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Optimal general inequalities for Lagrangian submanifolds in complex space forms ☆

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ABSTRACT

Lagrangian submanifolds appear naturally in the context of classical mechanics. Moreover, they play some important roles in supersymmetric field theories as well as in string theory. In this paper we establish general inequalities for Lagrangian submanifolds in complex space forms. We also provide examples showing that these inequalities are the best possible. Moreover, we provide simple non-minimal examples which satisfy the equality case of the improved inequalities.

1. Introduction

Let $\widetilde{M}^n$ be a complex n-dimensional Kähler manifold endowed with the complex structure J and the metric g. The Kähler 2-form ω is defined by $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$. An isometric immersion $\psi : M^n \to \widetilde{M}^n(4c)$ of a Riemannian n-manifold M^n into $\widetilde{M}^n$ is called *Lagrangian* if $\psi^*\omega = 0$. Lagrangian submanifolds appear naturally in the context of classical mechanics and mathematical physics. For instance, the systems of partial differential equations of Hamilton–Jacobi type lead to the study of Lagrangian submanifolds and foliations in the cotangent bundle. Furthermore, Lagrangian submanifolds play some important roles in supersymmetric field theories as well as in string theory.

In differential geometry of submanifolds, theorems which relate intrinsic and extrinsic curvatures always play an important role. Related with the famous Nash embedding theorem [14], the first author introduced in early 1990s a new type of Riemannian invariants, denoted by $\delta(n_1,\dots,n_k)$. He then established sharp general inequalities relating $\delta(n_1,\dots,n_k)$ and the squared mean curvature H^2 for submanifolds in real space forms. Such invariants and inequalities have many nice applications to several areas in mathematics (see [8,9] for more details).

Immersions of submanifolds which attain one of the equalities at every point were called ideal immersions. Roughly speaking, an ideal immersion of a Riemannian manifold into a real space form is an immersion which produces the least possible amount of tension from the ambient space.

Similar inequalities also hold for Lagrangian submanifolds of complex space forms. In [7] the first author proved that, for any $\delta(n_1,\dots,n_k)$, the equality case holds only when the Lagrangian submanifold is minimal.

In [15] Oprea improved the inequality on $\delta(2)$ for Lagrangian submanifolds in complex space forms. In this paper we establish general inequalities which only involve the squared mean curvature and $\delta(n_1,\dots,n_k)$ for Lagrangian submanifolds in complex space forms. Also, we obtain the necessary and sufficient condition for a Lagrangian submanifold to attain the equality for arbitrary $\delta(n_1,\dots,n_k)$. Further, we provide some examples showing these new improved inequalities for La-

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grangian submanifolds are best possible. Finally, we provide some non-minimal simple examples which verify some equality cases of the improved inequalities.

2. Preliminaries

Let $\widetilde{M}^n(4c)$ be a complete, simply-connected, Kähler n-manifold with constant holomorphic sectional curvature $4c$ and M an n-dimensional Lagrangian submanifold of $\widetilde{M}^n(4c)$. We denote the Levi-Civita connections of M and $\widetilde{M}^n(4c)$ by ∇ and $\widetilde{\nabla}$, respectively.

The formulas of Gauss and Weingarten are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi, \tag{2.2}$$

for tangent vector fields X and Y and normal vector fields ξ, where D is the normal connection. The second fundamental form h is related to A_ξ by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle.$$

The mean curvature vector H of M is defined by

$$H = \frac{1}{n} \operatorname{trace} h.$$

For Lagrangian submanifolds, we have (cf. [12])

$$D_X JY = J\nabla_X Y, \tag{2.3}$$

$$A_{JX} Y = -Jh(X, Y) = A_{JY} X. \tag{2.4}$$

The above formulas immediately imply that $\langle h(X, Y), JZ \rangle$ is totally symmetric. If we denote the curvature tensors of ∇ and D by R and R^D, respectively, then the equations of Gauss and Codazzi are given by

$$\langle R(X, Y)Z, W \rangle = \langle A_{h(Y,Z)} X, W \rangle - \langle A_{h(X,Z)} Y, W \rangle + c\big(\langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle\big), \tag{2.5}$$

$$(\nabla h)(X, Y, Z) = (\nabla h)(Y, X, Z), \tag{2.6}$$

where X, Y, Z, W (respectively, η and ξ) are vector fields tangent (respectively, normal) to M; and ∇h is defined by

$$(\nabla h)(X, Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z). \tag{2.7}$$

For an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM at a point $p \in M$, we put

$$h^A_{BC} = \langle h(e_B, e_C), Je_A \rangle, \quad A, B, C = 1, \ldots, n.$$

It follows from (2.4) that

$$h^A_{BC} = h^B_{AC} = h^C_{AB}. \tag{2.8}$$

3. Invariants $\delta(n_1, \ldots, n_k)$

Let M be a Riemannian n-manifold. Denote by $K(\pi)$ the sectional curvature of M associated with a plane section $\pi \subset T_pM$, $p \in M$. For any orthonormal basis $e_1, \ldots, e_n$ of the tangent space T_pM, the scalar curvature τ at p is defined to be

$$\tau(p) = \sum_{i<j} K(e_i \wedge e_j). \tag{3.1}$$

Let L be a subspace of T_pM of dimension $r \geqslant 2$ and $\{e_1, \ldots, e_r\}$ an orthonormal basis of L. The scalar curvature $\tau(L)$ of the r-plane section L is defined by

$$\tau(L) = \sum_{\alpha<\beta} K(e_\alpha \wedge e_\beta), \quad 1 \leqslant \alpha, \beta \leqslant r. \tag{3.2}$$

For given integers $n \geqslant 3$ and $k \geqslant 1$, denote by $\mathcal{S}(n, k)$ the finite set consisting of all k-tuples $(n_1, \ldots, n_k)$ of integers satisfying

$$2 \leqslant n_1, \ldots, n_k < n \quad \text{and} \quad n_1 + \cdots + n_k < n.$$

Denote by $\mathcal{S}(n)$ the union $\bigcup_{k \geqslant 1} \mathcal{S}(n, k)$.

For each $(n_1,\dots,n_k)\in\mathcal{S}(n)$ and each point $p\in M$, the first author introduced in [5,6] a Riemannian invariant $\delta(n_1,\dots,n_k)(p)$ defined by

$$\delta(n_1,\dots,n_k)(p)=\tau(p)-\inf\bigl\{\tau(L_1)+\cdots+\tau(L_k)\bigr\}, \tag{3.3}$$

where $L_1,\dots,L_k$ run over all k mutually orthogonal subspaces of T_pM such that $\dim L_j=n_j$, $j=1,\dots,k$. The invariants $\delta(n_1,\dots,n_k)$ and the scalar curvature τ are very much different in nature (see [8] for a general survey on $\delta(n_1,\dots,n_k)$).

For a given $(n_1,\dots,n_k)\in\mathcal{S}(n)$, let $L_1,\dots,L_k$ be mutually orthogonal subspaces of T_pM with $\dim L_j=n_j$, $j=1,\dots,k$. We choose an orthonormal basis $e_1,\dots,e_m$ of T_pM such that

$$L_j=\mathrm{Span}\{e_{n_1+\cdots+n_{j-1}+1},\dots,e_{n_1+\cdots+n_j}\},\quad j=1,\dots,k. \tag{3.4}$$

We put

$$\begin{aligned}
&\Delta_1=\{1,\dots,n_1\},\\
&\quad\dots\\
&\Delta_k=\{n_1+\cdots+n_{k-1}+1,\dots,n_1+\cdots+n_k\},\\
&\Delta_{k+1}=\{n_1+\cdots+n_k+1,\dots,n\}.
\end{aligned} \tag{3.5}$$

For simplicity we put

$$N=n_1+\cdots+n_k.$$

Throughout this paper, we shall make use of the following convention on the ranges of indices unless mentioned otherwise:

$$\begin{aligned}
&\alpha_i,\beta_i,\gamma_i\in\Delta_i,\quad i,j\in\{1,\dots,k\};\\
&r,s,t\in\Delta_{k+1};\qquad u,v\in\{N+2,\dots,n\};\\
&A,B,C\in\{1,\dots,n\}.
\end{aligned} \tag{3.6}$$

An n-dimensional submanifold of a Kähler n-manifold $\widetilde{M}^n$ is called *Lagrangian* if the complex structure J of $\widetilde{M}^n$ interchanges each tangent space T_pM, $p\in M$, with the corresponding normal space $T_p^{\perp}(M)$.

The first author proved in [5,6] the following optimal relationship between $\delta(n_1,\dots,n_k)$ and the squared mean curvature H^2 for an arbitrary submanifold in a real space form.

Theorem A. *Let M^n be an n-dimensional submanifold in a real space form $R^m(c)$ of constant curvature c. Then, for each k-tuple $(n_1,\dots,n_k)\in\mathcal{S}(n)$, we have*

$$\delta(n_1,\dots,n_k)\leqslant\frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)}H^2+\frac{1}{2}\left(n(n-1)-\sum_{j=1}^k n_j(n_j-1)\right)c. \tag{3.7}$$

The equality case of inequality (3.7) *holds at a point $p\in M$ if and only if, there exists an orthonormal basis $\{e_1,\dots,e_m\}$ at p, such that the shape operators of M in $R^m(\epsilon)$ at p with respect to $\{e_1,\dots,e_m\}$ take the form*:

$$A_r=\begin{bmatrix} A_1^r & \dots & 0 & \\ \vdots & \ddots & \vdots & 0\\ 0 & \dots & A_k^r & \\ & 0 & & \mu_r I\end{bmatrix},\quad r=n+1,\dots,m, \tag{3.8}$$

where I is an identity matrix and A_j^r is a symmetric $n_j\times n_j$ submatrix satisfying

$$\mathrm{trace}\bigl(A_1^r\bigr)=\cdots=\mathrm{trace}\bigl(A_k^r\bigr)=\mu_r.$$

The same result holds for a Lagrangian submanifolds in a complex space form $\widetilde{M}^n(4c)$ of constant holomorphic sectional curvature $4c$. More precisely, we have

Theorem B. *Let M^n be an n-dimensional Lagrangian submanifold in a complex space form $\widetilde{M}^n(4c)$ of constant holomorphic sectional curvature* $4c$. *Then, for each k-tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$, we have*

$$\delta(n_1, \dots, n_k) \leqslant \frac{n^2(n+k-1-\sum n_j)}{2(n+k-\sum n_j)} H^2 + \frac{1}{2}\left(n(n-1) - \sum_{j=1}^{k} n_j(n_j-1)\right)c. \tag{3.9}$$

The equality case of inequality (3.7) *holds at a point $p \in M$ if and only if, there exists an orthonormal basis $\{e_1, \dots, e_m\}$ at p, such that the shape operators of M in $\widetilde{M}^n(4c)$ at p with respect to $\{e_1, \dots, e_m\}$ take the form of* (3.8).

Remark 3.1. It was proved in [7] that every Lagrangian submanifold of a complex space form $\widetilde{M}^n(4c)$ that satisfies the equality case of inequality (3.9) identically for some k-tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$ is minimal; extending a result in [10,11] on $\delta(2)$.

4. A general inequality for Lagrangian submanifolds

Theorem 4.1. *Let M^n be an n-dimensional Lagrangian submanifold of a complex space form $\widetilde{M}^n(4c)$. Then, for any k-tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$, we have*

$$\delta(n_1, \dots, n_k) \leqslant \frac{n^2\{(n - \sum_{i=1}^k n_i + 3k - 1) - 6\sum_{i=1}^k (2+n_i)^{-1}\}}{2\{(n - \sum_{i=1}^k n_i + 3k + 2) - 6\sum_{i=1}^k (2+n_i)^{-1}\}} H^2 + \frac{1}{2}\left\{n(n-1) - \sum_{i=1}^{k} n_i(n_i-1)\right\}c. \tag{4.1}$$

The equality sign holds at a point $p \in M^n$ if and only if there is an orthonormal basis $\{e_1, \dots, e_n\}$ at p such that with respect to this basis the second fundamental form h takes the following form

$$\begin{aligned}
&h(e_{\alpha_i}, e_{\beta_i}) = \sum_{\gamma_i} h^{\gamma_i}_{\alpha_i\beta_i} J e_{\gamma_i} + \frac{3\delta_{\alpha_i\beta_i}}{2+n_i}\lambda J e_{N+1}, \qquad \sum_{\alpha_i=1}^{n_i} h^{\gamma_i}_{\alpha_i\alpha_i} = 0,\\
&h(e_{\alpha_i}, e_{\alpha_j}) = 0, \quad i \neq j,\\
&h(e_{\alpha_i}, e_{N+1}) = \frac{3\lambda}{2+n_i} J e_{\alpha_i}, \qquad h(e_{\alpha_i}, e_u) = 0,\\
&h(e_{N+1}, e_{N+1}) = 3\lambda J e_{N+1}, \qquad h(e_{N+1}, e_u) = \lambda J e_u,\\
&h(e_u, e_v) = \lambda \delta_{uv} J e_{N+1}, \quad N = n_1 + \cdots + n_k,
\end{aligned} \tag{4.2}$$

for $i, j = 1, \dots, k$; $u, v = N+2, \dots, n$ and $\lambda = \frac{1}{3} h^{N+1}_{N+1N+1}$.

Proof. Let $(n_1, \dots, n_k) \in \mathcal{S}(n)$ and let $L_1, \dots, L_k$ be mutually orthogonal subspaces of T_pM with $\dim L_j = n_j$, $j = 1, \dots, k$. We choose an orthonormal basis $\{e_1, \dots, e_n\}$ at a point $p \in M$ which satisfies (3.4). Since

$$\tau = \sum_{A=1}^{n} \sum_{B<C} \big(h^A_{BB} h^A_{CC} - \big(h^A_{BC}\big)^2\big), \tag{4.3}$$

$$\tau(L_i) = \sum_{A} \sum_{\alpha_i<\beta_i} \big(h^A_{\alpha_i\alpha_i} h^A_{\beta_i\beta_i} - \big(h^A_{\alpha_i\beta_i}\big)^2\big), \tag{4.4}$$

we have

$$\begin{aligned}
\tau - \sum_{i=1}^{k} \tau(L_i) &= \sum_{A} \sum_{r<s} \big(h^A_{rr} h^A_{ss} - \big(h^A_{rs}\big)^2\big) + \sum_{A,i} \sum_{\alpha_i,r} \big(h^A_{\alpha_i\alpha_i} h^A_{rr} - \big(h^A_{\alpha_i r}\big)^2\big) + \sum_{A} \sum_{i<j} \sum_{\alpha_i,\alpha_j} \big(h^A_{\alpha_i\alpha_i} h^A_{\alpha_j\alpha_j} - \big(h^A_{\alpha_i\alpha_j}\big)^2\big)\\
&\leqslant \sum_{A} \left\{ \sum_{r<s} h^A_{rr} h^A_{ss} + \sum_{i} \sum_{\alpha_i,r} h^A_{\alpha_i\alpha_i} h^A_{rr} + \sum_{i<j} \sum_{\alpha_i,\alpha_j} h^A_{\alpha_i\alpha_i} h^A_{\alpha_j\alpha_j} \right\} - \sum_{i} \sum_{\alpha_i,s} \big(h^{\alpha_i}_{ss}\big)^2 - \sum_{B\neq r} \sum_{r=N+1}^{n} \big(h^r_{BB}\big)^2,
\end{aligned} \tag{4.5}$$

with the equality sign holding if and only if

$$h^{\alpha_i}_{\alpha_j\alpha_\ell} = h^{\alpha_j}_{\alpha_i\beta_i} = h^r_{\alpha_i\alpha_j} = h^{\alpha_i}_{st} = h^r_{st} = 0 \tag{4.6}$$

for distinct $i, j, \ell \in \{1, \dots, k\}$ and distinct $r, s, t \in \Delta_{k+1}$.

For a given $i \in \{1, \ldots, k\}$ and a given $\gamma_i \in \Delta_i$, we have

$$\begin{aligned}
0 &\leqslant \sum_{j=1}^{k} \sum_{r \in \Delta_{k+1}}^{n} \left(\sum_{\alpha_j \in \Delta_j} h^{\gamma_i}_{\alpha_j\alpha_j} - 3h^{\gamma_i}_{rr} \right)^2 + 3\sum_{r<s} \left(h^{\gamma_i}_{rr} - h^{\gamma_i}_{ss}\right)^2 + 3\sum_{\ell<j} \left(\sum_{\alpha_\ell \in \Delta_\ell} h^{\gamma_i}_{\alpha_\ell\alpha_\ell} - \sum_{\alpha_j \in \Delta_j} h^{\gamma_i}_{\alpha_j\alpha_j} \right)^2 \\
&= (n - N + 3k - 3) \sum_j \left(\sum_{\alpha_j} h^{\gamma_i}_{\alpha_j\alpha_j} \right)^2 - 6 \sum_j \sum_{\alpha_j, r} h^{\gamma_i}_{\alpha_j\alpha_j} h^{\gamma_i}_{rr} - 6 \sum_{r<s} h^{\gamma_i}_{rr} h^{\gamma_i}_{ss} \\
&\quad - 6 \sum_{\ell<j} h^{\gamma_i}_{\alpha_\ell\alpha_\ell} \sum_{\alpha_j \in \Delta_j} h^{\gamma_i}_{\alpha_j\alpha_j} + 3(n - N + 3k - 1) \sum_r \left(h^{\gamma_i}_{rr}\right)^2 \\
&= (n - N + 3k - 3)\left(h^{\gamma_i}_{11} + \cdots + h^{\gamma_i}_{nn}\right)^2 - 2(n - N + 3k) \\
&\quad \times \left\{ \sum_{r<s} h^{\gamma_i}_{rr} h^{\gamma_i}_{ss} + \sum_{j=1}^{k} \sum_{\alpha_j, r} h^{\gamma_i}_{\alpha_j\alpha_j} h^{\gamma_i}_{rr} + \sum_{\ell<j} h^{\gamma_i}_{\alpha_\ell\alpha_\ell} h^{\gamma_i}_{\alpha_j\alpha_j} - \sum_s \left(h^{\gamma_i}_{ss}\right)^2 \right\}.
\end{aligned}$$

Thus we find

$$\sum_{r<s} h^{\gamma_i}_{rr} h^{\gamma_i}_{ss} + \sum_{j=1}^{k} \sum_{\alpha_j, r} h^{\gamma_i}_{\alpha_j\alpha_j} h^{\gamma_i}_{rr} + \sum_{\ell<j} h^{\gamma_i}_{\alpha_\ell\alpha_\ell} h^{\gamma_i}_{\alpha_j\alpha_j} - \sum_s \left(h^{\gamma_i}_{ss}\right)^2 \leqslant \frac{n - N + 3k - 3}{2(n - N + 3k)} \left(h^{\gamma_i}_{11} + \cdots + h^{\gamma_i}_{nn}\right)^2, \tag{4.7}$$

with the equality holding if and only if

$$\sum_{\alpha_j \in \Delta_j} h^{\gamma_i}_{\alpha_j\alpha_j} = 3h^{\gamma_i}_{ss}, \quad j = 1, \ldots, k,\ s \in \Delta_{k+1}. \tag{4.8}$$

Since

$$\frac{n - N + k - 1}{n - N + k + 2} < \frac{n - N + 3k - 1 - 6\sum_{i=1}^{k}(2 + n_i)^{-1}}{n - N + 3k + 2 - 6\sum_{i=1}^{k}(2 + n_i)^{-1}},$$

we get from (4.7) that

$$\begin{aligned}
&\sum_{r<s} h^{\gamma_i}_{rr} h^{\gamma_i}_{ss} + \sum_{j=1}^{k} \sum_{\alpha_j, r} h^{\gamma_i}_{\alpha_j\alpha_j} h^{\gamma_i}_{rr} + \sum_{\ell<j} h^{\gamma_i}_{\alpha_\ell\alpha_\ell} h^{\gamma_i}_{\alpha_j\alpha_j} - \sum_s \left(h^{\gamma_i}_{ss}\right)^2 \\
&\quad \leqslant \frac{n - N + 3k - 1 - 6\sum_{i=1}^{k}(2 + n_i)^{-1}}{2\{n - N + 3k + 2 - 6\sum_{i=1}^{k}(2 + n_i)^{-1}\}} \left(\sum_{A=1}^{n} h^{\gamma_i}_{AA} \right)^2,
\end{aligned} \tag{4.9}$$

with the equality sign holding if and only if, for each $i \in \{1, \ldots, k\}$, we have

$$\sum_{\alpha_j \in \Delta_j} h^{\gamma_i}_{\alpha_j\alpha_j} = 3h^{\gamma_i}_{ss} = 0, \quad j = 1, \ldots, k,\ s \in \Delta_{k+1}. \tag{4.10}$$

Let us put

$$w = \frac{2}{3} \left\{ n - N + 3k + 2 - \sum_{j=1}^{k} \frac{6}{2 + n_j} \right\}.$$

Since

$$\sum_{i=1}^{k} \frac{n_i}{2 + n_i} = k - \sum_{i=1}^{k} \frac{2}{2 + n_i}, \qquad \sum_{j \neq i} \frac{n_j}{2 + n_j} = k - \sum_j \frac{2}{2 + n_j} - \frac{n_i}{2 + n_i},$$

we find for each $t \in \{N + 1, \ldots, n\}$ that

$$\begin{aligned}
0 \leqslant{}& \sum_i \sum_{r\neq t} \frac{2+n_i}{3n_i}\left(\sum_{\alpha_i} h^t_{\alpha_i\alpha_i} - \frac{3n_i}{2+n_i} h^t_{rr}\right)^2 + \sum_i \sum_{\alpha_i<\beta_i} \frac{w}{n_i}\left(h^t_{\alpha_i\alpha_i} - h^t_{\beta_i\beta_i}\right)^2 \\
&+ \sum_{i<j}\left(\frac{\sqrt{(2+n_i)n_j}}{\sqrt{(2+n_j)n_i}} \sum_{\alpha_i} h^t_{\alpha_i\alpha_i} - \frac{\sqrt{(2+n_j)n_i}}{\sqrt{(2+n_i)n_j}} \sum_{\alpha_j} h^t_{\alpha_j\alpha_j}\right)^2 + \sum_{\substack{r<s\\ r,s\neq t}} \left(h^t_{rr} - h^t_{ss}\right)^2 \\
&+ \frac{1}{3}\sum_{r\neq t}\left(h^t_{tt} - 3h^t_{rr}\right)^2 + \sum_i \frac{n_i}{2+n_i}\left(h^t_{tt} - \frac{2+n_i}{n_i}\sum_{\alpha_i} h^t_{\alpha_i\alpha_i}\right)^2 \\
={}& \sum_i \left\{ (n-N+2)\frac{2+n_i}{3n_i} - \frac{w}{n_i} + \sum_{j\neq i} \frac{(2+n_i)n_j}{(2+n_j)n_i}\right\}\left(\sum_{\alpha_i} h^t_{\alpha_i\alpha_i}\right)^2 \\
&- 2\sum_i \sum_{\alpha_i} h^t_{\alpha_i\alpha_i} h^t_{rr} + \left\{ n-N+1+\sum_i \frac{3n_i}{2+n_i}\right\} \sum_{r\neq t}\left(h^t_{rr}\right)^2 + w\sum_i\sum_{\alpha_i}\left(h^t_{\alpha_i\alpha_i}\right)^2 - 2\sum_{i<j} h^t_{\alpha_i\alpha_i} h^t_{\alpha_j\alpha_j} \\
&- 2\sum_{\substack{r<s\\ r,s\neq t}} h^t_{rr}h^t_{ss} + \left\{ \frac{n-N-1}{3} + \sum_i \frac{n_i}{2+n_i}\right\}\left(h^t_{tt}\right)^2 - 2h^t_{tt}\sum_{r\neq t} h^t_{rr} - 2h^t_{tt}\sum_i\left(\sum_{\alpha_i} h^t_{\alpha_i\alpha_i}\right)^2 \\
={}& \frac{1}{3}\left\{ n-N+3k-1-\sum_i \frac{6}{2+n_i}\right\}\left(\sum_{\alpha_i} h^t_{\alpha_i\alpha_i}\right)^2 - 2\sum_i\sum_{\alpha_i} h^t_{\alpha_i\alpha_i}h^t_{rr} \\
&+ \left\{ n-N+3k+1-\sum_i \frac{6}{2+n_i}\right\}\sum_{r\neq t}\left(h^t_{rr}\right)^2 - 2h^t_{tt}\sum_i\left(\sum_{\alpha_i} h^t_{\alpha_i\alpha_i}\right)^2 - 2\sum_{i<j} h^t_{\alpha_i\alpha_i}h^t_{\alpha_j\alpha_j} \\
&- 2\sum_{\substack{r<s\\ r,s\neq t}} h^t_{rr}h^t_{ss} - 2h^t_{tt}\sum_{r\neq t} h^t_{rr} + w\sum_i\sum_{\alpha_i}\left(h^t_{\alpha_i\alpha_i}\right)^2 + \frac{1}{3}\left\{ n-N+3k-1-\sum_i \frac{6}{2+n_i}\right\}\left(h^t_{tt}\right)^2 \\
={}& w\left\{ \frac{n-N+3k-1-6\sum_{i=1}^k (2+n_i)^{-1}}{2\{n-N+3k+2-6\sum_{i=1}^k (2+n_i)^{-1}\}}\left(\sum_{A=1}^n h^t_{AA}\right)^2 - \sum_{r<s} h^t_{rr}h^t_{ss} \right. \\
&\left. - \sum_{i<j}\sum_{\alpha_i,\alpha_j} h^t_{\alpha_i\alpha_i}h^t_{\alpha_j\alpha_j} - \sum_i\sum_{\alpha_i,r} h^t_{\alpha_i\alpha_i}h^t_{rr} + \sum_{s\neq t}\left(h^t_{ss}\right)^2 + \sum_i\sum_{\alpha_i}\left(h^t_{\alpha_i\alpha_i}\right)^2 \right\}.
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\sum_{r<s} h^t_{rr}h^t_{ss} + \sum_i\sum_{\alpha_i,r} h^t_{\alpha_i\alpha_i}h^t_{rr} + \sum_{i<j}\sum_{\alpha_i,\alpha_j} h^t_{\alpha_i\alpha_i}h^t_{\alpha_j\alpha_j} - \sum_{B\neq t}\left(h^t_{BB}\right)^2 \\
&\quad \leqslant \frac{n-N+3k-1-6\sum_{i=1}^k (2+n_i)^{-1}}{2\{n-N+3k+2-6\sum_{i=1}^k (2+n_i)^{-1}\}}\left(\sum_{A=1}^n h^r_{AA}\right)^2,
\end{aligned} \tag{4.11}$$

with equality holding if and only if

$$h^t_{tt} = (2+n_i)h^t_{\alpha_i\alpha_i} = 3h^t_{ss}, \quad i=1,\dots,k,\ N+1 \leqslant s\neq t \leqslant n. \tag{4.12}$$

Thus, by combining (4.5), (4.9) and (4.11), we obtain inequality (4.1).

Equality in (4.1) implies that the inequalities (4.5), (4.9) and (4.11) become equalities. Thus, we have

$$\sum_{\alpha_j\in\Delta_j} h^{\gamma_i}_{\alpha_j\alpha_j} = 3h^{\gamma_i}_{ss} = 0, \quad j=1,\dots,k,\ s\in\Delta_{k+1}, \tag{4.13}$$

$$h^t_{tt} = (2+n_i)h^t_{\alpha_i\alpha_i} = 3h^t_{ss}, \quad i=1,\dots,k,\ N+1\leqslant s\neq t\leqslant n, \tag{4.14}$$

$$h^{\alpha_i}_{\alpha_j\alpha_\ell} = h^{\alpha_j}_{\alpha_i\beta_i} = h^r_{\alpha_i\alpha_j} = h^{\alpha_i}_{st} = h^r_{st} = 0 \tag{4.15}$$

for distinct $i, j, \ell \in \{1,\dots,k\}$ and distinct $r, s, t \in \Delta_{k+1}$.

It follows from (4.10) that the mean curvature vector lies in $\mathrm{Span}\{e_{N+1},\dots,e_n\}$. Thus, we may choose e_{N+1} in the direction of H. Then we conclude that conditions (4.13)–(4.15) are equivalent to (4.2) due to the totally symmetry of h. □

Remark 4.1. When $k=1$ and $n_1=2$, inequality (4.1) is due to Oprea [15] (see also [13]). The equality case for this special case have been investigated rather detailed in [1–3].

5. Lagrangian graphs attaining the equality at a point

Consider the product $\mathbb{E}^n \times \mathbb{E}^n$ of two Euclidean n-spaces equipped with the Euclidean metric and the natural coordinates $(x_1,\dots,x_n,y_1,\dots,y_n)$. The product $\mathbb{E}^n \times \mathbb{E}^n$ has a natural complex structure J defined by

$$J\frac{\partial}{\partial x_i}=\frac{\partial}{\partial y_i},\qquad J\frac{\partial}{\partial y_i}=-\frac{\partial}{\partial x_i},\quad i=1,\dots,n.$$

We denote the pair $(\mathbb{E}^n \times \mathbb{E}^n, J)$ by $\mathbf{C}^n$, which is known as the complex Euclidean n-space.

Consider the graph in $\mathbf{C}^n=(\mathbb{E}^n \times \mathbb{E}^n, J)$ of a smooth map $f: D\to\mathbb{E}^n$ defined on an open domain $D\subset\mathbb{E}^n$. Then the graph is a Lagrangian submanifold of $\mathbf{C}^n$ if and only if the matrix $(\frac{\partial f^i}{\partial x_j})$ is a symmetric matrix. In particular, if D is simply connected, then there exists a function $F: D\to\mathbf{R}$ with $f=\nabla F$. Therefore, the Lagrangian graph takes the form

$$L(x_1,\dots,x_n)=(x_1,\dots,x_n,F_{x_1},\dots,F_{x_n}),\qquad F_{x_i}=\frac{\partial F}{\partial x_i},\quad i=1,\dots,n. \tag{5.1}$$

The next result shows that, for each k-tuple $(n_1,\dots,n_k)\in\mathcal{S}(n)$, (4.1) is sharp.

Theorem 5.1. *For each k-tuple $(n_1,\dots,n_k)\in\mathcal{S}(n)$, there exists a non-minimal Lagrangian submanifold in $\mathbf{C}^n$ which satisfies the equality case of* (4.1) *at a point.*

Proof. For a given nonzero real number λ, let us consider the Lagrangian graph M in $\mathbf{C}^n$ defined by (5.1) with

$$F=\sum_{i=1}^k\frac{3\lambda}{2(2+n_i)}\sum_{\alpha_i\in\Delta_i}x_{\alpha_i}^2x_{N+1}+\frac{\lambda}{2}\sum_{r=N+1}^n x_{N+1}x_r^2,\quad N=\sum_{i=1}^k n_i. \tag{5.2}$$

Then

$$e_1=(1,0,\dots,0),\quad \dots,\quad e_n=\bigl(0,\dots,0,\overbrace{1}^{n\text{-th}},0,\dots,0\bigr) \tag{5.3}$$

form an orthonormal basis of $T_o(M)$ at $o=(0,\dots,0)$. It is easy to verify that the coefficients of the second fundamental form h of the Lagrangian graph satisfies

$$h_{AB}^C=\frac{\partial^3F}{\partial x_A\partial x_B\partial x_C},\quad A,B,C=1,\dots,n.$$

Thus, we find from (5.2) that, at the point o given by $(x_1,\dots,x_n)=(0,\dots,0)$, the second fundamental form satisfies

$$\begin{aligned}
&h(e_{\alpha_i},e_{\beta_i})=\frac{3\delta_{\alpha_i\beta_i}}{2+n_i}\lambda Je_{N+1},\\
&h(e_{\alpha_i},e_{\alpha_j})=0,\quad i\neq j,\ i,j=1,\dots,k,\\
&h(e_{\alpha_i},e_{N+1})=\frac{3\lambda}{2+n_i}Je_{\alpha_i},\\
&h(e_{\alpha_i},e_u)=0,\\
&h(e_{N+1},e_{N+1})=3\lambda Je_{N+1},\\
&h(e_{N+1},e_u)=\lambda Je_u,\\
&h(e_u,e_v)=\lambda\delta_{uv}Je_{N+1},\quad u,v=N+2,\dots,n,
\end{aligned} \tag{5.4}$$

at the point o. Consequently, by Theorem 4.1, we conclude that M is a non-minimal Lagrangian graph satisfies the equality case of inequality (4.1) at the point o. □

Remark 5.1. Theorem 5.1 shows that the constants in (4.1) cannot be improved.

6. Minimality

Theorem 6.1. *Let M^n be an n-dimensional Lagrangian submanifold of a complex space form $\widetilde{M}^n(4c)$. Then for any integer $n_1 \in [2, n-1]$ we have*

$$\delta(n_1) \leqslant \frac{n^2\{n_1(n-n_1)+2n-2\}}{2\{n_1(n-n_1)+2n+3n_1+4\}} H^2 + \frac{1}{2}\big\{n(n-1) - n_1(n_1-1)\big\}c. \tag{6.1}$$

Moreover, if M^n satisfies the equality case of (6.1) *for some $n_1 \leqslant n-2$, then M^n is a minimal submanifold of $\widetilde{M}^n(4c)$.*

Proof. Inequality (6.1) is special case of inequality (4.1) with $k = 1$. Now, let us assume that M satisfies the equality case of (6.1) identically on M. Then, according to Theorem 4.1, there exists a local orthonormal frame $\{e_1, \ldots, e_n\}$ such that

$$\begin{aligned}
&h(e_\alpha, e_\beta) = \sum_{\gamma=1}^{n_1} h^\gamma_{\alpha\beta} J e_\gamma + \frac{3\lambda\delta_{\alpha\beta}}{2+n_1} J e_{n_1+1}, \qquad \sum_{\alpha=1}^{n_1} h^\beta_{\alpha\alpha} = 0,\\
&h(e_\alpha, e_{n_1+1}) = \frac{3\lambda}{2+n_1} J e_\alpha, \qquad h(e_\alpha, e_v) = 0,\\
&h(e_{n_1+1}, e_{n_1+1}) = 3\lambda J e_{n_1+1},\\
&h(e_{n_1+1}, e_v) = \lambda J e_j,\\
&h(e_u, e_v) = \lambda \delta_{ij} J e_{n_1+1},
\end{aligned} \tag{6.2}$$

for some functions $h^\gamma_{\alpha\beta}$ and λ, where $\alpha, \beta = 1, \ldots, n_1$; $u, v = n_1+2, \ldots, n$.

To prove this theorem, we take the same approach as in [1,2]. So now assume that $n \geqslant n_1 + 2$ and that M has no minimal points, i.e. λ is nowhere zero. In this case Je_{n_1+1} is a multiple of the mean curvature vector implying that λ is a globally defined differentiable function. In accordance to [1,2] we denote by T the vector field corresponding to e_{n+1}, which is also a globally defined differentiable vector field, and by $\mathcal{D}_1$ the distribution spanned by T.

At each point, A_{JT} has three distinct eigenvalues of multiplicity 1, n_1 and $n - n_1 - 1$ with eigenvalues given respectively by

$$\lambda_1 = 3\lambda, \qquad \lambda_2 = \frac{3\lambda}{2+n_1}, \qquad \lambda_3 = \lambda. \tag{6.3}$$

Let $\mathcal{D}_2$ and $\mathcal{D}_3$ be distributions of dimension n_1 and $n - n_1 - 1$ corresponding to λ_2 and λ_3, respectively.

From [2] with $\lambda_1, \lambda_2, \lambda_3$ given above, we have the following lemma.

Lemma 6.1. *We have*

$$(2+n_1)\nabla\lambda - n_1\lambda\nabla_T T \in \mathcal{D}_2^\perp, \tag{6.4}$$

$$3\nabla\lambda - \lambda\nabla_T T \in \mathcal{D}_3^\perp, \tag{6.5}$$

$$n_1\lambda\nabla_V T - (\nabla\lambda)V - \frac{2+n_1}{3} Jh(V, \nabla_T T) \in \mathcal{D}_2^\perp, \tag{6.6}$$

$$\lambda\nabla_W T - (T\lambda)W - Jh(W, \nabla_T T) \in \mathcal{D}_3^\perp, \tag{6.7}$$

$$(2+n_1)\nabla_V T - (1-n_1)\nabla_T V \in \mathcal{D}_3^\perp, \tag{6.8}$$

$$3n_1\nabla_W T + (1-n_1)\nabla_T W \in \mathcal{D}_2^\perp, \tag{6.9}$$

$$(1-n_1)\lambda\nabla_V W + (2+n_1)Jh(V, \nabla_T W) \in \mathcal{D}_2^\perp, \tag{6.10}$$

$$(1-n_1)\lambda\nabla_V W + 3(W\lambda)V + (2+n_1)Jh(V, \nabla_W T) \in \mathcal{D}_2^\perp, \tag{6.11}$$

$$(1-n_1)\lambda\nabla_W V - (2+n_1)Jh(W, \nabla_T V) \in \mathcal{D}_3^\perp, \tag{6.12}$$

$$(1-n_1)\lambda\nabla_W V - (2+n_1)(V\lambda)W - (2+n_1)Jh(W, \nabla_V T) \in \mathcal{D}_3^\perp, \tag{6.13}$$

$$T\big(\big\langle h(V, \widetilde{V}), JV^*\big\rangle\big) - \frac{3V\lambda}{2+n_1}\big\langle \widetilde{V}, V^*\big\rangle = \sigma\big(\big\langle h(V, \widetilde{V}), J\nabla_T V^*\big\rangle\big) - \big\langle h\big(\widetilde{V}, V^*\big), J\nabla_V T\big\rangle, \tag{6.14}$$

$$T\big(\big\langle h(W, \widetilde{W}), JW^*\big\rangle\big) - (W\lambda)\big\langle \widetilde{W}, W^*\big\rangle = \sigma\big(\big\langle h(W, \widetilde{W}), J\nabla_T W^*\big\rangle\big) - \big\langle h\big(\widetilde{W}, W^*\big), J\nabla_W T\big\rangle, \tag{6.15}$$

$$\big\langle h(V, \widetilde{V}), J\nabla_W \widetilde{W}\big\rangle = \big\langle h(W, \widetilde{W}), J\nabla_V \widetilde{V}\big\rangle, \tag{6.16}$$

$$18n_1(W\lambda)V - (1+2n_1)(2+n_1)\sum_{\alpha=1}^{n_1} \langle \nabla_T W, e_\alpha\rangle Jh(V, e_\alpha) \in \mathcal{D}_2^\perp, \tag{6.17}$$

$$2(2+n_1)(V\lambda)W - n_1(1+2n_1)\sum_{u=1}^{n-n_1-1}\langle\nabla_T V, e_u\rangle Jh(W, e_u) \in \mathcal{D}_3^{\perp}, \tag{6.18}$$

$$\begin{aligned}\frac{2(1-n_1)}{2+n_1}(T\lambda)\langle V,\widetilde{V}\rangle\langle W,\widetilde{W}\rangle &= \frac{3n_1(1+2n_1)}{(1-n_1)(2+n_1)\lambda}\big\langle h(W,\widetilde{W}), J\widetilde{W}\big\rangle\langle V,\widetilde{V}\rangle \\ &\quad + \frac{(2+n_1)^2(1+2n_1)}{9n_1(1-n_1)\lambda}\big\langle h(V,\widetilde{V}), J\widetilde{V}\big\rangle\langle W,\widetilde{W}\rangle,\end{aligned} \tag{6.19}$$

for vector fields $V, \widetilde{V}, V^*$ in $\mathcal{D}_2$ and $W, \widetilde{W}, W^*$ in $\mathcal{D}_3$ respectively, where σ in (6.14) *and* (6.15) *denotes cyclic summation over $V, \widetilde{V}, V^*$ and $W, \widetilde{W}, W^*$, respectively.*

We choose a local orthonormal frame $\{e_1,\dots,e_{n_1},T,e_{n_1+2},\dots,e_n\}$ such that $T\in\mathcal{D}_1$, $e_1,\dots,e_{n_1}\in\mathcal{D}_2$ and $e_{n_1+2},\dots,e_n\in\mathcal{D}_3$.

In order to determine the connection coefficients of M as in [2]. We use the following notations: $\alpha,\beta,\gamma\in\{1,\dots,n_1\}$ and $t,u,v\in\{n_1+2,\dots,n\}$. We let

(i) T_α denote the $\mathcal{D}_3$ component of $\nabla_T e_\alpha$,
(ii) $\bar{V}$ denote the $\mathcal{D}_2$ component of $\nabla\lambda$,
(iii) $\bar{W}$ denote the $\mathcal{D}_3$ component of $\nabla\lambda$.

To show that λ is constant, first we observe that (6.4) gives

$$(2+n_1)V\lambda = n_1\lambda\langle V,\nabla_T T\rangle \quad \text{for } V\in\mathcal{D}_2. \tag{6.20}$$

On the other hand, by taking $V^*=\widetilde{V}=e_\alpha$ in (6.14) and summing up on α and by using (2.8) and (6.2), we have

$$\begin{aligned}\frac{3n_1}{2+n_1}V\lambda &= -2\sum_\alpha\big\langle h(V,e_\alpha), J\nabla_T e_\alpha\big\rangle \\ &= -2\sum_{\alpha,\beta}\big\langle h(V,e_\alpha), \omega_\alpha^\beta(T)Je_\beta\big\rangle - 2\sum_\alpha\omega_\alpha^{n_1+1}(T)\big\langle h(V,e_\alpha), JT\big\rangle \\ &= 2\Big\langle h(V,T), J\Big(\sum_\alpha\omega_{n_1+1}^\alpha(T)e_\alpha\Big)\Big\rangle \\ &= 2\big\langle h(V,T), J\nabla_T T\big\rangle = \frac{6\lambda}{2+n_1}\langle V,\nabla_T T\rangle.\end{aligned} \tag{6.21}$$

By combining this with (6.20), we obtain

$$V\lambda = \langle V,\nabla_T T\rangle = 0 \quad \text{for } V\in\mathcal{D}_2. \tag{6.22}$$

Similarly, it follows from (6.2), (6.4) and (6.15) that

$$W\lambda = \langle W,\nabla_T T\rangle = 0 \quad \text{for } W\in\mathcal{D}_3. \tag{6.23}$$

Since $\langle\nabla_T T, T\rangle = 0$, (6.22) and (6.23) imply that $\nabla_T T = 0$. Hence, it follows from (6.4) and (6.5) that $\nabla\lambda = 0$, i.e., λ is constant.

Next, we claim that

$$\nabla_{\mathcal{D}_2}\mathcal{D}_2\subset\mathcal{D}_2, \qquad \nabla_{\mathcal{D}_3}\mathcal{D}_3\subset\mathcal{D}_3. \tag{6.24}$$

To prove this, first we observe from $\nabla_T T = 0$ that we have

$$\langle\nabla_T W, T\rangle = -\langle W,\nabla_T T\rangle = 0. \tag{6.25}$$

Thus, it follows from (6.17) that

$$0 = \sum_\alpha\langle\nabla_T W, e_\alpha\rangle\big\langle Jh(V,e_\alpha), V^*\big\rangle = -\big\langle h\big(V,V^*\big), J\nabla_T W\big\rangle. \tag{6.26}$$

Now, from (6.2) and (6.13), we find

$$\begin{aligned}0 &= (1-n_1)\lambda\big\langle\nabla_W V, W^*\big\rangle + (2+n_1)\big\langle h(W,\nabla_V T), JW^*\big\rangle \\ &= (n_1-1)\lambda\big\langle V,\nabla_W W^*\big\rangle + (2+n_1)\big\langle h\big(W,W^*\big), J(\nabla_V T)\big\rangle \\ &= (n_1-1)\lambda\big\langle V,\nabla_W W^*\big\rangle.\end{aligned} \tag{6.27}$$

Thus, we have $\nabla_{\mathcal{D}_3}\mathcal{D}_3\in\mathcal{D}_2^{\perp}$.

On the other hand, it follows from (6.7) that

$$\big\langle T, \nabla_W W^*\big\rangle = -\big\langle \nabla_W T, W^*\big\rangle = 0. \tag{6.28}$$

Therefore, we find $\nabla_{\mathcal{D}_3}\mathcal{D}_3 \subset \mathcal{D}_3$. Similarly, we have $\nabla_{\mathcal{D}_2}\mathcal{D}_2 \subset \mathcal{D}_2$. This proves the second claim.

Since $n \geqslant n_1 + 2$, locally there exists a unit vector field $W \in \mathcal{D}_3$. Because $n_1 \geqslant 2$, there is a unit vector field $V \in \mathcal{D}_2$ such that

$$-\langle \nabla_T V, W\rangle = \langle V, \nabla_T W\rangle = 0. \tag{6.29}$$

Combining this with (6.8) and (6.9) gives

$$\langle \nabla_V W, T\rangle = \langle \nabla_W V, T\rangle = 0. \tag{6.30}$$

From (6.24) and (6.30), we have

$$\nabla_{\mathcal{D}_2}\mathcal{D}_3 \subset \mathcal{D}_3, \qquad \nabla_{\mathcal{D}_3}\mathcal{D}_2 \subset \mathcal{D}_2, \qquad \nabla_T\mathcal{D}_2 \subset \mathcal{D}_2, \qquad \nabla_T\mathcal{D}_3 \subset \mathcal{D}_3. \tag{6.31}$$

Consequently, after applying (6.24) and (6.31) we find

$$\big\langle R(V, W)W, V\big\rangle = \langle \nabla_V\nabla_W W, V\rangle - \langle \nabla_W\nabla_V W, V\rangle - \langle \nabla_{[V,W]}W, V\rangle = 0. \tag{6.32}$$

On the other hand, it follows from the equation of Gauss and (6.2) that

$$\big\langle R(V, W)W, V\big\rangle = \frac{3\lambda^2}{2+n_1} + c. \tag{6.33}$$

Eqs. (6.32) and (6.33) imply that $c < 0$, since $\lambda \neq 0$.

For $c < 0$, it follows from (6.24) and (6.31) that $\langle R(W, T)T, W\rangle = 0$. On the other hand, it follows from (6.2) that $\langle R(W, T)T, W\rangle = 3\lambda^2$. Thus, again we find that $\lambda = 0$ which gives a contradiction. □

Remark 6.1. Theorem 6.1 extends a result of [3].

7. Non-minimal Lagrangian submanifolds satisfying the equality

A Lagrangian submanifold of $\mathbf{C}^n$ without totally geodesic points is called a *Lagrangian H-umbilical submanifold* if its second fundamental form takes the following simple form (cf. [4]):

$$\begin{aligned} &h(\bar e_1, \bar e_1) = \varphi J\bar e_1, \qquad h(\bar e_j, \bar e_j) = \mu J\bar e_1, \quad j > 1, \\ &h(\bar e_1, \bar e_j) = \mu J\bar e_j, \qquad h(\bar e_j, \bar e_k) = 0, \quad 2 \leqslant j \neq k \leqslant n \end{aligned} \tag{7.1}$$

for some functions φ, μ with respect to a suitable orthonormal local frame field $\{\bar e_1, \ldots, \bar e_n\}$. Such submanifolds are the simplest Lagrangian submanifolds next to the totally geodesic ones.

Let $G : N^{n-1} \to \mathbb{E}^n$ be an isometric immersion of a Riemannian $(n-1)$-manifold into the Euclidean n-space $\mathbb{E}^n$ and let $F : I \to \mathbf{C}^*$ be a unit speed curve in $\mathbf{C}^* = \mathbf{C} - \{0\}$. We may extend $G : N^{n-1} \to \mathbb{E}^n$ to an immersion of $I \times N^{n-1}$ into $\mathbf{C}^n$ as

$$F \otimes G : I \times N^{n-1} \to \mathbf{C} \otimes \mathbb{E}^n = \mathbf{C}^n, \tag{7.2}$$

where $(F \otimes G)(s, p) = F(s) \otimes G(p)$ for $s \in I$, $p \in N^{n-1}$. This extension $F \otimes G$ of G via tensor product is called the *complex extensor* of G via F.

The following result was proved in [4].

Proposition 7.1. *Let $\iota : S^{n-1} \to \mathbb{E}^n$ be the inclusion of a hypersphere of $\mathbb{E}^m$ centered at the origin. Then every complex extensor $\phi = F \otimes \iota$ of ι via a unit speed curve $F : I \to \mathbf{C}^*$ is a Lagrangian H-umbilical submanifold of $\mathbf{C}^n$ unless F is contained in a line through the origin* (*which gives a totally geodesic Lagrangian submanifold*).

For $F \otimes \iota$, let us choose a unit vector field $\bar e_1$ tangent to the first factor and $\bar e_2, \ldots, \bar e_n$ tangent to the second factor of $I \times S^{n-1}$. If we put $F'(s) = e^{i\zeta(s)}$ and $F(s) = r(s)e^{i\theta(s)}$, then it follows from [4] that the second fundamental form of $F \otimes \iota$ satisfies (7.1) with

$$\varphi = \zeta'(s) = \kappa, \qquad \mu = \frac{\langle F', iF\rangle}{\langle F, F\rangle} = \theta'(s), \tag{7.3}$$

where κ is the curvature function of F. Therefore, by applying (7.1), (7.3) and Theorem 4.1, we conclude that if the unit speed curve $F : I \to \mathbf{C}$ satisfies

$$\kappa(s) = (n+1)\theta'(s) \neq 0, \quad s \in I, \tag{7.4}$$

then the complex extensor $F \otimes \iota : I \times S^{n-1} \to \mathbf{C}^n$ is a non-minimal Lagrangian submanifold of $\mathbf{C}^n$ which verifies the equality case of (6.1) with $n_1 = n-1$. Consequently, Theorem 6.1 is sharp.

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