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Limit cycles bifurcating from isochronous surfaces of revolution in $\mathbb{R}^3$ ☆

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ABSTRACT

In this paper we study the number of limit cycles bifurcating from isochronous surfaces of revolution contained in $\mathbb{R}^3$, when we consider polynomial perturbations of arbitrary degree. The method for studying these limit cycles is based on the averaging theory and on the properties of Chebyshev systems. We present a new result on averaging theory and generalizations of some classical Chebyshev systems which allow us to obtain the main results.

1. Introduction

Consider a differential system

$$\dot{\mathbf{x}} = X_0(\mathbf{x}) + \varepsilon X(\mathbf{x}), \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^3$, $X_0, X : \mathbb{R}^3 \to \mathbb{R}^3$ are vector fields and ε is a real small parameter; the dot denotes the derivative with respect to the time. If we suppose that $(1)_{\varepsilon=0}$ has an *isochronous invariant surface* $\mathcal{S} \subset \mathbb{R}^3$, that is, $\mathcal{S}$ is foliated by periodic orbits with the same period, then natural questions are: For $\varepsilon \neq 0$ sufficiently small does the differential system (1) possess limit cycles emerging from the periodic orbits of $\mathcal{S}$? How to compute them? How many? These questions are analogous to the following about planar differential systems: How many limit cycles emerge under a perturbation from a planar center? In this last case many results has been obtained (see for example [2] and the references therein). Recall that a *limit cycle* of a differential system is a periodic orbit which is isolated in the set of all periodic orbits of the system.

A tool for studying these kind of problems is the averaging theory. For instance, perturbations of isochronous sets of periodic orbits as planes, cylinders and tori in $\mathbb{R}^3$ has been studied, see [4–6]. For a general introduction to this theory see [9] and [11].

In this paper we consider differential systems $(1)_{\varepsilon=0}$ in $\mathbb{R}^3$ which contain an isolated isochronous invariant revolution surface of the form

$$\mathcal{S}_F = \left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; F(x, y, z) = x^2 + y^2 - f(z) = 0\right\}, \tag{2}$$

where $f(z) > 0$ in a nonempty open subset $\mathcal{U}_f$ of $\mathbb{R}$. Mainly we consider the quadratic case, that is, when X_0 is quadratic vector field and $f(z)$ is a polynomial of degree at most 2. The set of all these $\mathcal{S}_F$ contains the main quadratic surfaces of $\mathbb{R}^3$: the sphere $\{x^2 + y^2 + z^2 - 1 = 0\}$, the cylinder $\{x^2 + y^2 - 1 = 0\}$, the hyperboloid of one sheet $\{x^2 + y^2 - z^2 - 1 = 0\}$, the hyperboloid of two sheets $\{x^2 + y^2 - z^2 + 1 = 0\}$, the cone $\{x^2 + y^2 - z^2 = 0\}$ and the paraboloid $\{x^2 + y^2 - z = 0\}$.

☆ Limit cycles bifurcating from isochronous surfaces.

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We also consider some additional cases, for instance when either $f(z)$ is a polynomial of arbitrary degree, or $f(z) = z^{p/q}$ with p and q positive integers such that $(p, q) = 1$, or $f(z) = e^z$, or $f(z) = \log z$.

The paper is devoted to the study of the number of limit cycles of system (1) for $\varepsilon \neq 0$ sufficiently small bifurcating from the periodic orbits of $\mathcal{S}_F$ under polynomial perturbations X of degree $d \geqslant 1$.

Frequently it is necessary to use an appropriate system of coordinates in an open neighborhood of $\mathcal{S}_F$ for reducing the differential system (1) to the standard form for applying the results of the averaging theory. In addition it is well known that the study of limit cycles of differential systems is in general a hard problem. So some restrictions have to be imposed to system $(1)_{\varepsilon=0}$ for obtaining satisfactory results.

We will say that a quadratic differential system $(1)_{\varepsilon=0}$ with an isochronous invariant quadratic surface $\mathcal{S}_F$ have an *invariant dynamic by cylinders* if when we transform the differential system $(1)_{\varepsilon=0}$ by using cylindrical coordinates (θ, r, z) then we get that $\dot{r} = 0$. This condition and the assumption of isochronism on the periodic orbits of $\mathcal{S}_F$ are imposed in order to apply in a simple way the results of averaging theory for studying the limit cycles bifurcating from the periodic orbits of $\mathcal{S}_F$. We note that the natural invariant surfaces of a differential system having an invariant dynamic by cylinders are the revolution ones.

The bifurcation of limit cycles for the cases treated in [4,5] was studied by applying Theorem 3.1 of [1]. In [6] an improvement of such a theorem (see Theorem 4 in Section 2) was proved and applied. When both results cannot be applied (as for instance for system (1) when $f(z) \equiv 1$) we need other analogous results. In this paper we give a new result on the periodic orbits studied by averaging theory (see Theorem 5 in Section 2) that will allow us to study the bifurcation of limit cycles of (1) for ε sufficiently small and for any $f(z)$.

As we will see in Section 2 the number of limit cycles of system (1) bifurcating from the periodic orbits of the invariant isochronous surface $\mathcal{S}_F$ of $(1)_{\varepsilon=0}$ are controlled by the isolated zeros of a function $\delta(\alpha, \varepsilon)$ which is defined in a transversal section to the surface $\mathcal{S}_F$. If (1) is at least of class $\mathcal{C}^2$, then such function can be written as

$$\delta(\alpha, \varepsilon) = \varepsilon \mathcal{G}(\alpha) + \varepsilon^2 \widetilde{\mathcal{G}}(\alpha, \varepsilon).$$

In this paper we only study the bifurcated limit cycles which are controlled up to first order in ε, that is, assuming that $\mathcal{G}(\alpha)$ does not vanish identically. Hence as we shall see our problem reduce to study how many isolated zeros has the function $\mathcal{G}(\alpha)$. The results from the averaging theory guaranty the existence of a limit cycle for each simple zero of this function. If $\mathcal{G}(\alpha)$ vanish identically, then results using higher order averaging theory in ε must be applied.

The main result of this paper is the following one.

Theorem 1. *Any polynomial perturbation* (1) *of degree $d \geqslant 1$ of a quadratic differential system $(1)_{\varepsilon=0}$, which has an invariant isochronous quadratic surface $\mathcal{S}_F$ given by* (2) *and an invariant dynamic by cylinders, can be written as*

$$\dot{x} = -y + \varepsilon P(x, y, z), \qquad \dot{y} = x + \varepsilon Q(x, y, z), \qquad \dot{z} = \lambda F(x, y, z) + \varepsilon R(x, y, z), \tag{3}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$, $F(x, y, z) = x^2 + y^2 - f(z)$, $f(z)$ is a polynomial of degree $s \leqslant 2$ and $P(x, y, z)$, $Q(x, y, z)$ and $R(x, y, z)$ are polynomials of degree d. For $\varepsilon \neq 0$ sufficiently small the following statements hold:

(i) *If $\mathcal{G}(\alpha)$ does not vanish identically, then $d - 1$ is an upper bound for the number of limit cycles of system* (3) *that can bifurcate from the periodic orbits of the invariant isochronous surface $\mathcal{S}_F$ of system $(3)_{\varepsilon=0}$.*

(ii) *For any ν in $\{0, 1, 2, \ldots, d - 1\}$ we can find polynomials P, Q and R of degree d such that system* (3) *has exactly ν limit cycles, bifurcating from the periodic orbits of the isochronous surface $\mathcal{S}_F$ of system $(3)_{\varepsilon=0}$.*

Theorem 1 implies that the maximum number of limit cycles that can bifurcate from $(3)_{\varepsilon=0}$ inside the space of polynomial perturbations of degree d is at least $d - 1$. In other words the *cyclicity* of $(3)_{\varepsilon=0}$ is at least $d - 1$.

Inspired in Theorem 1 we consider systems (3) with $f(z)$ either a polynomial, or $z^{p/q}$ with p and q positive integers such that $(p, q) = 1$, or $\exp(z)$, or $\log z$.

Theorem 2. *Consider system* (3) *with $f(z)$ a polynomial of degree $s \geqslant 3$. For $\varepsilon \neq 0$ sufficiently small the following statements hold*:

(i) *If $\mathcal{G}(\alpha)$ does not vanish identically, then*

$$D = d - 1 + (s - 2)\left[\frac{d - 1}{2}\right]$$

is an upper bound for the number of limit cycles of (3) *that can bifurcate from periodic orbits of the isochronous invariant surface $\mathcal{S}_F$ of system $(3)_{\varepsilon=0}$. Moreover there exist polynomials P, Q and R of degree d and a polynomial f of degree s such that system* (3) *has D limit cycles, bifurcating from the periodic orbits of the invariant isochronous surface $\mathcal{S}_F$ of system $(3)_{\varepsilon=0}$.*

(ii) *If either $1 \leqslant d \leqslant 4$, or $s = 3$, or $s = 4$ and d even, or $s = 4$ and $d \leqslant 29$ odd, or $s = 5$ and $d \leqslant 27$, or $s = 6$ and $d \leqslant 24$ even, then for every $\nu \in \{0, 1, 2, \ldots, D\}$ we can find polynomials P, Q, R of degree d and a polynomial f of degree s such that system* (3) *has exactly ν limit cycles, bifurcating from the periodic orbits of the invariant isochronous surface $\mathcal{S}_F$ of system $(3)_{\varepsilon=0}$.*

As we will see in Section 4 the proof of some cases of Theorem 2(ii) is computational and cannot be generalized to arbitrary s and d.

Theorem 3. *Consider system* (3) *when $f(z)$ is the function $z^{p/q}$ with p and q positive integers such that $(p,q)=1$, or $\exp(z)$, or $\log z$. Then for $\varepsilon \neq 0$ sufficiently small*

$$D=\left(\left[\frac{d-1}{2}\right]+1\right)\left(\left[\frac{d}{2}\right]+1\right)$$

is an upper bound for the number of limit cycles of (3) *that can bifurcate from periodic orbits of the isochronous invariant surface $\mathcal{S}_F$ of system* $(3)_{\varepsilon=0}$. *Moreover we can find polynomials P, Q and R of degree d such that system* (3) *has exactly D limit cycles, bifurcating from the periodic orbits of the invariant isochronous surface $\mathcal{S}_F$ of system* $(3)_{\varepsilon=0}$.

The paper is structured as follows. In Section 2 we summarize the result from the averaging theory that we will use for proving Theorems 1 and 2. In Section 3 we prove some general results that can be applied to any f. Section 4 is devoted to prove Theorems 1 and 2, and in Section 5 we find some new families of Chebyshev systems and we prove Theorem 3.

2. A new result in averaging theory

We consider the problem of the bifurcation of T-periodic orbits from the differential system

$$\dot{\mathbf{x}} = F_0(t,\mathbf{x}) + \varepsilon F_1(t,\mathbf{x}) + \varepsilon^2 F_2(t,\mathbf{x},\varepsilon), \tag{4}$$

with $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and $\varepsilon_0 > 0$ sufficiently small. The functions $F_0, F_1 : \mathbb{R}\times\Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R}\times\Omega\times(-\varepsilon_0,\varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^k$ functions, with $k \geqslant 2$ and T-periodic in the variable t, where Ω is an open subset of $\mathbb{R}^n$. We assume that the unperturbed system $(4)_{\varepsilon=0}$ has a submanifold of dimension m with $1 \leqslant m \leqslant n$, foliated by T-periodic orbits.

Let $\mathbf{x}(t,\mathbf{z})$ be the solution of the unperturbed system $(4)_{\varepsilon=0}$ such that $\mathbf{x}(0,\mathbf{z}) = \mathbf{z}$. We write the linearization of $(4)_{\varepsilon=0}$ along the solution $\mathbf{x}(t,\mathbf{z})$ as

$$\dot{y}(t) = D_{\mathbf{x}} F_0\big(t, \mathbf{x}(t,\mathbf{z})\big) y. \tag{5}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ a fundamental matrix of the linear differential system (5), by $\xi : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ and $\xi^{\perp} : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ the projections of $\mathbb{R}^n$ onto its first m and $n-m$ coordinates respectively; i.e. $\xi(x_1,\dots,x_n) = (x_1,\dots,x_m)$, and $\xi^{\perp}(x_1,\dots,x_n) = (x_{m+1},\dots,x_n)$.

The result used in [4–6] can be stated as follows.

Theorem 4. *(See [6].) Let $V \subset \mathbb{R}^m$ be open and bounded, let $\beta_0 : \mathrm{Cl}(V) \to \mathbb{R}^{n-m}$ be a $\mathcal{C}^k$ function and $\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)) \mid \alpha \in \mathrm{Cl}(V)\} \subset \Omega$ its graphic in $\mathbb{R}^n$. Assume that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t,\mathbf{z}_\alpha)$ of $(4)_{\varepsilon=0}$ is T-periodic and that there exists a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of* (5) *such that the matrix $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$:*

(a) *has in the lower right corner the $(n-m)\times(n-m)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$, and*
(b) *has in the upper right corner the $m \times (n-m)$ zero matrix.*

Consider the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^m$ defined by

$$\mathcal{F}(\alpha) = \xi\left(\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1\big(t, \mathbf{x}(t,\mathbf{z}_\alpha)\big)\,dt\right). \tag{6}$$

Suppose that there is $\alpha_0 \in V$ with $\mathcal{F}(\alpha_0) = 0$, then the following statements hold for $\varepsilon \neq 0$ sufficiently small:

(i) *If $\det((\partial\mathcal{F}/\partial\alpha)(\alpha_0)) \neq 0$, then there is a unique T-periodic solution $\varphi_1(t,\varepsilon)$ of system* (4) *such that $\varphi_1(t,\varepsilon) \to \mathbf{x}(t,\mathbf{z}_{\alpha_0})$ as $\varepsilon \to 0$.*
(ii) *If $m = 1$ and $\mathcal{F}'(\alpha_0) = \dots = \mathcal{F}^{(s-1)}(\alpha_0) = 0$ and $\mathcal{F}^{(s)}(\alpha_0) \neq 0$ with $s \leqslant k$, then there are at most s T-periodic solutions $\varphi_1(t,\varepsilon),\dots,\varphi_s(t,\varepsilon)$ of system* (4) *such that $\varphi_i(t,\varepsilon) \to \mathbf{x}(t,\mathbf{z}_{\alpha_0})$ as $\varepsilon \to 0$ for $i = 1,\dots,s$.*

As we shall see in Section 3 this result cannot be applied in some cases for studying the bifurcation of limit cycles from the invariant isochronous surface $\mathcal{S}_F$. Then a natural question is: *there exists an analogous result for studying the periodic orbits of* (4) *bifurcating from an isochronous set of* $(4)_{\varepsilon=0}$?

The answer to the previous question is the following new result.

Theorem 5. *Let $V \subset \mathbb{R}^m$ be open and bounded, let $\beta_0 : \mathrm{Cl}(V) \to \mathbb{R}^m$ be a $\mathcal{C}^k$ function and $\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta_0(\alpha)) \mid \alpha \in \mathrm{Cl}(V)\} \subset \Omega$ its graphic in $\mathbb{R}^{2m}$. Assume that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha)$ of $(4)_{\varepsilon=0}$ is T-periodic and that there exists a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (5) such that the matrix $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$:*

(a) *has in the upper right corner the $m \times m$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$, and*
(b) *has in the lower right corner the $m \times m$ zero matrix.*

Consider the function $\mathcal{G} : \mathrm{Cl}(V) \to \mathbb{R}^m$ defined by

$$\mathcal{G}(\alpha) = \xi^{\perp}\left(\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1\big(t, \mathbf{x}(t, \mathbf{z}_\alpha)\big)\, dt\right). \tag{7}$$

Suppose that there is $\alpha_0 \in V$ with $\mathcal{G}(\alpha_0) = 0$, then the following statements hold for $\varepsilon \neq 0$ sufficiently small:

(i) *If $\det((\partial\mathcal{G}/\partial\alpha)(\alpha_0)) \neq 0$, then there is a unique T-periodic solution $\varphi_1(t, \varepsilon)$ of system (4) such that $\varphi_1(t, \varepsilon) \to \mathbf{x}(t, \mathbf{z}_{\alpha_0})$ as $\varepsilon \to 0$.*
(ii) *If $m = 1$ and $\mathcal{G}'(\alpha_0) = \cdots = \mathcal{G}^{(s-1)}(\alpha_0) = 0$ and $\mathcal{G}^{(s)}(\alpha_0) \neq 0$ with $s \leqslant k$, then there are at most s T-periodic solutions $\varphi_1(t, \varepsilon), \ldots, \varphi_s(t, \varepsilon)$ of system (4) such that $\varphi_i(t, \varepsilon) \to \mathbf{x}(t, \mathbf{z}_{\alpha_0})$ as $\varepsilon \to 0$ for $i = 1, \ldots, s$.*

Instead Theorem 5 is an analogous result to Theorem 4 and consequently their proofs are similar, we include it proof for completeness and for increasing the readability of the paper.

Proof of Theorem 5. Since $\mathcal{Z}$ is a compact set and $\mathbf{x}(t, \mathbf{z}_\alpha)$ is T-periodic for each $\mathbf{z}_\alpha \in \mathcal{Z}$, there is an open neighborhood D of $\mathcal{Z}$ in Ω and $0 < \varepsilon_1 \leqslant \varepsilon_0$ such that any solution $\mathbf{x}(t, \mathbf{z}, \varepsilon)$ of (4) with initial conditions in $D \times (-\varepsilon_1, \varepsilon_1)$ is well defined in $[0, T]$. We consider the function $L : D \times (-\varepsilon_1, \varepsilon_1) \to \mathbb{R}^{2m}$, $(\mathbf{z}, \varepsilon) \mapsto \mathbf{x}(T, \mathbf{z}, \varepsilon) - \mathbf{z}$. If $(\bar{\mathbf{z}}, \bar{\varepsilon}) \in D \times (-\varepsilon_1, \varepsilon_1)$ is such that $L(\bar{\mathbf{z}}, \bar{\varepsilon}) = 0$, then $\mathbf{x}(t, \bar{\mathbf{z}}, \bar{\varepsilon})$ is a T-periodic solution of $(4)_{\varepsilon=\bar{\varepsilon}}$. Clearly the converse is true. Hence the problem of finding T-periodic orbits of (4) close to the periodic orbits with initial conditions in $\mathcal{Z}$ is reduced to find the zeros of $L(\mathbf{x}, \varepsilon)$.

The sets of zeros of $L(\mathbf{z}, \varepsilon)$ and $\widetilde{L}(\mathbf{z}, \varepsilon) = M_{\mathbf{z}}^{-1}(T) L(\mathbf{z}, \varepsilon)$ are the same, since $M_{\mathbf{z}}(T)$ is a fundamental matrix. Moreover following the proof of [1] we can compute that

$$D_{\mathbf{z}}\widetilde{L}(\mathbf{z}, \varepsilon) = \big(M_{\mathbf{z}}^{-1}(0) - M_{\mathbf{z}}^{-1}(T)\big) + D_{\mathbf{z}}\left(\int_0^T M_{\mathbf{z}}^{-1}(t) F_1\big(t, \mathbf{x}(t, \mathbf{z}, 0)\big)\, dt\right)\varepsilon + O\big(\varepsilon^2\big). \tag{8}$$

We note that $\widetilde{L}^{-1}(0) = (\xi^{\perp} \circ \widetilde{L})^{-1}(0) \cap (\xi \circ \widetilde{L})^{-1}(0)$. From (8) we obtain $D_{\mathbf{z}}\widetilde{L}(\mathbf{z}_\alpha, 0) = M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$. If we write $\mathbf{z} \in \mathbb{R}^{2m}$ as $\mathbf{z} = (u, v)$ with $u, v \in \mathbb{R}^m$, then $D_v(\xi \circ \widetilde{L})(\mathbf{z}_\alpha, 0)$ is the upper right corner of $M_{\mathbf{z}}^{-1}(0) - M_{\mathbf{z}}^{-1}(T)$. Then from (a) we can apply the Implicit Function Theorem, thus it follows that there exist an open neighborhood $U \times (-\varepsilon_2, \varepsilon_2)$ of $\mathrm{Cl}(V)$ in $\xi(D) \times (-\varepsilon_1, \varepsilon_1)$, an open neighborhood $\mathcal{O}$ of $\beta_0(\mathrm{Cl}(V))$ in $\mathbb{R}^m$ and a unique $\mathcal{C}^k$ function $\beta(\alpha, \varepsilon) : U \times (-\varepsilon_2, \varepsilon_2) \to \mathcal{O}$ such that $(\xi \circ \widetilde{L})^{-1}(0) \cap (U \times \mathcal{O} \times (-\varepsilon_2, \varepsilon_2))$ is exactly the graphic of $\beta(\alpha, \varepsilon)$. Now if we define the function $\delta : U \times (-\varepsilon_2, \varepsilon_2) \to \mathbb{R}$ as $\delta(\alpha, \varepsilon) = (\xi^{\perp} \circ \widetilde{L})(\alpha, \beta(\alpha, \varepsilon), \varepsilon)$, then δ is a function of class $\mathcal{C}^k$ and $\widetilde{L}^{-1}(0) \cap (U \times \mathcal{O} \times (-\varepsilon_2, \varepsilon_2)) = \{(\alpha, \beta(\alpha, \varepsilon), \varepsilon) \mid (\alpha, \varepsilon) \in \delta^{-1}(0)\}$. Therefore for describing the set $\widetilde{L}^{-1}(0)$ in an open neighborhood of $\mathcal{Z}$ in $\mathbb{R}^n \times (-\varepsilon_0, \varepsilon_0)$, it is sufficient to describe $\delta^{-1}(0)$ in an open neighborhood of $\mathrm{Cl}(V)$ in $\mathbb{R} \times (-\varepsilon_0, \varepsilon_0)$.

Since $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$ has in the lower right corner the $m \times m$ zero matrix and $\delta(\alpha, 0) = 0$ in $V \times (-\varepsilon_2, \varepsilon_2)$, the function $\delta(\alpha, \varepsilon)$ can be written as $\delta(\alpha, \varepsilon) = \varepsilon\mathcal{G}(\alpha) + \varepsilon^2\widetilde{\mathcal{G}}(\alpha, \varepsilon)$ in $V \times (-\varepsilon_2, \varepsilon_2)$, where $\mathcal{G}(\alpha)$ is the function given in (7), see [1]. In addition if $\widetilde{\delta}(\alpha, \varepsilon) = \mathcal{G}(\alpha) + \varepsilon\widetilde{\mathcal{G}}(\alpha, \varepsilon)$, then $\delta^{-1}(0) = \widetilde{\delta}^{-1}(0)$.

If there is $\alpha_0 \in V$ such that $\widetilde{\delta}(\alpha_0, 0) = \mathcal{G}(\alpha_0) = 0$ and $\det((\partial\mathcal{G}/\partial\alpha)(\alpha_0)) \neq 0$, then from the Implicit Function Theorem there exist $\varepsilon_3 > 0$ small, an open neighborhood V_0 of α_0 in V and a unique function of class $\mathcal{C}^k$ $\alpha(\varepsilon) : (-\varepsilon_3, \varepsilon_3) \to V_0$ such that $\widetilde{\delta}^{-1}(0) \cap (V_0 \times (-\varepsilon_3, \varepsilon_3))$ is the graphic of $\alpha(\varepsilon)$, which also represents the set $\delta^{-1}(0) \cap (V_0 \times (-\varepsilon_3, \varepsilon_3))$. This prove statement (i).

Moreover if $m = 1$ the function $\mathcal{G}(\alpha)$ is of one variable, so we can consider higher order derivatives of it. Suppose that

$$\frac{\partial\widetilde{\delta}}{\partial\alpha}(\alpha_0, 0) = \mathcal{G}'(\alpha_0) = 0, \ldots, \frac{\partial^{s-1}\widetilde{\delta}}{\partial\alpha^{s-1}}(\alpha_0, 0) = \mathcal{G}^{(s-1)}(\alpha_0) = 0$$

and $\frac{\partial^s\widetilde{\delta}}{\partial\alpha^s}(\alpha_0, 0) = \mathcal{G}^{(s)}(\alpha_0) \neq 0$. We want to prove that there are at most s T-periodic solutions of system (4) bifurcating from $\mathbf{x}(t, \mathbf{z}_{\alpha_0})$. Suppose the contrary, that is suppose that there are at least $s+1$ T-periodic solutions of system (4) bifurcating from $\mathbf{x}(t, \mathbf{z}_{\alpha_0})$, then for any integer j there exist $\varepsilon_j > 0$ and $\eta_j > 0$, $\varepsilon_j \to 0$ and $\eta_j \to 0$ as $j \to \infty$, such that the function $\widetilde{L}(\mathbf{z}, \varepsilon_j)$ has at least $s+1$ zeros in $|\mathbf{z} - \mathbf{z}_{\alpha_0}| < \eta_j$. Equivalently the function $\widetilde{\delta}(\alpha, \varepsilon)$ has at least $s+1$ zeros in $|\alpha - \alpha_0| < \eta_j$. By using the Rolle Theorem we find an α_j such that $|\alpha_j - \alpha_0| < \eta_j$ and

$$\mathcal{G}^{(s)}(\alpha_j) + \varepsilon_j \frac{\partial^s \widetilde{\delta}}{\partial \alpha^s}(\alpha_j, \varepsilon_j) = 0,$$

which implies $\mathcal{G}^{(s)}(\alpha_0) = 0$ by taking limit as $j \to \infty$, which is a contradiction. Hence statement (ii) is proved. □

From Theorems 4 and 5 and using classical arguments from averaging theory we get the following result.

Proposition 6. *Suppose that a differential system* (1) *can be written in the form* (4) *and that the subset $\mathcal{S}$ corresponds to a manifold foliated by periodic orbits of period T. Under the hypothesis of Theorem* 4 *or* 5 *each T-periodic solution given by one of these theorems corresponds with a limit cycle of system* (1) *for ε small.*

Therefore, if $\mathcal{F}(\alpha)$ or $\mathcal{G}(\alpha)$ does not vanish identically, then the number of isolated zeros of them is an upper bound for the number of limit cycles of (1) bifurcating from the periodic orbits of the surface $\mathcal{S}$ and each simple zero corresponds with a limit cycle of (1) for ε small.

In general to know the maximum number of limit cycles that system (5) can have is a very difficult problem. The result described in the previous paragraph gives a partial answer in that direction. That is the upper bound provided by the simple zeros of the function $\mathcal{F}(\alpha)$ or $\mathcal{G}(\alpha)$, when it is reached, is a lower bound for the maximum number of limit cycles of system (1).

3. General results

Lemma 7. *Any quadratic differential system $(1)_{\varepsilon=0}$ in $\mathbb{R}^3$ with the invariant isochronous quadratic surface $\mathcal{S}_F$ of the form* (2) *with $f(z)$ of degree at most* 2 *having an invariant dynamic by cylinders can be written as system $(3)_{\varepsilon=0}$.*

Proof. Let $P_2(x,y,z)$, $Q_2(x,y,z)$ and $R_2(x,y,z)$ be polynomials of degree 2. Consider in $\mathbb{R}^3$ the quadratic differential system

$$\frac{dx}{d\tau} = P_2, \qquad \frac{dy}{d\tau} = Q_2, \qquad \frac{dz}{d\tau} = R_2. \tag{9}$$

By using cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, we obtain that

$$\frac{d\theta}{d\tau} = \frac{xQ_2 - yP_2}{r^2} \quad \text{and} \quad \frac{dr}{d\tau} = \frac{xP_2 + yQ_2}{r}.$$

Therefore $dr/d\tau = 0$ if and only if $xP_2 = -yQ_2$, whence $P_2 = -yT$ and $Q_2 = xT$ for a polynomial $T = T(x,y,z)$ of degree 1. By using this, we have that $d\theta/d\tau = T$. Now as $\{F=0\}$ is an invariant isochronous surface we have that the linear polynomial T restricted to the quadratic surface $\{F=0\}$ must be a nonzero constant, i.e. we conclude that T is a constant $\mu \neq 0$, thus $d\theta/d\tau = \mu$. In addition since $\{F=0\}$ is invariant then there is a polynomial $K = K(x,y,z)$ of degree 1 such that

$$P_2\frac{\partial F}{\partial x} + Q_2\frac{\partial F}{\partial y} + R_2\frac{\partial F}{\partial z} = xP_2 + yQ_2 - R_2 f'(z) = KF.$$

Since $xP_2 + yQ_2 = 0$ then $R_2 f'(z) = -KF$, and it follows that $R_2 = \lambda_1 F$ with $\lambda_1 \in \mathbb{R}$. By using the rescaling $\tau = t/\mu$ in time (9) is transformed into $(3)_{\varepsilon=0}$. □

Lemma 8. *By using cylindrical coordinates (θ, r, z) system* (3) *can be written into the form* (4); *the transformed system satisfies the conditions* (a) *and* (b) *of Theorem* 5; *and* (7) *takes the form*

$$\mathcal{G}(\alpha) = \int_0^{2\pi} \left(\widetilde{P}\big(\theta, \sqrt{f(\alpha)}, \alpha\big)\cos\theta - \widetilde{Q}\left(\theta, \sqrt{f(\alpha)}, \alpha\right)\sin\theta\right) d\theta, \tag{10}$$

where $\alpha \in \mathbb{R}$, and $\widetilde{P}(\theta, \sqrt{f(\alpha)}, \alpha)$ and $\widetilde{Q}(\theta, \sqrt{f(\alpha)}, \alpha)$ are the expressions in the new coordinates of $P(x,y,z)$ and $Q(x,y,z)$ restricted to the surface $\mathcal{S}_F$ respectively.

Proof. In cylindrical coordinates system (3) becomes

$$\begin{aligned} &\dot{\theta} = 1 + \varepsilon\big(\widetilde{Q}(\theta,r,z)\cos\theta - \widetilde{P}(\theta,r,z)\sin\theta\big)/r, \qquad \dot{r} = \varepsilon\big(\widetilde{P}(\theta,r,z)\cos\theta + \widetilde{Q}(\theta,r,z)\sin\theta\big), \\ &\dot{z} = \lambda\big(r^2 - f(z)\big) + \varepsilon\widetilde{R}(\theta,r,z), \end{aligned} \tag{11}$$

where $\widetilde{Y}(\theta,r,z) := Y(r\cos\theta, r\sin\theta, z)$ for $Y \in \{P, Q, R\}$.

We change the independent variable t of system (11) by the variable θ, and we obtain the equivalent 2-dimensional system

$$z' = \lambda\bigl(r^2 - f(z)\bigr) + \varepsilon S(\theta, r, z) + O\bigl(\varepsilon^2\bigr), \qquad r' = \varepsilon\bigl(\widetilde{P}(\theta, r, z)\cos\theta + \widetilde{Q}(\theta, r, z)\sin\theta\bigr) + O\bigl(\varepsilon^2\bigr), \tag{12}$$

where

$$S(\theta, r, z) = \bigl(\widetilde{R}(\theta, r, z) - \lambda\bigl(r^2 - f(z)\bigr)\bigl(\widetilde{Q}(\theta, r, z)\cos\theta - \widetilde{P}(\theta, r, z)\sin\theta\bigr)/r\bigr),$$

which is defined in $\mathbb{R}^3 \setminus \{(0,0,z) \mid z \in \mathbb{R}\}$. The prime denotes the derivative with respect to the variable θ.

If we use the notation $\mathbf{x} = \binom{z}{r}$, then system (12) can be written as

$$\mathbf{x}'(\theta) = F_0(\mathbf{x}) + \varepsilon F_1(\theta, \mathbf{x}) + \varepsilon^2 F_2(\theta, \mathbf{x}, \varepsilon),$$

where

$$F_0(\mathbf{x}) = \begin{pmatrix} \lambda(r^2 - f(z)) \\ 0 \end{pmatrix},$$

and

$$F_1(\theta, \mathbf{x}) = \begin{pmatrix} S(\theta, r, z) \\ \widetilde{P}(\theta, r, z)\cos\theta + \widetilde{Q}(\theta, r, z)\sin\theta \end{pmatrix}. \tag{13}$$

It is clear that $F_0, F_1 : \mathbb{R} \times \Omega \to \Omega$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \Omega$ are 2π-periodic in θ and analytic. Thus system (12) has the form (4).

Consider the subset

$$\mathcal{Z} = \bigl\{\mathbf{z}_\alpha = \bigl(\alpha, \sqrt{f(\alpha)}\bigr) \bigm| \alpha \in \mathbb{R},\ f(\alpha) > 0\bigr\} \subset \Omega.$$

The solution of $(12)_{\varepsilon=0}$ through the point $\mathbf{z}_\alpha$ is $\mathbf{x}(\theta, \mathbf{z}_\alpha) = \binom{\alpha}{\sqrt{f(\alpha)}}$ which is constant, hence 2π-periodic in θ. Therefore $\mathcal{Z}$ is an invariant 1-dimensional manifold foliated by periodic orbits of the unperturbed system $(12)_{\varepsilon=0}$ (in fact, singular points), which corresponds to the invariant isochronous surface $\mathcal{S}_F$ of system $(11)_{\varepsilon=0}$.

The variational system corresponding to the unperturbed system $(12)_{\varepsilon=0}$ along the solutions of $\mathcal{Z}$ is

$$\begin{pmatrix} z' \\ r' \end{pmatrix} = \begin{pmatrix} -\lambda f'(\alpha) & 2\lambda\sqrt{f(\alpha)} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z \\ r \end{pmatrix}.$$

For obtaining a fundamental matrix $M_{\mathbf{z}_\alpha}$ of the previous system we consider two cases.

Case $f'(\alpha) \neq 0$. In this case we compute that

$$M_{\mathbf{z}_\alpha}(\theta) = \begin{pmatrix} e^{-\lambda f'(\alpha)\theta} & \frac{-2\sqrt{f(\alpha)}(-1+e^{-\lambda f'(\alpha)\theta})}{f'(\alpha)} \\ 0 & 1 \end{pmatrix}.$$

Then we have

$$M_{\mathbf{z}_\alpha}^{-1}(\theta) = \begin{pmatrix} e^{\lambda f'(\alpha)\theta} & \frac{2\sqrt{f(\alpha)}(1-e^{\lambda f'(\alpha)\theta})}{f'(\alpha)} \\ 0 & 1 \end{pmatrix}, \tag{14}$$

and consequently

$$M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(2\pi) = \begin{pmatrix} 1 - e^{2\pi\lambda f'(\alpha)} & \frac{2\sqrt{f(\alpha)}(1-e^{2\pi\lambda f'(\alpha)})}{f'(\alpha)} \\ 0 & 0 \end{pmatrix}. \tag{15}$$

It is clear that $2\sqrt{f(\alpha)}(1 - e^{2\pi\lambda f'(\alpha)})/f'(\alpha) \neq 0$.

Case $f'(\alpha) = 0$. As $\lim_{x\to 0} \frac{1-e^{\theta\lambda x}}{x} = -\lambda\theta$ then for $f'(\alpha) = 0$ we have that

$$M_{\mathbf{z}_\alpha}(\theta) = \begin{pmatrix} 1 & -2\sqrt{f(\alpha)}\theta\lambda \\ 0 & 1 \end{pmatrix}.$$

Then

$$M_{\mathbf{z}_\alpha}^{-1}(\theta) = \begin{pmatrix} 1 & 2\sqrt{f(\alpha)}\lambda\theta \\ 0 & 1 \end{pmatrix}, \tag{16}$$

and

$$M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(2\pi) = \begin{pmatrix} 0 & -4\pi\sqrt{f(\alpha)}\lambda \\ 0 & 0 \end{pmatrix}. \tag{17}$$

Hence the right upper matrix of $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(2\pi)$ does not vanish for any α in the domain of f.

Therefore taking V any compact subset of $\mathbb{R}$, the function $\beta_0 : \mathrm{Cl}(V) \to \mathbb{R}^+$, $\alpha \mapsto \sqrt{f(\alpha)}$, and using (15) and (17) it is clear that system (12) satisfies hypotheses of Theorem 5.

By using (13) and (14) or (16) we have that (7) takes the form (10):

$$\mathcal{G}(\alpha) = \int_0^{2\pi} \big(\widetilde{P}\big(\theta, \sqrt{f(\alpha)}, \alpha\big)\cos\theta - \widetilde{Q}\left(\theta, \sqrt{f(\alpha)}, \alpha, \right)\sin\theta\big)\,d\theta. \qquad \Box$$

We note that from (15) and (17) it is clear that if $f'(\alpha) = 0$ then Theorem 4 cannot be applied, even changing the order of the coordinates (z, r). In particular in the case $f(z) \equiv 1$ ($\mathcal{S}_F$ the cylinder) Theorem 4 cannot be applied. However Theorem 5 can be used.

Lemma 9. *If $P(x, y, z)$ and $Q(x, y, z)$ are polynomials of degree at most d, then expression* (10) *takes the form*

$$\mathcal{G}(\alpha) = \sqrt{f(\alpha)} \sum_{l=0}^{[\frac{d-1}{2}]} h_{d-(2l+1)}(\alpha)\big(f(\alpha)\big)^l = \sqrt{f(\alpha)}\widehat{\mathcal{G}}(\alpha), \tag{18}$$

where $h_{d-(2l+1)}(\alpha)$ is a polynomial of degree $d - (2l + 1)$ and whose coefficients are functions on the coefficients of $P(x, y, z)$ and $Q(x, y, z)$. Moreover if $Q(x, y, z) \equiv 0$ then there exists a polynomial $P(x, y, z)$ such that all the coefficients of every polynomial $h_{d-(2l+1)}(\alpha)$ of (18) *are independent.*

Proof. Suppose that

$$P(x, y, z) = \sum_{i+j+k=0}^{d} p_{ijk}x^i y^j z^k, \qquad Q(x, y, z) = \sum_{i+j+k=0}^{d} q_{ijk}x^i y^j z^k,$$

then

$$\widetilde{P}\big(\theta, \sqrt{f(\alpha)}, \alpha\big) = \sum_{i+j=0}^{d}\left(\sum_{k=0}^{d-i-j} p_{ijk}\alpha^k\right)\big(\sqrt{f(\alpha)}\big)^{i+j}\cos^i\theta\sin^j\theta$$

and

$$\widetilde{Q}\left(\theta, \sqrt{f(\alpha)}, \alpha\right) = \sum_{i+j=0}^{d}\left(\sum_{k=0}^{d-i-j} q_{ijk}\alpha^k\right)\big(\sqrt{f(\alpha)}\big)^{i+j}\cos^i\theta\sin^j\theta.$$

Therefore

$$\int_0^{2\pi} \widetilde{P}\big(\theta, \sqrt{f(\alpha)}, \alpha\big)\cos\theta\,d\theta = \sum_{i+j=0}^{d} \widetilde{p}_{d-(i+j)}(\alpha)\big(f(\alpha)\big)^{\frac{i+j}{2}} I_{i,j},$$

$$\int_0^{2\pi} \widetilde{Q}\left(\theta, \sqrt{f(\alpha)}, \alpha\right)\sin\theta\,d\theta = \sum_{i+j=0}^{d} \widetilde{q}_{d-(i+j)}(\alpha)\big(f(\alpha)\big)^{\frac{i+j}{2}} J_{i,j},$$

where $\widetilde{p}_{d-(i+j)}(\alpha)$ and $\widetilde{q}_{d-(i+j)}(\alpha)$ are polynomials in α of degree $d - (i + j)$ and

$$I_{i,j} = \int_0^{2\pi} \cos^{i+1}\theta\sin^j\theta\,d\theta \quad \text{and} \quad J_{i,j} = \int_0^{2\pi} \cos^i\theta\sin^{j+1}\theta\,d\theta.$$

It is well known that $I_{i,j}$ does no vanish identically if and only if $i + 1$ and j are even. Suppose that $i = 2\mu + 1$ and $j = 2\nu$. As $i + j \leqslant d$ then $2(\mu + \nu) + 1 \leqslant d$, therefore $\mu + \nu \leqslant [\frac{d-1}{2}]$. If we consider $l = \mu + \nu$ and $I_{2(l+1)} = \sum_{\mu+\nu=l} I_{2\mu+1,2\nu}$

then

$$\int_0^{2\pi} \widetilde{P}\big(\theta, \sqrt{f(\alpha)}, \alpha\big) \cos\theta \, d\theta = \sum_{l=0}^{[\frac{d-1}{2}]} I_{2(l+1)} \widetilde{p}_{d-(2l+1)}(\alpha) \big(f(\alpha)\big)^{\frac{2l+1}{2}}. \tag{19}$$

Analogously $J_{i,j}$ does no vanish identically if and only if i and $j+1$ are even. Suppose that $i = 2\mu$ and $j = 2\nu + 1$. As $i + j \leqslant d$ then $2(\mu + \nu) + 1 \leqslant d$, therefore $\mu + \nu \leqslant [\frac{d-1}{2}]$. If we consider $l = \mu + \nu$ and $J_{2(l+1)} = \sum_{\mu+\nu=l} J_{2\mu,(2\nu+1)}$ then

$$\int_0^{2\pi} \widetilde{Q}\big(\theta, \sqrt{f(\alpha)}, \alpha\big) \sin\theta \, d\theta = \sum_{l=0}^{[\frac{d-1}{2}]} J_{2(l+1)} \widetilde{q}_{d-(2l+1)}(\alpha) \big(f(\alpha)\big)^{\frac{2l+1}{2}}. \tag{20}$$

Then from (19) and (20) we obtain that $\mathcal{G}(\alpha)$ given in (10) takes the form (18):

$$\mathcal{G}(\alpha) = \sqrt{f(\alpha)} \sum_{l=0}^{[\frac{d-1}{2}]} h_{d-(2l+1)}(\alpha) \big(f(\alpha)\big)^{l}.$$

Now we will prove the second assertion of the lemma. If $Q(x, y, z) \equiv 0$ and

$$P(x, y, z) = \sum_{l=0}^{[\frac{d-1}{2}]} \left(\sum_{k=0}^{d-1-2l} p_{1,2l,k} z^k \right) \frac{xy^{2l}}{c_l} \tag{21}$$

where $p_{1,2l,k} \in \mathbb{R}$ and

$$c_l := I_{1,2l} = \int_0^{2\pi} \cos^2\theta \sin^{2l}\theta \, d\theta \neq 0, \quad l = 0, 1, \ldots, \big[(d-1)/2\big]. \tag{22}$$

Then if we replace $\widetilde{P}(\theta, \sqrt{f(\alpha)}, \alpha)\cos\theta$ and $\widetilde{Q}(\theta, \sqrt{f(\alpha)}, \alpha)\sin\theta$ in (10) and by using (21) and (22) we obtain

$$\mathcal{G}(\alpha) = \sqrt{f(\alpha)} \sum_{l=0}^{[\frac{d-1}{2}]} \left(\sum_{k=0}^{d-1-2l} p_{1,2l,k} \alpha^k \right) \big(f(\alpha)\big)^{l} = \sqrt{f(\alpha)} \widehat{\mathcal{G}}(\alpha). \tag{23}$$

Since the coefficients $p_{1,2l,k}$, with $l = 0, 1, \ldots, [(d-1)/2]$ and $k = 0, 1, \ldots, d-1-2l$, were chosen independent the statement is proved. □

As we will interested in the number of zeros of $\mathcal{G}(\alpha)$ contained in the set $\mathcal{U}_f$, then from now on we can work with $\widehat{\mathcal{G}}$ instead of $\mathcal{G}$ because the number and the multiplicity of their zeros coincide in $\mathcal{U}_f$.

Remark 10. From Lemma 8 and the second assertion of Lemma 9 it follows that if $\mathcal{G}(\alpha)$ does not vanish identically, then for finding limit cycles of system (3) bifurcating from the periodic orbits of $\mathcal{S}_F$ it is sufficient to consider $Q \equiv 0$, $R \equiv 0$ and P arbitrary, in other words is sufficient to study the zeros of $\widehat{\mathcal{G}}$ given in (23).

4. The polynomial case

The proofs of Theorems 1 and 2 are based in the results of the previous section and some technical lemmas.

Lemma 11. *Let $d \geqslant 1$ be the maximum of the degrees of the polynomials $P(x, y, z)$, $Q(x, y, z)$ and $R(x, y, z)$ in (3). If $f(z)$ is a polynomial of degree s in $F = x^2 + y^2 - f(z)$, then $\widehat{\mathcal{G}}(\alpha)$, defined in (18), is a polynomial function of degree at most D with $D = d - 1$ for $s = 0, 1, 2$, and $D = d - 1 + (s-2)[\frac{d-1}{2}]$ for $s \geqslant 3$.*

Proof. Suppose that $f(z)$ is a polynomial of degree s. Then, from (18), $\widehat{\mathcal{G}}(\alpha) = h_{d-1}(\alpha) + \widetilde{h}(\alpha)$, where

$$\widetilde{h}(\alpha) = \sum_{l=1}^{[\frac{d-1}{2}]} h_{d-(2l+1)}(\alpha) \big(f(\alpha)\big)^{l}.$$

For each $l \in \{1, 2, \ldots, [\frac{d-1}{2}]\}$ the term of maximal degree in $h_{d-(2l+1)}(\alpha)(f(\alpha))^l$ is $\alpha^{(d-(2l+1)+sl)}$. Since $l \leqslant [\frac{d-1}{2}]$ then $d - (2l+1) + sl \leqslant d - 1 + (s-2)[\frac{d-1}{2}]$. Therefore $\widetilde{h}(\alpha)$ is a polynomial of degree at most $d - 1 + (s-2)[\frac{d-1}{2}]$. Therefore $\widehat{\mathcal{G}}(\alpha) = h_{d-1}(\alpha) + \widetilde{h}(\alpha)$ is a polynomial of degree at most $d-1$ if $s = 0, 1, 2$, and of degree at most $d - 1 + (s-2)[\frac{d-1}{2}]$ if $s \geqslant 3$. □

Lemma 12. *If $f(z)$ is a polynomial of degree s, with $s = 0, 1, 2$, and $f(z) > 0$ in a nonempty open subset $\mathcal{U}_f$ of $\mathbb{R}$, then for any positive integer d and any ν in $\{1, 2, \ldots, d-1\}$, we can find a polynomial $P(x, y, z)$ of degree d such that the function $\widehat{\mathcal{G}}(\alpha)$ given in (23) has exactly ν zeros in $\mathcal{U}_f$ and each one of them is simple.*

Proof. If we consider the polynomial $P(x, y, z)$ defined in (21) with $p_{1,2l,k} = 0$ for $l = 1, 2, \ldots, [\frac{d-1}{2}]$ and $p_{1,0,k} \neq 0$ then $\widehat{\mathcal{G}}(\alpha)$ given in (23) reduces to

$$\widehat{\mathcal{G}}(\alpha) = \sum_{k=0}^{d-1} p_{1,0,k}\alpha^k \tag{24}$$

which is a polynomial of degree $d-1$ with all its coefficients independents. Therefore for every $\nu \in \{1, 2, \ldots, d-1\}$ we can choose the coefficients $p_{1,0,k}$ in such a way that $\widehat{\mathcal{G}}(\alpha)$ has exactly ν simple zeros in $\mathcal{U}_f$. □

Proof of Theorem 1. From Lemma 7 any quadratic differential system $(1)_{\varepsilon=0}$ in $\mathbb{R}^3$ with the invariant isochronous quadratic surface $\mathcal{S}_F$ ($f(z)$ is a polynomial of degree at most 2) having an invariant dynamic by cylinders can be written in the form $(3)_{\varepsilon=0}$. From Lemma 8 the perturbed system (3), with $f(z)$ a polynomial of degree at most 2, is reduced to the form (4) and we can apply Theorem 5 for studying the limit cycles of the original system (3).

If d is the maximum of the degrees of $P(x, y, z)$ and $Q(x, y, z)$ and the function $\widehat{\mathcal{G}}(\alpha)$, given by (18), does not vanish identically, then it has at most $d-1$ isolated zeros, counting multiplicities, as we have proved in Lemma 11. Therefore the statement (i) follows from Theorem 5(ii) and Proposition 6.

Statement (ii) follows from Remark 10, Lemma 12, Theorem 5(i) and Proposition 6. □

Proof of Theorem 2. Proof of statement (i). The first assertion of this statement follows from Lemma 11, Theorem 5(ii) and Proposition 6.

For the second assertion we can suppose that $Q \equiv 0$, $R \equiv 0$ (see Remark 10) and we need to prove that there are a polynomial f of degree $s \geqslant 3$ and a polynomial P of degree d such that $\widehat{\mathcal{G}}(\alpha)$ given in (23) has D simple zeros contained in the set $\mathcal{U}_f$.

The second assertion of this statement is trivial if $d = 1$ and if $d = 2$, $D = 1$ and from (23) we have that $\widehat{\mathcal{G}}(\alpha) = p_{1,0,0} + p_{1,0,1}\alpha$, hence the assertion is valid also in this case. If $d = 3$ (respectively $d = 4$) then $D = s$ (respectively $D = s+1$). By considering $p_{1,2,0} = 1$ (respectively $p_{1,2,0} = 0$ and $p_{1,2,1} = 1$) and $f(\alpha) = a_0 + a_3\alpha^3 + \cdots + a_s\alpha^s$, with $a_s > 0$, from (23) we have that

$$\widehat{\mathcal{G}}(\alpha) = (p_{1,0,0} + a_0) + p_{1,0,1}\alpha + p_{1,0,2}\alpha^2 + \sum_{l=3}^{s} a_l\alpha^s$$

(respectively $\widehat{\mathcal{G}}(\alpha) = p_{1,0,0} + (p_{1,0,1} + a_0)\alpha + p_{1,0,2}\alpha^2 + p_{1,0,3}\alpha^3 + \sum_{l=3}^{s} a_l\alpha^{s+1}$) has independent coefficients. Thus $\widehat{\mathcal{G}}(\alpha)$ can have ν zeros, with $\nu = 0, 1, \ldots, D$, in $(0, \infty)$ and since a_0 is a free parameter, we can assume that $f(\alpha) > 0$ in $(0, \infty)$. Therefore for the cases $d = 3$ and $d = 4$ also the second assertion of (i) is true.

If $d \geqslant 5$ then the expression of $\widehat{\mathcal{G}}(\alpha)$ has powers of the polynomial $f(\alpha)$ (see (23)), hence we cannot ensure a priori that the polynomial $\widehat{\mathcal{G}}(\alpha)$ has all its coefficients independent as in the previous cases. For proving that the second assertion of (i) is also true for $d \geqslant 5$ we will use another approach. We only need to prove the existence of f and P such that $\widehat{\mathcal{G}}(\alpha)$ has D simple zeros contained in the set $\mathcal{U}_f$. We will do that in two steps. In the first step we construct an auxiliary function, which will allow us to find f and P. In a second step we obtain the expressions for f and P.

First step. We can split $d \geqslant 5$ as $d = 5 + 4j + i$, with $j = 0, 1, \ldots$ and $i = 0, 1, 2, 3$. In such a case the expression of $D = d - 1 + (s-2)[\frac{d-1}{2}]$ can be written as

$$D_{j,i,s} = 2(j+1)s + \left[\frac{i}{2}\right]s + i - 2\left[\frac{i}{2}\right]. \tag{25}$$

For each pair (j, i) and $s \geqslant 3$ we will construct a polynomial $h_{j,i,s}(\alpha)$ of degree which will have $D_{j,i,s}$ simple zeros in $[0, \infty)$. With $h_{j,i,s}(\alpha)$ we will find f and P.

We choose a polynomial $\tilde{f}(\alpha) = a_0 + a_1\alpha + \cdots + a_s\alpha^s$ of degree $s \geqslant 3$, with $a_s > 0$, such that it has s simple zeros in $(0, \infty)$. Thus $(\tilde{f}(\alpha))^2 \geqslant 0$ has a double zero at each zero of $\tilde{f}(\alpha)$. For any integer $j \geqslant 0$ we can choose $j+1$ small enough real numbers $A_0, \ldots, A_j$, with $0 < A_j < A_{j-1} < \cdots < A_0$, such that each function $(\tilde{f}(\alpha))^2 - A_l$, for $l = 0, \ldots, j$, has exactly $2s$ simple zeros in $(0, \infty)$. Then for the pair (j, i) we define the polynomial

$$h_{j,i,s}(\alpha) = (\alpha)^{i-2[\frac{i}{2}]}\big(\tilde{f}(\alpha)\big)^{[\frac{i}{2}]}\prod_{l=0}^{j}\big(\big(\tilde{f}(\alpha)\big)^2 - A_l\big),$$

of degree $D_{j,i,s}$ (see (25)) with $D_{j,i,s}$ simple zeros in $[0,\infty)$, since the $2s$ zeros of $(\tilde{f}(\alpha))^2 - A_k$ are all different from the $2s$ zeros of $(\tilde{f}(\alpha))^2 - A_l$ for all $k \neq l$ and the s zeros of $\tilde{f}(\alpha)$ are all different from the zeros of $(\tilde{f}(\alpha))^2 - A_l$ for all $l = 0,\dots,j$ and of course $\alpha = 0$ is not a zero neither of $\tilde{f}(\alpha)$ nor of $(\tilde{f}(\alpha))^2 - A_l$ for all $l = 0,\dots,j$.

Second step. There is a constant $K > 0$ such that the polynomial $f(\alpha) = \tilde{f}(\alpha) + K$ of degree s is strictly positive in $[0,\infty)$, and by using this relation we can write

$$h_{j,i,s}(\alpha) = \sum_{l=0}^{2(j+1)+[\frac{i}{2}]} (\alpha)^{i-2[\frac{i}{2}]} A_{[\frac{i}{2}],l}\big(f(\alpha)\big)^l, \tag{26}$$

where the coefficients $A_{[\frac{i}{2}],l}$ are functions of $A_0, A_1, \dots, A_j$ and K.

Now, if $d = 5 + 4j + i$ as above and let $\widehat{\mathcal{G}}_{j,i}(\alpha)$ be the resulting expression of considering all the coefficients in (23) nulls, except $p_{1,2l,k_i}$ with $k_i = i - 2[\frac{i}{2}]$. Then

$$\widehat{\mathcal{G}}_{j,i}(\alpha) = \sum_{l=0}^{2(j+1)+[\frac{i}{2}]} p_{1,2l,k_i}(\alpha)^{i-2[\frac{i}{2}]}\big(f(\alpha)\big)^l.$$

By comparing $\widehat{\mathcal{G}}_{j,i}$ with (26) it is clear that if we choose $p_{1,2l,k_i} = A_{[\frac{i}{2}],l}$, then $\widehat{\mathcal{G}}_{i,j}$ has $D_{j,i,s}$ simple zeros in $[0,\infty) \subset \mathcal{U}_f$. Hence, Theorem 5(i) and Proposition 6 completes the proof of statement (i).

Proof of statement (ii). Again we can suppose that $Q \equiv 0$, $R \equiv 0$ (see Remark 10). The case $1 \leqslant d \leqslant 4$ follows from third paragraph of the proof of statement (i).

For $s = 3$, or $s = 4$, we consider $f(\alpha) = \alpha^s$. Hence from (23) we have that

$$\widehat{\mathcal{G}}(\alpha) = \sum_{l=0}^{[\frac{d-1}{2}]} \left(\sum_{k=0}^{d-1-2l} p_{1,2l,k}\alpha^{sl+k} \right).$$

As the coefficients of the previous expression are all independent we only need to show that every monomial α^μ, $\mu = 0, 1, \dots, D$ appears.

For $s = 3$ it is enough to consider $p_{1,2l,k} = 0$ when $k \geqslant 3$. Then $sl + k = 3l + k$ for $k = 0, 1, 2$ cover all the naturals hence all the monomials appear.

For $s = 4$ if we consider $p_{1,2l,k} = 0$ when $k \geqslant 4$ we have that $sl + k = 4l + k$ for $k = 0, 1, 2, 3$ cover all naturals only when d is even because when d is odd the corresponding monomial to power $4l + 3$ when $l = [\frac{d-1}{2}]$ does not appear. Hence $\widehat{\mathcal{G}}$ has all the monomials.

The proof of the other cases follows arranging all the monomials of (23), as we have done in the proof of previous statement when $d = 3$, and changing the coefficients of $\widehat{\mathcal{G}}$ by new independent parameters. This is possible when the rank of the corresponding linear system of the coefficients is maximal. This happens, choosing a concrete f for each case, when $s = 4$ and $d \leqslant 24$ odd, or $s = 5$ and $d \leqslant 24$, or $s = 6$ and $d \leqslant 24$ even. All the computations for these concrete values of d and s have been done with a computer and MAPLE as algebraic manipulator.

The proof ends applying Theorem 5(i) and Proposition 6. □

Remark 13. We cannot improve the values of d and s in Theorem 2(ii). Because, for example, the computations involved in the case $d = 26$ and $s = 6$ are too big for a computer with 32 GB of Ram.

5. Particular cases

When f is not a polynomial, the techniques for controlling the number of zeros of (23) usually use the properties on the Chebyshev systems, that is the natural generalization concept of knowing the number of zeros, like polynomials, of a linear combination of functions. We first recall some properties on them.

The set of $j + 1$ real functions $\{f_0(x), f_1(x), \dots, f_j(x)\}$ defined in a closed interval A forms a *Chebyshev system* in A if any nontrivial linear combination $a_0 f_0(x) + a_1 f_1(x) + \dots + a_j f_j(x)$ has at most j zeros in A counting multiplicities.

Proposition 14. *Suppose that the set of real functions $\{f_0(x), f_1(x), \dots, f_j(x)\}$ forms a Chebyshev system in A. If $x_0, x_1, \dots, x_j$ are $j + 1$ different points in A and $c_0, c_1, \dots, c_j$ are $j + 1$ arbitrary real numbers, then the system of equations*

$$a_0 f_0(x_i) + a_1 f_1(x_i) + \dots + a_j f_j(x_i) = c_i, \quad i = 0, 1, \dots, j,$$

has a unique solution for $a_0, a_1, \dots, a_j$.

For a proof of Proposition 14 see [7, p. 24].

Proposition 15. *Suppose that the set of real functions $\{f_0(x), f_1(x), \ldots, f_j(x)\}$ forms a Chebyshev system in A. Then the set $\{\int f_0(x)\,dx, \int f_1(x)\,dx, \ldots, \int f_j(x)\,dx, 1\}$ also forms a Chebyshev system in A.*

For a proof of Proposition 15 see [10, p. 589].

Now we will introduce three families of Chebyshev systems that are used for proving Theorem 3. Proposition 16 can be found, without proof, in [8, p. 138] but for completeness we include its proof. It is well known that for any real number α and any natural number n the set $\{x^\alpha, x^{\alpha+1}, \ldots, x^{\alpha+n}\}$ is a Chebyshev system. Some generalizations of this family can be found in [3] or in Proposition 17. Finally Proposition 18 is a generalization of the well-known Chebyshev family $\{1, \log(x), x, x\log(x), \ldots, x^n, x^n\log(x)\}$ where n is a natural number.

Proposition 16. *Let $\alpha_1 < \cdots < \alpha_L$ be real numbers and let $n_1, \ldots, n_L$ be positive integers. The set of $L + n_1 + \cdots + n_L$ functions*

$$T^{\alpha_1,\ldots,\alpha_L}_{n_1,\ldots,n_L} = \left\{e^{\alpha_1 x}, xe^{\alpha_1 x}, \ldots, x^{n_1}e^{\alpha_1 x}, \ldots, e^{\alpha_L x}, xe^{\alpha_L x}, \ldots, x^{n_L}e^{\alpha_L x}\right\}$$

is a Chebyshev system in any closed interval A.

Proof. Given a positive integer n_1 and a real number β_1, from the definition of a Chebyshev system, we observe that the set $T^{\beta_1}_{n_1} = \{e^{\beta_1 x}, xe^{\beta_1 x}, \ldots, x^{n_1}e^{\beta_1 x}\}$ is also a Chebyshev system in any closed interval A. Computing the primitive, see Proposition 15, of every element of the set $T^{\beta_1}_{n_1}$ we obtain the Chebyshev system

$$\left\{\frac{1}{\beta_1}e^{\beta_1 x}, \left(-\frac{1}{\beta_1^2} + \frac{1}{\beta_1}x\right)e^{\beta_1 x}, \ldots, \left(p_{n_1-1}(x) + \frac{1}{\beta_1}x^{n_1}\right)e^{\beta_1 x}, 1\right\},$$

where $p_{n_1-1}(x)$ is an explicit polynomial of degree $n_1 - 1$. Now adding to each one of the first $n_1 + 1$ elements of the above set a precise linear combination of the previous elements, and after multiplying by β_1 we obtain the next Chebyshev system

$$T^{\beta_1,0}_{n_1,0} = \left\{e^{\beta_1 x}, xe^{\beta_1 x}, \ldots, x^{n_1}e^{\beta_1 x}, 1\right\}.$$

By doing the similar procedure n_2 times more and after multiplying by $e^{\beta_2 x}$ we obtain that

$$T^{\beta_1+\beta_2,\beta_2}_{n_1,n_2} = \left\{e^{(\beta_1+\beta_2)x}, xe^{(\beta_1+\beta_2)x}, \ldots, x^{n_1}e^{(\beta_1+\beta_2)x}, e^{\beta_2 x}, xe^{\beta_2 x}, \ldots, x^{n_2}e^{\beta_2 x}\right\}$$

is a Chebyshev system. The proof that $T^{\alpha_1,\ldots,\alpha_L}_{n_1,\ldots,n_L}$ is a Chebyshev system follows from the application of the previous procedure until β_L and doing the change of parameters $\beta_L = \alpha_L$, $\beta_{L-1} + \beta_L = \alpha_{L-1}$, $\ldots$, $\beta_1 + \cdots + \beta_L = \alpha_1$.

The other statements follow from the Chebyshev system properties. □

Proposition 17. *Let $\alpha_1, \ldots, \alpha_L$ be different positive nonintegers and $n_1, \ldots, n_L$ be positive integers. The set of $L + n_1 + \cdots + n_L$ functions*

$$V^{\alpha_1,\ldots,\alpha_L}_{n_1,\ldots,n_L} = \left\{x^{\alpha_1}, \ldots, x^{\alpha_1+n_1}, x^{\alpha_2}, \ldots, x^{\alpha_2+n_2}, \ldots, x^{\alpha_L}, \ldots, x^{n_L+\alpha_L}\right\}$$

is a Chebyshev system in any closed interval A contained in the positive axis.

Proof. Given a positive integer n_1 and a noninteger number β_1, it is well known that the set $V^{\beta_1}_{n_1} = \{x^{\beta_1}, \ldots, x^{n_1+\beta_1}\}$ is a Chebyshev system in any closed interval A contained in the positive axis. Computing the primitive, see Proposition 15, of every element of the set $V^{\beta_1}_{n_1}$ we obtain also the Chebyshev system

$$\left\{\frac{1}{\beta_1+1}x^{\beta_1+1}, \frac{1}{\beta_1+2}x^{\beta_1+2}, \ldots, \frac{1}{\beta_1+n_1+1}x^{n_1+\beta_1+1}, 1\right\}.$$

Now multiplying any element for a nonzero constant, all can be different, we obtain also the Chebyshev system

$$V^{\beta_1+1,0}_{n_1,0} = \left\{x^{\beta_1+1}, x^{\beta_1+2}, \ldots, x^{n_1+\beta_1+1}, 1\right\}.$$

Using the same procedure n_2 times more and after multiplying by x^{β_2} we obtain that

$$V^{\beta_1+1+\beta_2+n_2,\beta_2}_{n_1,n_2} = \left\{x^{\beta_2+\beta_1+n_2+1}, x^{\beta_2+1+\beta_1+n_2+1}, \ldots, x^{\beta_2+n_1+\beta_1+n_2+1}, x^{\beta_2}, \ldots, x^{\beta_2+n_2}\right\}$$

is a Chebyshev system. The proof that $V^{\alpha_1,\ldots,\alpha_L}_{n_1,\ldots,n_L}$ is a Chebyshev system follows repeating the previous procedure until β_L, assuming every time that $\beta_1 + \beta_2, \ldots, \beta_1 + \cdots + \beta_L$ are nonintegers numbers, and by doing the change of parameters $\beta_L = \alpha_L$, $\beta_{L-1} + (\beta_L + n_L) + 1 = \alpha_{L-1}, \ldots, (\beta_1) + (\beta_2 + n_2) + \cdots + (\beta_L + n_L) + (L-1) = \alpha_1$. □

Proposition 18. *The sets* $U_1 = \{1\}$, $U_2 = \{1, x\}$ *and*

$$U_d = \bigcup_{l=0}^{[\frac{d-1}{2}]} \left\{(\log x)^l, x(\log x)^l, \ldots, x^{d-1-2l}(\log x)^l\right\}$$

for $d \geqslant 3$ *are Chebyshev systems in any closed interval* A *contained in the positive axis.*

Proof. For $d = 1, 2, 3, 4$ it is well known that the corresponding set U_d is a Chebyshev system in any closed interval contained in the positive axis. Now we will prove the case $d = 5$ using that U_4 is a Chebyshev system. Computing the primitive, see Proposition 15, of every element of the set

$$U_4 = \left\{1, x, x^2, x^3, \log x, x \log x\right\}$$

we obtain also the Chebyshev system

$$\left\{1, x, \frac{1}{2}x^2, \frac{1}{3}x^3, \frac{1}{4}x^4, x \log x - x, \frac{1}{2}x^2 \log x - \frac{1}{4}x^2\right\}.$$

Now adding a specific multiple of some element of the set and multiplying any element for a nonzero constant, all can be different, we obtain also the next Chebyshev system

$$\left\{1, x, x^2, x^3, x^4, x \log x, x^2 \log x\right\}.$$

Dividing by x, doing a primitive and arranging all the elements with the previous procedure we obtain that

$$\left\{1, x, x^2, x^3, x^4, \log x, x \log x, x^2 \log x\right\}.$$

Now repeating the previous argument, a division by x, a primitive and an arranging of the elements, we obtain that

$$U_5 = \left\{1, x, x^2, x^3, x^4, \log x, x \log x, x^2 \log x, (\log x)^2\right\}$$

is also a Chebyshev system.

To prove the statement for any d we can use the previous arguments starting with the set U_{d-1}. Then the proof follows doing one primitive, then $[\frac{d-1}{2}]$ times the procedure of a division by x, a primitive and an arranging of the elements. As it is shown in the previous case $[\frac{d-1}{2}] = 2$ when $d = 5$. □

Proposition 18 could be improved in some sense considering polynomials of different degrees multiplying the function $\log x$, but they cannot be chosen in an arbitrary way like in Propositions 16 or 17. The difficulties to control the number of zeros of this family can be showed with the next examples. The family

$$\left\{1, x, x^2, \log x, x \log x, x^2 \log x, (\log x)^2, x(\log x)^2, x^2(\log x)^2\right\}$$

is a Chebyshev system because the corresponding Wronskian has no zeros but

$$\left\{1, x, x^2, \log x, (\log x)^2, x(\log x)^2, x^2(\log x)^2\right\}$$

is not a Chebyshev system because the function

$$f(x) = -\frac{56939}{2500} + \frac{75109}{2000}x - \frac{36927}{2500}x^2 - \frac{27797}{4000}\log x - \frac{17757}{31250}(\log x)^2 + \frac{23137}{2000}x(\log x)^2 + x^2(\log x)^2 \tag{27}$$

has 7 zeros, one more than it would have in case that the Chebyshev property was satisfied. We can prove that the seven zeros are localized inside the seven intervals

$$\left[\frac{1}{1000}, \frac{1}{100}\right], \left[\frac{1}{100}, \frac{1}{20}\right], \left[\frac{1}{20}, \frac{1}{10}\right], \left[\frac{1}{10}, \frac{1}{2}\right], \left[\frac{1}{2}, 2\right], [2, 6], [6, 11].$$

The size of the coefficients in the previous example can be chosen as an indication of the difficulties that appear in the problem of controlling the number of zeros of such functions. Moreover the high sensibility on the coefficients can be also considered.

Proof of Theorem 3. Given a function $f(\alpha)$, from Lemma 9, Theorem 5(ii) and Proposition 6, the limit cycles of (3) that bifurcate from the periodic orbits of $(3)_{\varepsilon=0}$ for ε small enough are controlled by the zeros of $\widehat{\mathcal{G}}(\alpha)$ of (18).

When the sets $W_1 = \{1\}$, $W_2 = \{1, \alpha\}$ and

$$W_d = \bigcup_{l=0}^{[\frac{d-1}{2}]} \{ (f(\alpha))^l, \alpha (f(\alpha))^l, \ldots, \alpha^{d-1-2l} (f(\alpha))^l \}$$

for $d \geqslant 3$ are Chebyshev systems in any closed interval A the number of zeros of (18) is at most the number of elements of the set W_d minus one, that is

$$D = \left(\left[\frac{d-1}{2} \right] + 1 \right) \left(\left[\frac{d}{2} \right] + 1 \right).$$

Moreover this bound is reached as ensures Proposition 14. Therefore Theorem 3 is proved when $f(z) = \log z$ using Proposition 18 and when $f(z) = e^z$ (respectively $f(z) = z^{p/q}$) we apply Proposition 16 (respectively 17) taking $L = [\frac{d-1}{2}] + 1$, $\alpha_l = l - 1$ (resp. $\alpha_l = p/q + l - 1$) and $n_l = d - 2l + 1$ for $l = 1, \ldots, [\frac{d-1}{2}] + 1$. $\square$

As it can be seen in the previous proof, the statements of Theorem 3 can be extended to any f such that the set W_d satisfies the Chebyshev property. In fact, as it can be showed in this section, this is the main difficulty.

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