypergroup (see [14], [20, pp. 243–263]), on which the involution and the Haar measure are respectively given by the homeomorphism $(x,t) \to (x,t)^- = (x,-t)$ and the Radon positive measure $dm_\alpha(x,t) = \frac{x^{2\alpha+1}}{\pi\Gamma(\alpha+1)}\,dx\,dt$. The unity element of $(\mathbb{K}, *_\alpha)$ is given by $e = (0,0)$, i.e. $\delta_{(x,t)} *_\alpha \delta_{(0,0)} = \delta_{(0,0)} *_\alpha \delta_{(x,t)} = \delta_{(x,t)}$ for all $(x,t) \in \mathbb{K}$.

The convolution product $*_\alpha$ is defined for two bounded Radon measures μ and ν on $\mathbb{K}$ as follows

$$\langle \mu *_\alpha \nu, f\rangle = \int_{\mathbb{K}\times\mathbb{K}} T^{(\alpha)}_{(x,t)} f(y,s)\,d\mu(x,t)\,d\nu(y,s), \tag{1}$$

where α is a fixed nonnegative real number and $\{T^{(\alpha)}_{(x,t)}\}_{(x,t)\in\mathbb{K}}$ are the translation operators on the Laguerre hypergroup, given by

$$T^{(\alpha)}_{(x,t)} f(y,s) = \langle \delta_{(x,t)} *_\alpha \delta_{(y,s)}, f\rangle = \begin{cases} \frac{\alpha}{\pi}\int_0^1\int_0^{2\pi} f((\xi,\eta)_{r,\theta}) r(1-r^2)^{\alpha-1}\,d\theta\,dr, & \text{if } \alpha > 0, \\ \frac{1}{2\pi}\int_0^{2\pi} f((\xi,\eta)_{1,\theta})\,d\theta, & \text{if } \alpha = 0, \end{cases} \tag{2}$$

where $(\xi,\eta)_{r,\theta} = (\sqrt{x^2+y^2+2xyr\cos\theta}, t+s+xyr\sin\theta)$.

Note that for the particular case $\mu = fm_\alpha$ and $\nu = gm_\alpha$, f and g being two suitable functions on $\mathbb{K}$, one has $\mu *_\alpha \nu = (f *_\alpha g)m_\alpha$, where $f *_\alpha g$ is the convolution product of f and g, given by

$$f *_\alpha g(x,t) = \int_{\mathbb{K}\times\mathbb{K}} T^{(\alpha)}_{(-y,s)} f(x,t) g(y,s)\,dm_\alpha(x,t). \tag{3}$$

Moreover, by K. Stempak [16, pp. 249–252], the normed Lebesgue space $(L^1_\alpha(\mathbb{K}), \|\cdot\|_{L^1_\alpha(\mathbb{K})})$ of integrable functions on $\mathbb{K}$ with respect to the Haar measure dm_α, endowed with the above convolution product, is a Banach commutative algebra, $\|\cdot\|_{L^1_\alpha(\mathbb{K})}$ being the usual norm on $L^1_\alpha(\mathbb{K})$ given by $\|f\|_{L^1_\alpha(\mathbb{K})} = \int_{\mathbb{K}} |f|\,dm_\alpha$.

The dual (see [2, p. 46]) of Laguerre hypergroup, i.e. the space of all bounded continuous and multiplicative functions $\chi : K \to \mathbb{C}$ such that $\tilde{\chi} = \chi$ where $\tilde{\chi}(x,t) = \overline{\chi(x,-t)}$; $(x,t) \in \mathbb{K}$, is given (see [12, Proposition 2.1]) by $\widehat{\mathbb{K}} = \{\varphi_{\lambda,m};\ (\lambda,m) \in \mathbb{R}^* \times \mathbb{N}\} \cup \{\varphi_\rho;\ \rho \geqslant 0\}$, where

$$\varphi_{\lambda,m}(x,t) = e^{-i\lambda t}\mathcal{L}^{(\alpha)}_m\big(|\lambda|x^2\big) \quad \text{and} \quad \varphi_\rho(x,t) = j_\alpha(\rho x); \quad (x,t) \in \mathbb{K}, \tag{4}$$

where $j_\alpha(x) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(x)}{x^\alpha}$ and $\mathcal{L}^{(\alpha)}_m(x) = e^{-\frac{x^2}{2}}L^{(\alpha)}_m(x)$, J_α being the Bessel function of first kind and order α [21] and $L^{(\alpha)}_m$ being the Laguerre polynomial of degree m and order α [8,19].

Identifying $\widehat{\mathbb{K}}$ and $(\mathbb{R}^* \times \mathbb{N}) \cup [0,+\infty[$, the Fourier transform of a bounded Radon measure μ on the Laguerre hypergroup is then, by [2, p. 80], the function defined on $(\mathbb{R}^* \times \mathbb{N}) \cup [0,+\infty[$ by

$$\mathcal{F}(\mu)(\lambda,m) = \int_{\mathbb{K}} \varphi_{-\lambda,m}(x,t)\,d\mu(x,t) \quad \text{and} \quad \mathcal{F}(\mu)(\rho) = \int_{\mathbb{K}} j_\alpha(\rho x)\,d\mu(x,t).$$

The Fourier transform of a suitable function $f : \mathbb{K} \to \mathbb{C}$, is given by $\mathcal{F}(f) = \mathcal{F}(f\,dm_\alpha)$, so that

$$\mathcal{F}(f)(\lambda,m) = \int_{\mathbb{K}} f(x,t)\varphi_{-\lambda,m}(x,t)\,dm_\alpha(x,t) \quad \text{and} \quad \mathcal{F}(\mu)(\rho) = \int_{\mathbb{K}} f(x,t) j_\alpha(\rho x)\,dm_\alpha(x,t).$$

The Laguerre Plancherel measure γ_α, associated to Laguerre hypergroup is, by [12, Remark 2.3], supported on $\mathbb{R}^* \times \mathbb{N}$ and is given by

$$\int_{\mathbb{R}^*\times\mathbb{N}} F(\lambda,m)\,d\gamma_\alpha(\lambda,m) = \sum_{m\geqslant 0} L^{(\alpha)}_m(0)\int_{\mathbb{R}} F(\lambda,m)|\lambda|^{\alpha+1}\,d\lambda. \tag{5}$$

Note that for $\alpha = n-1$, n being a positive integer, the functions $(z,t) \mapsto \varphi_{\lambda,m}(\|z\|,t)$ are spherical functions of the Gelfand pair $(G, \mathrm{U}(\mathbb{C}^n))$, where $G = \mathrm{U}(\mathbb{C}^n) \ltimes \mathbb{H}^n$ is the semi-direct product of the unitary group $\mathrm{U}(\mathbb{C}^n)$ by the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ with the multiplication law $(z,t)(z',t') = (z+z', t+t'-\Im(z_1\overline{z'_1}+\cdots+z_n\overline{z'_n}))$.

Moreover, the translation operators $\{T^{(\alpha)}_{(x,t)}\}_{(x,t)\in\mathbb{K}}$ can be derived from the ordinary convolution of radial functions on $\mathbb{H}^n$ (see [17]). More precisely, if F and G are two integrable and radial functions on $\mathbb{H}^n$, such that $F(z,t) = f(\|z\|,t)$ and

$G(z,t) = g(\|z\|, t)$; $\|z\| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$, then (see [14, p. 338]) we have $F \star G(z,t) = 2\pi^{n+1} f *_\alpha g(\|z\|, t)$, $\star$ being the ordinary convolution product on the Heisenberg group $\mathbb{H}^n$.

Let us now consider a radial probability measure on $\mathbb{H}^n$ taking the form $d\nu(z,t) = f(\|z\|, t)\,dz\,dt$, where $dz\,dt$ is the usual Lebesgue measure or still the Haar measure on the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$. A straightforward computation shows that for each suitable function $G : \mathbb{H}^n \to \mathbb{C}$, we have

$$\int_{\mathbb{H}^n} G(z,t)\,d\nu^{\star k}(z,t) = \int_{\mathbb{K}} \biggl(\int_{\mathrm{U}(\mathbb{C}^n)} G(x\omega, t)\,d\sigma(\omega) \biggr) d\mu^{*_{n-1} k}(x,t), \quad \text{for all } k \in \mathbb{N}^*, \tag{6}$$

where σ is the usual Haar measure on $\mathrm{U}(\mathbb{C}^n)$ normalized such that $\sigma(\mathrm{U}(\mathbb{C}^n)) = 1$, μ being the probability measure on the Laguerre hypergroup $(\mathbb{K}, *_{n-1})$; $d\mu(x,t) = 2\pi^{n+1} f(x,t)\,dm_{n-1}(x,t)$.

Assume further that the measure ν satisfies the condition

$$\nu\bigl(\{0\} \times \mathbb{R}\bigr) = 0, \qquad \int_{\mathbb{H}^n} t\,d\nu(z,t) = 0 \quad \text{and} \quad \int_{\mathbb{H}^n} \bigl(1 + \bigl(\|z\|^2 + t^2\bigr)\bigl(\|z\|^2 + t^2 + |t|\bigr)\bigr)\,d\nu(z,t) < \infty,$$

then we can assert, thanks to the local limit theorem on the Laguerre hypergroup $(\mathbb{K}, *_{n-1})$, that there is a positive constant $\tilde{C}_\mu^{(n-1)}$ such that for every compactly supported continuous function $G : \mathbb{K} \to \mathbb{C}$, we have

$$\lim_{k \to \infty} \biggl[k^{\alpha+2} \int_{\mathbb{H}^n} G(z,t)\,d\nu^{\star k}(z,t) \biggr] = \tilde{C}_\mu^{(n-1)} \int_{\mathbb{H}^n} G(z,t)\,dz\,dt,$$

or equivalently

$$\lim_{k \to \infty} \biggl[k^{\alpha+2} \int_{\mathbb{H}^n} G(z,t) F^{\star k}(z,t)\,dz\,dt \biggr] = \tilde{C}_\mu^{(n-1)} \int_{\mathbb{H}^n} G(z,t)\,dz\,dt.$$

This can be regarded as a central local limit theorem on the $(2n+1)$-dimensional Heisenberg group $\mathbb{H}^n$.

Note that in [5], it is proven a local limit theorem for compactly supported probability measures on the three-dimensional Heisenberg group. One can see also [15] for other limit theorems on the Heisenberg group.

2. Preliminaries

Throughout subsequently, μ will designate a fixed regular probability measure on the Laguerre hypergroup.

We shall, in this section, summarize all the results and tools we need for the proof of our main theorem.

Proposition 2.1. *Suppose that*

$$\int_{\mathbb{K}} t\,d\mu = 0 \quad \textit{and} \quad \rho_\mu = \int_{\mathbb{K}} \bigl(1 + \bigl(x^2 + |t|\bigr)^2 + t^2\bigl(x^2 + |t|\bigr)\bigr)\,d\mu < \infty, \tag{7}$$

then

$$\mathcal{F}(\mu)(\lambda, m) = 1 - \frac{\sigma_\mu}{\alpha+1} \xi_{\lambda,m}^{(\alpha)} + \bigl(\xi_{\lambda,m}^{(\alpha)}\bigr)^2 \widetilde{\mathcal{R}}_\mu^\alpha(\lambda, m), \tag{8}$$

with

$$\bigl|\widetilde{\mathcal{R}}_\mu^\alpha(\lambda, m)\bigr| \leqslant 4\bigl(1 + \xi_{\lambda,m}^{(\alpha)}\bigr)\rho_\mu, \tag{9}$$

where $\xi_{\lambda,m}^{(\alpha)} = |\lambda|\alpha_m$, $\alpha_m = m + \frac{\alpha+1}{2}$ *and* $\sigma_\mu = \int_{\mathbb{K}} x^2\,d\mu(x,t)$.

Proof. From [13, Proposition 7] we deduce that for every $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$ and $(x,t) \in \mathbb{K}$ we have

$$\varphi_{\lambda,m}(x,t) = 1 + i\lambda t - \frac{\xi_{\lambda,m}^{(\alpha)}}{\alpha+1} x^2 + \bigl(\xi_{\lambda,m}^{(\alpha)}\bigr)^2 \mathcal{R}_{\lambda,m}^\alpha(x,t), \tag{10}$$

with

$$\bigl|\mathcal{R}_{\lambda,m}^\alpha(x,t)\bigr| \leqslant 4\bigl(1 + \xi_{\lambda,m}^{(\alpha)}\bigr)\bigl(1 + \bigl[x^2 + |t|\bigr]^2 + t^2\bigl[x^2 + |t|\bigr]\bigr). \tag{11}$$

The result follows by a straightforward calculation. $\square$

Corollary 2.1. *If μ satisfies the condition* (7)*, the following properties hold*:

1. *There exists $\eta_\mu > 0$ such that for each $k \in \mathbb{N}^*$ and $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$; $\xi^{(\alpha)}_{\lambda,m} < k\eta_\mu$, we have*

$$\left|\mathcal{F}(\mu)\left(\frac{\lambda}{k}, m\right)\right|^k \leqslant \exp\left(-\frac{\sigma_\mu}{2(\alpha+1)}\xi^{(\alpha)}_{\lambda,m}\right). \tag{12}$$

2. *For any $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$ we have*

$$\lim_{k\to\infty}\left(\mathcal{F}(\mu)\left(\frac{\lambda}{k}, m\right)\right)^k = \exp\left(-\frac{\sigma_\mu}{\alpha+1}\xi^{(\alpha)}_{\lambda,m}\right). \tag{13}$$

Proof. The relation (8) can be written

$$\mathcal{F}(\mu)\left(\frac{\lambda}{k}, m\right) = 1 - \frac{\xi^{(\alpha)}_{\lambda,m}}{k}\Phi_\mu\left(\frac{\lambda}{k}, m\right), \tag{14}$$

where

$$\Phi_\mu(\lambda, m) = \frac{\sigma_\mu}{\alpha+1} - \xi^{(\alpha)}_{\lambda,m}\widetilde{\mathcal{R}}^\alpha_\mu(\lambda, m). \tag{15}$$

Now, by using the relation (9) we get

$$\frac{\xi^{(\alpha)}_{\lambda,m}}{k}\left|\Phi_\mu\left(\frac{\lambda}{k}, m\right)\right| \leqslant \frac{\sigma_\mu}{\alpha+1}\frac{\xi^{(\alpha)}_{\lambda,m}}{k} + 4\left(\frac{\xi^{(\alpha)}_{\lambda,m}}{k}\right)^2\left(1 + \frac{\xi^{(\alpha)}_{\lambda,m}}{k}\right)\rho_\mu. \tag{16}$$

Then there exists $\beta_\mu > 0$ such that

$$\frac{\xi^{(\alpha)}_{\lambda,m}}{k}\left|\Phi_\mu\left(\frac{\lambda}{k}, m\right)\right| \leqslant \frac{1}{2}, \quad \text{for } \frac{\xi^{(\alpha)}_{\lambda,m}}{k} \leqslant \beta_\mu, \tag{17}$$

and consequently $\Re(\mathcal{F}(\mu)(\frac{\lambda}{k}, m)) \geqslant \frac{1}{2} > 0$, for $\frac{\xi^{(\alpha)}_{\lambda,m}}{k} \leqslant \beta_\mu$, so that

$$\left(\mathcal{F}(\mu)\left(\frac{\lambda}{k}, m\right)\right)^k = \exp\left(k\log\left(1 - \frac{\xi^{(\alpha)}_{\lambda,m}}{k}\Phi_\mu\left(\frac{\lambda}{k}, m\right)\right)\right), \quad \text{for } \frac{\xi^{(\alpha)}_{\lambda,m}}{k} \leqslant \beta_\mu, \tag{18}$$

where $z \mapsto \log(z)$ is the principal value of the logarithm, $z \in \mathbb{C} \setminus]-\infty, 0]$.

By using the usual development $\log(1+z) = z + z\psi(z)$; $\lim_{z\to 0}\psi(z) = 0$, we deduce that for $\frac{\xi^{(\alpha)}_{\lambda,m}}{k} \leqslant \beta_\mu$, we have

$$\left(\mathcal{F}(\mu)\left(\frac{\lambda}{k}, m\right)\right)^k = \exp\left(-\xi^{(\alpha)}_{\lambda,m}\Phi_\mu\left(\frac{\lambda}{k}, m\right) - \xi^{(\alpha)}_{\lambda,m}\Phi_\mu\left(\frac{\lambda}{k}, m\right)\psi\left(-\frac{\xi^{(\alpha)}_{\lambda,m}}{k}\Phi_\mu\left(\frac{\lambda}{k}, m\right)\right)\right). \tag{19}$$

Suppose further $\sigma_\mu > 0$, we get

$$\left(\mathcal{F}(\mu)\left(\frac{\lambda}{k}, m\right)\right)^k = \exp\left(-\frac{\sigma_\mu}{\alpha+1}\xi^{(\alpha)}_{\lambda,m}\left(1 + \Psi_\mu\left(\frac{\lambda}{k}, m\right)\right)\right), \quad \text{for } \frac{\xi^{(\alpha)}_{\lambda,m}}{k} \leqslant \beta_\mu, \tag{20}$$

where

$$\Psi_\mu\left(\frac{\lambda}{k}, m\right) = -\frac{\alpha+1}{\sigma_\mu}\frac{\xi^{(\alpha)}_{\lambda,m}}{k}\widetilde{\mathcal{R}}^\alpha_\mu\left(\frac{\lambda}{k}, m\right) + \frac{\alpha+1}{\sigma_\mu}\Phi_\mu\left(\frac{\lambda}{k}, m\right)\psi\left(-\frac{\xi^{(\alpha)}_{\lambda,m}}{k}\Phi_\mu\left(\frac{\lambda}{k}, m\right)\right).$$

Now, thanks to (9) and (16), there is $\gamma_\mu > 0$ such that

$$\left|\Re\left(\Psi_\mu\left(\frac{\lambda}{k}, m\right)\right)\right| \leqslant \left|\Psi_\mu\left(\frac{\lambda}{k}, m\right)\right| < \frac{1}{2}, \quad \text{for } \frac{\xi^{(\alpha)}_{\lambda,m}}{k} \leqslant \gamma_\mu,$$

and consequently

$$-\left(1 + \Re\left(\Psi_\mu\left(\frac{\lambda}{k}, m\right)\right)\right) \leqslant \frac{1}{2}, \quad \text{for } \frac{\xi^{(\alpha)}_{\lambda,m}}{k} \leqslant \gamma_\mu.$$

Using the relation (20) we deduce

$$\left|\mathcal{F}(\mu)\left(\frac{\lambda}{k},m\right)\right|^k \leqslant \exp\left(-\frac{\sigma_\mu}{2(\alpha+1)}\right), \quad \text{for } \frac{\xi^{(\alpha)}_{\lambda,m}}{k} \leqslant \min(\beta_\mu,\gamma_\mu). \tag{21}$$

To obtain the inequality (12) for $\sigma_\mu > 0$, it suffices then to chose $\eta_\mu = \min(\beta_\mu,\gamma_\mu)$.

On the other hand, it is easy to see that $\sigma_\mu = 0$ if and only if $\mu(\mathbb{K}\setminus\{0\}) = 0$ or equivalently $\mu(\{0\}\times\mathbb{R}) = 1$. Thus if $\sigma_\mu = 0$, then $\mathcal{F}(\mu) \equiv 1$, since $\phi_{\lambda,m}(0,t) = L^{(\alpha)}_m(0) = 1$ for all $(\lambda,m) \in \mathbb{R}\times\mathbb{N}$. The inequality (12) is then obviously satisfied for this case.

It remains finally to prove the relation (13). Let so (λ,m) be fixed in $\mathbb{R}\times\mathbb{N}$ and let $k_{\lambda,m} \in \mathbb{N}$, chosen such that $\frac{\xi^{(\alpha)}_{\lambda,m}}{k} \leqslant \beta_\mu$ for all $k \geqslant k_{\lambda,m}$. Using the relation (19), we get

$$\left(\mathcal{F}(\mu)\left(\frac{\lambda}{k},m\right)\right)^k = \exp\left(-\frac{\sigma_\mu}{\alpha+1}\xi^{(\alpha)}_{\lambda,m} + \left(\frac{\xi^{(\alpha)}_{\lambda,m}}{k}\right)^2 \widetilde{\mathcal{R}}^\alpha_\mu\left(\frac{\lambda}{k},m\right) - \xi^{(\alpha)}_{\lambda,m}\Phi_\mu\left(\frac{\lambda}{k},m\right)\psi\left(-\frac{\xi^{(\alpha)}_{\lambda,m}}{k}\Phi_\mu\left(\frac{\lambda}{k},m\right)\right)\right).$$

Taking into account (9) and (16) we obtain the relation (13), which finishes the proof. □

Lemma 2.1.

1. *Let j_α be the modified Bessel function defined on $\mathbb{R}$ by*

$$j_\alpha(x) = \begin{cases} 2^\alpha \Gamma(\alpha+1)\frac{J_\alpha(x)}{x^\alpha}, & \text{if } x \neq 0,\\ 1, & \text{if } x = 0,\end{cases}$$

J_α being the Bessel function of first kind and order α [21], *then $|j_\alpha(x)| < j_\alpha(0) = 1$, for all $x \in \mathbb{R}\setminus\{0\}$.*

2. *$|\mathcal{L}^{(\alpha)}_m(x^2)| < \mathcal{L}^{(\alpha)}_m(0) = 1$, for all $x \in \mathbb{R}\setminus\{0\}$.*

Proof. 1. By [19], the function j_α possesses the following integral representation

$$j_\alpha(x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)} \int_0^1 \left(1-t^2\right)^{\alpha-1/2} \cos xt\,dt; \quad x \in \mathbb{R}. \tag{22}$$

Let now $x \in \mathbb{R}$ such that $|j_\alpha(x)| = 1$, so

$$1 \leqslant \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)} \int_0^1 \left(1-t^2\right)^{\alpha-1/2} |\cos xt|\,dt \leqslant \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma(\alpha+1/2)} \int_0^1 \left(1-t^2\right)^{\alpha-1/2} dt = 1,$$

and therefore $\int_0^1 (1-t^2)^{\alpha-1/2}[1-|\cos xt|]\,dt = 0$, so that $1-|\cos xt| = 0$ for each $t \in [0,1]$ or equivalently $|\cos y| = 1$ for each $y \in [0,|x|]$ which implies obviously that $x = 0$, and the property 1 is proved.

2. Since $\varphi_{1,m}(x,0) = \mathcal{L}^{(\alpha)}_m(x^2)$ for every $x \in \mathbb{R}$, then by using the product formula $T^{(\alpha)}_{(x,t)}\varphi_{1,m}(y,s) = \varphi_{1,m}(x,t)\varphi_{1,m}(y,s)$, we get

$$\left[\mathcal{L}^{(\alpha)}_m\left(x^2\right)\right]^2 = \begin{cases} \frac{\alpha}{\pi}\int_0^1\int_0^{2\pi} \mathcal{L}^{(\alpha)}_m(2x^2[1+r_\alpha\cos\theta])r(1-r^2)^{\alpha-1}\,d\theta\,dr, & \text{if } \alpha > 0,\\ \frac{1}{2\pi}\int_0^{2\pi} \mathcal{L}^{(\alpha)}_m(2x^2[1+r_\alpha\cos\theta])\,d\theta, & \text{if } \alpha = 0,\end{cases} \tag{23}$$

where $r_\alpha = (\chi_{\{0\}} + r\chi_{]0,+\infty[})(\alpha)$.

Let $x \in \mathbb{R}$ such that $|\mathcal{L}^{(\alpha)}_m(x^2) = 1$, so by a similar reasoning as in the previous property we get $|\mathcal{L}^{(\alpha)}_m(2x^2[1+r_\alpha\cos\theta])| = \mathcal{L}^{(\alpha)}_m(0) = 1$ for each (θ,r_α), and consequently $|\mathcal{L}^{(\alpha)}_m(y])| = 1$ for each $y \in [0,4x^2]$. Thus necessarily $x = 0$, since the set of zeros of the Laguerre polynomial $L^{(\alpha)}_m$ is finite, and the proof is achieved. □

Proposition 2.2. *Assume $\mu(\{0\}\times\mathbb{R}) = 0$, then for every $\eta > 0$ and $A > \frac{2\eta}{\alpha+1}$, there is $a^{A,\eta}_\mu \in]0,1[$ such that*

$$\left|\mathcal{F}(\mu)(\lambda,m)\right| \leqslant a^{A,\eta}_\mu, \quad \textit{for all } (\lambda,m) \in (\mathbb{R}\times\mathbb{N})_{A,\eta}, \tag{24}$$

where $(\mathbb{R}\times\mathbb{N})_{A,\eta} = \{(\lambda,m) \in \mathbb{R}\times\mathbb{N};\ |\lambda| \leqslant A \text{ and } \xi^{(\alpha)}_{\lambda,m} \geqslant \eta\}$.

Proof. Since $\lim_{x\to0^+}\mu([0,x]\times\mathbb{R}) = \mu(\{0\}\times\mathbb{R}) = 0$ and $\lim_{X\to\infty}\mu([X,\infty[\times\mathbb{R}) = \mu(\emptyset) = 0$, then for all $\varepsilon > 0$, there is $\omega_{\mu,\varepsilon}, \rho_{\mu,\varepsilon} > 0$ such that $\mu([0,\omega_{\mu,\varepsilon}]\times\mathbb{R}) + \mu([\omega_{\mu,\varepsilon}+\rho_{\mu,\varepsilon},\infty[\times\mathbb{R}) < \varepsilon$, and so

$$\left|\mathcal{F}(\mu)(\lambda,m)\right| < \varepsilon + \int\limits_{[\omega_{\mu,\varepsilon},\omega_{\mu,\varepsilon}+\rho_{\mu,\varepsilon}]\times\mathbb{R}} \left|\mathcal{L}_m^{(\alpha)}\big(|\lambda|x^2\big)\right| d\mu(x,t), \quad \text{for all } (\lambda,m)\in\mathbb{R}\times\mathbb{N}. \tag{25}$$

Now, by the relation (5.5) given in [16, p. 489], a simple calculation shows that for all $m\in\mathbb{N}$ and $\lambda,x\in\mathbb{R}$ such that $|\lambda|mx^2>0$ we have

$$\mathcal{L}_m^{(\alpha)}\big(|\lambda|x^2\big) = \left(\frac{m!m^\alpha}{\Gamma(m+\alpha+1)}\right)^{\frac{1}{2}} j_\alpha\big(2x\sqrt{|\lambda|m}\big) + \frac{\varepsilon_m^{(\alpha)}(x\sqrt{|\lambda|})}{\sqrt{m}(|\lambda|mx^2)^{\frac{2\alpha+1}{4}}}, \tag{26}$$

where $(\varepsilon_m^{(\alpha)})_{m\geqslant 1}$ denotes a sequence of real-valued functions uniformly bounded on each interval $[0,b]$; $b>0$, so that there is $c_{\mu,\varepsilon}^{A,\eta}>0$ such that

$$\int\limits_{[\omega_{\mu,\varepsilon},\omega_{\mu,\varepsilon}+\rho_{\mu,\varepsilon}]\times\mathbb{R}} \left|\frac{\varepsilon_m^{(\alpha)}(x\sqrt{|\lambda|})}{\sqrt{m}(|\lambda|mx^2)^{\frac{2\alpha+1}{4}}}\right| d\mu(x,t) < \frac{c_{\mu,\varepsilon}^{A,\eta}}{\sqrt{m}}, \quad \text{for all } (\lambda,m)\in(\mathbb{R}\times\mathbb{N})_{A,\eta};\ m\in\mathbb{N}^*, \tag{27}$$

because off $|\lambda|m\geqslant\frac{2\eta}{\alpha+3}$ for all $(\lambda,m)\in(\mathbb{R}\times\mathbb{N})_{A,\eta}$; $m\in\mathbb{N}^*$.

Combining the relations (25)–(27), we deduce that there is $m_\alpha\in\mathbb{N}$ such that

$$\left|\mathcal{F}(\mu)(\lambda,m)\right| < 2\varepsilon + \left(\frac{m!m^\alpha}{\Gamma(m+\alpha+1)}\right)^{\frac{1}{2}} \int\limits_{\mathbb{K}} \left|j_\alpha\big(2x\sqrt{|\lambda|m}\big)\right| d\mu(x,t), \tag{28}$$

for all $(\lambda,m)\in(\mathbb{R}\times\mathbb{N})_{A,\eta}$; $m>m_\alpha$.

Let us now show that there exists $\tilde{c}_\mu^{A,\eta}\in\,]0,1[$ such that

$$\int\limits_{\mathbb{K}} \left|j_\alpha\big(2x\sqrt{|\lambda|m}\big)\right| d\mu(x,t) < \tilde{c}_\mu^{A,\eta}, \quad \text{for all } (\lambda,m)\in(\mathbb{R}\times\mathbb{N})_{A,\eta},\ m\in\mathbb{N}^*. \tag{29}$$

Since $2\sqrt{|\lambda|m}\geqslant\eta_\alpha=2\sqrt{\frac{2\eta}{\alpha+3}}$ for all $(\lambda,m)\in(\mathbb{R}\times\mathbb{N})_{A,\eta}$, $m\in\mathbb{N}^*$, it suffices then to show that $\sup_{\rho\geqslant\eta_\alpha}\int_{\mathbb{K}}|j_\alpha(\rho x)|\,d\mu(x,t)<1$. Indeed, thanks to the well-known Riemann–Lebesgue Lemma we deduce from the relation (22) that $\lim_{\rho\to\infty} j_\alpha(\rho x)=0$, for any $x>0$. But for all $x\in\mathbb{R}$ we have $|j_\alpha(\rho x)|\leqslant 1$, so by applying Lebesgue Theorem we get $\lim_{\rho\to\infty}\int_{\mathbb{K}}|j_\alpha(\rho x)|\,d\mu(x,t)=0$, so that there is $\tilde{A}>\eta_\alpha$ such that $\int_{\mathbb{K}}|j_\alpha(\rho x)|\,d\mu(x,t)<\frac{1}{2}$, for all $\rho\geqslant\tilde{A}$. Now, by continuity of the function $\rho\mapsto\int_{\mathbb{K}}|j_\alpha(|\rho|x)|\,d\mu(x,t)$, there exists $\rho_0\in[\eta_\alpha,\tilde{A}]$ such that

$$\max_{\rho\in[\eta_\alpha,\tilde{A}]}\int\limits_{\mathbb{K}}\left|j_\alpha(\rho x)\right| d\mu(x,t) = \int\limits_{\mathbb{K}}\left|j_\alpha(\rho_0 x)\right| d\mu(x,t).$$

To obtain the relation (29) it remains to show that $\int_{\mathbb{K}}|j_\alpha(\rho_0 x)|\,d\mu(x,t)<1$ or equivalently $\int_{\mathbb{K}}|j_\alpha(\rho_0 x)|\,d\mu(x,t)\neq 1$, since we already have $\int_{\mathbb{K}}|j_\alpha(\rho_0 x)|\,d\mu(x,t)\leqslant\int_{\mathbb{K}}d\mu(x,t)=1$.

Indeed, let us assume that $\int_{\mathbb{K}}|j_\alpha(\rho_0 x)|\,d\mu(x,t)=1$, thus $\int_{\mathbb{K}}(1-|j_\alpha(\rho_0 x)|)\,d\mu(x,t)=0$, so that $\mu(\{(x,t)\in\mathbb{K};\ |j_\alpha(\rho_0 x)|\neq 1\})=0$. This implies, by Lemma 2.1.1, that $\mu(]0,\infty[\,\times\mathbb{R})=0$ which is impossible since $\mu(]0,\infty[\,\times\mathbb{R})=\mu(\mathbb{K})-\mu(\{0\}\times\mathbb{R})=\mu(\mathbb{K})=1$, and the relation (29) is proved. The relation (28) becomes then

$$\left|\mathcal{F}(\mu)(\lambda,m)\right| < 2\varepsilon + \tilde{c}_\mu^{A,\eta}\left(\frac{m!m^\alpha}{\Gamma(m+\alpha+1)}\right)^{\frac{1}{2}}, \quad \text{for all } (\lambda,m)\in(\mathbb{R}\times\mathbb{N})_{A,\eta};\ m>m_\alpha. \tag{30}$$

Now, using the well-known Stirling's formula, we easily verify that $\lim_{m\to\infty}\tilde{c}_\mu^{A,\eta}(\frac{m!m^\alpha}{\Gamma(m+\alpha+1)})^{\frac{1}{2}}=\tilde{c}_\mu^{A,\eta}$, so that there is $\tilde{m}_\alpha\in\mathbb{N}$, such that

$$\left|\mathcal{F}(\mu)(\lambda,m)\right| < 3\varepsilon + \tilde{c}_\mu^{A,\eta}, \quad \text{for all } (\lambda,m)\in(\mathbb{R}\times\mathbb{N})_{A,\eta};\ m>\tilde{m}_\alpha. \tag{31}$$

This implies, since ε can be arbitrarily chosen in $]0,1[$, that there is $\tilde{c}_{1,\mu}^{A,\eta}\in\,]0,1[$ and $\tilde{m}_{1,\alpha}\in\mathbb{N}$, such that

$$\left|\mathcal{F}(\mu)(\lambda,m)\right| < \tilde{c}_{1,\mu}^{A,\eta}, \quad \text{for all } (\lambda,m)\in(\mathbb{R}\times\mathbb{N})_{A,\eta};\ m>\tilde{m}_{1,\alpha}. \tag{32}$$

To finish the proof, we have to show finally that for each $m\leqslant\tilde{m}_{1,\alpha}$, there exists $b_\mu^{A,\eta}(m)\in\,]0,1[$ such that $|\mathcal{F}(\mu)(\lambda,m)|\leqslant b_\mu^{A,\eta}(m)$, for all $\lambda\in\mathbb{R}$; $\frac{\eta}{\alpha_m}\leqslant|\lambda|\leqslant A$, where $\alpha_m=m+\frac{\alpha+1}{2}$.

Indeed, by continuity on $\mathbb{R}$ of the function $\lambda\mapsto\int_{\mathbb{K}}|\mathcal{L}_m^{(\alpha)}(|\lambda|x^2)|\,d\mu(x,t)$, there exists for all $m\leqslant\tilde{m}_{1,\alpha}$, a real λ_m such that $|\lambda_m|\in[\frac{\eta}{\alpha_m},A]$ and

$$\sup_{\frac{\eta}{\alpha_m}\leqslant|\lambda|\leqslant A}\int_{\mathbb{K}}\left|\mathcal{L}_m^{(\alpha)}\left(|\lambda|x^2\right)\right|d\mu(x,t)=\int_{\mathbb{K}}\left|\mathcal{L}_m^{(\alpha)}\left(|\lambda_m|x^2\right)\right|d\mu(x,t).$$

Now, since $|\lambda_m|>0$ and $|\mathcal{L}_m^{(\alpha)}(|\lambda_0|x^2)|\leqslant 1$ for all $(x,t)\in\mathbb{K}$, we deduce by a similar reasoning as above (for the proof of $\int_{\mathbb{K}}|j_\alpha(\rho_0 x)|\,d\mu(x,t)<1$), that $\int_{\mathbb{K}}|\mathcal{L}_m^{(\alpha)}(|\lambda_0|x^2)|\,d\mu(x,t)<1$, which finishes the proof. □

We will use in the sequel the following notations

- $M_b^+(\mathbb{K})$: The space of regular bounded and positive measures on $\mathbb{K}$.
- $\mathcal{C}_c(\mathbb{K})$: The space of compactly supported continuous functions $f:\mathbb{K}\to\mathbb{C}$.
- $L_\alpha^1(\mathbb{K})$: The Lebesgue space of integrable functions on $\mathbb{K}$ with respect to the Haar measure m_α.
- $L_\alpha^1(\mathbb{R}\times\mathbb{N})$: The Lebesgue space of integrable functions on $\mathbb{R}\times\mathbb{N}$ with respect to the measure γ_α.
- $\mathcal{H}_\alpha(\mathbb{K})$: The space of continuous functions $h\in L_\alpha^1(\mathbb{K})$ such that $\mathcal{F}(h)\in L_\alpha^1(\mathbb{R}\times\mathbb{N})$ and there exists $A_h>0$ such that

$$\mathcal{F}(h)(\lambda,m)=0,\quad \text{for any } |\lambda|\geqslant A_h \text{ and } m\in\mathbb{N}. \tag{33}$$

Proposition 2.3.

1. *The positive function defined on $\mathbb{K}$ by*

$$h_0(x,t)=e^{-x^2}\left(\frac{\sin t}{t}\right)^2+e^{-\pi x^2}\left(\frac{\sin\pi t}{\pi t}\right)^2 \tag{34}$$

belongs to $\mathcal{H}_\alpha(\mathbb{K})$ and we have

$$\mathcal{F}(h_0)(\lambda,m)=F_0(\lambda,m)+\frac{1}{\pi^2}F_0\left(\frac{\lambda}{\pi},m\right),\quad \textit{for all } (\lambda,m)\in\mathbb{R}\times\mathbb{N}, \tag{35}$$

where $F_0(\lambda,m)=\frac{2^{\alpha-1}(2-|\lambda|)^{m+1}}{(2+|\lambda|)^{m+\alpha+1}}\chi_{[-2,+2]}(\lambda)$, $\chi_{[-2,+2]}$ being the characteristic function of the interval $[-2,+2]$.

2. *Let $h\in\mathcal{H}_\alpha(\mathbb{K})$ then $\varphi_{-\lambda,m}h\in\mathcal{H}_\alpha(\mathbb{K})$, for all $(\lambda,m)\in\mathbb{R}\times\mathbb{N}$.*

Proof. 1. It is clear that $h_0\in Ł_\alpha^1(\mathbb{K})$ and the relation (35) can be obtained by a straightforward computation via the classical formula

$$\mathcal{F}_0(\chi_{[-1,+1]}*_0\chi_{[-1,+1]})(\lambda)=\left(\mathcal{F}_0(\chi_{[-1,+1]})(\lambda)\right)^2=\left(\frac{\sin\lambda}{\lambda}\right)^2,\quad \text{for all } \lambda\in\mathbb{R},$$

where $\chi_{[-2,+2]}$ denotes the characteristic function of the interval $[-1,+1]$, $\mathcal{F}_0$ and $*_0$ being respectively the usual Fourier transform and convolution product on $\mathbb{R}$.

It remains to show that $\mathcal{F}(h_0)\in L_\alpha^1(\mathbb{R}\times\mathbb{N})$, but by the variables changing $\lambda\mapsto\pi\lambda$ one can easily see that $\|F_0(\frac{\lambda}{\pi},m)\|_{L_\alpha^1(\mathbb{R}\times\mathbb{N})}=\pi^{\alpha+2}\|F_0(\lambda,m)\|_{L_\alpha^1(\mathbb{R}\times\mathbb{N})}$ so that it suffices to verify that $\|F_0(\lambda,m)\|_{L_\alpha^1(\mathbb{R}\times\mathbb{N})}<\infty$. Indeed, by a simple computation we obtain

$$\begin{aligned}\left\|F_0(\lambda,m)\right\|_{L_\alpha^1(\mathbb{R}\times\mathbb{N})}&=2^{\alpha+2}\sum_{m=0}^{\infty}L_m^\alpha(0)\int_0^1\left(\frac{1-u}{1+u}\right)^{m+1}\frac{u^{\alpha+1}}{(1+u)^\alpha}\,du\\&=2^{\alpha+2}\sum_{m=0}^{\infty}L_m^\alpha(0)\int_0^1\left(1-\frac{2u}{1+u}\right)^{m+1}\left(\frac{u}{1+u}\right)^\alpha u\,du.\end{aligned}$$

Now, by the variable changing $u\mapsto v=\frac{2u}{1+u}$ and the well-known formula $\int_0^1(1-x)^{m+1}x^{\alpha+1}\,dx=B(m+2,\alpha+2)=\frac{\Gamma(m+2)\Gamma(\alpha+2)}{\Gamma(m+\alpha+4)}$, one easily gets

$$\int_0^1\left(1-\frac{2u}{1+u}\right)^{m+1}\left(\frac{u}{1+u}\right)^\alpha u\,du=2\int_0^1(1-v)^{m+1}\left(\frac{v}{2}\right)^\alpha\frac{v}{(2-v)^3}\,dv\leqslant 2\frac{\Gamma(m+2)\Gamma(\alpha+2)}{\Gamma(m+\alpha+4)},$$

and since $L_m^\alpha(0)=\frac{\Gamma(m+\alpha+1)}{m!\Gamma(\alpha+1)}$ we deduce

$$\|F_0(\lambda,m)\|_{L^1_\alpha(\mathbb{R}\times\mathbb{N})} \leqslant 2^{\alpha+3}\sum_{m=0}^{\infty} L_m^\alpha(0)\frac{\Gamma(m+2)\Gamma(\alpha+2)}{\Gamma(m+\alpha+4)}$$
$$\leqslant 2^{\alpha+3}(\alpha+1)\sum_{m=0}^{\infty}\frac{m+1}{(m+\alpha+3)(m+\alpha+2)(m+\alpha+1)},$$

which obviously shows that $\|F_0(\lambda,m)\|_{L^1_\alpha(\mathbb{R}\times\mathbb{N})} < +\infty$, and the property 1 is proved.

2. By [7], for each $((\lambda,m),(\xi,p)) \in (\mathbb{R}\times\mathbb{N})^2$; $\lambda+\xi \neq 0$, we have the following dual product formula

$$\varphi_{\lambda,m}(x,t)\varphi_{\xi,p}(x,t) = \sum_{k=0}^{\infty} C_k^{(\alpha)}\big((\lambda,m),(\xi,p)\big)\varphi_{\lambda+\xi,k}(x,t), \quad \text{for all } (x,t)\in\mathbb{K}, \tag{36}$$

where

$$C_k^{(\alpha)}\big((\lambda,m),(\xi,p)\big) = \frac{L_k^\alpha(0)}{\Gamma(\alpha+1)}\int_0^\infty \mathcal{L}_m^{(\alpha)}\left(\frac{|\lambda|x}{|\lambda+\xi|}\right)\mathcal{L}_p^{(\alpha)}\left(\frac{|\xi|x}{|\lambda+\xi|}\right)\mathcal{L}_k^{(\alpha)}(x)x^\alpha\,dx. \tag{37}$$

Note that if furthermore $\lambda\xi > 0$, then $C_k^{(\alpha)}((\lambda,m),(\xi,p)) = 0$ for all $k \geqslant m+n+1$. Indeed, for this case we have $|\lambda+\xi| = |\lambda|+|\xi|$, so that by [6] we get

$$\mathcal{L}_m^{(\alpha)}\left(\frac{|\lambda|x}{|\lambda+\xi|}\right)\mathcal{L}_p^{(\alpha)}\left(\frac{|\xi|x}{|\lambda+\xi|}\right) = \mathcal{L}_m^{(\alpha)}\left(\frac{|\lambda|x}{|\lambda|+|\xi|}\right)\mathcal{L}_p^{(\alpha)}\left(\frac{|\xi|x}{|\lambda|+|\xi|}\right) = \sum_{k=0}^{m+n}\tilde{C}_k^{(\alpha)}\mathcal{L}_k^{(\alpha)}(x), \quad \text{for all } x\geqslant 0, \tag{38}$$

where $(\tilde{C}_k^{(\alpha)})_{0\leqslant k\leqslant m+n}$ is a sequence of nonnegative numbers.

By using the orthogonality of the sequence $(\mathcal{L}_k^{(\alpha)})_{k\in\mathbb{N}}$ on $\mathbb{R}$ with respect to the measure $x^\alpha\,dx$, we deduce easily from (37) and (38) that in the case $\lambda\xi > 0$, we effectively have $C_k^{(\alpha)}((\lambda,m),(\xi,p)) = 0$ for all $k \geqslant m+n+1$.

Note also that, by [7], for all $((\lambda,m),(\xi,p)) \in (\mathbb{R}^*\times\mathbb{N})^2$; $\lambda+\xi\neq 0$, the sequence $(C_k^{(\alpha)}((\lambda,m),(\xi,p)))_{k\in\mathbb{N}}$ satisfies the following properties

(i) $C_k^{(\alpha)}((\lambda,m),(\xi,p)) \geqslant 0$, for every $k\in\mathbb{N}$.
(ii) $\sum_{k=0}^{\infty} C_k^{(\alpha)}((\lambda,m),(\xi,p)) = 1$.
(iii) $|\xi|^{\alpha+1}L_p^{(\alpha)}(0)C_k^{(\alpha)}((\lambda,m),(\xi,p)) = |\lambda+\xi|^{\alpha+1}L_k^{(\alpha)}(0)C_p^{(\alpha)}((-\lambda,m),(\xi+\lambda,k))$, for every $k\in\mathbb{N}$.

Let now $h\in\mathcal{H}_\alpha(\mathbb{K})$ and $(\lambda,m)\in\mathbb{R}^*\times\mathbb{N}$. It is clear that $\varphi_{-\lambda,m}h \in L^1_\alpha(\mathbb{K})$, since $|\varphi_{-\lambda,m}(x,t)| \leqslant 1$ for all $(x,t)\in\mathbb{K}$. On the other hand by using the dual product formula (36) we deduce by a straightforward computation that for all $(\xi,p)\in\mathbb{R}^*\times\mathbb{N}$ we have

$$\begin{aligned}\mathcal{F}(\varphi_{-\lambda,m}h)(\xi,p) &= \int_{\mathbb{K}} h\varphi_{-\lambda,m}\varphi_{-\xi,p}\,dm_\alpha = \int_{\mathbb{K}}\left[h\sum_{k=0}^{\infty}C_k^{(\alpha)}\big((\lambda,m),(\xi,p)\big)\varphi_{-(\lambda+\xi),k}\right]dm_\alpha \\ &= \sum_{k=0}^{\infty}C_k^{(\alpha)}\big((\lambda,m),(\xi,p)\big)\int_{\mathbb{K}} h\varphi_{-(\lambda+\xi),k}\,dm_\alpha = \sum_{k=0}^{\infty}C_k^{(\alpha)}\big((\lambda,m),(\xi,p)\big)\mathcal{F}(h)(\lambda+\xi,k).\end{aligned} \tag{39}$$

It follows that $\mathcal{F}(\varphi_{-\lambda,m}h)(\xi,p) = 0$ for all $(\xi,p)\in\mathbb{R}\times\mathbb{N}$ such that $|\xi| \geqslant |\lambda|+A_h$. Moreover, from the last relation (39) together with the properties (i), (ii) and (iii) we get

$$\begin{aligned}\|\mathcal{F}(\varphi_{-\lambda,m}h)\|_{L^1_\alpha(\mathbb{R}\times\mathbb{N})} &\leqslant \sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\int_{\mathbb{R}} L_p^{(\alpha)}(0)C_k^{(\alpha)}\big((\lambda,m),(\xi,p)\big)\mathcal{F}(h)(\lambda+\xi,k)|\xi|^{\alpha+1}\,d\xi \\ &\leqslant \sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\int_{\mathbb{R}} L_k^{(\alpha)}(0)C_p^{(\alpha)}\big((-\lambda,m),(\xi+\lambda,k)\big)\mathcal{F}(h)(\lambda+\xi,k)|\lambda+\xi|^{\alpha+1}\,d\xi \\ &\leqslant \sum_{k=0}^{\infty}\int_{\mathbb{R}} L_k^{(\alpha)}(0)\mathcal{F}(h)(\lambda+\xi,k)|\lambda+\xi|^{\alpha+1}\,d\xi \\ &\leqslant \|\mathcal{F}(h)\|_{L^1_\alpha(\mathbb{R}\times\mathbb{N})} < \infty,\end{aligned} \tag{40}$$

which achieves the proof. □

Theorem 2.1. *Let $(\mu_k)_k$ be a sequence in $M_b^+(\mathbb{K})$ such that $\lim_{k\to\infty}\langle\mu_k,h\rangle = 0$ for every $h\in\mathcal{H}_\alpha$, then the sequence $(\mu_k)_k$ converges vaguely to 0, that is $\lim_{k\to\infty}\langle\mu_k,h\rangle = 0$ for every $h\in\mathcal{C}_c(\mathbb{K})$.*

Proof. Remark first that the positive function h_0 given in the previous Proposition 2.3 is bounded on $\mathbb{K}$, so that for every $k\in\mathbb{N}$, the measure $\nu_k = h_0\mu_k$ belongs to $M_b^+(\mathbb{K})$. Moreover, since $\varphi_{-\lambda,m}h_0\in\mathcal{H}_\alpha(\mathbb{K})$ and $\langle\mu_k,\varphi_{-\lambda,m}h_0\rangle = \langle\nu_k,\varphi_{-\lambda,m}\rangle = \mathcal{F}(\nu_k)(\lambda,m)$ for all $(\lambda,m)\in\mathbb{R}\times\mathbb{N}$ and $k\in\mathbb{N}$, we get

$$\lim_{k\to\infty}\mathcal{F}(\nu_k)(\lambda,m)=0,\quad \text{for all } (\lambda,m)\in\mathbb{R}\times\mathbb{N},$$

and consequently the sequence $(\nu_k)_k$ converges vaguely to 0, by virtue of Lévy-continuity theorem (see [13]).

This gives the result, since for all $h\in\mathcal{C}_c(\mathbb{K})$ and $k\in\mathbb{N}$, one readily has $\frac{h}{h_0}\in\mathcal{C}_c(\mathbb{K})$ and $\langle\mu_k,h\rangle = \langle\nu_k,\frac{h}{h_0}\rangle$. $\square$

3. Local central limit theorem on $\mathbb{K}$

Theorem 3.1. *Suppose that*

(i) $\mu(\{0\}\times\mathbb{R}) = 0$.
(ii) $\int_{\mathbb{K}} t\,d\mu = 0$ *and* $\int_{\mathbb{K}}(1+[x^2+|t|]^2+t^2[x^2+|t|])\,d\mu<\infty$.

Then the sequence $(\frac{k^{\alpha+2}}{C_\mu^{(\alpha)}}\mu^{*_\alpha k})_k$ converges vaguely on $\mathbb{K}$ to the Haar measure m_α, where $C_\mu^{(\alpha)} = \int_{\mathbb{R}\times\mathbb{N}} e^{-\frac{|\lambda|\alpha m}{\alpha+1}\sigma_\mu}\,d\gamma_\alpha(\lambda,m)$.

Proof. We shall use the last Theorem 2.1. Let so $h\in\mathcal{H}_\alpha(\mathbb{K})$ then by the Fourier Laguerre inversion formula (see [14, Theorem II.3]), we have

$$h=\int_{\mathbb{R}\times\mathbb{N}}\mathcal{F}(h)(\lambda,m)\varphi_{\lambda,m}\,d\gamma_\alpha(\lambda,m).$$

It follows

$$\begin{aligned}\langle\mu^{*_\alpha k},h\rangle &= \int_{\mathbb{K}} h(x,t)\,d\mu^{*_\alpha k}(x,t) = \int_{\mathbb{K}}\Bigg(\int_{\mathbb{R}\times\mathbb{N}}\mathcal{F}(h)(\lambda,m)\varphi_{\lambda,m}(x,t)\,d\gamma_\alpha(\lambda,m)\Bigg)\,d\mu^{*_\alpha k}(x,t)\\ &= \int_{\mathbb{R}\times\mathbb{N}}\Bigg(\int_{\mathbb{K}}\varphi_{\lambda,m}(x,t)\,d\mu^{*_\alpha k}(x,t)\Bigg)\mathcal{F}(h)(\lambda,m)\,d\gamma_\alpha(\lambda,m)\\ &= \int_{\mathbb{R}\times\mathbb{N}}\mathcal{F}\big(\mu^{*_\alpha k}\big)(\lambda,m)\mathcal{F}(h)(\lambda,m)\,d\gamma_\alpha(\lambda,m).\end{aligned}\tag{41}$$

But by the relation (II.11) in [14, p. 346], we have $\mathcal{F}(\mu^{*_\alpha k})(\lambda,m) = [\mathcal{F}(\mu)(\lambda,m)]^k$, so

$$\langle\mu^{*k},h\rangle = \int_{\mathbb{R}\times\mathbb{N}}\big[\mathcal{F}(\mu)(\lambda,m)\big]^k\mathcal{F}(h)(\lambda,m)\,d\gamma_\alpha(\lambda,m).\tag{42}$$

Let now $\eta>0$ then by the variables changing $\lambda\mapsto\tilde\lambda = k\lambda$; $(k\in\mathbb{N}^*)$, we obtain

$$\int_{\{\xi^{(\alpha)}_{\lambda,m}\leqslant\eta\}\times\mathbb{N}}\big[\mathcal{F}(\mu)(\lambda,m)\big]^k\mathcal{F}(h)(\lambda,m)\,d\gamma_\alpha(\lambda,m) = \frac{1}{k^{\alpha+2}}\int_{\{\xi^{(\alpha)}_{\tilde\lambda,m}\leqslant k\eta\}\times\mathbb{N}}\bigg[\mathcal{F}(\mu)\bigg(\frac{\tilde\lambda}{k},m\bigg)\bigg]^k\mathcal{F}(h)\bigg(\frac{\tilde\lambda}{k},m\bigg)\,d\gamma_\alpha(\tilde\lambda,m).\tag{43}$$

On the other hand, by Corollary 2.1 together with the inequality $|\mathcal{F}(h)|\leqslant\|h\|_{L^1_\alpha(\mathbb{K})}$ (see [14, p. 346]), one can choose $\eta>0$ such that for all $k\in\mathbb{N}^*$ and $(\tilde\lambda,m)\in\mathbb{R}\times\mathbb{N}$; $(\xi^{(\alpha)}_{\tilde\lambda,m}<k\eta)$ we have

$$\bigg|\bigg[\mathcal{F}(\mu)\bigg(\frac{\tilde\lambda}{k},m\bigg)\bigg]^k\mathcal{F}(h)\bigg(\frac{\tilde\lambda}{k},m\bigg)\bigg|\leqslant\exp\bigg(-\frac{\sigma_\mu}{2(\alpha+1)}\xi^{(\alpha)}_{\tilde\lambda,m}\bigg)\|h\|_{L^1_\alpha(\mathbb{K})}.\tag{44}$$

In addition, by continuity of the function $\zeta\mapsto\mathcal{F}(h)(\zeta,m)$ and by using again Corollary 2.1, we deduce that for all $(\tilde\lambda,m)\in\mathbb{R}\times\mathbb{N}$, we have

$$\lim_{k\to\infty}\bigg(\mathcal{F}(\mu)\bigg(\frac{\tilde\lambda}{k},m\bigg)\bigg)^k\mathcal{F}(h)\bigg(\frac{\tilde\lambda}{k},m\bigg) = \exp\bigg(-\frac{\sigma_\mu}{\alpha+1}\xi^{(\alpha)}_{\tilde\lambda,m}\bigg)\mathcal{F}(h)(0,m).\tag{45}$$

But $\sigma_\mu > 0$ since otherwise $\mu(\{0\} \times \mathbb{R}) = 1$, which contradicts the hypothesis. Thus, by a straightforward computation we get

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{N}} \exp\left(-\frac{\sigma_\mu}{2(\alpha+1)}\xi^{(\alpha)}_{\tilde{\lambda},m}\right) &= \sum_{k=0}^{\infty} L_m^{(\alpha)}(0) \int_{\mathbb{R}} \exp\left(-\frac{\sigma_\mu|\tilde{\lambda}|\alpha_m}{2(\alpha+1)}\right)|\tilde{\lambda}|^{\alpha+1}\,d\tilde{\lambda} \\ &= \left[\int_0^\infty \exp(-\zeta)\zeta^{\alpha+1}\,d\zeta\right] \sum_{k=0}^{\infty}\left(\frac{2(\alpha+1)}{\sigma_\mu|\zeta|\alpha_m}\right)^{\alpha+2} L_m^{(\alpha)}(0) \\ &= \Gamma(\alpha+2)\sum_{k=0}^{\infty}\left(\frac{2(\alpha+1)}{\sigma_\mu(m+\frac{\alpha+1}{2})}\right)^{\alpha+2} L_m^{(\alpha)}(0) < \infty, \end{aligned} \tag{46}$$

because off $L_m^{(\alpha)}(0) = \frac{\Gamma(m+\alpha+1)}{m!\Gamma(\alpha+1)} \sim \frac{m^\alpha}{\Gamma(\alpha+1)}$ $(m \to \infty)$ by Stirling's formula.

Hence by applying Lebesgue Theorem we deduce from (43)–(46) that

$$\begin{aligned}\lim_{k\to\infty}\left[k^{\alpha+2}\int_{\{\xi^{(\alpha)}_{\lambda,m}\leqslant\eta\}\times\mathbb{N}} \left[\mathcal{F}(\mu)(\lambda,m)\right]^k \mathcal{F}(h)(\lambda,m)\,d\gamma_\alpha(\lambda,m)\right] &= \int_{\mathbb{R}\times\mathbb{N}} \exp\left(-\frac{\sigma_\mu|\tilde{\lambda}|\alpha_m}{\alpha+1}\right)\mathcal{F}(h)(0,m)\,d\gamma_\alpha(\tilde{\lambda},m) \\ &= \int_{\mathbb{R}\times\mathbb{N}} \exp\left(-\frac{\sigma_\mu|\tilde{\lambda}|\alpha_m}{\alpha+1}\right)\left(\int_{\mathbb{K}} h(x,t)\,dm_\alpha(x,t)\right)d\gamma_\alpha(\tilde{\lambda},m) \\ &= C_\mu^{(\alpha)}\int_{\mathbb{K}} h(x,t)\,dm_\alpha(x,t), \end{aligned} \tag{47}$$

where $C_\mu^{(\alpha)} = \int_{\mathbb{R}\times\mathbb{N}} \exp(-\frac{\sigma_\mu|\tilde{\lambda}|\alpha_m}{\alpha+1})\,d\gamma_\alpha(\tilde{\lambda},m) = \Gamma(\alpha+2)(\frac{\alpha+1}{\sigma_\mu})^{\alpha+2}\sum_{k=0}^{\infty}\frac{L_m^{(\alpha)}(0)}{(m+\frac{\alpha+1}{2})^{\alpha+2}}$.

To obtain the result it remains to show that

$$\lim_{k\to\infty} k^{\alpha+2}\int_{\{\xi^{(\alpha)}_{\lambda,m}>\eta\}\times\mathbb{N}} \left[\mathcal{F}(\mu)(\lambda,m)\right]^k \mathcal{F}(h)(\lambda,m)\,d\gamma_\alpha(\lambda,m) = 0. \tag{48}$$

Indeed, since $h \in \mathcal{H}_\alpha(\mathbb{K})$ then there exists $A_h > 0$ such that $\mathcal{F}(h)(\lambda,m) = 0$ for all $|\lambda| \geqslant A_h$ and $m \in \mathbb{N}$, so that

$$\int_{\{\xi^{(\alpha)}_{\lambda,m}>\eta\}\times\mathbb{N}} \left[\mathcal{F}(\mu)(\lambda,m)\right]^k \mathcal{F}(h)(\lambda,m)\,d\gamma_\alpha(\lambda,m) = \int_{\{\frac{\eta}{\alpha_m}<|\lambda|\leqslant A_h\}\times\mathbb{N}} \left[\mathcal{F}(\mu)(\lambda,m)\right]^k \mathcal{F}(h)(\lambda,m)\,d\gamma_\alpha(\lambda,m). \tag{49}$$

It follows, by Proposition 2.2, that there is $a \in]0,1[$ such that

$$k^{\alpha+2}\int_{\{\xi^{(\alpha)}_{\lambda,m}>\eta\}\times\mathbb{N}} \left[\mathcal{F}(\mu)(\lambda,m)\right]^k \mathcal{F}(h)(\lambda,m)\,d\gamma_\alpha(\lambda,m) \leqslant k^{\alpha+2}a^k \int_{\mathbb{R}\times\mathbb{N}} \left|\mathcal{F}(h)(\lambda,m)\right|d\gamma_\alpha(\lambda,m), \tag{50}$$

which finishes the proof, since $\int_{\mathbb{R}\times\mathbb{N}} |\mathcal{F}(h)(\lambda,m)|\,d\gamma_\alpha(\lambda,m) = \|\mathcal{F}(h)\|_{L^1_\alpha(\mathbb{R}\times\mathbb{N})} < \infty$ for every $h \in \mathcal{H}_\alpha(\mathbb{K})$. □

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