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Multiple solutions for semilinear elliptic equations with Neumann boundary condition and jumping nonlinearities

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ABSTRACT

We obtain nonconstant solutions of semilinear elliptic Neumann boundary value problems with jumping nonlinearities when the asymptotic limits of the nonlinearity fall in the type (I_l), $l > 2$ and (II_l), $l \geqslant 1$ regions formed by the curves of the Fucik spectrum. Furthermore, we have at least two nonconstant solutions in every order interval under resonance case. In this paper, we apply the sub–sup solution method, Fucik spectrum, mountain pass theorem in order intervals, degree theory and Morse theory to get the conclusions.

1. Introduction and statement of the results

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$, we consider the equation with Neumann boundary condition

$$\begin{cases} -\Delta u + \alpha u = f(u), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\alpha > 0$, f satisfies $f(0) = 0$.

In [9], the authors have solved the case that $f'(x)$ exists for some special points. In this paper, we consider the more general problem with a jumping nonlinearity at some special points, i.e., the left and the right derivatives of f at some points are different.

Now we give the following conditions and our main results.

(a) $\exists c > 0$ such that

$$\overline{\lim_{\Delta x \to 0}} \frac{f(x+\Delta x) - f(x)}{\Delta x} \leqslant c\left(1 + |x|^{\beta-1}\right), \quad x \in R,$$

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and

$$\varliminf_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \geqslant -c\big(1+|x|^{\beta-1}\big), \quad x \in R,$$

where $1<\beta<2^*-1$, $2^*=2n/(n-2)$, if $n \geqslant 3$, and $2^*=\infty$, if $n=1,2$.

(b) $\exists$ sequences $\{a_i\}$ and $\{b_i\}$, where $a_i, b_i \in R$, $i=1,2,\ldots$, which satisfy $a_i>0$, $b_i<0$ and $a_i \nearrow +\infty$, $b_i \searrow -\infty$ as $i \to \infty$. And at the same time $\{a_i\}$, $\{b_i\}$ satisfy

$$f(a_i)=\alpha a_i, \qquad f(b_i)=\alpha b_i,$$

which means $\{a_i\}$, $\{b_i\}$ are constant solution sequences of (1.1).

Let $a_0=b_0=0$, $f(t)<\alpha t$ if $t \in (a_i,a_{i+1})$, where i is odd number, $i \geqslant 1$; $f(t)>\alpha t$ if $t \in (a_i,a_{i+1})$, where i is even number, $i \geqslant 0$; $f(t)<\alpha t$ if $t \in (b_{i+1},b_i)$, where i is even number, $i \geqslant 0$; $f(t)>\alpha t$ if $t \in (b_{i+1},b_i)$, where i is odd number, $i \geqslant 1$.

(c) f is C^1 for all $t \neq a_i, b_i$, i is even number, $i \geqslant 2$. And $f'_-(a_i) \neq f'_+(a_i)$, $f'_-(b_i) \neq f'_+(b_i)$ for i even number, $i \geqslant 2$, where $f'_-(t)$, $f'_+(t)$ denote the left and the right derivatives of f at t, respectively.

(d) Let $(a,b)=(f'_-(a_i)-\alpha, f'_+(a_i)+\alpha)$ for i even number, $i \geqslant 2$. For $(a,b) \in R^2$, the problem

$$\begin{cases} -\Delta u = b(u-c)^+ - a(u-c)^-, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \end{cases}$$

only has constant solution c, where $(u-c)^{\pm}(x)=\max\{\pm(u-c),0\}$ and c is a constant.

And $\lambda_{l+1} > f'_{i-}(a_{2k})-\alpha > \lambda_l$, $\lambda_{l+1} > f'_{i+}(a_{2k})-\alpha > \lambda_l$, $l \geqslant 2$, $k=1,2,\ldots$, where

$$f_i(t)=\begin{cases} 0, & t<0, \\ f(t), & 0 \leqslant t \leqslant a_i, \\ f(a_i), & t>a_i, \end{cases}$$

and $f'_{i-}(a_{2k})$, $f'_{i+}(a_{2k})$ denote the left and the right derivatives of f_i at a_{2k}, respectively and λ_l, $l \geqslant 2$ are the eigenvalues of Neumann problem of $-\Delta$.

(e) $\exists m>\alpha$, such that $f(x)+mx$ is increasing, $x \in R$.

Then we have the main result of this paper:

Theorem 1.1. *Suppose f satisfies* (a)–(e). *Then there are infinitely many nonconstant solutions of problem* (1.1). *Moreover, if we choose some order intervals which have two pairs of strict constant sub–sup solutions, then there are at least two nonconstant solutions.*

Furthermore, if we assume that $f'_-(0) \neq f'_+(0)$ under conditions (a)–(e), we can have at least one sign-changing solution which is of mountain pass type from the mountain pass theorem in order interval (see Lemma 2.8 below). When we discuss multiple solutions of (1.1), we notice that there may be infinitely many sign-changing solutions under stronger assumptions. In fact, if we give more assumptions, we can obtain infinitely many sign-changing solutions.

We give the following assumption:

(f) $F(t) > ((\mu_2+\varepsilon_0)/2)t^2$, $|t| \geqslant M$, M is large enough, where μ_2 is the second eigenvalue of $(-\Delta+\alpha)$ and $\varepsilon_0>0$.

Corollary 1.2. *Under the assumptions* (a)–(f) *and* $f'_-(0) \neq f'_+(0)$, *we can get infinitely many sign-changing solutions which are of mountain pass type or not mountain pass type but with positive local degree.*

Our technique is based on the mountain pass theorem in order interval, computing the critical groups and Fucik spectrum.

2. Preliminaries

At first, we recall some notions and known results of the critical point theory and Morse theory. Let M be a Banach space, $J \in C^1(M,R)$, the set $J^a=\{u \in M \mid J(u) \leqslant a\}$ is called a level set. We denote the set of all critical points by K, i.e., $K=\{u \in M \mid J'(u)=0\}$. A real number c is called a critical value of J if $J^{-1}(c) \cap K \neq \emptyset$. Denote $K_c=\{u \in K\colon\ J(u)=c\}$, $c \in R$. Assume that $J \in C^2(M,R)$, a critical point u is called nondegenerate if the Hessian $J''(u)$ at this point has a bounded inverse. Let u be a nondegenerate critical point of J, we call the dimension of the negative space corresponding to the spectral decomposition of $J''(u)$, i.e., the dimension of the subspace of negative eigenvectors of $J''(u)$, the Morse index of u, and denote it by $ind(J''(u))$.

Let us use singular relative homology groups $H_q(X, Y; G)$ with an Abelian coefficient group G to describe the topological difference between the topological spaces X and Y, with $Y \subset X, q = 0, 1, 2, \dots$. Use $H^q(X; G)$ to stand for the qth singular cohomology group with an Abelian coefficient group G, from now on we denote it by $H^q(X)$. Readers are referred to [1,2], pp. 252–257 for the definitions of $H_q(X, Y; G)$ and $H^q(X; G)$.

Definition 2.1. (See [3].) Let J be a C^1 function defined on M, let u be an isolated critical point of J, and let $c = J(u)$,

$$C_q(J, u) = H_q\big(J^c \cap U, \big(J^c \setminus \{u\}\big) \cap U; G\big)$$

is called the qth critical group of J at u, $q = 0, 1, 2, \dots$, where U is an isolated neighborhood of u, i.e., $K \cap U = \{u\}$.

From the definition, we can compute the critical group of J at some special points. For example, from Section 5.1.3, Example 1 in [3], we already know that if u is an isolated minimum point of J, then $C_q(J, u) = \delta_{q0}G$. Next, we give the proof of this example.

Proposition 2.2. *If u is an isolated minimum point of J, then*

$$C_q(J, u) = \delta_{q0}G.$$

Proof. In fact, from the assumption, we conclude that

$$J^c \cap U = \big\{v \in M \;\big|\; J(v) \leqslant c\big\} \cap U = \{u\} \cap U = \{u\}.$$

And by the definition of the critical group (see Definition 2.1 above), we have that

$$C_q(J, u) = H_q\big(J^c \cap U, \big(J^c \setminus \{u\}\big) \cap U; G\big) = H_q\big(\{u\}, \emptyset; G\big) = H_q\big(\{u\}; G\big) = \begin{cases} G, & q = 0, \\ 0, & q \neq 0, \end{cases}$$

where $c = J(u)$, U is an isolated neighborhood of u, i.e., $K \cap U = \{u\}$. The proposition holds. □

Proposition 2.3. *(See [3].) If $J \in C^2(M, R)$ and u is a nondegenerate critical point of J with Morse index j, then*

$$C_q(J, u) = \delta_{qj}G.$$

Proof. See 5.1.3, Example 3 of Chang [3]. □

Definition 2.4 *(Mountain pass point).* (See [3].) If $C_1(J, u) \neq 0$, then we call an isolated critical point u of J a mountain pass point.

Definition 2.5. (See [9].) If any sequence $\{u_k\} \subset M$ which satisfies $J(u_k) \to c$ and $J'(u_k) \to 0 (k \to \infty)$ has a convergent subsequence, we say that J satisfies the $(PS)_c$ condition. If J satisfies $(PS)_c$ condition for all $c \in R$, we say that J satisfies the (PS) condition.

Definition 2.6. (See [9].) We say that J satisfies deformation property, if $\forall \varepsilon^* > 0$, $\forall N$, $\exists \varepsilon \in (0, \varepsilon^*)$ and a continuous map $\eta : [0, 1] \times M \to M$, such that

(i) $\eta(0, \cdot) = id$,
(ii) $\eta(t, u) = u$, $\forall u \in M \setminus \{u \in M\colon\ c - \varepsilon^* \leqslant J(u) \leqslant c + \varepsilon^*\}$, $t \in [0, 1]$,
(iii) $J(\eta(\cdot, u))$ is nonincreasing, $\forall u \in M$,
(iv) $\eta(1, J^{c+\varepsilon} \setminus N) \subset J^{c-\varepsilon}$,

where $J \in C^1(M, R)$, $c \in R$, N is a closed neighborhood of K_c.

Lemma 2.7 *(First deformation theorem). (See [16].) If $J \in C^1(M, R)$ satisfies $(PS)_c$ condition for all $c \in R$, N is a closed neighborhood of $K_c \triangleq K \cap J^{-1}(c)$, $\forall \bar{\varepsilon} > 0$, then there is a continuous map $\eta : [0, 1] \times M \to M$ and $\varepsilon \in (0, \bar{\varepsilon})$, such that*

(i) $\eta(0, \cdot) = id$,
(ii) $\eta(t, u) = u$, $\forall u \in M \setminus J^{-1}[c - \bar{\varepsilon}, c + \bar{\varepsilon}]$, $t \in [0, 1]$,
(iii) $\eta(t, \cdot) : M \to M$ *is homeomorphism*, $\forall t \in [0, 1]$,

(iv) $\eta(1, J^{c+\varepsilon} \setminus N) \subset J^{c-\varepsilon}$,
(v) $J(\eta(t,u))$ *is nonincreasing*, $\forall u \in M$, $t \in [0,1]$,

where $J^{c+\varepsilon} = \{u \in M \mid J(u) \leqslant c + \varepsilon\}$ *is a level set.*

Proof. See Theorem 2.2 of [16]. □

Consider the functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\alpha}{2}\int_\Omega u^2\,dx - \int_\Omega F(u)\,dx,$$

where $F(u) = \int_0^u f(s)\,ds$. From the variational point of view, solutions of (1.1) are the critical points of J defined on $W^{1,2}(\Omega)$. Let $[u_1, u_2] = \{u \in X \mid u_1 \leqslant u \leqslant u_2, x \in \Omega\}$ be the order interval in $X = \{u \in C^1(\overline{\Omega}) \mid \frac{\partial u}{\partial \nu} = 0,\ x \in \partial\Omega\}$.

From the deformation theorem (see Lemma 2.7 above), we know that J satisfies deformation property when J satisfies (PS) condition.

Let E be a Hilbert space and $P_E \subset E$ a closed convex cone such that X is densely embedded to E. Assume that $P = X \cap P_E$, P has nonempty interior $\dot{P}$ and any order interval is bounded.

Then $J : E \to R$ satisfies the following assumptions (see [9]):

(J_1) $J \in C^2(E, R)$ and satisfies (PS) condition in E and deformation property in X.
(J_2) $\nabla J = id - K_E$, where $K_E : E \to E$ is compact. $K_E(X) \subset X$ and the restriction $K = K_E|_X : X \to X$ is of class C^1 and strongly preserving, i.e., $u \gg v \Leftrightarrow u - v \in \dot{P}$.
(J_3) J is bounded from below on any order interval in X.

Lemma 2.8 *(Mountain pass theorem in order intervals). (See [11].) Suppose* J *satisfies* (J_1)–(J_3) *and* $\{v_1, v_2\}$, $\{\omega_1, \omega_2\}$ *are two pairs of strict sub–sup solutions of* $\nabla J = 0$ *in* X *with* $v_1 < \omega_2$, $[v_1, v_2] \cap [\omega_1, \omega_2] = \emptyset$. *Then* J *has a mountain pass point* u_0, $u_0 \in [v_1, \omega_2] \setminus ([v_1, v_2] \cup [\omega_1, \omega_2])$. *More precisely, let* v_0 *be the maximal minimizer of* J *in* $[v_1, v_2]$ *and* ω_0 *be the minimal minimizer of* J *in* $[\omega_1, \omega_2]$. *Then* $v_0 \ll u_0 \ll \omega_0$. *Moreover,* $C_1(J, u_0)$, *the critical group of* J *at* u_0, *is nontrivial.*

Proof. See Theorem 1.3 of Li and Wang [11]. □

Remark 2.9. Lemma 2.8 still holds if $J \in C^1(E, R)$, K is of class C^0 and J has infinitely many isolated critical points.

This was known from the results in Li and Wang [12,11].

Now, let us recall some notions and known results on Fucik spectrum (see Perera and Schechter [13] and Definition 2.10 below).

Consider the problem

$$\begin{cases} -\Delta u = b(u-c)^+ - a(u-c)^-, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

from the variational point of view, solutions of (2.1) are the critical points of the functional

$$I(u) = I(u,a,b) = \int_\Omega |\nabla u|^2 - a\big[(u-c)^-\big]^2 - b\big[(u-c)^+\big]^2, \quad u \in X = \left\{C^1(\overline{\Omega}) \;\middle|\; \frac{\partial u}{\partial \nu} = 0\right\}$$

where c is a constant and $\Omega \in R^n$ is a bounded domain with smooth boundary $\partial\Omega$ and $(u-c)^{\pm}(x) = \max\{\pm(u-c), 0\}$.

Recall that Fucik spectrum of $-\Delta$ is the set of those points $(c,d) \in R^2$ for which the problem

$$\begin{cases} -\Delta u = du^+ - cu^-, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

has nontrivial solutions (see Perera and Schechter [13]).

Definition 2.10. The set Σ is denoted by the points $(a,b) \in R^2$ for which (2.1) has nonconstant solutions, we call Σ the Fucik spectrum of $-\Delta$ with Neumann boundary problem.

Denote by $0 = \lambda_1 < \lambda_2 < \cdots$ the distinct Neumann eigenvalues of $-\Delta$ on Ω. It was shown in Schechter [14] and Garbuza [6] that Σ has two strictly decreasing curves C_{l_1}, C_{l_2} passing through the (λ_l, λ_l) such that the region I_{l_1} in the square

$Q_l = (\lambda_{l-1}, \lambda_{l+1})^2$ below the lower curve C_{l_1} and the region I_{l_2} above the upper curve C_{l_2} are free of Σ, while the points on the curves are in Σ. We denote by II_l the regions between the curves, then the points in II_l may or may not belong to Σ. Denote by $I_l = I_{l_1} \cup I_{(l-1)_2}$.

If (a, b) doesn't belong to Σ, c is the constant solution of (2.1), i.e, c is an isolated critical point of I, then from the definition of critical group (see Definition 2.1 above), we have the $C_q(I, c)$ defined, $q = 0, 1, 2, \ldots$.

Now, we give some results relative to the computation of the critical groups when (a, b) falls in certain parts of Q_l.

Lemma 2.11. *(See [4,13].) Let $(a, b) \in Q_l \setminus \Sigma$ and let d_l denote the dimension of the subspace N_l spanned by the eigenfunctions corresponding to $\lambda_1, \ldots, \lambda_l$.*

(i) *If $(a, b) \in I_l$, then*

$$C_q(I, c) = \begin{cases} Z, & q = d_{l-1}, \\ 0, & q \neq d_{l-1}. \end{cases}$$

(ii) *If $(a, b) \in II_l$, then $C_q(I, c) = 0$ for $q \leqslant d_{l-1}$ or for $q \geqslant d_l$.*

In particular, $C_q(I, c) = 0$ for all q when λ_l is a simple eigenvalue.

Proof. See Theorem 1 of Dancer [4]. □

We observe that when $(a, b) \in II_l$ and λ_l is a multiple eigenvalue, the above theorem doesn't determine $C_q(I, c)$ for $d_{l-1} < q < d_l$. Let K_p denote the set of critical points of $I_p = I(\cdot, p)$ and $\hat{K}_p = \{u \in K_p\colon \|u\| = 1\}$. Recall that Γ_p is the set of $(a, b) \in R^2 \setminus \Sigma$ for which there is a curve $\gamma = (\gamma_1, \gamma_2) \in C([0, 2]) \cap C^1([0, 1])$, $\gamma(0) = p = (p_1, p_2)$, $\gamma(2) = (a, b)$, for $p \in Q_l \cap C_{l_2}$, such that

(1) $\gamma((0, 2]) \cap \Sigma = \emptyset$,
(2) $\gamma_1', \gamma_2' \leqslant 0$ on $[0, 1]$,
(3) $\gamma(0) + \gamma'(0) = \gamma(1)$ (see Definition 1.2 of Perera and Schechter [13]).

It was shown in [13] that if γ intersects Σ only at $p = (p_1, p_2)$ and $a \leqslant p_1$, $b \leqslant p_2$, we can take γ to be the line segment joining $p = (p_1, p_2)$ and (a, b). Then we have $II_l \subset \Gamma_p$ when the region II_l is free of Σ.

Lemma 2.12. *(See [13].) If $p \in Q_l \cap C_{l_2}$, and $(a, b) \in \Gamma_p$, then*

$$C_q(I, c) \cong \begin{cases} H^{d_l - q - 1}(\hat{K}_p), & q \neq d_{l-1}, \\ H^0(\hat{K}_p)/Z, & q = d_{l-1}. \end{cases}$$

Proof. See Theorem 1.3 of Perera and Schechter [13]. □

Moreover, set $A_l = I - \lambda_l(-\Delta)^{-1}$, let N_{l-1}, $E(\lambda_l)$, M_l denote the negative, zero and positive subspaces of A_l, respectively, and for p, let $I_p = I(\cdot, p)$,

$$I_p(v + \omega_0) = \inf_{\omega \in M_l} I_p(v + \omega), \quad v \in N_l, \tag{2.3}$$

$$I_p(v_0 + \omega) = \sup_{v \in N_{l-1}} I_p(v + \omega), \quad \omega \in M_{l-1}. \tag{2.4}$$

It was shown in Schechter [15] that there are continuous and positive homogeneous functions

$$\tau_l : N_l \to M_l, \qquad \gamma_{l-1} : M_{l-1} \to N_{l-1}$$

such that $\omega_0 = \tau_l(v)$, $v_0 = \gamma_{l-1}(\omega)$ are the unique solutions of (2.3), (2.4), respectively.

Let

$$T_l = \{v + \tau_l(v)\colon v \in N_l\},$$
$$R_{l-1} = \{\gamma_{l-1}(\omega) + \omega\colon \omega \in M_{l-1}\},$$
$$S_l = T_l \cap R_{l-1}, \qquad \hat{S}_l = \{u \in S_l\colon \|u\| = 1\}.$$

We also have the following conclusion:

Lemma 2.13. *(See [13].)*

$$C_q(I,c) \cong \begin{cases} H^{d_l-q-1}(\hat{S}_l^+), & q \neq d_{l-1}, \\ H^0(\hat{S}_l^+)/Z, & q = d_{l-1}, \end{cases}$$

where $\hat{S}_l^+ = \{u \in \hat{S}_l\colon I(u) > 0\}$, *for* $(a,b) \in II_l \setminus \Sigma$.

Proof. See Theorem 1.6 of Perera and Schechter [13]. □

3. The proof of the main results

Consider the equation

$$\begin{cases} -\Delta u + \alpha u = f_i(u), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where

$$f_i(t) = \begin{cases} 0, & t < 0, \\ f(t), & 0 \leqslant t \leqslant a_i, \\ f(a_i), & t > a_i \end{cases}$$

is a truncation function, the corresponding functional

$$J_i(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{\alpha}{2}\int_\Omega u^2\,dx - \int_\Omega F_i(u)\,dx,$$

where $F_i(u) = \int_0^u f_i(s)\,ds$.

We notice that $f_i(t) \in C^0(R,R)$ and $J_i \in C^1(E,R)$. We can discuss a similar case for b_i.

Li [8] shows that $J_i(u)$ satisfies coercive condition on $E = W^{1,2}(\Omega)$. So if $J_i'(u) = u - K_i u$, $u \in E$, where K_i is a compact operator, then J_i satisfies (PS) condition in E and deformation property in X.

Next, we give the relation of the solutions of (3.1) and the solutions of (1.1), i.e., Lemma 3.2 below. In order to prove Lemma 3.2, we firstly give the weak maximum principle.

Lemma 3.1 *(Weak maximum principle). (See [5].) Assume* $u \in C^2(U) \cap C(\overline{U})$ *and*

$$c \equiv 0 \quad \textit{in } U.$$

(I) *If* $Lu \leqslant 0$ *in* U, *then*

$$\max_{\overline{U}} u = \max_{\partial U} u.$$

(II) *If* $Lu \geqslant 0$ *in* U, *then*

$$\min_{\overline{U}} u = \min_{\partial U} u,$$

where $U \subset R^n$ *is open and bounded,* L *is elliptic operator having the nondivergence form*

$$Lu = -\sum_{i,j=1}^n a^{ij} u_{x_i x_j} + \sum_{i=1}^n b^i u_{x_i} + cu,$$

the coefficients a^{ij}, b^i, c *are continuous.*

Proof. See Section 6.4.1, Theorem 1 in Evans [5]. □

Lemma 3.2. *If* $u_i(x)$ *is a solution of* (3.1), *then* $u_i(x)$ *is also a solution of* (1.1) *and satisfies* $0 \leqslant u_i(x) \leqslant a_i$, $i = 1, 2, \ldots.$

Proof. Suppose the conclusion is false. Now, consider the domain $U_i = \{x \in \Omega \mid u_i(x) < 0\}$, then we have

$$\begin{cases} -\Delta u = f_i(u) - \alpha u \geqslant 0, & \text{in } U_i, \\ u = 0, & \text{on } \partial U_i, \end{cases}$$

where $-\Delta u = f_i(u) - \alpha u = 0 - \alpha u = -\alpha u \geqslant 0$, $x \in U_i$ by the definition of $f_i(u)$.

By the maximum principle (see Lemma 3.1 above), we have $u_i(x) \geqslant 0$ in U_i. It is a contradiction, so it follows that $U_i = \emptyset$, i.e., $u_i(x) \geqslant 0$. Similarly, we consider $V_i = \{x \in \Omega \mid u_i(x) > a_i\}$, we have the equation

$$\begin{cases} -\Delta u = f_i(u) - \alpha u \leqslant 0, & \text{in } V_i, \\ u = 0, & \text{on } \partial V_i, \end{cases}$$

where $-\Delta u = f_i(u) - \alpha u = f(a_i) - \alpha u \leqslant f(a_i) - \alpha a_i = 0$, $x \in V_i$ by the definition of $f_i(u)$. By the maximum principle, we can conclude that $u_i(x) \leqslant a_i$ in V_i. It is a contradiction, so we have that $V_i = \emptyset$, i.e., $u_i(x) \leqslant a_i$. From the above discussion, we have that $0 \leqslant u_i(x) \leqslant a_i$, $i = 1, 2, \ldots$ and $f_i(u) = f(u_i)$, so $u_i(x)$ is a solution of (1.1). This completes the proof of the lemma. □

From the above discussion, by applying Lemma 3.2, we know that in order to prove the main result of Theorem 1.1 we only need to prove that (3.1) has infinitely many nonconstant solutions under the assumptions (a)–(e) and (3.1) has two nonconstant solutions in every order interval.

Theorem 3.3. *Assume that f_i satisfies* (a)–(e)*, then there are infinitely many nonconstant solutions of* (3.1)*. Moreover, if there exist some order intervals which have two pairs of strict constant sub–sup solutions, then there are at least two nonconstant solutions.*

Proof. According to (b), we can conclude that $\{a_i\}$ are all positive constant solutions of (3.1). Assume i is large enough and i is an even number, from assumption (b), we infer that $\{a_{2k-1}\}$ are local minima, $k = 1, 2, \ldots, \frac{i}{2}$. So we get $\underline{u}_{2k-1}$ and $\overline{u}_{2k-1}$ a strict sub-solution and sup-solution pair for (3.1), satisfying $\underline{u}_{2k-1} < a_{2k-1} < \overline{u}_{2k-1}$ for each k, $k = 1, 2, \ldots, \frac{i}{2}$.

Now, we study the order interval $[\underline{u}_1, \overline{u}_3]$ in X which includes two sub-order intervals $[\underline{u}_1, \overline{u}_1]$ and $[\underline{u}_3, \overline{u}_3]$.

We infer that $J_i(u)$ satisfies deformation properties and is bounded from below on $[\underline{u}_1, \overline{u}_3]$, so we get a mountain pass point $u_1 \in [\underline{u}_1, \overline{u}_3] \setminus ([\underline{u}_1, \overline{u}_1] \cup [\underline{u}_3, \overline{u}_3])$ according to the mountain pass theorem in order interval (see Lemma 2.8 above). From the definition of mountain pass point (see Definition 2.4 above), we have that $C_1(J_i, u_1)$ is nontrivial.

From assumption (c), we know that the left and the right derivatives of f_i at a_2 are different. We consider the problem

$$\begin{cases} -\Delta u = f_i(u) - \alpha u, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where $f_i \in C(\overline{\Omega} \times R)$ and as $u \to a_{2k}$ we have

$$f_i(u) - \alpha u = \big(f'_{i+}(a_{2k}) - \alpha\big)(u - a_{2k})^+ - \big(f'_{i-}(a_{2k}) - \alpha\big)(u - a_{2k})^- + \circ(u - a_{2k}).$$

We take $a = f'_{i-}(a_{2k}) - \alpha$, $b = f'_{i+}(a_{2k}) - \alpha$, then from assumption (d) and the definition of Σ (see Definition 2.10 above), we know that (a, b) doesn't belong to Σ.

From Lemma 2.11, we know that if $(a, b) \in I_l$, $l \neq 2$, we have

$$C_q(J_i, a_2) = \begin{cases} Z, & q = d_{l-1}, \\ 0, & q \neq d_{l-1}, \end{cases}$$

and

$$C_q(J_i, a_{2k}) = \begin{cases} Z, & q = d_{l-1}, \\ 0, & q \neq d_{l-1}, \end{cases} \quad k = 2, 3, \ldots, \frac{i}{2} - 1,$$

then if $l \neq 2$, we have $C_q(J_i, a_2) \ncong C_q(J_i, u_1)$, where i is an even number, so $u_1 \neq a_2$.

If $(a, b) \in II_l$, $l \geqslant 1$, then from Lemma 2.13, we have

$$C_q(J_i, a_2) \cong \begin{cases} H^{d_l - q - 1}(\hat{S}_l^+), & q \neq d_{l-1}, \\ H^0(\hat{S}_l^+)/Z, & q = d_{l-1}, \end{cases}$$

for $(a, b) \in II_l \setminus \Sigma$. If $l = 2$, then $d_{l-1} = d_1 = 1$, so for $q = 1$, we have $C_1(J_i, a_2) \cong H^0(\hat{S}_2^+)/Z$. Furthermore, for a point p, we have

$$H^q(p; G) \cong \begin{cases} Z, & q = 0, \\ 0, & q \neq 0. \end{cases}$$

Then we have $C_q(J_i, a_2) \cong 0 \ncong C_q(J_i, u_1)$, so $u_1 \neq a_2$. If $l > 2$, then $d_{l-1} > 1$, and from Lemma 2.11, we have for $q = 1$ that $C_1(J_i, a_2) \cong 0$. Then we have $C_q(J_i, a_2) \ncong C_q(J_i, u_1)$, so $u_1 \neq a_2$.

Similar to the previous discussion, applying the mountain pass theorem in order interval to $[\underline{u}_3, \overline{u}_5]$ which contains two sub-order intervals $[\underline{u}_3, \overline{u}_3]$ and $[\underline{u}_5, \overline{u}_5]$, we get a mountain pass point u_2 and prove that $u_2 \neq a_4$ from Lemmas 2.11, 2.12 and 2.13.

We let the procedure go on.... So $\frac{i}{2}-1$ mountain pass points are available which are nonconstant solutions of (3.1), where i is large enough and i is an even number. Then we have infinitely many nonconstant positive solutions of (3.1) by the arbitrariness of i.

We can discuss the similar case for b_i and get infinitely many nonconstant negative solutions.

Now, we discuss the solutions in $[\underline{u}_1, \overline{u}_3]$ more deeply. Since u_1 is a mountain pass point, for the Leray–Schauder degree of $id - K^i$, we have the computing formula

$$deg\big(id - K^i, B(u_1, r), 0\big) = -1,$$

where $r > 0$ is small enough, $K^i = K^i_E|_X = (-\Delta + (m+\alpha)id)^{-1} f_i^*|_X : X \to X$ is of class C^0 and strongly preserving, $f_i^*(u) = f_i(u) + mu$ (see Hofer [7]). Then according to the Poincaré–Hopf formula for C^1 case (see [10]) and the computation of $C_q(J_i, a_2)$, we have

$$index(J_i, a_2) = (-1)^l.$$

Furthermore, for minimum points a_1, a_3,

$$C_q(J_i, a_1) \cong \delta_{q0} G, \qquad C_q(J_i, a_3) \cong \delta_{q0} G.$$

From the additivity of Leray–Schauder degree and Theorem 1.1 in [11], we can get

$$\begin{aligned} 1 &= deg\big(id - K^i, [\underline{u}_1, \overline{u}_3], 0\big) \\ &= deg\big(id - K^i, [\underline{u}_1, \overline{u}_1], 0\big) + deg\big(id - K^i, [\underline{u}_3, \overline{u}_3], 0\big) + deg\big(id - K^i, B(a_2, r), 0\big) + deg\big(id - K^i, B(u_1, r), 0\big) \\ &= 1 + 1 + (-1)^l + (-1). \end{aligned}$$

So we have $(-1)^l = 0$. It is impossible. From the above discussion, we conclude that there must exist another critical point $u_1^* \in [\underline{u}_1, \overline{u}_3]$, which satisfies $u_1^* \neq u_1$ and is nonconstant.

Similarly, we can discuss the order interval $[\underline{u}_3, \overline{u}_5]$, we get another critical point $u_2^* \neq u_2$. We let the procedure go on....

This completes the proof of Theorem 3.3. □

Thus, we prove that the conclusion of Theorem 1.1 holds.

Proof of Corollary 1.2. See Theorem 3.5 of Li and Li [8]. □

Remark 3.4. In Theorem 1.1, we can deal with the case in which $(a, b) \in I_l$, $l > 2$, and $(a, b) \in II_l$, $l \geqslant 1$, but, when $(a, b) \in I_2$, then

$$C_q(J_i, a_2) = \begin{cases} Z, & q = 1 \\ 0, & q \neq 1 \end{cases} = C_q(J_i, u_1),$$

we cannot distinguish u_1 from a_2, then there may not nonconstant solutions.

Remark 3.5. Let the following assumption holds:

(f)′ $\int_\Omega F(u)\,dx > ((\mu_2 + \varepsilon_0)/2) \int_\Omega u^2\,dx$, as $\|u\| \geqslant M$, $u \in E_2$, where $E_2 = \{u \in E \mid u = k_1 e_1 + k_2 e_2\}$, e_1, e_2 are the first and the second eigenfunctions of $(-\Delta + \alpha)$ with Neumann boundary, respectively, $\forall k_1, k_2 \in R$, $\|e_1\| = \|e_2\| = 1$, $\varepsilon_0 > 0$ and M is large enough.

Then under (a)–(c) and (f)′, we can obtain infinitely many nonconstant positive, negative and sign-changing solutions of (1.1).

As a matter of fact, we can infer (f)′ from (f).

Corollary 3.6. *Moreover,* (1.1) *has infinitely many nonconstant negative energy solutions* $\{u_k\}$, *which are of mountain pass type, if* (a)–(e) *hold and* $J(a_{2k}) \to -\infty$ *or* $J(b_{2k}) \to -\infty$ *as* $k \to +\infty$.

Proof. Assume that $J(a_{2k}) \to -\infty$ as $k \to +\infty$. Let $c = \inf_{\gamma\in\Gamma} \max_{\gamma(I)\cap S} J(u(t))$, where $\Gamma = \{\gamma \in C(I, W) \mid \gamma(0) = a_{2k-1}, \gamma(1) = a_{2k+1}\}$, and $I = [0, 1]$, $S = W \setminus (W_1 \cup W_2)$, $W = [\underline{u}_{2k-1}, \overline{u}_{2k+1}]$, $W_1 = [\underline{u}_{2k-1}, \overline{u}_{2k-1}]$, $W_2 = [\underline{u}_{2k+1}, \overline{u}_{2k+1}]$, $c^* = J(a_{2k})$, $k = 1, 2, \ldots$. We discuss the problem in W which has two minimum points a_{2k-1} and a_{2k+1}. We have that a_{2k-1} and a_{2k+1} are in the same radial direction $A = \{ke_1 \mid k \in R\}$, e_1 is the first eigenvalue function of $(-\Delta + \alpha)$ with Neumann boundary. In fact, e_1 is a constant. We conclude that $c^* \geqslant c$ (see Corollary 3.4 of Li and Li [9]). Furthermore, if (c), (d) hold, then $c^* > c$. In fact, if $c^* = c$, then $c^* = \max_{u\in\gamma^*(I)\cap S} J(u) = \inf_{\gamma\in\Gamma} \max_{\gamma(I)\cap S} J(u(t)) = J(a_{2k})$, where γ^* is a special path between a_{2k-1} and a_{2k+1}, which is a path of radial direction $A = \{ke_1 \mid k \in R\}$, e_1 is the first eigenvalue function of $(-\Delta + \alpha)$ with Neumann boundary. So a_{2k} is a mountain pass point. But according to the assumptions (c) and (d) and Lemma 2.11, we know that $C_1(J, a_{2k}) = 0$ $(l \neq 2)$, i.e., a_{2k} is not of mountain pass type. This is a contradiction. We draw the conclusion. □

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