



# Regularity and Rothe method error estimates for parabolic hemivariational inequality

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## ABSTRACT

This paper deals with regularity of solutions to the abstract operator version of parabolic partial differential inclusion of the form  $u'(t) + Au(t) + \iota^* \partial J(\iota u(t)) \ni f(t)$  with the multivalued term given in the form of Clarke subdifferential of a locally Lipschitz functional  $J$ . Using the Rothe method, it is shown that under appropriate assumptions on the data, the solution has the increased regularity. Three regularity theorems are given. The first one concerns the regularity of solution in the Besov space  $B_{2\infty}^{1/2}(0, T; H)$ . The second theorem provides assumption under which the solution lies in the space  $H^1(0, T; H) \cap L^\infty(0, T; V) \cap C([0, T]; V_{weak})$ . The last one shows that if the operator  $A$  is strongly monotone and the multivalued term satisfies the relaxed monotonicity assumption then the unique solution also belongs to  $H^1(0, T; V) \cap W^{1,\infty}(0, T; H)$ . In the last case the error estimates on the Rothe method are also proved.

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## 1. Introduction

Partial differential inclusions with the multivalued and nonmonotone term in the form of the Clarke subdifferential of a locally Lipschitz functional are known as hemivariational inequalities (HVIs). They are the natural extension of variational inequalities where the multivalued term is the subdifferential of a convex functional and were firstly considered by Panagiotopoulos in early 1980s. Since then, several monographs on HVIs have appeared (see for instance [30,15,10]). As their most important application one has to mention the contact problems in the theory of (visco)elasticity where the multivalued laws have often the nonsmooth and nonmonotone nature.

In this article we deal with the initial value problem for the parabolic type abstract inclusion of the form

$$u'(t) + Au(t) + \iota^* \partial J(\iota u(t)) \ni f(t) \quad \text{a.e. } t \in (0, T) \text{ with } u(0) = u_0.$$

We look for weak solutions in the standard framework based on the evolution triple of spaces  $V \subset H \subset V^*$ . The operator  $A: V \rightarrow V^*$  is nonlinear and pseudomonotone (for simplicity we assume that it does not depend on time directly), the function  $f$  belongs to  $L^2(0, T; V^*)$  and  $J: U \rightarrow \mathbb{R}$  where  $U$  is a reflexive Banach space associated with the space  $V$  through the linear and compact mapping  $\iota: V \rightarrow U$ . The existence of weak solutions to this problem in the space  $\mathcal{W} = \{u \in L^2(0, T; V), u' \in L^2(0, T; V^*)\}$  was shown in [19] where the Rothe method (known also as the time approximation method) is exploited. The key idea of this method is to use a backward difference scheme to approximate the time derivative and solve the elliptic problem in every time step for consecutive values of the solution in the points of the time mesh. The novelty of the result in [19] relies on the fact that the multivalued term is defined on the space  $U$ , other than the spaces in the evolution triple, which allows to put in the unified framework the problems where the multivalued term comes from the semilinear source term and from the nonlinear and multivalued Neumann–Robin boundary condition.

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It must be remarked that the Rothe method appears to be the tool which can be used to effectively deal with time dependent HVIs in a general setup. Recently, in [33], the Rothe approach was used to obtain the existence result for another generalization of parabolic type HVI, which consists in the maximal monotone nonlinearity in the term with the first time derivative.

The second advantage of the Rothe method is that it gives an efficient numerical tool for the solution approximation without the need of additional regularizing terms in the original inclusion, as it was needed in [25,26,20,21,23] and [24]. The other approaches used to show the existence of weak solutions for the considered problem rely either on the surjectivity theorems which are not constructive (see [22,27,28]) or on the notion of upper and lower solutions (see [6–8,11]).

The results on the convergence of numerical methods for contact problems often require the increased solution regularity (cf. for example [2]). However, although there are many papers on the existence of solutions to HVIs, only few results are known on their increased regularity. Note that the time regularity of the solution cannot be shown to arbitrarily high. Indeed, consider the simple example  $u' + \partial(|u|) \ni 0$  with  $u(0) = -1$ , for which the solution is given by  $u(t) = \min\{0, t - 1\}$ . Putting  $V = H = U = \mathbb{R}$  it can be easily shown that all the assumptions formulated below hold and  $u \notin C^1(0, T; H)$ . For the elliptic case it is shown in Section 6.2 of [15], that the increased space regularity of source term (namely  $f \in L^2(\Omega)$ ) implies that the solution of elliptic variational–hemivariational inequality with the additional multivalued semilinear source term (see formula (6.1.18) in [15]) also has the increased regularity (namely  $u \in W^{2,p}(\Omega)$ ). As for time dependent problems all aforementioned papers show the existence of solution in the space  $\mathcal{W}$ . According to the author's knowledge there is no result on the increased time regularity of solutions for parabolic HVIs. In [32] a theorem analogous to Theorem 3 of the present paper, for the simplified case of linear operator  $A$  and multivalued source term, is shown using the geometric assumptions on the problem domain (see (1.6) in [32]) which are not needed in the present article.

The first important result of this paper is Theorem 2 which shows that if, in addition to the assumptions needed for existence of solutions, the operator  $A$  is of type  $(S_+)$  then the solution belongs to the Besov space  $B_{2\infty}^{1/2}(0, T; H)$ . This result relies on the ideas of Baiocchi [3] developed later by Savaré in [37] and [38] who prove the mentioned regularity for the case of parabolic variational inequalities.

Next two regularity results delivered in this paper are Theorems 3 and 4. In Theorem 3 it is shown that under increased regularity of the initial condition and the source term ( $u_0 \in V$  and  $f \in L^2(0, T; H)$  in contrast to  $u_0 \in H$  and  $f \in L^2(0, T; V^*)$  needed for existence) and if the operator  $A$  is decomposed into a sum of a gradient of convex functional and a lower order perturbation term (see  $H(A)_2$  in the sequel) and moreover the mapping  $\iota$  can be represented as a composition of the embedding operator  $i: V \rightarrow H$  with a mapping  $p: H \rightarrow U$  (this assumption is denoted  $H_{\text{aux}}(i)$  in the sequel) then the solution belongs to  $H^1(0, T; H) \cap L^\infty(0, T; V) \cap C([0, T]; V_{\text{weak}})$ . Theorem 4 shows that if the source term  $f \in H^1(0, T; V^*)$  and the initial condition  $u_0 \in V$  with the compatibility condition  $Au_0 - \iota^* \xi_0 - f(0) \in H$ , where  $\xi_0 \in \partial J(\iota u_0)$  and under the strong monotonicity assumption on  $A$  (see  $H(A)_3$  in the sequel) as well as the relaxed monotonicity assumption on  $\partial J$  (see  $H(J)_1$  in the sequel), and the additional assumption  $H_{\text{const}}$  (which is an alternative between  $H_{\text{aux}}(i)$  and some inequality involving the problem constants) then we have  $u \in H^1(0, T; V) \cap W^{1,\infty}(0, T; H)$ .

Analogous results for parabolic equations as well as variational inequalities are shown using the similar technique in the monograph [35]. Regularity results for parabolic equations can be also found in [18]. Another important paper on the time approximation technique for parabolic variational inequalities is [12].

Despite the fact that there are several results on existence of solutions to parabolic HVIs, there are only few results known to the author, on the convergence of approximate solution obtained by numerical methods for the considered problem. The convergence of fully discrete schemes for the case of linear operator  $A$  and the multivalued source term was proved in [17]. The convergence of finite difference schemes was proved in [16]. There are no results, known to the author, on the error estimate of numerical solutions to parabolic HVIs. This gap is filled in by Theorem 5 where under the same assumptions as in Theorem 4, it is established that the error between the semidiscrete solution obtained by the Rothe method and the solution of the original problem is of order  $O(\sqrt{\tau})$ , in the norm  $C([0, T]; H)$  for the piecewise linear interpolant and in the norm  $L^2(0, T; V)$  for the piecewise constant interpolant, where  $\tau$  denotes the length of the time step. The results on error estimates for parabolic equations can be found in [34,31] and [39] and for variational inequalities in [29,36] and [4].

The structure of the paper is the following. In Section 2 the preliminary definitions are recalled. Section 3 is devoted to a summary of existence results from [19]. Section 4 contains the auxiliary result on the  $(S_+)$  condition of the Nemytskii operator. This result is needed in Section 5 where the regularity of the solution in the Besov space is shown. Further results on the solution regularity are contained in Section 6. Finally, the error estimates are proved in Section 7.

## 2. Preliminaries

For a locally Lipschitz functional  $j: X \rightarrow \mathbb{R}$ , where  $X$  is a Banach space, generalized directional derivative in the sense of Clarke at  $x \in X$  in the direction  $z \in X$  is defined as

$$j^0(x; z) = \limsup_{y \rightarrow x, \lambda \rightarrow 0^+} \frac{f(y + \lambda z) - f(y)}{\lambda}.$$

The generalized gradient of  $j$  in the sense of Clarke is the multifunction  $\partial j: X \rightarrow 2^{X^*}$  defined by

$$\partial j(x) = \{\xi \in X^*: j^0(x; y) \geq \langle \xi, y \rangle, \text{ for all } y \in X\}.$$

For more details on those definitions and the properties of generalized directional derivative and generalized gradient see [13].

Recall that a multifunction  $A : X \rightarrow 2^{X^*}$ , where  $X$  is a real and reflexive Banach space is pseudomonotone if

- (i)  $A$  has values which are nonempty, weakly compact and convex,
- (ii)  $A$  is upper semicontinuous from every finite dimensional subspace of  $X$  into  $X^*$  furnished with weak topology,
- (iii) if  $v_n \rightarrow v$  weakly in  $X$  and  $v_n^* \in A(v_n)$  is such that  $\limsup_{n \rightarrow \infty} \langle v_n^*, v_n - v \rangle \leq 0$  then for every  $y \in X$  there exists  $u(y) \in A(v)$  such that  $\langle u(y), v - y \rangle \leq \liminf_{n \rightarrow \infty} \langle v_n^*, v_n - y \rangle$ .

Note that sometimes it is useful to check the pseudomonotonicity of an operator via the following sufficient condition (see Proposition 1.3.66 in [14] or Proposition 3.1 in [9]).

**Proposition 1.** *Let  $X$  be a real reflexive Banach space, and assume that  $A : X \rightarrow 2^{X^*}$  satisfies the following conditions:*

- (i) *For each  $v \in X$  we have that  $A(v)$  is a nonempty, closed and convex subset of  $X^*$ .*
- (ii)  *$A$  is bounded.*
- (iii) *If  $v_n \rightarrow v$  weakly in  $X$  and  $v_n^* \rightarrow v^*$  weakly in  $X^*$  with  $v_n^* \in A(v_n)$  and if  $\limsup_{n \rightarrow \infty} \langle v_n^*, v_n - v \rangle \leq 0$ , then  $v^* \in A(v)$  and  $\langle v_n^*, v_n \rangle \rightarrow \langle v^*, v \rangle$ .*

Then the operator  $A$  is pseudomonotone.

We also recall (see for instance Proposition 1.3.68 [14]) that the sum of two pseudomonotone multifunctions is pseudomonotone.

For a Banach space  $X$  and finite time interval  $I = (0, T)$  we consider the standard spaces  $L^p(I; X)$ . Furthermore we denote by  $BV(I; X)$  the space of functions of bounded total variation on  $I$ . Let  $\pi$  denote any finite partition of  $I$  by a family of disjoint subintervals  $\{\sigma_i = (a_i, b_i)\}$  such that  $\bar{I} = \bigcup_{i=1}^n \bar{\sigma}_i$ . Let  $\mathcal{F}$  denote the family of all such partitions. Then we define the total variation as

$$\|x\|_{BV(I; X)} = \sup_{\pi \in \mathcal{F}} \left\{ \sum_{\sigma \in \pi} \|x(b_i) - x(a_i)\|_X \right\}.$$

As a generalization of above definition for  $1 \leq q < \infty$  we can define a seminorm

$$\|x\|_{BV^q(I; X)}^q = \sup_{\pi \in \mathcal{F}} \left\{ \sum_{\sigma \in \pi} \|x(b_i) - x(a_i)\|_X^q \right\}.$$

The functions with bounded  $BV^q$  seminorm are known as the functions of bounded  $q$ -variation. Note that since the  $BV$  and  $BV^q$  seminorms are sensitive to modification in single points, sometimes one defines them using the essential  $q$ -variation, i.e. as infima over the equivalence classes with respect to the relation of  $\mathcal{L}$ -a.e. equality. In the sequel we will use these norms for piecewise constant functions or continuous functions when the two definitions are equivalent. For Banach spaces  $X$  and  $Z$  such that  $X \subset Z$  we introduce a new vector space

$$M^{p,q}(I; X, Z) = L^p(I; X) \cap BV^q(I; Z).$$

Then  $M^{p,q}(I; X, Z)$  is also a Banach space.

Recall that for a Banach space  $X$  the Besov space  $B_{2\infty}^{1/2}(0, T; X)$  is defined as a set of functions for which the following seminorm is finite:

$$\|u\|_{B_{2\infty}^{1/2}(0, T; X)}^2 = \sup_{h \in (0, T)} \frac{1}{h} \int_0^{T-h} \|u(t+h) - u(t)\|_X^2 dt < \infty.$$

The other possible definition of Besov space is via the interpolation theory (see [38,41]). Note that for small  $\varepsilon > 0$  we have  $B_{2\infty}^{1/2}(0, T; X) \subset H^{1/2-\varepsilon}(0, T; X)$  (see [37,38]).

### 3. Problem formulation, existence and convergence results

In this section we formulate the parabolic operator inclusion, present the Rothe method used for the problem solution and recall the existence and convergence results of [19].

Let  $V \subset H \subset V^*$  be a classical evolution triple, where  $V$  is a reflexive and separable Banach space and  $H$  is a separable Hilbert space with the embeddings being continuous, dense and compact. The embedding between  $V$  and  $H$  will be denoted by  $i$ . Furthermore let  $U$  be a reflexive Banach space on which the multivalued term will be defined. We use the notation  $\mathcal{V} = L^2(0, T; V)$ ,  $\mathcal{H} = L^2(0, T; H)$ ,  $\mathcal{U} = L^2(0, T; U)$  and  $\mathcal{W} = \{u \in \mathcal{V}, u' \in \mathcal{V}^*\}$ , where the derivative is understood in the sense

of distributions. Duality pairings and norms for all spaces will be denoted by the appropriate subscripts and for the space  $V$  no subscript will be used. The scalar product in  $H$  will be denoted by  $(\cdot, \cdot)$  and norm in  $\mathbb{R}^n$  by  $|\cdot|$ . We consider the operator  $A: V \rightarrow V^*$  and the functional  $J: U \rightarrow \mathbb{R}$  such that the following assumptions hold

- $H(A)$ : (i)  $A$  is pseudomonotone,  
(ii)  $A$  satisfies the growth condition  $\|A(v)\|_{V^*} \leq a + b\|v\|$  for every  $v \in V$  with  $a \geq 0$ ,  $b > 0$ ,  
(iii)  $A$  is coercive  $\langle A(v), v \rangle \geq \alpha\|v\|^2 - \beta\|v\|_H^2$  for every  $v \in V$  with  $\alpha > 0$ ,  $\beta \geq 0$ .  
 $H(J)$ : (i)  $J$  is locally Lipschitz,  
(ii)  $\partial J$  satisfies the growth condition  $\|\xi\|_{U^*} \leq c(1 + \|u\|_U)$  for every  $u \in U$  and  $\xi \in \partial J(u)$  with  $c > 0$ .

Moreover we assume that

$$H_0: f \in \mathcal{V}^* \text{ and } u_0 \in H.$$

We also impose the assumption concerning the space  $U$

$H(U)$ : There exists the linear, continuous and compact mapping  $\iota: V \rightarrow U$  such that the associated Nemytskii mapping  $\bar{\iota}: M^{2,2}(0, T; V, V^*) \rightarrow \mathcal{U}$  defined by  $(\bar{\iota}v)(t) = \iota(v(t))$  is also compact.

Finally we impose the last assumption

- $H_{aux}$ : One of the following three assertions holds  
(i) There exists the linear and continuous mapping  $p: H \rightarrow U$  such that for  $v \in V$  we have  $p(i(v)) = \iota(v)$ .  
(ii) The constants  $\alpha$  and  $c$  satisfy the inequality  $\alpha > c\|\iota\|_{\mathcal{L}(V, U)}^2$ .  
(iii) For every  $u \in U$  we have  $J^0(u; -u) \leq d(1 + \|u\|_U^\sigma)$  with  $d > 0$  and  $1 \leq \sigma < 2$ .

The discussion of the assumptions can be found in [19]. In particular it is shown that  $H(U)$  is satisfied in two typical settings for hemivariational inequalities: case of multivalued nonmonotone source term and multivalued Neumann–Robin boundary conditions.

The problem under consideration is stated as follows:

$$\begin{aligned} &\text{find } u \in \mathcal{W} \text{ such that } u(0) = u_0 \text{ and for a.e. } t \in (0, T) \text{ we have} \\ &u'(t) + Au(t) + \iota^* \partial J(\iota u(t)) \ni f(t). \end{aligned} \quad (1)$$

The last inclusion is understood in the following sense

$$\text{there exists } \eta \in \mathcal{V}^* \text{ such that } u'(t) + Au(t) + \eta(t) = f(t) \text{ for a.e. } t \in (0, T) \text{ and} \quad (2)$$

$$\langle \eta(t), v \rangle \in \langle \partial J(\iota u(t)), \iota v \rangle_{U^* \times U} \text{ for a.e. } t \in (0, T) \text{ and } v \in V. \quad (3)$$

We define the sequence of time steps  $\tau_n \searrow 0$  such that the value  $T/\tau_n$  is an integer, which we denote by  $N_n$ . For simplicity of notation the subscript  $n$  is omitted in the sequel.

We need to define the appropriate approximation of the source term and the initial condition. The piecewise constant approximation of the function  $f$  is given by

$$V^* \ni \bar{f}_\tau(t) := f_\tau^k = \frac{1}{\tau} \int_{(k-1)\tau}^{k\tau} f(t) dt \quad \text{for } t \in ((k-1)\tau, k\tau], k \in \{1, \dots, N\}. \quad (4)$$

We have  $\bar{f}_\tau \rightarrow f$  in  $\mathcal{V}^*$  when  $\tau \rightarrow 0$ . The initial condition must be approximated by elements of  $V$ . Let  $\{u_{0\tau}\} \subset V$  be a sequence, such that  $u_{0\tau} \rightarrow u_0$  strongly in  $H$ , and  $\|u_{0\tau}\| \leq C/\sqrt{\tau}$  for some constant  $C > 0$ .

The Rothe problem for the problem (1) is defined as follows

$$\begin{aligned} &\text{find the sequence } u_\tau^k \in V, k = 0, \dots, N \text{ such that } u_\tau^0 = u_{0\tau} \text{ and for } k = 1, \dots, N \text{ we have} \\ &\left( \frac{u_\tau^k - u_\tau^{k-1}}{\tau}, v \right) + \langle Au_\tau^k, v \rangle + \langle \partial J(\iota u_\tau^k), \iota v \rangle_{U^* \times U} \ni \langle \bar{f}_\tau^k, v \rangle \text{ for } v \in V. \end{aligned} \quad (5)$$

The above formula is known as the implicit or backward Euler scheme. We recall the following result (see Lemma 3 in [19]).

**Lemma 1.** Under assumptions  $H(A)$ ,  $H(J)$ ,  $H_0$ ,  $H(U)$  and  $H_{aux}$ , there exists  $\tau_0 > 0$  such that for all  $\tau \in (0, \tau_0)$ , the problem (5) has a solution  $u_\tau^k \in V, k \in \{1, \dots, N\}$  which satisfy

$$\max_{k=1, \dots, N} \|u_\tau^k\|_H \leq \text{const}, \quad (6)$$

$$\sum_{k=1}^N \|u_\tau^k - u_\tau^{k-1}\|_H^2 \leq \text{const}, \quad (7)$$

$$\tau \sum_{k=1}^N \|u_\tau^k\|^2 \leq \text{const}, \quad (8)$$

with the constants independent of  $\tau$ .

Note that there is no guarantee of uniqueness of the solution to the Rothe problem. Given  $u_\tau^{k-1} \in V$  there may exist more than one  $u_\tau^k$  satisfying (5).

The piecewise linear and piecewise constant interpolants  $u_\tau \in C([0, T]; V)$  and  $\bar{u}_\tau \in L^\infty(0, T; V)$  are defined by the formulae

$$u_\tau(t) = \left(\frac{t}{\tau} - k + 1\right)u_\tau^k + \left(k - \frac{t}{\tau}\right)u_\tau^{k-1} \quad \text{for } t \in [(k-1)\tau, k\tau] \text{ and } k = 1, \dots, T/\tau, \quad (9)$$

$$\bar{u}_\tau(t) = u_\tau^k \quad \text{for } t \in ((k-1)\tau, k\tau] \text{ and } k = 1, \dots, T/\tau. \quad (10)$$

Observe, that  $u_\tau$  has a distributional derivative  $u'_\tau \in L^\infty(0, T; V)$  given by  $u'_\tau(t) = \frac{u_\tau^k - u_\tau^{k-1}}{\tau}$  for almost every  $t \in ((k-1)\tau, k\tau)$ . So, since  $u_\tau^k$  solve the Rothe problem, we have

$$\langle u'_\tau(t), v \rangle + \langle A\bar{u}_\tau(t), v \rangle + \langle \xi_\tau(t), \iota v \rangle_{U^* \times U} = \langle \bar{f}_\tau(t), v \rangle \quad \text{for } v \in V, \text{ a.e. } t \in (0, T) \quad (11)$$

with  $u_\tau(0) = u_{0\tau}$  and  $\xi_\tau(t) = \xi_\tau^k \in \partial J(u_\tau^k) = \partial J(\bar{u}_\tau(t))$  for  $t \in ((k-1)\tau, k\tau]$ .

We quote the following results from [19].

**Lemma 2.** Under assumptions  $H(A)$ ,  $H(J)$ ,  $H_0$ ,  $H(U)$  and  $H_{\text{aux}}$ , there exists  $\tau_0 > 0$  such that for all  $\tau \in (0, \tau_0)$ , the functions  $\bar{u}_\tau$  and  $u_\tau$  satisfy

$$\|\bar{u}_\tau\|_{\mathcal{V}} \leq \text{const}, \quad (12)$$

$$\|\bar{u}_\tau\|_{B^2(0, T; V^*)} \leq \text{const}, \quad (13)$$

$$\|\bar{u}_\tau\|_{L^\infty(0, T; H)} \leq \text{const}, \quad (14)$$

$$\|u_\tau\|_{C([0, T]; H)} \leq \text{const}, \quad (15)$$

$$\|u_\tau\|_{\mathcal{V}} \leq \text{const}, \quad (16)$$

$$\|u'_\tau\|_{\mathcal{V}^*} \leq \text{const} \quad (17)$$

with the constants independent of  $\tau$ .

**Theorem 1.** Under assumptions  $H(A)$ ,  $H(J)$ ,  $H_0$ ,  $H(U)$  and  $H_{\text{aux}}$ , the problem (1) has a solution  $u$ . Furthermore if  $u_\tau$  and  $\bar{u}_\tau$  are piecewise constant and piecewise linear interpolants given by (9) and (10), respectively, then, for a subsequence, we have

$$u_\tau \rightarrow u \quad \text{weakly in } \mathcal{W} \text{ and weakly}^* \text{ in } L^\infty(0, T; H), \quad (18)$$

$$\bar{u}_\tau \rightarrow u \quad \text{weakly in } \mathcal{V} \text{ and weakly}^* \text{ in } L^\infty(0, T; H). \quad (19)$$

Moreover, any cluster point of  $u_\tau$  and  $\bar{u}_\tau$  in the sense of (18) and (19) solves the problem (1).

In other words if we denote by  $S_\tau$  the (nonempty) set of all  $u_\tau$  piecewise linear interpolants built on solutions of the Rothe problem and by  $\bar{S}_\tau$  the (nonempty) set of all  $\bar{u}_\tau$  respective piecewise constant interpolants, then it follows that  $K\text{-}\limsup_{\tau \rightarrow 0} S_\tau \subset S$  and  $K\text{-}\limsup_{\tau \rightarrow 0} \bar{S}_\tau \subset S$ , where  $S$  is the (nonempty) set of solutions to (1) and limits are understood in topologies as in (18) and (19), respectively, and taken in the sense of Kuratowski, cf. Definition 1.1.1 in [1].

#### 4. The condition $(S_+)$ for Nemytskii operator

We begin with the following definition (see [5], [42, Definition 27.1]).

**Definition 1.** The operator  $A: V \rightarrow V^*$  is of type  $(S_+)$  if for any sequence  $u_n \rightarrow u$  weakly in  $V$  with  $\limsup_{n \rightarrow \infty} \langle Au_n, u_n - u \rangle \leq 0$  we have  $u_n \rightarrow u$  strongly in  $V$ .

If  $A : V \rightarrow V^*$ , where  $V$  is a Banach space is an operator, then we will consider the Nemytskii operator  $\mathcal{A} : L^2(0, T; V) \rightarrow L^2(0, T; V^*)$  defined as  $(\mathcal{A}u)(t) = A(u(t))$ . We prove the lemma on  $(S_+)$  condition on Nemytskii operator with respect to the space  $M^{2,2}(0, T; V, V^*)$ . Note that Lemma 3 uses the same argument as Theorem 2(b) in [5], however the thesis here is more general since instead of the space  $\mathcal{W}$  more general space  $M^{2,2}(0, T; V, V^*)$  is considered here. Note furthermore that Lemma 3 is closely related with Lemma 1 in [19] where the pseudomonotonicity of Nemytskii operator with respect to the space  $M^{2,2}(0, T; V, V^*)$  is established.

**Lemma 3.** *Let  $A : V \rightarrow V^*$  satisfy  $H(A)$  and be of type  $(S_+)$ . Then if, for a uniformly bounded sequence  $\{u_n\} \subset M^{2,2}(0, T; V, V^*)$ , we have  $u_n \rightarrow u$  weakly in  $\mathcal{V}$  and  $\limsup_{n \rightarrow \infty} \langle \mathcal{A}u_n, u_n - u \rangle_{V^* \times V} \leq 0$  then  $u_n \rightarrow u$  strongly in  $\mathcal{V}$ .*

**Proof.** It is enough to obtain the thesis for a subsequence, which we denote by subscript  $n$  in the course of the proof. From the proof of Lemma 1 in [19] it follows that for a subsequence we have  $u_n \rightarrow u$  strongly in  $\mathcal{H}$ ,  $u_n(t) \rightarrow u(t)$  strongly in  $H$  and  $u_n(t) \rightarrow u(t)$  weakly in  $V$  (apart from some set of measure zero in  $(0, T)$ ). Furthermore, we have  $\lim_{n \rightarrow \infty} \langle \mathcal{A}u_n(t), u_n(t) - u(t) \rangle \rightarrow 0$  for a.e.  $t \in (0, T)$ . Thus, by the fact that  $A$  is of type  $(S_+)$ , we obtain that  $u_n(t) \rightarrow u(t)$  strongly in  $V$  for a.e.  $t \in (0, T)$ . Similarly as in the proof of Lemma 1 in [19], we have

$$\|u_n(t)\|^2 \leq \frac{1}{\alpha} (\langle \mathcal{A}u_n(t), u_n(t) - u(t) \rangle + a\|u(t)\| + b\|u_n(t)\|\|u(t)\| + \beta\|u_n(t)\|_H^2).$$

Using the elementary inequality  $2ab \leq a^2 + b^2$  for  $a, b \in \mathbb{R}$ , we get

$$\|u_n(t)\|^2 \leq \frac{2}{\alpha} \langle \mathcal{A}u_n(t), u_n(t) - u(t) \rangle + \frac{2a}{\alpha} \|u(t)\| + \frac{b^2\|u(t)\|^2}{\alpha^2} + \frac{2\beta\|u_n(t)\|_H^2}{\alpha}.$$

Hence

$$-\frac{4a}{\alpha} \|u(t)\| - \left( \frac{2b^2}{\alpha^2} + 2 \right) \|u(t)\|^2 \leq \frac{4}{\alpha} \langle \mathcal{A}u_n(t), u_n(t) - u(t) \rangle - \|u(t) - u_n(t)\|^2 + \frac{4\beta\|u_n(t)\|_H^2}{\alpha},$$

where the left-hand side of this inequality is an integrable function. Therefore, we can invoke the Fatou lemma to obtain

$$0 \leq \liminf_{n \rightarrow \infty} \int_0^T \frac{4}{\alpha} \langle \mathcal{A}u_n(t), u_n(t) - u(t) \rangle - \|u(t) - u_n(t)\|^2 dt. \quad (20)$$

From the proof of Lemma 1 in [19] we have  $\int_0^T \langle \mathcal{A}u_n(t), u_n(t) - u(t) \rangle dt \rightarrow 0$ . Hence and from (20), we get

$$\limsup_{n \rightarrow \infty} \|u_n - u\|_{\mathcal{V}}^2 \leq 0,$$

and the thesis follows.  $\square$

## 5. Regularity in the Besov space

Throughout this section we need the following assumption

$H(A)_1$ : the operator  $A$  is of type  $(S_+)$  and satisfies  $H(A)$ .

We will prove the following result on increased regularity of the solution to problem (1).

**Theorem 2.** *Under assumptions  $H(A)_1$ ,  $H(J)$ ,  $H_0$ ,  $H(U)$  and  $H_{\text{aux}}$ , the solution of the problem (1) belongs to the space  $B_{2\infty}^{1/2}(0, T; H)$ .*

The proof of Theorem 2 is based on three lemmas. The first one, being the main ingredient of the proof, is due to Savaré, cf. Lemma 4.13 in [38] (see also [37]).

**Lemma 4.** *Let  $v_\tau$  be a sequence which converges to  $v$  in  $\mathcal{H}$  as  $\tau \rightarrow 0^+$ . If  $v_\tau$  is bounded in  $B_{2\infty}^{1/2}(0, T; H)$  and*

$$\lim_{h \rightarrow 0^+, \tau \rightarrow 0^+} \frac{1}{h} \int_0^{T-h} \|v_\tau(t+h) - v_\tau(t)\|_H^2 dt = 0, \quad (21)$$

*then  $v_\tau \rightarrow v$  in  $B_{2\infty}^{1/2}(0, T; H)$  and, in particular,  $v \in B_{2\infty}^{1/2}(0, T; H)$ .*

The following two lemmas will be used in the proof of Theorem 2.

**Lemma 5.** Let  $X$  be a Banach space. If  $v_n \rightarrow v$  in  $L^2(0, T; X)$ , then

$$\lim_{n \rightarrow \infty, h \rightarrow 0^+} \sup_{s \in [0, h]} \int_0^{T-s} \|v_n(t+s) - v_n(t)\|_X^2 dt = 0.$$

**Proof.** Observe that

$$\sup_{s \in [0, h]} \int_0^{T-s} \|v_n(t+s) - v_n(t)\|_X^2 dt \leq 6\|v_n - v\|_{L^2(0, T; X)}^2 + 3 \sup_{s \in [0, h]} \int_0^{T-s} \|v(t+s) - v(t)\|_X^2 dt.$$

It suffices to prove that the last term tends to zero as  $h \rightarrow 0^+$ . Let  $\varepsilon > 0$ . We can find a simple function  $\bar{v}$  such that  $\|\bar{v} - v\|_{L^2(0, T; X)}^2 \leq \frac{\varepsilon}{9}$ . Assume that the simple function  $\bar{v}$  takes the values  $\bar{v}^k$  on the interval  $(t^{k-1}, t^k)$  for  $k = 1, \dots, K$  with  $t^0 = 0$  and  $t^K = T$ . We denote  $M = \sum_{k=1}^{K-1} \|\bar{v}^{k+1} - \bar{v}^k\|_X^2$  and we take

$$h \leq \min \left\{ \frac{\varepsilon}{9M}, \min_{k=1, \dots, K} (t^k - t^{k-1}) \right\}.$$

For  $s \in [0, h]$  we have

$$\int_0^{T-s} \|\bar{v}(t+s) - \bar{v}(t)\|_X^2 dt \leq \sum_{k=1}^{K-1} s \|\bar{v}^{k+1} - \bar{v}^k\|_X^2 \leq hM \leq \frac{\varepsilon}{9}.$$

Thus

$$\sup_{s \in [0, h]} \int_0^{T-s} \|v(t+s) - v(t)\|_X^2 dt \leq 6\|v - \bar{v}\|_{L^2(0, T; X)}^2 + 3 \int_0^{T-s} \|\bar{v}(t+s) - \bar{v}(t)\|_X^2 dt \leq \varepsilon.$$

The proof of the lemma is complete.  $\square$

**Lemma 6.** Under assumptions  $H(A)$ ,  $H(J)$ ,  $H_0$ ,  $H(U)$  and  $H_{\text{aux}}$ , if  $K, p$  are natural numbers such that  $1 \leq K < K + p \leq N$ , then there exists a constant  $C$  independent on  $K, p$  and  $\tau_0 > 0$  such that for every  $\tau \in (0, \tau_0)$

$$\sum_{k=1}^K \|u_\tau^{k+p} - u_\tau^k\|_H^2 \leq Cp \sup_{s \in [0, p\tau]} \|\bar{u}_\tau(\cdot + s) - \bar{u}_\tau(\cdot)\|_{L^2(0, T-s; V)}.$$

**Proof.** Let us choose  $\tau > 0$  and consider  $u_\tau^k$  for  $k = 0, \dots, N$  the solution of (5). Choose natural numbers  $k$  and  $p$  such that  $0 \leq k < k + p \leq N$ . Using the elementary equality  $2(y, y - x) = \|y\|_H^2 - \|x\|_H^2 + \|x - y\|_H^2$ , we have

$$\begin{aligned} & 2 \sum_{j=0}^{p-1} (u_\tau^{k+j+1} - u_\tau^{k+j}, u_\tau^{k+j+1} - u_\tau^k) \\ &= 2 \sum_{j=0}^{p-1} (u_\tau^{k+j+1} - u_\tau^{k+j}, u_\tau^{k+j+1}) - 2 \sum_{j=0}^{p-1} (u_\tau^{k+j+1} - u_\tau^{k+j}, u_\tau^k) \\ &= \sum_{j=0}^{p-1} (\|u_\tau^{k+j+1}\|_H^2 - \|u_\tau^{k+j}\|_H^2 + \|u_\tau^{k+j+1} - u_\tau^{k+j}\|_H^2) - 2(u_\tau^{k+p} - u_\tau^k, u_\tau^k) \\ &= \sum_{j=0}^{p-1} \|u_\tau^{k+j+1} - u_\tau^{k+j}\|_H^2 + \|u_\tau^{k+p} - u_\tau^k\|_H^2 \geq \|u_\tau^{k+p} - u_\tau^k\|_H^2. \end{aligned}$$

Now let us take  $K$  such that  $K + p \leq N$ . We have

$$\sum_{k=1}^K \|u_\tau^{k+p} - u_\tau^k\|_H^2 \leq 2\tau \sum_{k=1}^K \sum_{j=1}^p \left( \frac{u_\tau^{k+j} - u_\tau^{k+j-1}}{\tau}, u_\tau^{k+j} - u_\tau^k \right) = 2\tau \sum_{j=1}^p \sum_{k=1}^K \langle f_\tau^{k+j} - Au_\tau^{k+j} - \iota^* \xi_\tau^{k+j}, u_\tau^{k+j} - u_\tau^k \rangle,$$

where in the last expression  $\xi_\tau^{k+j} \in \partial J(u_\tau^{k+j})$ . Further calculations lead to the following estimate

$$\begin{aligned} \sum_{k=1}^K \|u_\tau^{k+p} - u_\tau^k\|_H^2 &\leq 2 \sum_{j=1}^p \int_0^{K\tau} \langle \bar{f}_\tau(t+j\tau) - A\bar{u}_\tau(t+j\tau) - \iota^* \xi_\tau(t+j\tau), \bar{u}_\tau(t+j\tau) - \bar{u}_\tau(t) \rangle \\ &\leq 2 \sum_{j=1}^p (\|\bar{f}_\tau\|_{V^*} + \|A\bar{u}_\tau\|_{V^*} + \|\iota\|_{\mathcal{L}(V;U)} \|\xi_\tau\|_U) \|\bar{u}_\tau(\cdot+j\tau) - \bar{u}_\tau(\cdot)\|_{L^2(0,K\tau;V)} \\ &\leq Cp \sup_{s \in [0,p\tau]} \|\bar{u}_\tau(\cdot+s) - \bar{u}_\tau(\cdot)\|_{L^2(0,T-s;V)} \end{aligned}$$

with a constant  $C > 0$ . In order to obtain the thesis of the lemma we used growth conditions  $H(A)(ii)$  and  $H(j)(ii)$  as well as the estimate (12).  $\square$

**Proof of Theorem 2.** We know that, for a subsequence,  $u_\tau \rightarrow u$  in  $\mathcal{H}$ . We prove that also  $\bar{u}_\tau \rightarrow u$  in  $\mathcal{H}$ . Indeed  $\|\bar{u}_\tau - u\|_{\mathcal{H}} \leq \|u_\tau - \bar{u}_\tau\|_{\mathcal{H}} + \|u_\tau - u\|_{\mathcal{H}}$ . We need to estimate the first component of the sum

$$\|u_\tau - \bar{u}_\tau\|_{\mathcal{H}} = \sum_{k=1}^N \int_{(k-1)\tau}^{k\tau} (k\tau - t)^2 \left\| \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \right\|_H^2 dt = \frac{\tau}{3} \sum_{k=1}^N \|u_\tau^k - u_\tau^{k-1}\|_H^2 \leq \text{const} \cdot \tau \rightarrow 0 \quad \text{as } \tau \rightarrow 0.$$

Thus, by Lemma 4, in order to prove the theorem we need to show that  $\|\bar{u}_\tau\|_{B_{2\infty}^{1/2}(0,T;H)}$  is bounded and that (21) holds for  $\bar{u}_\tau$ . To this end, let  $\tau > 0$ ,  $h > 0$  and let  $p = \lfloor \frac{h}{\tau} \rfloor$ . We compute the quantity

$$\int_0^{T-h} \|\bar{u}_\tau(t+h) - \bar{u}_\tau(t)\|_H^2 dt = \sum_{k=1}^{N-p} ((p+1)\tau - h) \|u_\tau^{k+p} - u_\tau^k\|_H^2 + \sum_{k=1}^{N-p-1} (h - p\tau) \|u_\tau^{k+p+1} - u_\tau^k\|_H^2.$$

Invoking Lemma 6 we get

$$\begin{aligned} &\int_0^{T-h} \|\bar{u}_\tau(t+h) - \bar{u}_\tau(t)\|_H^2 dt \\ &\leq ((p+1)\tau - h) Cp \sup_{s \in [0,p\tau]} \|\bar{u}_\tau(\cdot+s) - \bar{u}_\tau(\cdot)\|_{L^2(0,T-s;V)} \\ &\quad + (h - p\tau) C(p+1) \sup_{s \in [0,(p+1)\tau]} \|\bar{u}_\tau(\cdot+s) - \bar{u}_\tau(\cdot)\|_{L^2(0,T-s;V)} \\ &\leq Ch \sup_{s \in [0,(p+1)\tau]} \|\bar{u}_\tau(\cdot+s) - \bar{u}_\tau(\cdot)\|_{L^2(0,T-s;V)}, \end{aligned}$$

with a constant  $C > 0$  independent of  $h, \tau$ . We now estimate the Besov seminorm of  $\bar{u}_\tau$ , as follows

$$\|\bar{u}_\tau\|_{B_{2\infty}^{1/2}(0,T;H)} \leq C \sup_{h \in (0,T)} \sup_{s \in [0,(p+1)\tau]} \|\bar{u}_\tau(\cdot+s) - \bar{u}_\tau(\cdot)\|_{L^2(0,T-s;V)} \leq 2C \|\bar{u}_\tau\|_{\mathcal{V}} \leq \text{const}.$$

In order to finish the proof of theorem, it suffices to show (21). We have

$$\lim_{h \rightarrow 0^+, \tau \rightarrow 0^+} \frac{1}{h} \int_0^{T-h} \|v_\tau(t+h) - v_\tau(t)\|_H^2 dt \leq C \lim_{h \rightarrow 0^+, \tau \rightarrow 0^+} \sup_{s \in [0,(p+1)\tau]} \|\bar{u}_\tau(\cdot+s) - \bar{u}_\tau(\cdot)\|_{L^2(0,T-s;V)}. \quad (22)$$

We observe that  $0 < (p+1)\tau \leq h + \tau \rightarrow 0$ . Furthermore, by Lemma 3, since the operator  $A$  is of type  $(S_+)$ , we know that the Nemytskii operator  $\mathcal{A}$  is of type  $(S_+)$  with respect to the space  $M^{2,2}(0,T;V,V^*)$ . Moreover by Theorem 4 in [19], we have  $\bar{u}_\tau \rightarrow u$  strongly in  $\mathcal{V}$ . Thus, using Lemma 5, we note that the last limit in (22) is equal to zero, and we have the thesis.  $\square$

## 6. Regularity results for smooth data

In this section we prove some additional estimates on the solutions of the Rothe problem and, in consequence, we establish the increased regularity results for the solution of the problem (1).



### 6.1. First estimate

For the first estimate we need the following assumptions.

$H(A)_2$ :  $H(A)$  holds and the operator  $A$  is represented as  $A = A_1 + A_2$ , where  $A_1 = \partial\psi$  where  $\psi: V \rightarrow \mathbb{R}$  is a convex functional with  $\psi(v) \geq c_0\|v\|^2 - c_1\|v\|_H^2$  for  $v \in V$  with  $c_0 > 0$ , and  $A_2: V \rightarrow H$  is the lower order term which satisfies the growth condition  $\|A_2v\|_H \leq c_2(1 + \|v\|)$  for  $v \in V$  with  $c_2 > 0$ .

$H_1$ :  $f \in \mathcal{H}$  and  $u_0 \in V$ .

Note that in  $H(A)_2$  the subdifferential is understood in the sense of convex analysis.

**Theorem 3.** Under assumptions  $H(A)_2$ ,  $H(J)$ ,  $H_1$ ,  $H(U)$  and  $H_{\text{aux}}(i)$  the problem (1) has a solution which belongs to the space  $L^\infty(0, T; V) \cap H^1(0, T; H) \cap C([0, T]; V_{\text{weak}})$ .

**Proof.** Observe that since  $u_0 \in V$  we can take  $u_\tau^0 = u_0$  and therefore we are in position to apply Lemmas 1 and 2 and Theorem 1. We take the duality in (5) with  $\frac{u_\tau^k - u_\tau^{k-1}}{\tau}$ . This gives the following relation which holds for  $k = 1, \dots, N$

$$\left\| \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \right\|_H^2 + \left\langle Au_\tau^k, \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \right\rangle + \left\langle \xi_\tau^k, \iota \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \right\rangle_{U^* \times U} = \left\langle f_\tau^k, \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \right\rangle,$$

where  $\xi_\tau^k \in \partial J(u_\tau^k)$ . Note that by  $H_1$  we have  $f_\tau^k \in H$ . We obtain

$$\left\| \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \right\|_H^2 + \frac{1}{\tau} \langle A_1 u_\tau^k, u_\tau^k - u_\tau^{k-1} \rangle \leq (\|\xi_\tau^k\|_{U^*} \|p\|_{\mathcal{L}(H; U)} + \|f_\tau^k\|_H + \|A_2 u_\tau^k\|_H) \left\| \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \right\|_H.$$

Now we use the growth conditions  $H(J)(ii)$  and  $H(A)_2$  to obtain

$$\begin{aligned} & \left\| \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \right\|_H^2 + \frac{1}{\tau} (\psi(u_\tau^k) - \psi(u_\tau^{k-1})) \\ & \leq (c(1 + \|\iota\|_{\mathcal{L}(V; U)} \|u_\tau^k\|) \|p\|_{\mathcal{L}(H; U)} + \|f_\tau^k\|_H + c_2(1 + \|u_\tau^k\|)) \left\| \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \right\|_H. \end{aligned}$$

Applying the elementary inequality  $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$  for  $a, b \in \mathbb{R}$  we have

$$\frac{1}{2} \left\| \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \right\|_H^2 + \frac{1}{\tau} (\psi(u_\tau^k) - \psi(u_\tau^{k-1})) \leq D_1 + D_2 \|u_\tau^k\|^2 + D_3 \|f_\tau^k\|_H^2,$$

where the positive constants  $D_1$ ,  $D_2$  and  $D_3$  depend only on  $c$ ,  $c_2$ ,  $\|\iota\|_{\mathcal{L}(V; U)}$  and  $\|p\|_{\mathcal{L}(H; U)}$ . Adding the above inequalities for  $k = 1, \dots, n$  and multiplying by  $\tau$ , we arrive at the inequality

$$\frac{1}{2} \sum_{k=1}^n \frac{\|u_\tau^k - u_\tau^{k-1}\|_H^2}{\tau} + \psi(u_\tau^n) \leq \tau D_1 + D_2 \tau \sum_{k=1}^n \|u_\tau^k\|^2 + D_3 \tau \sum_{k=1}^n \|f_\tau^k\|_H^2 + \psi(u_0), \quad (23)$$

which holds for all  $n = 1, \dots, N$ . Observe that all terms on the right-hand side of above inequality are bounded. Indeed, by Lemma 1 we know that  $\tau \sum_{k=1}^N \|u_\tau^k\|^2$  is bounded. As for the term with  $f_\tau^k$  let us calculate

$$\tau \sum_{k=1}^N \|f_\tau^k\|_H^2 = \frac{1}{\tau} \sum_{k=1}^N \left\| \int_{(k-1)\tau}^{k\tau} f(t) dt \right\|_H^2 \leq \frac{1}{\tau} \sum_{k=1}^N \left( \int_{(k-1)\tau}^{k\tau} \|f(t)\|_H dt \right)^2 \leq \sum_{k=1}^N \int_{(k-1)\tau}^{k\tau} \|f(t)\|_H^2 dt = \|f\|_{\mathcal{H}}^2.$$

Thus we have  $\psi(u_\tau^n) \leq D_4$  for  $n = 0, \dots, N$ , where  $D_4 > 0$  is independent of the choice of  $\tau$ . By  $H(A)_2$  this means that

$$c_0 \|u_\tau^n\|^2 \leq D_4 + c_1 \|u_\tau^n\|_H^2.$$

By the estimate (6), this means that

$$\|u_\tau^n\| \leq \text{const} \quad \text{for } n = 0, \dots, N,$$

where the constant is independent of  $\tau$ . In other words the sequence  $\bar{u}_\tau$  is bounded in  $L^\infty(0, T; V)$ . Then, for a subsequence, we may assume that

$$\bar{u}_\tau \rightarrow \eta_1 \quad \text{weakly}^* \text{ in } L^\infty(0, T; V),$$

where  $\eta_1 \in L^\infty(0, T; V)$ . By (19) we know that (for another subsequence)  $\bar{u}_\tau \rightarrow u$  weakly\* in  $L^\infty(0, T; H)$ , where  $u$  solves (1). Therefore,  $\eta_1 = u$  and  $u \in L^\infty(0, T; V)$ .

From the estimate (23) observe that we also have

$$\frac{1}{2} \sum_{k=1}^N \frac{\|u_\tau^k - u_\tau^{k-1}\|_H^2}{\tau} \leq \text{const.}$$

This estimate allows to obtain a uniform bound on  $\|u'_\tau\|_{\mathcal{H}}$ . Indeed

$$\|u'_\tau\|_{\mathcal{H}}^2 = \sum_{k=1}^N \int_{(k-1)\tau}^{k\tau} \left\| \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \right\|_H^2 dt = \sum_{k=1}^N \frac{\|u_\tau^k - u_\tau^{k-1}\|_H^2}{\tau}.$$

Then, for a subsequence  $u'_\tau \rightarrow \eta_2$  weakly in  $\mathcal{H}$ , where  $\eta_2 \in \mathcal{H}$ . Since by (18) we have (possibly for another subsequence)  $u'_\tau \rightarrow u'$  weakly in  $\mathcal{V}^*$ , then we conclude that  $\eta_2 = u'$  and  $u' \in \mathcal{H}$ . Since  $u \in \mathcal{V} \subset \mathcal{H}$ , we conclude then  $u \in H^1(0, T; H)$ .

Now we prove that  $u \in C([0, T]; V_{\text{weak}})$ . First let us note that, since  $u \in L^\infty(0, T; V)$ , then, apart from the set  $S$  of measure zero,  $u(t)$  is bounded in  $V$ . Let  $t \in S$ . There exists the sequence  $t_i \rightarrow t$  such that  $t_i \notin S$  and therefore, for a subsequence, still denoted by  $i$ ,  $u(t_i) \rightarrow v$  weakly in  $V$ . Since  $u \in C([0, T]; H)$ , then  $v = u(t)$ , and  $\|u(t)\| \leq \liminf_{i \rightarrow \infty} \|u(t_i)\| \leq \|u\|_{L^\infty(0, T; V)}$ . Therefore  $\|u(t)\|$  is bounded for all  $t \in [0, T]$ . Let  $t \in [0, T]$  and the sequence  $t_i \rightarrow t$  be such that  $\{t_i\} \subset [0, T]$ . Since  $u \in C([0, T]; H)$  obviously we have  $u(t_i) \rightarrow u(t)$  in  $H$ . But, since  $\|u(t_i)\|$  is bounded, then, for a subsequence  $u(t_i) \rightarrow v$  weakly in  $V$  and  $v = u(t)$ . Since the limit is unique, it follows that the whole sequence  $u(t_i)$  converges to  $u(t)$  weakly in  $V$  and the assertion is proved.  $\square$

**Remark 1.** Observe that  $H_1$  implies that the term  $f$  must originate from the source term defined inside the problem domain in the concrete problem setup. This means that no nonhomogeneous Neumann conditions are allowed. Similarly in Theorem 4 we require  $H_{\text{aux}}(i)$  which means that the multivalued term cannot be defined on the boundary.

## 6.2. Second estimate

In the sequel of the paper we will use the following notation

$$m(\partial J) = \inf_{u, v \in U, u \neq v, \xi \in \partial J(u), \eta \in \partial J(v)} \frac{\langle \xi - \eta, u - v \rangle_{U^* \times U}}{\|u - v\|_U^2}.$$

The quantity  $m(\partial J)$  is in fact the logarithmic lower Lipschitz constant of the nonlinear and multivalued mapping  $\partial J$  (see [40]).

In this section we impose the following assumptions on the data.

$H_2$ :  $u_0 \in V$ ,  $f \in H^1(0, T; V^*)$  and  $Au_0 + \iota^* \xi_0 - f(0) \in H$  with some  $\xi_0 \in \partial J(\iota u_0)$ .

$H(A)_3$ :  $H(A)$  holds and there exist positive constants  $m_1$  and  $m_2$  such that for all  $u, v \in V$  we have

$$\langle Au - Av, u - v \rangle \geq m_1 \|u - v\|^2 - m_2 \|u - v\|_H^2.$$

$H(J)_1$ :  $H(J)$  holds and  $m(\partial J) > -\infty$ .

$H_{\text{const}}$ : Either  $H_{\text{aux}}(i)$  holds or  $m_1 > m(\partial J) \|\iota\|_{\mathcal{L}(V; U)}^2$ .

**Theorem 4.** Under assumptions  $H(A)_3$ ,  $H(J)_1$ ,  $H(U)$ ,  $H_{\text{aux}}$ ,  $H_2$  and  $H_{\text{const}}$ , the unique solution of (1) belongs to  $H^1(0, T; V) \cap W^{1, \infty}(0, T; H)$ .

**Proof.** Note that under listed assumptions Theorem 3 in [19] guarantees the existence and uniqueness of a solution to (1).

Since  $u_0 \in V$  we take  $u_\tau^0 = u_0$ . Moreover we define  $u_\tau^{-1} = u_0 + \tau(Au_0 + \iota^* \xi_0 - f(0))$ . Then  $u_\tau^{-1} \in H$  and if we put  $f_\tau^0 = f(0)$  then the inclusion (5) holds also for  $k = 0$ . We choose  $k \in \{1, \dots, N\}$  and subtract the inclusions (5) for  $k$  and  $k - 1$ , and take  $v = u_\tau^k - u_\tau^{k-1}$ . We arrive at

$$\begin{aligned} & \frac{1}{\tau} (u_\tau^k - u_\tau^{k-1} - (u_\tau^{k-1} - u_\tau^{k-2}), u_\tau^k - u_\tau^{k-1}) + \langle Au_\tau^k - Au_\tau^{k-1}, u_\tau^k - u_\tau^{k-1} \rangle + \langle \xi_\tau^k - \xi_\tau^{k-1}, \iota u_\tau^k - \iota u_\tau^{k-1} \rangle_{U^* \times U} \\ &= \langle f_\tau^k - f_\tau^{k-1}, u_\tau^k - u_\tau^{k-1} \rangle. \end{aligned}$$

Now we use the relation  $\|a\|_H^2 - \|b\|_H^2 \leq 2(a, a - b)$  for  $a, b \in H$  as well as assumptions  $H(A)_3$  and  $H(J)_1$  to obtain

$$\begin{aligned} & \frac{1}{2\tau} (\|u_\tau^k - u_\tau^{k-1}\|_H^2 - \|u_\tau^{k-1} - u_\tau^{k-2}\|_H^2) + m_1 \|u_\tau^k - u_\tau^{k-1}\|^2 - m_2 \|u_\tau^k - u_\tau^{k-1}\|_H^2 - m(\partial J) \|\iota(u_\tau^k - u_\tau^{k-1})\|_U^2 \\ & \leq \frac{1}{4\varepsilon} \|f_\tau^k - f_\tau^{k-1}\|_{V^*}^2 + \varepsilon \|u_\tau^k - u_\tau^{k-1}\|^2 \end{aligned}$$

for every  $\varepsilon > 0$ . Using  $H_{\text{const}}$ , in the case  $H_{\text{aux}}(i)$  we get

$$\begin{aligned} & \frac{1}{2\tau} (\|u_\tau^k - u_\tau^{k-1}\|_H^2 - \|u_\tau^{k-1} - u_\tau^{k-2}\|_H^2) + (m_1 - \varepsilon) \|u_\tau^k - u_\tau^{k-1}\|^2 - (m_2 + m(\partial J) \|p\|_{\mathcal{L}(H;U)}^2) \|u_\tau^k - u_\tau^{k-1}\|_H^2 \\ & \leq \frac{1}{4\varepsilon} \|f_\tau^k - f_\tau^{k-1}\|_{V^*}^2, \end{aligned}$$

and in the case  $m_1 > m(\partial J) \|\iota\|_{\mathcal{L}(V;U)}^2$  we have

$$\begin{aligned} & \frac{1}{2\tau} (\|u_\tau^k - u_\tau^{k-1}\|_H^2 - \|u_\tau^{k-1} - u_\tau^{k-2}\|_H^2) + (m_1 - \varepsilon - m(\partial J) \|\iota\|_{\mathcal{L}(V;U)}^2) \|u_\tau^k - u_\tau^{k-1}\|^2 - m_2 \|u_\tau^k - u_\tau^{k-1}\|_H^2 \\ & \leq \frac{1}{4\varepsilon} \|f_\tau^k - f_\tau^{k-1}\|_{V^*}^2. \end{aligned}$$

Therefore in both cases, we can choose  $\varepsilon > 0$  such that the positive constants stand in front of the term  $\|u_\tau^k - u_\tau^{k-1}\|^2$  on the left-hand side. Thus for positive constants  $D_1$ ,  $D_2$  and  $D_3$  dependent only on  $m_1$ ,  $m_2$ ,  $m(\partial J)$ ,  $\|\iota\|_{\mathcal{L}(V;U)}$  and  $\|p\|_{\mathcal{L}(H;U)}$  we have

$$\frac{1}{2\tau} (\|u_\tau^k - u_\tau^{k-1}\|_H^2 - \|u_\tau^{k-1} - u_\tau^{k-2}\|_H^2) + D_1 \|u_\tau^k - u_\tau^{k-1}\|^2 \leq D_2 \|u_\tau^k - u_\tau^{k-1}\|_H^2 + D_3 \|f_\tau^k - f_\tau^{k-1}\|_{V^*}^2.$$

Now we choose  $n \in \{1, \dots, N\}$  and add above inequalities for  $k = 1, \dots, n$ . We get

$$\begin{aligned} & \left\| \frac{u_\tau^n - u_\tau^{n-1}}{\tau} \right\|_H^2 + 2D_1 \tau \sum_{k=1}^n \left\| \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \right\|^2 \\ & \leq \|Au_0 + \iota^* \xi_0 - f(0)\|_H^2 + 2D_2 \tau \sum_{k=1}^n \left\| \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \right\|_H^2 + 2D_3 \tau \sum_{k=1}^n \left\| \frac{f_\tau^k - f_\tau^{k-1}}{\tau} \right\|_{V^*}^2. \end{aligned} \quad (24)$$

Taking  $f(t) = f(0)$  for  $t \in (-\tau, 0)$  we extend  $f$  to the interval  $(-\tau, T)$  in a way such that  $f \in H^1(-\tau, T; V^*)$ . By the definition of  $f_\tau^k$  we have

$$\left\| \frac{f_\tau^k - f_\tau^{k-1}}{\tau} \right\|_{V^*}^2 = \frac{1}{\tau^4} \left\| \int_{(k-1)\tau}^{k\tau} f(s) - f(s - \tau) ds \right\|_{V^*}^2 = \frac{1}{\tau^4} \left\| \int_{(k-1)\tau}^{k\tau} \int_0^\tau \frac{d}{dt} f(s - z) dz ds \right\|_{V^*}^2.$$

By the Jensen inequality the last relation implies

$$\left\| \frac{f_\tau^k - f_\tau^{k-1}}{\tau} \right\|_{V^*}^2 \leq \frac{1}{\tau^2} \int_{(k-1)\tau}^{k\tau} \int_0^\tau \left\| \frac{d}{dt} f(s - z) \right\|_{V^*}^2 dz ds.$$

Summing up the last relations for  $k = 1$  to  $N$  and multiplying the result by  $\tau$ , we get

$$\tau \sum_{k=1}^N \left\| \frac{f_\tau^k - f_\tau^{k-1}}{\tau} \right\|_{V^*}^2 \leq \sum_{k=1}^N \frac{1}{\tau} \int_{(k-1)\tau}^{k\tau} \int_0^\tau \left\| \frac{d}{dt} f(s - z) \right\|_{V^*}^2 dz ds \quad (25)$$

$$= \frac{1}{\tau} \int_0^\tau \int_0^T \left\| \frac{d}{dt} f(s - z) \right\|_{V^*}^2 ds dz = \frac{1}{\tau} \int_0^\tau \int_{-z}^{T-z} \left\| \frac{d}{dt} f(s) \right\|_{V^*}^2 ds dz. \quad (26)$$

Since on  $(-\tau, 0)$  we have  $\frac{d}{dt} f(s) = 0$  for a.e.  $t$ , we obtain

$$\tau \sum_{k=1}^N \left\| \frac{f_\tau^k - f_\tau^{k-1}}{\tau} \right\|_{V^*}^2 \leq \frac{1}{\tau} \int_0^\tau \int_0^T \left\| \frac{d}{dt} f(s) \right\|_{V^*}^2 ds dz = \|f'\|_{L^2(0,T;V^*)}^2.$$

From (24), we obtain

$$\begin{aligned} & \left\| \frac{u_\tau^n - u_\tau^{n-1}}{\tau} \right\|_H^2 + 2D_1 \tau \sum_{k=1}^n \left\| \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \right\|^2 \\ & \leq \|Au_0 + \iota^* \xi_0 - f(0)\|_H^2 + 2D_2 \tau \sum_{k=1}^n \left\| \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \right\|_H^2 + 2D_3 \|f'\|_{L^2(0,T;V^*)}^2. \end{aligned} \quad (27)$$

By the discrete Gronwall inequality it follows that there exists  $\tau_0 > 0$  (it suffices to take  $0 < \tau_0 < \frac{1}{2D_2}$ ) such that for all  $\tau \in (0, \tau_0)$ , we have

$$\left\| \frac{u_\tau^n - u_\tau^{n-1}}{\tau} \right\|_H \leq \text{const} \quad \text{for all } n \in \{1, \dots, N\}, \quad (28)$$

$$\tau \sum_{k=1}^N \left\| \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \right\|^2 \leq \text{const}. \quad (29)$$

This means that, for a subsequence, we have

$$u'_\tau \rightarrow \xi \quad \text{weakly}^* \text{ in } L^\infty(0, T; H), \quad (30)$$

$$u'_\tau \rightarrow \eta \quad \text{weakly in } L^2(0, T; V). \quad (31)$$

Now, since, by Theorem 1, we have  $u'_\tau \rightarrow u'$  weakly in  $\mathcal{V}^*$ , where  $u$  is the solution of (1) then it follows that  $\xi = \eta = u'$  and thus  $u' \in L^2(0, T; V)$  and  $u' \in L^\infty(0, T; H)$ . This concludes the proof of the theorem.  $\square$

## 7. Error estimate

In this section we provide the error estimate between the unique solution of the original problem and the solution of the Rothe problem.

**Theorem 5.** Under assumptions of Theorem 4 there exist constants  $C_1$  and  $C_2$  depending only on the problem setup (including  $T$ ) and independent of  $\tau$  and a constant  $\tau_0 > 0$  such that for all  $0 < \tau < \tau_0$  we have

$$\|u - u_\tau\|_{C([0, T]; H)} + \|u - \bar{u}_\tau\|_{\mathcal{V}} \leq C_1 \|f - \bar{f}_\tau\|_{\mathcal{V}^*} + C_2 \sqrt{\tau}. \quad (32)$$

**Proof.** We subtract (11) from (1) and obtain

$$\langle (u - u_\tau)'(t), v \rangle + \langle Au(t) - A\bar{u}_\tau(t), v \rangle + \langle \eta(t) - \xi_\tau(t), \iota v \rangle_{U \times U^*} = \langle f(t) - \bar{f}_\tau(t), v \rangle \quad \text{for all } v \in V \text{ a.e. } t \in (0, T)$$

with  $\eta(t) \in \partial J(\iota u(t))$  and  $\xi_\tau(t) \in \partial J(\iota \bar{u}_\tau(t))$  for a.e.  $t \in (0, T)$ . We take  $v = u(t) - \bar{u}_\tau(t)$  and use  $H(A)_3$  and  $H(J)_1$ . For  $\varepsilon > 0$  we obtain

$$\begin{aligned} & \langle (u - u_\tau)'(t), u(t) - \bar{u}_\tau(t) \rangle + m_1 \|u(t) - \bar{u}_\tau(t)\|^2 - m(\partial J) \|\iota(u(t) - \bar{u}_\tau(t))\|_U^2 \\ & \leq \frac{1}{4\varepsilon} \|f(t) - \bar{f}_\tau(t)\|_{\mathcal{V}^*}^2 + \varepsilon \|u(t) - \bar{u}_\tau(t)\|_V^2 + m_2 \|u(t) - \bar{u}_\tau(t)\|_H^2. \end{aligned} \quad (33)$$

Similarly as in the proof of Theorem 4 using  $H_{\text{const}}$  we find with the positive constants  $D_1$ ,  $D_2$  and  $D_3$  depending only on  $m_1$ ,  $m_2$ ,  $m(\partial J)$ ,  $\|\iota\|_{\mathcal{L}(V; U)}$  and  $\|p\|_{\mathcal{L}(H; U)}$  such that

$$\langle (u - u_\tau)'(t), u(t) - \bar{u}_\tau(t) \rangle + D_1 \|u(t) - \bar{u}_\tau(t)\|^2 \leq D_2 \|f(t) - \bar{f}_\tau(t)\|_{\mathcal{V}^*}^2 + D_3 \|u(t) - \bar{u}_\tau(t)\|_H^2. \quad (34)$$

We compute the first term in the last inequality as follows

$$\langle (u - u_\tau)'(t), u(t) - \bar{u}_\tau(t) \rangle = \frac{d}{dt} \|u(t) - u_\tau(t)\|_H^2 + (u'(t), u_\tau(t) - \bar{u}_\tau(t)) + (u'_\tau(t), \bar{u}_\tau(t) - u_\tau(t)). \quad (35)$$

In the last formula we used the fact that  $u' \in L^\infty(0, T; H)$ . We can estimate for  $t \in ((k-1)\tau, k\tau]$

$$(u'_\tau(t), \bar{u}_\tau(t) - u_\tau(t)) = \left( \frac{u_\tau^k - u_\tau^{k-1}}{\tau}, \left( k - \frac{t}{\tau} \right) (u_\tau^k - u_\tau^{k-1}) \right) = \left( \frac{k\tau - t}{\tau^2} \right) \|u_\tau^k - u_\tau^{k-1}\|_H^2 \geq 0. \quad (36)$$

Applying (35) and (36) in (34) we get

$$\begin{aligned} & \frac{d}{dt} \|u(t) - u_\tau(t)\|_H^2 + D_1 \|u(t) - \bar{u}_\tau(t)\|^2 \\ & \leq D_2 \|f(t) - \bar{f}_\tau(t)\|_{\mathcal{V}^*}^2 + D_3 \|u(t) - \bar{u}_\tau(t)\|_H^2 + \|u'\|_{L^\infty(0, T; H)} \|\bar{u}_\tau(t) - u_\tau(t)\|_H \quad \text{for a.e. } t \in (0, T). \end{aligned} \quad (37)$$

We integrate the last inequality (note that, by  $H_2$ , we have  $u_0 \in V$  and  $u_\tau(0) = u(0) = u_0$ ) and get

$$\begin{aligned}
& \|u(t) - u_\tau(t)\|_H^2 + D_1 \int_0^t \|u(s) - \bar{u}_\tau(s)\|_H^2 ds \\
& \leq D_2 \|f - \bar{f}_\tau\|_{V^*}^2 + D_3 \int_0^t \|u(s) - \bar{u}_\tau(s)\|_H^2 ds + \|u'\|_{L^\infty(0,T;H)} \|\bar{u}_\tau - u_\tau\|_{L^1(0,T;H)}.
\end{aligned} \tag{38}$$

Using (28), we have

$$\|\bar{u}_\tau - u_\tau\|_{L^1(0,T;H)} = \frac{1}{2} \tau \sum_{k=1}^N \|u_\tau^k - u_\tau^{k-1}\|_H \leq \frac{1}{2} T \tau \max_{k=1,\dots,N} \left\| \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \right\|_H \leq D_4 \tau,$$

with a positive constant  $D_4$ . Dropping the second term on the left-hand side of (38), by the Gronwall inequality we get

$$\|u(t) - u_\tau(t)\|_H^2 \leq e^{D_3 T} (D_2 \|f - \bar{f}_\tau\|_{V^*}^2 + D_4 \tau \|u'\|_{L^\infty(0,T;H)}). \tag{39}$$

From the last inequality we obtain the estimate

$$\|u - u_\tau\|_{C^\infty([0,T];H)} \leq D_5 \|f - \bar{f}_\tau\|_{V^*} + D_6 \sqrt{\tau},$$

where the constants  $D_5$  and  $D_6$  depend on the problem setup (including the time  $T$ ) and are independent of  $\tau$ . Coming back to (38) and dropping the first term on the left-hand side, we obtain the estimate

$$\|u - \bar{u}_\tau\|_V \leq D_7 \|f - \bar{f}_\tau\|_{V^*} + D_8 \sqrt{\tau},$$

where the constants again depend only on the problem setup (including  $T$ ) and not on  $\tau$ . Summing up the last two estimates ends the proof of the theorem.  $\square$

**Remark 2.** The order of the quantity  $\|f - \bar{f}_\tau\|_{V^*}$  depends on the smoothness of  $f$ . In particular if  $f \in C^1([0, T]; V^*)$  then we have  $\|f - \bar{f}_\tau\|_{V^*} = \mathcal{O}(\tau^{\frac{1}{2}})$ .

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