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New existence of periodic solutions for second order non-autonomous Hamiltonian systems ☆

Qiongfen Zhang *, X.H. Tang

School of Mathematical Sciences and Computing Technology, Central South University, Changsha, Hunan 410083, PR China

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ABSTRACT

By using mountain pass theorem and local link theorem, some existence theorems are obtained for periodic solutions of second order non-autonomous Hamiltonian systems under local superquadratic condition and other suitable conditions.

1. Introduction

Consider the following second order Hamiltonian systems

$$\begin{cases} \ddot{x}(t) - B(t)x(t) + \nabla H(t,x) = 0, & \forall t \in [0,T], \\ x(0) - x(T) = \dot{x}(0) - \dot{x}(T) = 0, \end{cases} \tag{1.1}$$

where $T > 0$, $B \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric matrix-valued function. Let $H : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $(t,x) \to H(t,x)$ be measurable in t for each $x \in \mathbb{R}^N$ and continuously differentiable in x for almost every $t \in \mathbb{R}$, H is T-periodic in t and $H(t,0) = 0$, $\nabla H(t,x) = \frac{\partial H(t,x)}{\partial x}$.

When $B(t) \equiv 0$ for all $t \in \mathbb{R}$, the existence of periodic solutions for systems (1.1) had been extensively studied and a lot of important existence and multiplicity results had been obtained, for example, see [1,4–8,11–14,24] and references cited therein. Particularly, Fei [5] got the existence of 1-periodic solutions of systems (1.1) under some new superquadratic conditions; Schechter [6] studied the existence of non-constant T-periodic solutions of systems (1.1) and got a new saddle point theorem; Schechter [7] obtained non-constant T-periodic solutions of systems (1.1) with local superquadratic condition by using linking methods; Wu [13] studied multiplicity of periodic solutions; Tao and Tang [12] researched the subharmonic solutions; and Wang [24] obtained the existence of periodic solutions for systems (1.1) under local superquadratic condition and other conditions. When $B(t) \neq 0$, Zou and Li [15] studied the existence of infinitely many T-periodic solutions under the assumption that $H(t,x)$ was even in x; Ou and Tang [9] got the existence of homoclinic solutions; Faraci [3] studied the existence of multiple periodic solutions; Tang and Lin [18] studied the homoclinic solutions for systems (1.1); and Xiao and Tang [17] investigated the existence of periodic solutions for systems (1.1) with potential indefinite in sign by employing

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* Corresponding author.
E-mail addresses: zqf5150718@sina.com (Q. Zhang), tangxh@mail.csu.edu.cn (X.H. Tang).

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liking methods when $H(t,x) = b(t)V'(x)$. There are also some more general Hamiltonian systems which are considered in the paper [19–21].

In 2008, He and Wu [16] had obtained some results of the nontrivial T-periodic solutions for systems (1.1) under much weaker assumptions, which greatly generalized the corresponding results in [5]. More precisely, they established the following two main theorems.

Theorem 1.1. *(See [16].) Suppose that H satisfies the following conditions*:

(H1) $H(t,x)/|x|^2 \to +\infty$ *as* $|x| \to \infty$ *uniformly for all* t.
(H2) $|\nabla H(t,x)|/|x| \to 0$ *as* $|x| \to 0$ *uniformly for all* t.
(H3) *There exist constants* $\alpha_0 > 0$ *and* $d_0 > 0$ *such that*

$$\left|\nabla H(t,x)\right| \leqslant d_0\bigl(|x|^{\alpha_0} + 1\bigr), \quad \forall (t,x) \in [0,T] \times \mathbb{R}^N.$$

(H4) *There exist constants* $\beta_0 \geqslant \alpha_0 \geqslant 1$, $d_0' > 0$ *and* $L_0 > 0$ *such that*

$$\bigl(\nabla H(t,x), x\bigr) - 2H(t,x) \geqslant d_0'|x|^{\beta_0}, \quad \forall |x| \geqslant L_0,\ t \in [0,T].$$

(H5)

$$\int_0^T H(t,x)\,dt \geqslant 0, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^N.$$

And $B(t)$ satisfies the condition:

(L1) *For the smallest eigenvalue* $b(t) = \inf_{|x|=1}(B(t)x,x)$ *of* $B(t)$, *there exists a constant* $\gamma < 1$ *such that* $b(t)/|t|^{\gamma-1} \to \infty$ *as* $|t| \to \infty$.

Then there exists a nontrivial T-periodic solution of systems (1.1).

Theorem 1.2. *(See [16].) Suppose that* (L1), (H1)–(H4) *and the following condition hold*:

(H5)$'$ *There exists a constant* $R_1 > 0$ *such that*
  (i) $H(t,x) \geqslant 0$, $\forall |x| \leqslant R_1$, $t \in [0,T]$; *or*
  (ii) $H(t,x) \leqslant 0$, $\forall |x| \leqslant R_1$, $t \in [0,T]$.

If 0 is an eigenvalue of $-(d^2/dt^2) + B(t)$, *then there exists at least one nontrivial T-periodic solution of systems* (1.1).

However, we must point out that the condition (H3) contradicts with condition (H1) when $0 < \alpha_0 \leqslant 1$. In fact, if $0 < \alpha_0 \leqslant 1$, from (H3), we have $|H(t,x)| \leqslant d_0(|x|^{\alpha_0+1} + |x|)$ for all $(t,x) \in [0,T] \times \mathbb{R}^N$, thus $\frac{H(t,x)}{|x|^2} \to 0$ (when $0 < \alpha_0 < 1$) or d_0 (when $\alpha_0 = 1$) as $|x| \to \infty$ uniformly for all t. Therefore, (H3) holds only when $\alpha_0 > 1$. As is known, the following so-called global Ambrosetti–Rabinowitz condition on $H(t,x)$ introduced by Ambrosetti and Rabinowitz in [22] is very important in many proofs: there is a constant μ such that

$$0 < \mu H(t,x) \leqslant \bigl(\nabla H(t,x), x\bigr), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R}^N \backslash \{0\}, \tag{1.2}$$

which implies that $H(t,x)$ is of superquadratic growth as $|x| \to \infty$, that is

$$\lim_{|x|\to\infty} \frac{H(t,x)}{|x|^2} = \infty \quad \text{uniformly for all } t.$$

As is pointed out in [23], by (H3) the nonlinearity grows subcritically, and the condition (H4) guarantees that $H(t,x)$ grows non-quadratic in x as $|x| \to \infty$ for all $x \in \mathbb{R}^N$. In a word, the conditions (H1), (H3) and (H4) all guarantee the superquadratic condition hold.

Motivated by the ideas of [5,7,16,24], we will use the following conditions to generalize the results of [16] in another direction.

(H1)$'$ There exists a subset E_0 of $[0,T]$ with $\mathrm{meas}(E_0) > 0$ such that

$$\lim_{|x|\to\infty} \inf \frac{H(t,x)}{|x|^2} > 0, \quad \text{a.e. } t \in E_0.$$

(H3)′ There exist constants $\alpha > 1$ and $d_1 > 0$ such that

$$\left|\nabla H(t,x)\right| \leqslant d_1\left(|x|^\alpha + 1\right), \quad \forall (t,x) \in [0,T] \times \mathbb{R}^N.$$

(H4)′ There exist constants $\mu > 2$, $0 < \beta < 2$, $L > 0$ and a function $a(t) \in L^1(0,T;R^+)$ such that

$$\mu H(t,x) \leqslant \nabla H(t,x)x + a(t)|x|^\beta, \quad \forall |x| \geqslant L,\ x \in \mathbb{R}^N \text{ and a.e. } t \in [0,T].$$

Here are our main results.

Theorem 1.3. *Suppose* (H1)′, (H2), (H3)′, (H4)′, (H5) *and* (L1) *hold. Then there exists a nontrivial T-periodic solution of systems* (1.1).

Theorem 1.4. *Suppose* (H1)′, (H2), (H3)′, (H4)′, (L1) *and the following condition* (H5)″ *hold*:

(H5)″ $H(t,x) \geqslant 0$, $\forall (t,x) \in [0,T] \times \mathbb{R}^N$.

Then there exists a nontrivial T-periodic solution of systems (1.1).

Theorem 1.5. *Suppose* (H1)′, (H2), (H3)′, (H4)′, (L1) *and* (H5)′ *hold. If* 0 *is an eigenvalue of* $-(d^2/dt^2) + B(t)$, *then there exists at least one nontrivial T-periodic solution of systems* (1.1).

Remark 1.1. The condition (H1)′ is a local superquadratic condition. As far as we know, only [7,24] considered this situation. In [7], the author obtained a new results by using linking methods. In [24], the authors studied the non-constant T-periodic solutions by employing the generalized mountain pass theorem. When we take $a(t) \equiv 0$, the condition (H4)′ reduces to (1.2), it is easy to see that (H4)′ is weaker than (1.2). From (H1)′, we only need $\lim_{|x|\to\infty}\inf \frac{H(t,x)}{|x|^2} > 0$ holds in a subset E_0 of $[0,T]$, but it is not by (H1), thus (H1)′ is weaker than (H1). There are functions that satisfy our theorems but not satisfy those results in [5,16]. For example, let

$$H(t,x) = \frac{1}{6} f(t)|x|^4, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^N,$$

where

$$f(t) = \begin{cases} \sin\frac{2\pi t}{T}, & t \in [0, \frac{T}{2}], \\ 0, & t \in [\frac{T}{2}, T]. \end{cases}$$

Let $E_0 = [0, \frac{T}{4}]$, an easy computation shows that (H1)′ and (H4)′ hold. It is also clear that the other conditions of our theorems hold. Therefore, $H(t,x)$ satisfies all the conditions of our theorems, but doesn't satisfy (H1) and (H4), thus doesn't satisfy the corresponding results in [5] and [16].

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proofs of our theorems.

2. Preliminaries

Now, let us give some concepts which appeared in [4].

Let X be a real Banach space with direct decomposition $X = X_1 \oplus X_2$. Consider two sequences of subspaces $X^1 = \overline{\bigcup_{n\in N} X_n^1}$ and $X^2 = \overline{\bigcup_{n\in N} X_n^2}$ such that $X_0^j \subset X_1^j \cdots X^j$, $j = 1,2$. For every multi-index $a = (a_1, a_2) \in N^2$, we denote by X_a the space $X_{a_1}^1 \oplus X_{a_2}^2$. We say $a \leqslant b$ if $a_1 \leqslant b_1$, $a_2 \leqslant b_2$.

Definition 2.1. A sequence $\{a_n\} \subset N^2$ is said to be admissible if, for every $a \in N^2$ there is $m \in N$ such that $n \geqslant m \Rightarrow a_n > a$.

Definition 2.2. Let $c \in \mathbb{R}$ and $\varphi \in C^1(X,\mathbb{R})$. The functional φ is said to satisfy the (PS)* condition if every sequence $\{x_{a_n}\}$ such that $\{a_n\}$ is admissible and

$$x_{a_n} \in X_{a_n}, \qquad c = \sup \varphi_{a_n}(x_{a_n}) < \infty, \qquad \varphi'_{a_n}(x_{a_n}) \to 0$$

contains a subsequence which converges to a critical point of φ, where $\varphi_a = \varphi|_{X_a}$.

Definition 2.3. Let X be a Banach space with a direct sum decomposition $X = X_1 \oplus X_2$. The function $f \in C^1(X, \mathbb{R})$ has a local linking at 0, with respect to (X^1, X^2), if, for some $r > 0$,

$$f(x) \geqslant 0, \quad \forall x \in X^1,\ \|x\| \leqslant r,$$
$$f(x) \leqslant 0, \quad \forall x \in X^2,\ \|x\| \leqslant r.$$

The following two theorems are very useful in our proofs.

Theorem A. *(See [10].) Let E be a real Banach space with $E = V \oplus X$, where V is finite-dimensional. Suppose $I \in C^1(E, \mathbb{R})$ satisfies* (PS) *condition and the following conditions*:

(A1) *There are constants $\rho, \alpha_1 > 0$ such that $I|_{\partial B_\rho \cap X} \geqslant \alpha_1$, where $B_\rho := \{u \in E \mid \|u\| \leqslant \rho\}$, ∂B_ρ denotes the boundary of B_ρ.*
(A2) *There is $e \in \partial B_1 \cap X$ and $r_0 > \rho$ such that if $Q \equiv (\bar{B}_{r_0} \cap V) \oplus \{re \mid 0 < r < r_0\}$, then $I|_{\partial Q} \leqslant 0$.*

Then I possesses a critical value $c \geqslant \alpha_1$ which can be characterized as

$$c = \inf_{h \in \Gamma} \max_{u \in Q} I\big(h(u)\big),$$

where $\Gamma = \{h \in C(\overline{Q}, E) \mid h = id \text{ on } \partial Q\}$.

Theorem B. *(See [4].) Suppose that $f \in C^1(X, \mathbb{R})$ satisfies the following assumptions*:

(B1) *f has a local linking at 0 and $X^1 \neq \{0\}$;*
(B2) *f satisfies* (PS)* *condition*;
(B3) *f maps bounded sets into bounded sets*;
(B4) *for every $m \in N$, $f(x) \to -\infty$ as $\|x\| \to \infty$ on $X^1_m \oplus X^2$.*

Then f has at least one nonzero critical point.

Write $L^\lambda = L^\lambda([0, T], \mathbb{R}^N)$ for $1 \leqslant \lambda \leqslant \infty$ and denote by A the self-adjoint extension of the operator $-(d^2/dt^2) + B(t)$. Let $|A|$ be the absolute value of A and $|A|^{\frac{1}{2}}$ be the square of $|A|$. Let $E = D(|A|^{\frac{1}{2}})$, where $D(|A|^{\frac{1}{2}})$ denotes the domain of $|A|^{\frac{1}{2}}$. The following lemma which is due to Ding [2] (see also [9,16]) is needed in our proofs.

Lemma 2.1. *Suppose that $B(t)$ satisfies* (L1)*. Then E is compactly embedded into L^p for any $1 \leqslant p \leqslant \infty$, which implies that there exists a constant $C > 0$ such that*

$$\|x\|_p \leqslant C\|x\|$$

for all $x \in E$ and all $p = 1, 2, \beta + 2, \frac{\beta}{\beta - \alpha}$ (when $\beta > \alpha$), ∞.

Ding [2] pointed out that the spectrum $\sigma(A)$ consists of eigenvalues numbered in

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \to \infty$$

by Lemma 2.1 (counted in their multiplicities), and a corresponding system of eigenfunctions $\{e_i\}$ forms an orthogonal basis in L^2. Let n^- (respectively n^0) be the number of λ_i satisfying $\lambda_i < 0$ (respectively $\lambda_i = 0$), $\bar{n} = n^- + n^0$, and set $E^- = \operatorname{span}\{e_1, \ldots, e_{n^-}\}$, $E^0 = \operatorname{span}\{e_{n^-+1}, \ldots, e_{\bar{n}}\} = \ker A$, $E^+ = \operatorname{span}\{e_{\bar{n}+1}, \ldots\}$. Then $E = E^- \oplus E^0 \oplus E^+$. The inner product and norm on E are the following:

$$(x, y) = \big(|A|^{\frac{1}{2}} x, |A|^{\frac{1}{2}} y\big)_{L^2} + \big(x^0, y^0\big)_{L^2}, \qquad \|x\|^2 = (x, x) = \big\||A|^{\frac{1}{2}} x\big\|^2_{L^2} + \big\|x^0\big\|^2_{L^2},$$

where $x = x^- + x^0 + x^+$, $y = y^- + y^0 + y^+ \in E = E^- \oplus E^0 \oplus E^+$. Then E is a Hilbert space.

3. Proofs of theorems

The functional φ corresponding to (1.1) on E is given by

$$\varphi(x) = \frac{1}{2}\int_0^T |\dot{x}|^2\, dt + \frac{1}{2}\int_0^T \big(B(t)x, x\big)\, dt - \int_0^T H(t, x)\, dt, \tag{3.1}$$

which is continuously differentiable on E, and

$$\big(\varphi'(x), y\big) = \int_0^T (\dot{x}, \dot{y})\, dt + \int_0^T \big(B(t)x, y\big)\, dt - \int_0^T \big(\nabla H(t, x), y\big)\, dt$$

for all $x, y \in E$. It follows from (3.1) that

$$\varphi(x) = \frac{1}{2}\|x^+\|^2 - \frac{1}{2}\|x^-\|^2 - \int_0^T H(t, x)\, dt \tag{3.2}$$

for all $x = x^- + x^0 + x^+ \in E^- \oplus E^0 \oplus E^+$, and

$$\big(\varphi'(x), y\big) = \big(x^+, y^+\big) - \big(x^-, y^-\big) - \int_0^T \big(\nabla H(t, x), y\big)\, dt \tag{3.3}$$

for all $x = x^- + x^0 + x^+$, $y = y^- + y^0 + y^+ \in E = E^- \oplus E^0 \oplus E^+$ (see [2,9]).

In the following, we denote C_i $(i = 0, 1, 2, \ldots)$ for different positive constants.

Proof of Theorem 1.3. *Step* 1. We prove that φ satisfies the (PS) condition. For $x \in E$, let $\bar{x} = \frac{1}{T}\int_0^T x(t)\, dt$, $x = \tilde{x} + \bar{x}$. It is well known that there exists a constant $C_0 > 0$ such that

$$\|x\|_\infty \leqslant C_0\|x\| \quad \text{for all } x \in E, \tag{3.4}$$

where $\|x\|_\infty = \max_{0\leqslant t\leqslant T} |x(t)|$. Let $\{x_n\} \subset E$ satisfy $\varphi'(x_n) \to 0$ as $n \to \infty$ and $\{\varphi(x_n)\}$ is bounded. We prove that $\{x_n\}$ is a bounded sequence in E. Otherwise, going to a subsequence if necessary, we can assume that $\|x_n\| \to \infty$ as $n \to \infty$. Set $y_n = \frac{x_n}{\|x_n\|}$, then $\{y_n\}$ is bounded in E. Hence, there exists a subsequence, still denoted by $\{y_n\}$, such that

$$y_n \rightharpoonup y_0 \quad \text{weakly in } E,$$
$$y_n \to y_0 \quad \text{strongly in } C\big(0, T; \mathbb{R}^N\big).$$

Then, we have

$$\bar{y}_n \to \bar{y}_0. \tag{3.5}$$

By (H3)$'$, for $|x| \leqslant L$, one has

$$\big|H(t, x)\big| \leqslant d_1\big(|x|^{1+\alpha} + |x|\big) \leqslant d_1\big(L^{1+\alpha} + L\big),$$

together with (H4)$'$, one has

$$\mu H(t, x) \leqslant \nabla H(t, x)x + a(t)|x|^\beta + \mu d_1\big(L^{1+\alpha} + L\big) \tag{3.6}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$. It follows from (3.4), (3.6), (L1) and $\mu > 2$ that

$$\begin{aligned}
\left(\frac{\mu}{2} - 1\right)\|\dot{x}_n\|_{L^2}^2 &= \mu\varphi(x_n) - \big(\varphi'(x_n), x_n\big) + \left(-\frac{\mu}{2} + 1\right)\int_0^T \big(B(t)x_n, x_n\big)\, dt + \int_0^T \big(\mu H(t, x_n) - \nabla H(t, x_n)x_n\big)\, dt \\
&\leqslant \mu\varphi(x_n) - \big(\varphi'(x_n), x_n\big) + \int_0^T \big(\mu H(t, x_n) - \nabla H(t, x_n)x_n\big)\, dt \\
&\leqslant C_1 + \int_0^T \big[a(t)|x_n|^\beta + \mu d_1\big(L^{1+\alpha} + L\big)\big]\, dt \\
&\leqslant C_2 + C_3\|x_n\|^\beta. \qquad (3.7)
\end{aligned}$$

Notice that $0 < \beta < 2$, we have

$$\|\dot{x}_n\|_{L^2} \leqslant C_4\|x_n\|^\nu + C_5, \tag{3.8}$$

where $0 < \nu < 1$, which implies that

$$\|\dot{y}_n\|_{L^2} \to 0 \quad \text{as } n \to \infty.$$

Together with (3.5), we have

$$y_n \to \bar{y}_0 \quad \text{as } n \to \infty.$$

Hence, we obtain

$$y_0 = \bar{y}_0 \quad \text{and} \quad T|\bar{y}_0|^2 = \|\bar{y}_0\|^2 \to 1.$$

Consequently, $|x_n| \to \infty$ as $n \to \infty$ uniformly for a.e. $t \in [0, T]$. From (H1)$'$ and (H5), we get

$$\lim_{|x_n|\to\infty} \inf \frac{\int_0^T H(t, x_n)\,dt}{\|x_n\|^2} \geqslant \frac{\int_0^T [\lim_{|x_n|\to\infty} \inf H(t, x_n)]\,dt}{\|x_n\|^2} \geqslant \int_{E_0} \left[\lim_{|x_n|\to\infty} \inf \frac{H(t, x_n)}{|x_n|^2} |y_n|^2 \right] dt$$
$$= \int_{E_0} \left[\lim_{n\to\infty} \inf \frac{H(t, x_n)}{|x_n|^2} |y_0|^2 \right] dt > 0. \tag{3.9}$$

From (3.2)–(3.4) and (3.6), we have

$$\left(\frac{\mu}{2} - 1\right)\left(\|x_n^+\|^2 - \|x_n^-\|^2\right) = \mu\varphi(x_n) - \left(\varphi'(x_n), x_n\right) + \int_0^T \left(\mu H(t, x_n) - \nabla H(t, x_n)x_n\right) dt$$
$$\leqslant C_1 + \int_0^T \left[a(t)|x_n|^\beta + \mu d_1\left(L^{1+\alpha} + L\right)\right] dt$$
$$\leqslant C_2 + C_3\|x_n\|^\beta.$$

Notice that $\mu > 2$, we obtain

$$\|x_n^+\|^2 - \|x_n^-\|^2 \leqslant \frac{2C_2}{\mu - 2} + \frac{2C_3}{\mu - 2}\|x_n\|^\beta. \tag{3.10}$$

By the boundedness of $\varphi(x_n)$ and (3.10), we have

$$\frac{\varphi(x_n)}{\|x_n\|^2} = \frac{\frac{1}{2}(\|x_n^+\|^2 - \|x_n^-\|^2)}{\|x_n\|^2} - \frac{\int_0^T H(t, x_n)\,dt}{\|x_n\|^2} \leqslant \frac{\frac{C_2}{\mu-2}}{\|x_n\|^2} + \frac{\frac{C_3}{\mu-2}\|x_n\|^\beta}{\|x_n\|^2} - \frac{\int_0^T H(t, x_n)\,dt}{\|x_n\|^2},$$

which together with $0 < \beta < 2$ implies that

$$\lim_{|x_n|\to\infty} \inf \frac{\int_0^T H(t, x_n)\,dt}{\|x_n\|^2} = 0.$$

This contradicts with (3.9). Thus, $\{x_n\}$ is bounded in E. Hence, there exists a constant $C_6 > 0$ such that

$$\|x_n\| \leqslant C_6 \quad \text{for all } n. \tag{3.11}$$

Going if necessary to a subsequence, we can assume that

$$x_n \rightharpoonup x \quad \text{in } E. \tag{3.12}$$

It follows from Lemma 2.1 that

$$x_n \to x \quad \text{in } L^2\left(0, T; \mathbb{R}^N\right) \text{ as } n \to \infty. \tag{3.13}$$

From (H2), there is $\delta > 0$ such that

$$\left|\nabla H(t, x)\right| \leqslant |x| \quad \text{for all } t \in [0, T] \text{ and } |x| \leqslant \delta. \tag{3.14}$$

Choose a positive integer k_0 such that

$$\frac{C_6}{\sqrt{2k_0 + 3}} \leqslant \frac{1}{2}\delta. \tag{3.15}$$

It follows from (L1) that there exists $\delta_1 > 1$ such that

$$\left(\frac{k_0+1}{\sqrt{2k_0+1}}\frac{2C_6}{\delta}\right)^2 \leqslant b(s) \quad \text{for all } |s| \geqslant \delta_1 - 1. \tag{3.16}$$

It follows from Hölder's inequality, (3.11), (3.15) and (3.16) that

$$\begin{aligned}
|x_n| &= \left|\int_t^{t+1} \left[-\dot{x}_n(s)(t+1-s)^{k_0+1} + x_n(s)(k_0+1)(t+1-s)^{k_0}\right] ds\right| \\
&\leqslant \frac{1}{\sqrt{2k_0+3}}\left(\int_t^{t+1} \left|\dot{x}_n(s)\right|^2 ds\right)^{\frac{1}{2}} + \frac{k_0+1}{\sqrt{2k_0+1}}\left(\int_t^{t+1} \left|x_n(s)\right|^2 ds\right)^{\frac{1}{2}} \\
&\leqslant \frac{1}{\sqrt{2k_0+3}}\|x_n\| + \frac{k_0+1}{\sqrt{2k_0+1}}\left(\int_{|s|\geqslant\delta_1-1} \left|x_n(s)\right|^2 ds\right)^{\frac{1}{2}} \\
&\leqslant \frac{1}{\sqrt{2k_0+3}}C_6 + \frac{k_0+1}{\sqrt{2k_0+1}}\left(\int_{|s|\geqslant\delta_1-1} \frac{(B(s)x_n(s), x_n(s))}{b(s)} ds\right)^{\frac{1}{2}} \\
&\leqslant \frac{1}{2}\delta + \frac{k_0+1}{\sqrt{2k_0+1}}\frac{\|x_n\|}{(\inf_{|s|\geqslant\delta_1-1} b(s))^{\frac{1}{2}}} \\
&\leqslant \frac{1}{2}\delta + \frac{k_0+1}{\sqrt{2k_0+1}}\frac{C_6}{(\inf_{|s|\geqslant\delta_1-1} b(s))^{\frac{1}{2}}} \\
&\leqslant \delta
\end{aligned} \tag{3.17}$$

for all $n \in N$ and all $|t| \geqslant \delta_1$. Hence, by (3.14), (3.17) and Hölder's inequality, one has

$$\begin{aligned}
&\left|\int_0^T \left(\nabla H(t,x_n) - \nabla H(t,x), x_n^+ - x^+\right) dt\right| \\
&\quad \leqslant \int_0^T \left(\left|\nabla H(t,x_n)\right| + \left|\nabla H(t,x)\right|\right)\left|x_n^+ - x^+\right| dt \leqslant \int_{|t|\geqslant\delta_1} \left(|x_n| + |x|\right)\left|x_n^+ - x^+\right| dt + 2C_7 \int_{|t|\leqslant\delta_1} \left|x_n^+ - x^+\right| dt \\
&\quad \leqslant \left(2\delta T^{\frac{1}{2}} + 2(2\delta_1)^{\frac{1}{2}} C_7\right)\left\|x_n^+ - x^+\right\|_{L^2} \leqslant C_8\left\|x_n^+ - x^+\right\|_{L^2}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\left\|x_n^+ - x^+\right\|^2 &= \left(\varphi'(x_n) - \varphi'(x), x_n^+ - x^+\right) + \int_0^T \left(\nabla H(t,x_n) - \nabla H(t,x), x_n^+ - x^+\right) dt \\
&\leqslant \left\|\varphi'(x_n)\right\|\left\|x_n^+ - x^+\right\| - \left(\varphi'(x), x_n^+ - x^+\right) + C_8\left\|x_n^+ - x^+\right\|_{L^2} \\
&\leqslant \left\|\varphi'(x_n)\right\|\left\|C_6 + x^+\right\| - \left(\varphi'(x), x_n^+ - x^+\right) + C_8\left\|x_n^+ - x^+\right\|_{L^2}
\end{aligned}$$

for all n, which implies that $x_n^+ \to x^+$ in E as $n \to \infty$ by (3.12) and (3.13). From (3.13) and the equivalence of the norms on the finite-dimensional subspace $E^- \oplus E^0$, we obtain that $x_n^0 \to x^0$ and $x_n^- \to x^-$ in E as $n \to \infty$, which implies that $x_n \to x$ in E as $n \to \infty$. Hence $\{x_n\}$ has a convergent subsequence, which shows that the (PS) condition holds.

Step 2. We claim that there exist $\rho > 0$ and $\alpha_1 > 0$ such that

$$\varphi(x) \geqslant \alpha_1, \quad \forall x \in \partial B_\rho \cap E^+.$$

From (H2), for any $\varepsilon > 0$, there exists $\delta_2 > 0$ such that

$$\left|H(t,x)\right| \leqslant \varepsilon|x|^2, \quad \forall t \in [0,T],\ \forall |x| \leqslant \delta_2. \tag{3.18}$$

By (H3)$'$, we have

$$\left|H(t,x)\right| \leqslant d_1\left(|x|^{1+\alpha} + |x|\right), \quad \forall (t,x) \in [0,T] \times \mathbb{R}^N,$$

which together with (3.18) implies that there exists a constant $M > 0$ such that

$$\left|H(t,x)\right| \leqslant \varepsilon|x|^2 + M|x|^{1+\alpha}, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^N. \tag{3.19}$$

Hence, it follows from Lemma 2.1 and (3.19) that

$$\int_0^T \left|H(t,x)\right| dt \leqslant \int_0^T \left(\varepsilon|x|^2 + M|x|^{1+\alpha}\right) dt \leqslant \varepsilon\|x\|_{L^2}^2 + M\|x\|_{L^{1+\alpha}}^{1+\alpha} \leqslant \varepsilon C^2\|x\|^2 + MC^{1+\alpha}\|x\|^{1+\alpha}. \tag{3.20}$$

For any $x \in E^1 = E^+$, by (3.20), we obtain

$$\varphi(x) = \frac{1}{2}\|x\|^2 - \int_0^T H(t,x)\,dt \geqslant \frac{1}{2}\|x\|^2 - \varepsilon C^2\|x\|^2 - MC^{1+\alpha}\|x\|^{1+\alpha}.$$

Taking $\varepsilon = \frac{1}{4C^2}$ and noticing that $1 + \alpha > 2$, we can find a constant $\rho > 0$ small enough such that

$$\varphi(x) \geqslant \frac{1}{4}\rho^2 - MC^{1+\alpha}\rho^{1+\alpha} \geqslant \frac{1}{16}\rho^2 \equiv \alpha_1 > 0 \quad \text{for all } x \in E^1 = E^+ \text{ with } \|x\| = \rho.$$

Step 3. Let $e \in E^+$ with $\|e\| = 1$. By (H1)′, there exist constants $L_1 > 0$, $M_1 > 0$ such that

$$H(t,x) \geqslant M_1|x|^2 \quad \text{for all } |x| \geqslant L_1,\ x \in \mathbb{R}^N \text{ and a.e. } t \in E_0. \tag{3.21}$$

For $|x| \leqslant L_1$ and a.e. $t \in E_0$, from (H3)′, we get

$$\left|H(t,x)\right| \leqslant d_1\left(|x|^{1+\alpha} + |x|\right) \leqslant d_1\left(|L_1|^{1+\alpha} + |L_1|\right) := C_9. \tag{3.22}$$

Thus, it follows from (3.21) and (3.22) that

$$H(t,x) \geqslant M_1|x|^2 - C_9 \quad \text{for all } x \in \mathbb{R}^N \text{ and a.e. } t \in E_0. \tag{3.23}$$

Let $M_2 = \int_{E_0} |e|^2\,dt$, $M_3 = \int_{E_0} |x|^2\,dt$, choose M_1 sufficiently large such that $M_1M_2 > \frac{1}{2}$, by (3.23), we have

$$\begin{aligned}
\varphi(re + x) &= \frac{1}{2}\left(r^2 - \left\|x^-\right\|^2\right) - \int_0^T H(t, re + x)\,dt \leqslant \frac{1}{2}\left(r^2 - \left\|x^-\right\|^2\right) - \int_{E_0} H(t, re + x)\,dt \\
&\leqslant \frac{1}{2}\left(r^2 - \left\|x^-\right\|^2\right) - \int_{E_0} M_1|re + x|^2\,dt + C_9T = \frac{1}{2}\left(r^2 - \left\|x^-\right\|^2\right) - M_1r^2\int_{E_0} e^2\,dt - M_1\int_{E_0} x^2\,dt + C_9T \\
&= \frac{1}{2}\left(r^2 - \left\|x^-\right\|^2\right) - M_1M_2r^2 - M_1M_3 + C_9T
\end{aligned}$$

for all $r > 0$ and $x \in E^- \oplus E^0$. Hence, we obtain

$$\varphi(re + x) \leqslant 0, \quad \text{either } r \geqslant r_1 \text{ or } M_3 \geqslant r_2, \tag{3.24}$$

where $r_1 = \sqrt{\frac{2C_9T}{2M_1M_2 - 1}}$, $r_2 = \frac{C_9T}{M_1}$.

Since $E^- \oplus E^0$ is finite-dimensional, there exists $M_4 > 0$ such that

$$\|x\|_{L^2} \leqslant M_4\|x\| \quad \text{for all } x \in E^- \oplus E^0. \tag{3.25}$$

Notice that

$$\|x\| \geqslant \frac{1}{M_4}\|x\|_{L^2} = \frac{1}{M_4}\left(\int_0^T |x|^2\,dt\right)^{\frac{1}{2}} \geqslant \frac{1}{M_4}\left(\int_{E_0} |x|^2\,dt\right)^{\frac{1}{2}} dt \geqslant \frac{1}{M_4}r_2^{\frac{1}{2}},$$

thus (3.24) holds for all $\|x\| \geqslant \frac{1}{M_4}r_2^{\frac{1}{2}} := r_3$ whenever $x \in E^- \oplus E^0$. Let

$$Q = \{re \mid 0 \leqslant r \leqslant r_1\} \oplus \left\{x \in E^- \oplus E^0 \;\middle|\; \|x\| \leqslant r_3\right\}. \tag{3.26}$$

Then we have $\partial Q = Q_1 \cup Q_2 \cup Q_3$, where

$$Q_1 = \{x \in E^- \oplus E^0 \mid \|x\| \leqslant r_3\}, \qquad Q_2 = r_1 e \oplus \{x \in E^- \oplus E^0 \mid \|x\| \leqslant r_3\},$$
$$Q_3 = \{re \mid 0 \leqslant r \leqslant r_1\} \oplus \{x \in E^- \oplus E^0 \mid \|x\| = r_3\}.$$

By (3.24), one has

$$\varphi(x) \leqslant 0, \quad \forall x \in Q_2 \cup Q_3.$$

It follows from (H5) that $\varphi(x) \leqslant 0$ for all $x \in E^- \oplus E^0$, which implies that

$$\varphi(x) \leqslant 0, \quad \forall x \in Q_1.$$

Hence, we have

$$\varphi(x) \leqslant 0, \quad \forall x \in \partial Q.$$

Consequently, the proof of Theorem 1.3 is complete by Theorem A. □

Proof of Theorem 1.4. In fact, the proof of Theorem 1.4 is similar to the proof of Theorem 1.3, we omit the detail here. □

Proof of Theorem 1.5. We only consider the case (i). The other case is similar.

(1) We claim that φ has a local linking at zero with respect to (X^1, X^2), where $X^1 = E^+$ and $X^2 = E^0 \oplus E^-$. Indeed, for $x \in X^1$, by (3.20), we have

$$\varphi(x) \geqslant \frac{1}{2}\|x\| - \varepsilon C^2 \|x\|^2 - MC^{1+\alpha}\|x\|^{1+\alpha}.$$

Let $\varepsilon = \frac{1}{4C^2} > 0$, noting that $1+\alpha > 2$, we can choose a constant $\rho_0 > 0$ such that

$$\varphi(x) \geqslant 0, \quad \forall x \in X^1 \text{ with } \|x\| \leqslant \rho_0.$$

It follows from the equivalence of the norms on the finite-dimensional subspace E^0 that there exists a constant $M_5 > 0$ such that

$$\|x\|_{L^\infty} \leqslant M_5\|x\|, \qquad \|x\| \leqslant M_5\|x\|_{L^1}, \quad \forall x \in E^0. \tag{3.27}$$

Let $x = x^0 + x^- \in X^2$ satisfying $\|x\| \leqslant \rho_1 = \frac{R_1}{2M_5}$, where R_1 is the same in (H5)′. Let

$$\Omega_1 = \left\{t \in [0,T]\colon\ |x^-| \leqslant \frac{R_1}{2}\right\}, \qquad \Omega_2 = \left\{t \in [0,T]\colon\ |x^-| > \frac{R_1}{2}\right\}.$$

Then, by (3.27) we have

$$|x^0| \leqslant \|x^0\|_{L^\infty} \leqslant M_5\|x^0\| \leqslant M_5\|x\| \leqslant \frac{R_1}{2} \quad \text{for all } t \in [0,T].$$

On the one hand, one has

$$|x| \leqslant |x^0| + |x^-| \leqslant R_1 \quad \text{for all } t \in \Omega_1.$$

Hence, from (H5)′, we have

$$\int_{\Omega_1} H(t,x)\,dt \geqslant 0.$$

On the other hand, one has

$$|x| \leqslant |x^0| + |x^-| \leqslant \frac{R_1}{2} + |x^-| \leqslant 2|x^-| \quad \text{for all } t \in \Omega_2.$$

It follows from (3.19) and the above inequality that

$$|H(t,x)| \leqslant \varepsilon|x|^2 + M|x|^{1+\alpha} \leqslant 4\varepsilon|x^-|^2 + 2^{1+\alpha}M|x^-|^{1+\alpha}$$

for all $t \in \Omega_2$ and all $x \in X^2$ with $\|x\| \leqslant \rho_1$, which implies that

$$\begin{aligned}\left|\int\limits_{\Omega_2} H(t,x)\,dt\right| &\leqslant 4\varepsilon \int\limits_{\Omega_2} \left|x^-\right|^2 dt + 2^{1+\alpha} M \int\limits_{\Omega_2} \left|x^-\right|^{1+\alpha} dt \leqslant 4\varepsilon \left\|x^-\right\|_{L^2}^2 + 2^{1+\alpha} M \left|x^-\right|_{L^{1+\alpha}}^{1+\alpha} \\ &\leqslant 4C^2\varepsilon \left\|x^-\right\|^2 + (2C)^{1+\alpha} M \left|x^-\right|^{1+\alpha}.\end{aligned}$$

Let $\varepsilon = \frac{1}{16C^2}$, we obtain

$$\begin{aligned}\varphi(x) &= -\frac{1}{2}\left\|x^-\right\|^2 - \int\limits_0^T H(t,x)\,dt \leqslant -\frac{1}{2}\left\|x^-\right\|^2 + 4C^2\varepsilon\left\|x^-\right\|^2 + (2C)^{1+\alpha} M \left|x^-\right|^{1+\alpha} \\ &\leqslant -\frac{1}{4}\left\|x^-\right\|^2 + (2C)^{1+\alpha} M \left|x^-\right|^{1+\alpha}\end{aligned}$$

for all $x \in X^2$ with $\|x\| \leqslant \rho_1$, which implies that

$$\varphi(x) \leqslant 0, \quad \forall x \in X^2 \text{ with } \|x\| \leqslant \rho_2,$$

where $\rho_2 = \min\{\rho_0, \rho_1\}$ is small enough.

(2) We claim that φ satisfies the $(\mathrm{PS})^*$ condition. Let

$$X_n^1 = \operatorname{span}\{e_{\bar{n}+1}, \ldots, e_{\bar{n}+n}\}, \qquad X_n^2 = X^2 = \operatorname{span}\{e_1, \ldots, e_{\bar{n}}\}, \quad n \in N,$$

then $X^j = \overline{\bigcup_{n\in N} X_n^j}$, $j = 1, 2$. Let $\{x_{a_n}\}$ be a sequence such that $\{a_n\}$ is admissible and satisfying

$$x_{a_n} \in X_{x_{a_n}}, \qquad c = \sup \varphi_{a_n}(x_{a_n}) < \infty, \qquad \varphi'_{a_n}(x_{a_n}) \to 0.$$

In a similar way to the proof of Step 1 in Theorem 1.3, we can prove that $\{x_{a_n}\}$ is a bounded sequence, and then by a standard argument, $\{x_{a_n}\}$ has a convergent subsequence. We omit the detail here.

(3) We claim that for every $m \in N$, $\varphi(x) \to -\infty$ as $\|x\| \to \infty$ on $x \in X_m^1 \oplus X^2$. Since X_m^1 and X^2 are finite-dimensional, we can choose $M_6 > 0$ sufficiently large such that

$$\|x\| \leqslant M_6 \left(\int\limits_{E_0} |x|^2\,dt\right)^{\frac{1}{2}} \quad \text{for all } x \in X_m^1 \oplus X^2. \tag{3.28}$$

By $(\mathrm{H1})'$, there exists $M_7 > 0$ such that

$$H(t,x) \geqslant M_6^2 |x|^2 - M_7 \quad \text{for all } x \in \mathbb{R}^N \text{ and a.e. } t \in E_0. \tag{3.29}$$

Hence, it follows from (3.28) and (3.29) that

$$\begin{aligned}\varphi(x) &= \frac{1}{2}\left(\left\|x^+\right\|^2 - \left\|x^-\right\|^2\right) - \int\limits_0^T H(t,x)\,dt \\ &\leqslant \frac{1}{2}\left(\left\|x^+\right\|^2 - \left\|x^-\right\|^2\right) - \int\limits_{E_0} H(t,x)\,dt \\ &\leqslant \frac{1}{2}\left(\left\|x^+\right\|^2 - \left\|x^-\right\|^2\right) - M_6^2 \int\limits_{E_0} |x|^2\,dt + M_2 T \\ &\leqslant \frac{1}{2}\left(\left\|x^+\right\|^2 - \left\|x^-\right\|^2\right) - M_6^2\left(\frac{1}{M_6^2}\left\|x^+\right\|^2 + \frac{1}{M_6^2}\left\|x^0\right\|^2\right) + M_2 T \\ &\leqslant -\frac{1}{2}\left\|x^+\right\|^2 - \frac{1}{2}\left\|x^-\right\|^2 - \left\|x^0\right\|^2 + M_2 T \\ &\leqslant -\frac{1}{2}\|x\|^2 + M_2 T\end{aligned}$$

for $x \in X_m^1 \oplus X^2$ and a.e. $t \in E_0$, which implies that

$$\varphi(x) \to -\infty \quad \text{as } \|x\| \to \infty \text{ on } x \in X_m^1 \oplus X^2.$$

By the definition of φ and (3.19), we know that φ satisfied the condition (B3) of Theorem B, thus all the conditions in Theorem B are satisfied. Consequently, the proof of Theorem 1.5 is complete by Theorem B. □

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