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Limit cycles bifurcated from a class of quadratic reversible center of genus one ☆

Haihua Liang [a,*], Yulin Zhao [b]

[a] Department of Computer Science, Guangdong Polytechnic Normal University, Guangzhou, Guangdong 510665, PR China
[b] Department of Mathematics, Sun Yat-Sen University, Guangzhou 510275, PR China

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ABSTRACT

We investigate the bifurcation of limit cycles in a class of planar quadratic reversible (non-Hamiltonian) systems. This systems has a center of genus one. The exact upper bound of the number of limit cycles is given.

1. Introduction

This paper deals with the number of limit cycles for small quadratic perturbations of quadratic integrable systems:

$$\begin{aligned}\dot{x} &= H_y(x,y)/M(x,y) + \varepsilon f(x,y,\varepsilon),\\ \dot{y} &= -H_x(x,y)/M(x,y) + \varepsilon g(x,y,\varepsilon),\end{aligned} \tag{1.1}$$

where ε is a small parameter and $f(x,y,\varepsilon)$, $g(x,y,\varepsilon)$ are quadratic polynomials in x, y with coefficients depending analytically on ε, $H(x,y)$ is the first integral of the unperturbed system $(1.1)_{\varepsilon=0}$ with the integrating factor $M(x,y)$.

The plane quadratic systems with at least one center are always integrable, which can be classified into the following four classes: Hamiltonian Q_3^H, reversible Q_3^R, codimension four Q_4 and generalized Lotka–Volterra Q_3^{LV}. A natural problem is asking for a maximal number of limit cycles, bifurcated from the period annulus of these four classes of systems under small quadratic perturbations. For the Hamiltonian Q_3^H, this number is closely related to the weak Hilbert 16th problem in the quadratic case. It is well known that the weak Hilbert 16th problem was solved completely by many authors, see [2,9] and the references therein. The next natural step is to find the upper bound of the number of limit cycles which bifurcated from the period annulus of quadratic integrable but non-Hamiltonian systems, under quadratic perturbation. As usual, we use the notion *cyclicity* for the total number of limit cycles which can emerge from a configuration of trajectories (center, period annulus, a singular loop) under a perturbation. In [14,34], the authors proposed a conjecture about the cyclicity of the period annulus of quadratic centers Q_3^R, Q_4 and Q_3^{LV} under small quadratic perturbation.

Recently, by using the idea of [10], Zhao [31] proved that under quadratic perturbation the cyclicity of period annulus of Q_4 is greater than or equal to 3 and is less than or equal to 5 (the conjectural one is 3). The cyclicity of period annulus of

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* Corresponding author.
E-mail address: haiihuaa@tom.com (H. Liang).

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some classes of general Q_3^{LV} under quadratic perturbation has been studied by Zoladek in [34]. Quadratic perturbation of several class of reversible Lotka–Volterra systems are investigated by [11,15,18,27].

H. Zoladek pointed out in [34] that the reversible case is the most complicated case for the studying bifurcation of limit cycles. Taking a complex coordinate, the quadratic reversible center have the following form [14]:

$$\dot{z} = -iz + az^2 + 2|z|^2 + b\bar{z}^2, \quad a, b \in \mathbf{R},\ z = x + iy. \tag{1.2}$$

In [7], the authors determine all reversible centers with phase portrait formed by elliptic curves. Using the terminology from [8], the centers whose periodic orbits are elliptic curves will be called *centers of genus one*. Theorem 1 of [8] shows that the system (1.2) has a center of genus one if and only if a, b verify one of the following 18 conditions:

$$\begin{array}{lll}
(r1)\ a = 2b+1, & (r2)\ a = -1, & (r3)\ a = 5b+4 (b \neq -3),\\
(r4)\ a = -3b-4 (b \neq -3), & (r5)\ a = \frac{5}{3}b + \frac{2}{3}, & (r6)\ a = \frac{b}{3} - \frac{2}{3},\\
(r7)\ (a,b) = (\frac{5}{2}, -\frac{1}{2}), & (r8)\ (a,b) = (-\frac{7}{2}, -\frac{1}{2}), & (r9)\ (a,b) = (-8,-2),\\
(r10)\ (a,b) = (4,-2), & (r11)\ (a,b) = (-17,-5), & (r12)\ (a,b) = (7,-5),\\
(r13)\ (a,b) = (-7, -\frac{5}{3}), & (r14)\ (a,b) = (\frac{11}{3}, -\frac{5}{3}), & (r15)\ (a,b) = (-23,-7),\\
(r16)\ (a,b) = (9,-7), & (r17)\ (a,b) = (13,5), & (r18)\ (a,b) = (-3,5).
\end{array}$$

To our knowledge, until now, known results concerning the cyclicity of period annulus of these systems under small quadratic perturbations are: family ($r1$) was completely solved in several papers [6,16–18,25,28,30]. Family ($r2$) (the reversible Hamiltonian case) has been completely studied by many authors, see [2,9] and the references therein. Family ($r3$) is entirely investigated in a series of papers [20,22,33]. Family ($r4$) with $a \in (-\infty, -3) \cup (5, +\infty)$ and $a \in (-3, -1)$ is considered in [4] and [21] respectively. Family ($r5$) with $b \neq 2, \frac{1}{2}$ and family ($r6$) with $b \in (\frac{1}{2}, 2)$ are studied in [1]. Cases ($r7$), ($r14$), ($r15$) and ($r17$) are studied in [11]. Cases ($r8$), ($r13$) and ($r16$) are studied in [3]. Cases ($r9$), ($r10$) and ($r12$) are studied in [23,13] and [24] respectively. Cases ($r11$) and ($r18$) are studied in [8].

In the present paper, we will deal with the case ($r4$):

$$\dot{z} = -iz + az^2 + 2|z|^2 - \frac{a+4}{3}\bar{z}^2. \tag{1.3}$$

The authors of [14,8] conjecture that the cyclicity of the systems (1.3) under small quadratic perturbation is three with $4 < a < 5$ and is two otherwise. That is to say, for small ε, the maximum number of limit cycles in the system (rewritten in the (x, y) coordinates)

$$\begin{aligned}
\dot{x} &= y + \frac{2}{3}(a+1)x^2 - \frac{2}{3}(a-5)y^2 + \varepsilon f(x,y,\varepsilon),\\
\dot{y} &= -x + \frac{8}{3}(a+1)xy + \varepsilon g(x,y,\varepsilon),
\end{aligned} \tag{1.4}$$

which emerge from the period annulus of the unperturbed system is equal to three with $4 < a < 5$ and two otherwise. If $a = -1$, then system (1.3) belongs to ($r2$). If $a = 5$, then system (1.3) corresponding to an isochronous centers. Perturbations of the quadratic isochronous centers have been studied in [5]. Apart from $a = -1, 5$, as we have mentioned that, until now the conjecture has already been verified by [4] and [21] for the cases $a \in (-\infty, -3) \cup (5, +\infty)$ and $a \in (-3, -1)$ respectively. The main work of this paper is to investigate the remaining cases except that $a = -3$ (the outer boundary of the period annulus contains a degenerate critical point). We will obtain the following theorem.

Theorem 1.1. *The cyclicity of the period annulus of* (1.3) *under small quadratic perturbation is three for the case that* $4 < a < 5$ *and two for the case that* $-1 < a \leqslant 4$.

Combining Theorem 1.1 with the results of [2,4,5,21], it turns out that

Theorem 1.2. *Suppose that* $a \neq -3$, *then the cyclicity of the period annulus of* (1.3) *under small quadratic perturbation is three for the case* $4 < a < 5$ *and two for others.*

Moreover, the outer bounder of the period annulus of system (1.3) with $-1 < a < 5$ is a homoclinic loop (see Fig. 1 below). Using Roussarie's theorem [26], it is easy to see that the cyclicity of the period annulus covers the number of limit cycles bifurcated from the non-degenerate center, from the period annuls and from the homoclinic loop. Consequently, we have the following corollary.

Corollary 1.1. *For any sufficiently small* ε, *the exact upper bound for the number of limit cycles of system* (1.4) *in the finite phase plane is three for* $4 < a < 5$ *and two for* $-1 < a \leqslant 4$.

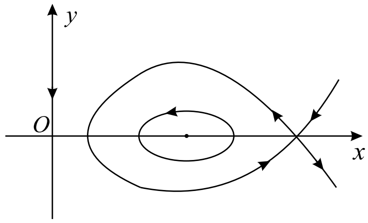

Fig. 1. The phase portrait of system (2.1).

Let us now describe the contents of the paper.

In Section 2 we firstly transform the system (1.3) to a more convenient system which possess a unique center at $(1,0)$. Secondly, by using Theorem 2 of [14], we derive the Melnikov function $I(h)$ whose zeros correspond to the limit cycles emerging from the period annulus under small quadratic perturbations. As usually, the question about the number of zeros of $I(h)$ can be transformed into a geometric form: to find in the plane the number of intersection points of a curve Σ with the straight line. This means that a careful study of the geometric property of Σ is necessary. The last part of this section is to derive the needed Picard–Fuchs equations satisfied by some integrals.

In Section 3 we estimate the upper bound for the number of zeros of Abelian integral $I(h)$. The main idea used here is based on [4]. However, the analysis used in that paper rely on the range of the parameter and some methods are not applicable for our paper. Hence some non-trivial computations and analysis are needed.

Finally, in Section 4 we establish the asymptotic behavior of the curve Σ near its two endpoints. And then, by using the results from Section 3, we obtain the geometric property of the curve Σ by which the main results of this paper is proved naturally.

2. Some preliminary results

As the same of [4], by using an affine transformation of coordinate and a rescaling of time, the system (1.3) with $-1 < a < 5$ is changed to

$$\begin{aligned} \dot{x} &= -2xy, \\ \dot{y} &= k - 1 - 2kx + (k+1)x^2 - \frac{1}{2}y^2, \end{aligned} \tag{2.1}$$

where $k = -\frac{a+7}{2a+2} < -1$.

To prove Theorem 1.1, we only need to prove the following result.

Theorem 2.1. *The cyclicity of the period annulus of* (1.3) *under small quadratic perturbations is three for the case that* $-\infty < k < -1.1$ *and is two for the case that* $-1.1 \leqslant k < -1$.

System (2.1) has a center at $(1,0)$ and a saddle at $(\frac{k-1}{k+1}, 0)$ respectively. Also, it has the invariant straight line $\{x = 0\}$ and the first integral

$$H(x,y) = x^{-\frac{1}{2}}\left(y^2 + \frac{2(k+1)}{3}x^2 - 4kx - 2(k-1)\right) = h \tag{2.2}$$

with the integrating factor $\mu(x,y) = |x|^{-\frac{3}{2}}$.

The phase portrait of the system (2.1) for $k < -1$ is shown in Fig. 1.

Using the results of [14], we have the following lemma.

Lemma 2.1. *If* $k < -1$, *then the cyclicity of the period annulus of system* (2.1) *under small quadratic perturbations is given by the maximal number of isolated zeros of the* (*generalized*) *Abelian integral*

$$I(h) = \oint_{\Gamma_h} x^{-\frac{3}{2}}\left(\alpha x + \beta x^{-1} + \gamma\right) y \, dx, \quad h \in (h_1, h_2), \tag{2.3}$$

where α, β, γ *are constants and*

$$h_1 = H(1,0) = -\frac{8(2k-1)}{3}, \qquad h_2 = H\left(\frac{k-1}{k+1}, 0\right) = -\frac{8}{3}(2k+1)\sqrt{\frac{k-1}{k+1}}.$$

The critical levels $h = h_1$ *and* $h = h_2$ *corresponding to the center* $(1,0)$ *and the homoclinic loop respectively.*

Denoted by

$$I_j(h) = \oint_{\Gamma_h} x^{j-\frac{3}{2}} y\,dx,$$

where the orientation of the integral over Γ_h is counterclockwise.

Obviously,

$$I(h) = \alpha I_1(h) + \beta I_{-1}(h) + \gamma I_0(h). \tag{2.4}$$

It is well known that the integrals $I_j(h)$ admit an analytic continuation in a neighborhood of $h = h_1$ and have no such continuation in a neighborhood of $h = h_2$ since their expansion along $h = h_2$ contains logarithmic terms. Noting that $I_1(h) < 0$, we rewrite $I(h)$ as $I(h) = I_1(h)(\alpha + \beta Q(h) + \gamma P(h))$, where

$$P(h) = \frac{I_0(h)}{I_1(h)}, \qquad Q(h) = \frac{I_{-1}(h)}{I_1(h)}, \quad h \in (h_1, h_2).$$

Hence, the number of zeros of $I(h)$ equals the number of intersection points of the straight line

$$L := \big\{(P, Q)\colon \alpha + \beta Q + \gamma P\big\}$$

with the planar parametric curve

$$\Sigma := \big\{(P, Q) = \big(P(h), Q(h)\big)\colon h \in (h_1, h_2)\big\},$$

taking the multiplicities into account. Usually one calls Σ the centroid curve.

Our first goal will be to establish some Picard–Fuchs systems related to the Abelian integral $I(h)$ which will be used in the next section. Performing the standard calculation procedure, we obtain that

$$\frac{4}{3}(k+1)I_1(h) + hI_{-\frac{1}{2}}(h) + 4(k-1)I_{-1}(h) = 0, \tag{2.5}$$

$$I_0(h) = -\frac{8}{3}(k+1)I'_{\frac{3}{2}}(h) + 8kI'_{\frac{1}{2}}(h) + hI'_0(h), \tag{2.6}$$

$$2I_{\frac{1}{2}}(h) = 8kI'_1(h) + 3hI'_{\frac{1}{2}}(h) + 8(k-1)I'_0(h), \tag{2.7}$$

$$3I_1(h) = 8kI'_{\frac{3}{2}}(h) + 3hI'_1(h) + 8(k-1)I'_{\frac{1}{2}}(h), \tag{2.8}$$

$$4I_{\frac{3}{2}}(h) = 8kI'_2(h) + 3hI'_{\frac{3}{2}}(h) + 8(k-1)I'_1(h), \tag{2.9}$$

$$2(k+1)I_{\frac{3}{2}}(h) = 3kI_{\frac{1}{2}}(h) - \frac{3}{8}hI_0(h) - 3(k-1)I_{-\frac{1}{2}}(h). \tag{2.10}$$

Eliminating $I_{\frac{3}{2}}(h)$ from Eqs. (2.6)–(2.10), we obtain a closed Picard–Fuchs system

$$\begin{pmatrix} I_{-\frac{1}{2}} \\ I_0 \\ I_{\frac{1}{2}} \\ I_1 \end{pmatrix} = \begin{pmatrix} \frac{h}{2} & 4k & 0 & -\frac{4}{3}(k+1) \\ 8(k-1) & 3h & 8k & 0 \\ 0 & 4(k-1) & \frac{3}{2}h & 4k \\ -\frac{8k(k-1)}{k+1} & -\frac{2kh}{k+1} & \frac{8}{3}(k-1) & h \end{pmatrix} \begin{pmatrix} I'_{-\frac{1}{2}} \\ I'_0 \\ I'_{\frac{1}{2}} \\ I'_1 \end{pmatrix}. \tag{2.11}$$

Taking derivative with respect to h on Eqs. (2.6)–(2.9), it follows that

Lemma 2.2. *The vector function $col(I'_{\frac{1}{2}}(h), I'_{\frac{3}{2}}(h))$ verifies the first order linear differential system*

$$\Delta(h)\begin{pmatrix} I''_{\frac{1}{2}}(h) \\ I''_{\frac{3}{2}}(h) \end{pmatrix} = \begin{pmatrix} a_{21}(h) & a_{22}(h) \\ a_{41}(h) & a_{42}(h) \end{pmatrix}\begin{pmatrix} I'_{\frac{1}{2}}(h) \\ I'_{\frac{3}{2}}(h) \end{pmatrix}, \tag{2.12}$$

where

$$\Delta(h) = (3h + 8 - 16k)(3h - 8 + 16k)\big[(9 + 9k)h^2 + 64 + 192k - 256k^3\big] = 81(k+1)\big(h^2 - h_1^2\big)\big(h^2 - h_2^2\big),$$

$$a_{21}(h) = -3(k+1)h\big[9h^2 - 256k(k-1)\big] = -a_{42}(h),$$

$$a_{22}(h) = 192(1+k)h, \qquad a_{41}(h) = -6h\big[9kh^2 - 32(k-1)\big(8k^2+1\big)\big].$$

Differentiating both sides of (2.12) we get that

Lemma 2.3. *The vector function* $col(I''_{\frac{1}{2}}(h), I''_{\frac{3}{2}}(h))$ *verifies the first order linear differential system*:

$$G(h)\begin{pmatrix} I'''_{\frac{1}{2}}(h) \\ I'''_{\frac{3}{2}}(h) \end{pmatrix}\begin{pmatrix} b_{11}(h) & b_{12}(h) \\ b_{21}(h) & b_{22}(h) \end{pmatrix}\begin{pmatrix} I''_{\frac{1}{2}}(h) \\ I''_{\frac{3}{2}}(h) \end{pmatrix}, \tag{2.13}$$

where

$$\begin{aligned}
&G(h) = 81(k+1)h\big(h^2-h_1^2\big)\big(h^2-h_2^2\big) = h\Delta(h),\\
&b_{11}(h) = -4\big[27(k+1)h^4 - 96k\big(1+2k^2\big)h^2 - 1024(k-1)\big(4k^2-1\big)^2\big],\\
&b_{12}(h) = -960(k+1)h^2, \qquad b_{21}(h) = -6h^2\big[9kh^2 - 32(k-1)\big(8k^2+7\big)\big],\\
&b_{22}(h) = -2\big[27(k+1)h^4 + 192k\big(1+2k^2\big)h^2 - 2048(k-1)\big(4k^2-1\big)^2\big].
\end{aligned}$$

3. Estimate for the number of zeros of $I(h)$

This section is devoted to estimate the number of zeros of $I(h)$, $h \in (h_1, h_2)$, counting the multiplicities. Firstly, we note that the zeros of $I(h)$ are just the zeros of $I(h)/h$. Since $I(h)/h$ vanishes at h_1, the number of zeros of it is not large than the number of zeros of $\frac{d}{dh}(\frac{I(h)}{h})$.

Lemma 3.1. *Denoted by* $\Psi(h) = -3(k-1)h^2\frac{d}{dh}(\frac{I(h)}{h}) = 3(k-1)(I(h) - hI'(h))$, *then*

$$\Psi(h) = \eta I'_{\frac{3}{2}} + \xi I'_{\frac{1}{2}} + \zeta h^2 I'_{-\frac{1}{2}}, \tag{3.1}$$

where

$$\begin{aligned}
&\eta = 8\alpha k(k-1) - \frac{8}{3}\beta k(k+1) - 8\big(k^2-1\big)\gamma,\\
&\xi = 8\alpha(k-1)^2 - \frac{8}{3}\beta\big(k^2-1\big) + 24k(k-1)\gamma, \qquad \zeta = \frac{3}{4}\beta.
\end{aligned}$$

Proof. It is direct follows from (2.6)–(2.10) by calculation. □

In order to simplify the expression of $\Psi(h)$ by putting $\zeta = 1$ it is first necessary to prove the following lemma.

Lemma 3.2. *If* $\beta = 0$, *then* $I(h)$ *has at most one zero, taking into account the multiplicities.*

Proof. By the relation between $I(h)$ and $\Psi(h)$ and the fact that $I(h_1) = 0$, it suffices to show that $\eta I'_{\frac{3}{2}}(h) + \xi I'_{\frac{1}{2}}(h)$ has at most one zero. To this end, we will prove that the function $v(h) := I'_{\frac{1}{2}}(h)/I'_{\frac{3}{2}}(h)$ is monotonic decreasing. It is worth pointing out here that the fact $I'_{\frac{3}{2}}(h) < 0$ guarantees that $v(h)$ is well-defined on (h_1, h_2).

By applying the Picard–Fuchs (2.12), $v(h)$ verifies

$$\Delta(h)v'(h) = -a_{41}(h)v^2(h) + 2a_{21}(h)v(h) + a_{22}(h). \tag{3.2}$$

Instead of this Riccati equation we study an equivalent autonomous differential system on the plane, namely

$$\frac{dh}{dt} = \Delta(h), \qquad \frac{dv}{dt} = -a_{41}(h)v^2 + 2a_{21}(h)v + a_{22}(h). \tag{3.3}$$

We shall first identify which solution of (3.3) corresponds to the $v(h)$ under consideration. Differentiating (3.2) and using the fact that $v(h_1) = 1$, we find $v'(h_1) = \frac{1}{8}(2k-1) < 0$.

To determine the feature of $v(h)$ near $h = h_2$, we performing the standard calculation procedure (for example, see [12,32]) and obtain that the Picard–Fuchs equation (2.12) has a fundamental solution matrix in a neighborhood of h_2 as follows

$$\mathbf{TP}(h)\begin{pmatrix} 1 & \ln(h_2-h) \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{T} = \begin{pmatrix} k+1 & k+1 \\ k-1 & k-13 \end{pmatrix} \tag{3.4}$$

where $\mathbf{P}(h)$ is analytic at $h = h_2$, $\mathbf{P}(h_2)$ is a unite matrix. Therefore $I'_{\frac{1}{2}}(h)$ and $I'_{\frac{3}{2}}(h)$ have the forms

$$I'_{\frac{1}{2}}(h) = (k+1)\big[c_2 \ln(h_2 - h) + (c_1 + c_2) + \cdots\big], \tag{3.5}$$

$$I'_{\frac{3}{2}}(h) = c_2(k-1)\ln(h_2 - h) + (k-1)c_1 + (k-13)c_2 + \cdots, \tag{3.6}$$

where c_1 and c_2 are constants. Note in addition that $-I'_{\frac{3}{2}}(h)$ is the period function of the Hamiltonian system

$$\dot{x} = -H_y(x, y), \qquad \dot{y} = H_x(x, y), \tag{3.7}$$

and that $h = h_2$ corresponding to the homoclinic orbit of system (3.7), it follows that $-I'_{\frac{3}{2}}(h) \to +\infty$ as $h \to h_2^-$. Hence $c_2 \neq 0$ (otherwise $I'_{\frac{3}{2}}(h)$ will doesn't contain logarithmic terms). Then

$$v(h) = \frac{I'_{\frac{1}{2}}(h)}{I'_{\frac{3}{2}}(h)} = \frac{1+k}{k-1} + \frac{12(k+1)}{(k-1)^2 \ln(h_2 - h)} + \cdots, \quad \text{as } h \to h_2^-. \tag{3.8}$$

This yields that $v(h) \to \frac{k+1}{k-1}$ and $v'(h) \to -\infty$ as $h \to h_2$.

We have already shown that $v(h)$ is monotonic decreasing near the two endpoints. We assert that $v(h)$ is monotonic decreasing on (h_1, h_2). If this is not true, then the inequality $v(h_2) < v(h_1)$ means that $v(h)$ would have at least one maximum point and one minimum point. Thus there exists a straight line $v = v_0$ in the (h, v)-plane cutting the graph of $v(h)$ at three times, which means that there are at least two points on the segment $\{(v, h) | v = v_0, h \in (h_1, h_2)\}$, at which the vector field (3.3) is horizontal. This is however impossible because the function

$$\begin{aligned}\frac{dv}{dt}\bigg|_{v=v_0} &= -a_{41}(h)v_0^2 + 2a_{21}(h)v_0 + a_{22}(h) \\ &= h\big[6\big(9kh^2 - 32(k-1)\big(8k^2+1\big)\big)v_0^2 - 6(k+1)\big(9h^2 - 256k(k-1)\big)v_0 + 192(1+k)\big]\end{aligned}$$

has at most one zero in (h_1, h_2). The proof is finished. □

By the above lemma, we can assume without loss of generality that $\zeta = 1$ in (3.1), i.e.,

$$\Psi(h) = \eta I'_{\frac{3}{2}}(h) + \xi I'_{\frac{1}{2}}(h) + h^2 I'_{-\frac{1}{2}}(h).$$

Unfortunately, it is very difficult to estimate directly the number of zeros of $\Psi(h)$ since the expression of $\Psi(h)$ contains three Abelian integrals. As [4] has done, to overcome this difficulty we will consider the derivative of $\Psi(h)/h^3$. By a straightforward manipulation we obtain the following result:

Lemma 3.3. *Let $\tilde{\Psi}(h) = \frac{9}{2}(k+1)h^5 \frac{d}{dh}(\frac{\Psi(h)}{h^3})$, then $\tilde{\Psi}(h) = (ah^2 + b)I''_{\frac{1}{2}}(h) - (ch^2 + d)I''_{\frac{3}{2}}(h)$, where*

$$\begin{aligned} a &= 45(k+1)\xi + 81k\eta + 32(k+1)\big(8k^2+1\big), \\ b &= -288(k-1)\big[4k(k+1)\xi + \big(8k^2+1\big)\eta\big], \\ c &= 4(k+1)\big[9\eta + 32k(k+1)\big], \\ d &= 288(k+1)\big[\xi - 4k(k-1)\eta\big]. \end{aligned} \tag{3.9}$$

The expression of $\tilde{\Psi}(h)$ inspired us to consider the function $w(h) := I''_{\frac{3}{2}}(h)/I''_{\frac{1}{2}}(h)$. We will illustrate that $I''_{\frac{1}{2}}(h) \neq 0$ for all $h \in (h_1, h_2)$ after the proof of Lemma 3.5, see Remark 3.1 below.

Taking advantage of Lemma 2.3, it is easy to see that $w(h)$ verifies the Riccati equation:

$$\Delta(h)w' = 6h\big[160(k+1)w^2 + \big(9(k+1)h^2 - 128k\big(1+2k^2\big)\big)w - 9kh^2 + 32(k-1)\big(8k^2+7\big)\big]. \tag{3.10}$$

Setting $u = h^2$, $u_1 = h_1^2$, $u_2 = h_2^2$, $\bar{w}(u) = w(h) = w(-\sqrt{u})$. By (3.10), it turns out that

Lemma 3.4. *$\bar{w}(u)$ verifies the following Riccati equation*:

$$G_2(u)\bar{w}' = 160(k+1)\bar{w}^2 + \big(9(k+1)u - 128k\big(1+2k^2\big)\big)\bar{w} - 9ku + 32(k-1)\big(8k^2+7\big), \tag{3.11}$$

where $G_2(u) = 27(k+1)(u-u_1)(u-u_2)$.

Lemma 3.5. *$\bar{w}(u)$ is monotonic decreasing and concave on (u_1, u_2).*

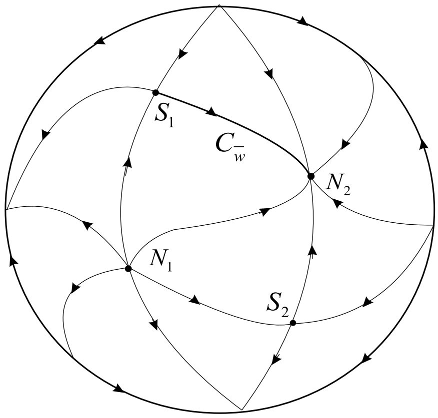

Fig. 2. Phase portraits of (3.12) and the graph of $\bar{w}(u)$ (denoted by $C_{\bar{w}}$).

Proof. By Lemma 3.4, $\bar{w}(u)$ is a solution of the system

$$\begin{aligned} \frac{du}{dt} &= G_2(u), \\ \frac{d\bar{w}}{dt} &= 160(k+1)\bar{w}^2 + \big(9(k+1)u - 128k\big(1+2k^2\big)\big)\bar{w} - 9ku + 32(k-1)\big(8k^2+7\big). \end{aligned} \tag{3.12}$$

This system has two invariant straight lines $u = u_1$ and $u = u_2$ as well as four critical points: saddle $S_1(u_1, w_1)$, saddle $S_2(u_2, w_{22})$ and node $N_1(u_1, 1)$, node $N_2(u_2, w_{21})$, where

$$w_1 = \frac{5k-7}{5(k+1)}, \qquad w_{21} = \frac{k-1}{k+1}, \qquad w_{22} = \frac{5k+7}{5(k+1)}.$$

Putting $h = h_1$ into (2.13) gives rise to $w(h_1) = w_1 = \bar{w}(u_1)$. Taking derivative with respect to u in (3.11) yields that $\bar{w}'(u_1) = -\frac{21}{1280}$. Applying the same procedure we get $\bar{w}''(u_1) = \frac{77(k+1)}{131\,072}$.

On the other hand, by applying the methods of [32], we obtain a fundamental solution matrix of (2.13) in a neighborhood of $h = h_2$

$$\begin{pmatrix} k+1 & 5k+5 \\ k-1 & 5k+7 \end{pmatrix} \begin{pmatrix} q_{11}(h) & q_{12}(h) \\ q_{21}(h) & q_{22}(h) \end{pmatrix} \begin{pmatrix} \frac{1}{h_2-h} & 0 \\ v\ln(h_2-h) & 1 \end{pmatrix}, \tag{3.13}$$

where $v = (2k+1)(k+1)\sqrt{(k-1)/(k+1)}/96$, $q_{ij}(h)$ is analytic at $h = h_2$ and $(q_{ij}(h_2))_{1\leqslant i,j\leqslant 2}$ is an unite matrix. This implies that there exist two constants d_1 and d_2, such that

$$\begin{aligned} I''_{\frac{1}{2}}(h) &= d_1(k+1)(h_2-h)^{-1} + 5d_1(k+1)v\ln(h_2-h) + 5d_2(k+1) + \cdots, \\ I''_{\frac{3}{2}}(h) &= d_1(k-1)(h_2-h)^{-1} + d_1(5k+7)v\ln(h_2-h) + d_2(5k+7) + \cdots, \end{aligned}$$

as $h \to h_2^-$.

Since $-I'_{\frac{3}{2}}(h)$ is the period function of system (3.7), $-I''_{\frac{3}{2}}(h) \to +\infty$ as $h \to h_2^-$. Thus $d_1 \neq 0$. Consequently $w(h)$ has the following asymptotic expansion

$$w(h) = \frac{k-1}{k+1} + \frac{12v}{k+1}(h_2-h)\ln(h_2-h) + \cdots, \quad h \to h_2^-.$$

It turns out that $w(h_2) = w_{21}$, $w'(h_2) = -\infty$, which means that $\bar{w}(u_2) = w_{21}$, $\bar{w}'(u_2) = -\infty$.

On the other hand, by the global analysis of the phase portraits of system (3.12) it is easy to see that $\bar{w}(u)$ is an orbit of system (3.12) connecting S_1 and N_2. See Fig. 2.

Since $\bar{w}'(u_1) < 0$, $\bar{w}''(u_1) < 0$ and $\bar{w}'(u_2) = -\infty$, $\bar{w}(u)$ is monotonous decreasing and concave near S_1 and N_2. We claim that $\bar{w}(u)$ is global monotonous decreasing and concave on (u_2, u_1).

In fact, if $\bar{w}(u)$ has an extremum in (u_1, u_2), then there exists a straight line $\bar{w} = w_0$ which cuts the graph of $\bar{w}(u)$ at three points at least. Thus we can find two points on $(u_1, u_2) \times \{w_0\}$, at them $\frac{d\bar{w}}{dt} = 0$. However, this contradicts that $\frac{d\bar{w}}{dt}|_{w=w_0}$ has at most one zero (see (3.12)). Consequently, we conclude that $\bar{w}(u)$ is global monotonous decreasing on (u_2, u_1).

Suppose that $\bar{w}(u)$ has an inflection point. Then we would find a straight line $\bar{w} = mu + n$ cuts $\bar{w}(u)$ at least at three points and intersects the invariant straight line $\{u = u_1\}$ below the saddle S_1, at these three points the vector field is tangent to this line. This contradicts to the fact that $\frac{d\bar{w}}{dt} - m\frac{\dot{u}}{dt}|_{\bar{w}=mu+n}$ is a quadratic polynomial in u. Therefore, $\bar{w}(u)$ is convex on (u_1, u_2). The proof is complete. $\square$

Remark 3.1. In the proof of the above lemma, we show that $\bar{w}(u)$ is bounded. This implies that $I''_{\frac{1}{2}}(h) \neq 0$ for all $h \in (h_1, h_2)$. In fact, taking derivative with respect to h in (2.6)–(2.8), we obtain

$$I''_0(h) = \frac{1}{h}\left(-8kI''_{\frac{1}{2}}(h) + \frac{8}{3}(k+1)I''_{\frac{3}{2}}(h)\right), \tag{3.14}$$

$$-I'_{\frac{1}{2}}(h) = 8(k-1)I''_0(h) + 3hI''_{\frac{1}{2}}(h) + 8kI''_1(h), \tag{3.15}$$

$$3hI''_1(h) = -8(k-1)I''_{\frac{1}{2}}(h) - 8kI''_{\frac{3}{2}}(h). \tag{3.16}$$

Substituting (3.14) and (3.16) into (3.15), we have

$$3hI'_{\frac{1}{2}}(h) = 64I''_{\frac{3}{2}}(h) + \left[256k(k-1) - 9h^2\right]I''_{\frac{1}{2}}(h). \tag{3.17}$$

By (3.17) and $hI'_{\frac{1}{2}}(h) < 0$, it turns out that $|I''_{\frac{3}{2}}(h)| + |I''_{\frac{1}{2}}(h)| \neq 0$. Consequently, the fact $|w(h)| < \infty$ implies that $I''_{\frac{1}{2}}(h) \neq 0$ for all $h \in (h_1, h_2)$.

We can now prove the following lemma.

Lemma 3.6. *For any real numbers α, β, γ, the integral $I(h)$ in* (2.4) *has at most three isolated zeros in (h_1, h_2), taking into account their multiplicities. This bound is exact.*

Proof. Since $I(h_1) = 0$, the relation between $I(h)$ and $\tilde{\Psi}(h)$ implies that it is enough to prove that $\tilde{\Psi}(h)$ has at most two isolated zeros, taking into account their multiplicities.

Recall that $\tilde{\Psi}(h) = (ah^2 + b)I''_{\frac{1}{2}}(h) - (ch^2 + d)I''_{\frac{3}{2}}(h)$. If $ad - bc = 0$, then there exists a real number μ such that $\tilde{\Psi}(h) = (ch^2 + d)I''_{\frac{1}{2}}(h)(\mu - w(h))$ which has, by Lemma 3.5, at most two positive zeros. Therefore, we will assume without loss of generality that $ad - bc \neq 0$. If there exists some $h_0 \in (h_1, h_2)$ such that $ch_0^2 + d = 0$, then $\tilde{\Psi}(h_0) = (ah_0^2 + b)I''_{\frac{1}{2}}(h_0) \neq 0$. So we may be limited to consider the number of zeros of $\tilde{\Psi}(h)$ on the set $(h_1, h_2) \setminus \{h\colon\ ch^2 + d = 0\}$, and rewrite $\tilde{\Psi}(h) = (ch^2 + d)I''_{\frac{1}{2}}(h)(r(h) - w(h))$, where $r(h) = \frac{ah^2+b}{ch^2+d}$.

Let $u = h^2$ and $\bar{r}(u) = r(h) = r(-\sqrt{u})$, then the number of zeros of $\tilde{\Psi}(h)$ equals to the number of intersection points of the graphs of $\bar{r}(u) = \frac{au+b}{cu+d}$ and $\bar{w}(u)$, denoted by $C_{\bar{r}}$ and $C_{\bar{w}}$ respectively.

If $c = 0$, then $C_{\bar{r}}$ is a straight line and by Lemma 3.5 it could has at most two intersection points with $C_{\bar{w}}$. Hence we can suppose that $c \neq 0$, and write $\bar{r}(u) = \frac{\bar{a}u+\bar{b}}{u+\bar{d}}$, where $\bar{a} = a/c, \bar{b} = b/c, \bar{d} = d/c$.

If $\bar{r}'(u) > 0$, then the monotonicity of $\bar{w}(u)$ means that $C_{\bar{r}}$ can intersect $C_{\bar{w}}$ at two points at most. Thus we suppose in the rest of the proof that $\bar{r}'(u) < 0$. In this case, we denote by $C_{\bar{r}}^+$ and $C_{\bar{r}}^-$ the right-upper and the left-lower branch of $C_{\bar{r}}$ respectively.

It is easy to see that only one branch of $C_{\bar{r}}$ could intersect $C_{\bar{w}}$. By the convexities, $C_{\bar{r}}^+$ can intersect $C_{\bar{w}}$ at two points at most. Hence it remains to study the number of intersection points of $C_{\bar{r}}^-$ with $C_{\bar{w}}$

The hyperbola $\bar{r}(u)$ has a horizontal asymptote $u = \bar{a}$ and a vertical one $u = -\bar{d}$. The equation for the contact points between $C_{\bar{r}}$ and the flow in (3.12) is:

$$0 = \frac{d}{dt}\big(\bar{w} - \bar{r}(u)\big)\bigg|_{\bar{w}=\bar{r}(u)} = -\frac{\bar{E}(u)}{3(u+\bar{d})^2} = \frac{4(k+1)^2E(u)}{(cu+d)^2} \tag{3.18}$$

where $\bar{E}(u) = \bar{e}_3u^3 + \bar{e}_2u^2 + \bar{e}_1u + \bar{e}_0$ with $\bar{e}_3 = -27(\bar{a} - k + \bar{a}k)$ and $E(u) = e_3u^3 + e_2u^2 + e_1u + e_0$ with

$$\begin{aligned} e_3 &= 9(k+1)\left[9\eta + 32k(k+1)\right]\left[45k\eta + 45(k+1)\xi + 32(k+1)\left(4k^2+1\right)\right], \\ e_2 &= -8\big[81\left(376k^4 - 317k^2 - 32\right)\eta^2 - 2835(k+1)^2\xi^2 + 648k(k+1)\left(47k^2 - 44\right)\xi\eta \\ &\quad + 1728k(k+1)\left(96k^4 - 100k^2 + 13\right)\eta + 2304(k+1)^2\left(92k^4 - 85k^2 - 4\right)\xi \\ &\quad + 1024(k+1)^2\left(4k^2 - 1\right)^2\left(8k^2 - 5\right)\big]. \end{aligned} \tag{3.19}$$

By using the saddle S_1 property and the fact that $\deg \bar{E}(u) \leqslant 3$, it follows that $C_{\bar{r}}^-$ could has at most three intersection points with $C_{\bar{w}}$, taking the multiplicity into account.

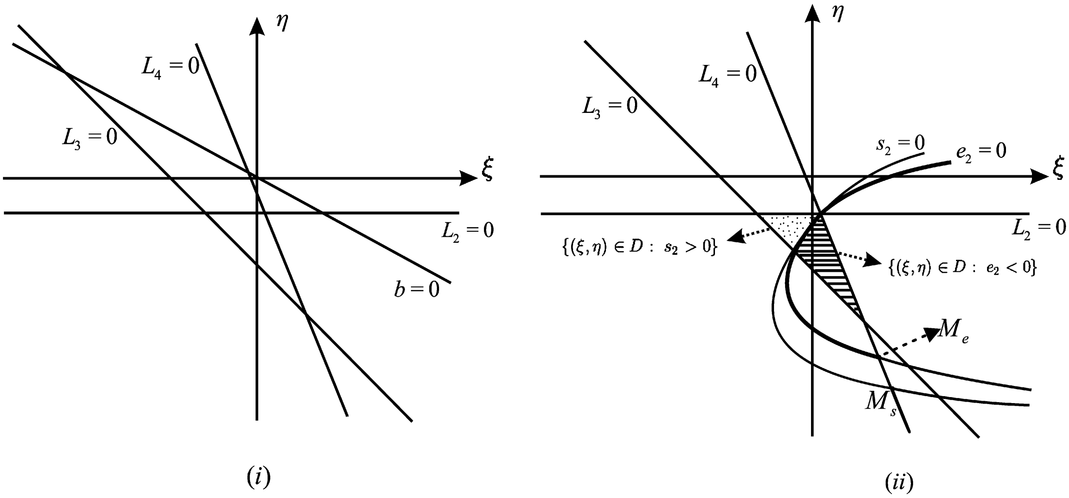

Fig. 3. (i): $\{(\xi,\eta)\colon\ L_2>0,\ L_3>0,\ L_4\leqslant 0,\ b>0\}$ is an empty set; (ii): $\{(\xi,\eta)\in D\colon\ e_2<0,\ s_2>0\}$ is an empty set.

If the vertical asymptote $u=-\bar d$ located on the left side of the straight line $u=u_1$, then $C_{\bar r}^-$ does not intersect $C_{\bar w}$ at all. This is also true if $-\bar d=u_1$. Therefore, in what follows we may suppose that $u_1<-\bar d$.

We will verify that $C_{\bar r}^-$ can intersect $C_{\bar w}$ at most twice. Assume the opposite. Then there are at least three contact points on $C_{\bar r}^-$ with the vector field, due to the behavior of the vector field near the hyperbolic saddle S_1. What is more, the relative position of horizontal asymptote $u=\bar a$ and the node N_2 means that $\bar a>w_{21}$, which yields $\bar e_3=-27(k+1)(\bar a-k/(k+1))>0$. This implies that on $C_{\bar r}^-$, the vector field for $u\ll 0$ is transversal to $C_{\bar r}^-$ in the positive direction. If $C_{\bar r}^-$ goes above S_1, then at the intersection points of $C_{\bar r}^-$ with the trajectory $u=u_1$ the vector field is transversal to $C_{\bar r}^-$ in the negative direction, which means an additional contact point on $C_{\bar r}^-$ for $u<u_1$, leading to a contradiction. Thus we may suppose that $C_{\bar r}^-$ goes below S_1.

If $-\bar d\leqslant u_2$ or $-\bar d>u_2$ but $C_{\bar r}^-$ goes below N_2, then the number of intersections of $C_{\bar r}^-$ and $C_{\bar w}$ is even. This implies that $C_{\bar r}^-$ can intersect $C_{\bar w}$ at most twice. Accordingly, it remains to consider the case that $-\bar d>u_2$ with that $C_{\bar r}^-$ goes above N_2. In this case, if $C_{\bar r}^-$ intersects $C_{\bar w}$ at three points, then the following conditions are satisfied:

$$\bar r'(u)<0,\qquad \bar r(0)>w_{21}>0,\qquad -\bar d>u_2,\qquad \bar r(u_2)>w_{21},\qquad \bar a\geqslant w_{21}. \tag{3.20}$$

Using (3.9) and $\bar a=a/c$, $\bar b=b/c$, $\bar d=d/c$, calculation gives

$$\bar r'(u)=\frac{ad-bc}{(cu+d)^2},\qquad \bar r(0)=\frac{b}{d},\qquad -\bar d-u_2=\frac{-8L_1}{9(1+k)L_2},$$
$$\bar r(u_2)-w_{21}=\frac{(k-1)(2k+1)^2L_3}{(k+1)L_1},\qquad \bar a-w_{21}=\frac{L_4}{4(1+k)L_2},$$

where

$$L_1=-36(k+2)(k-1)^2\eta+81(k+1)\xi+256k\big(k^2-1\big)(2k+1)^2,$$
$$L_2=9\eta+32k(k+1)=\frac{c}{4(k+1)},$$
$$L_3=9(k-1)\eta+9(k+1)\xi+64(k+1)(2k+1)^2,$$
$$L_4=9(5k+4)\eta+45(k+1)\xi+32(k+1)(2k+1)^2.$$

Thus (3.20) is equivalent to

$$ad<bc,\qquad bd>0,\qquad L_1L_2>0,\qquad L_1L_3>0,\qquad L_4L_2\leqslant 0. \tag{3.21}$$

Moreover, since $\bar a\geqslant w_{21}>0$ and $-\bar d>u_2>0$, we have $ac>0$ and $cd<0$. The graphics of the straight lines $L_i=0$ $(i=2,3,4)$ and $b=0$ is shown in Fig. 3.

If $L_2>0$, then $L_3>0, L_4\leqslant 0$ and $b>0$. One can easy check that in the (ξ,η)-plane, $\{(\xi,\eta)\colon\ L_2>0,\ L_3>0,\ L_4\leqslant 0,\ b>0\}$ is an empty set.

If $L_2<0$, then $L_3<0, L_4\geqslant 0$. Let $D=\{(\xi,\eta)\colon\ L_2<0,\ L_3<0,\ L_4\geqslant 0\}$. Below, we are going to show that for any $(\xi,\eta)\in D$, the polynomial $E(u)$ does not have three zeros in (u_1,u_2). Assume the opposite. Then it follows that $s_2:=2e_3u_2+e_2>0$ and $e_2e_3<0$.

Noting that $e_3 = 9(k+1)L_2[45k\eta + 45(k+1)\xi + 32(k+1)(4k^2+1)]$, one can easy check that $e_3 > 0$ for any $(\xi,\eta) \in D$. This means that $e_2 < 0$. Computation shows that

$$\begin{aligned} s_2 = 8\big[&81\big(104k^4 - 43k^2 - 120k + 32\big)\eta^2 + 2835(1+k)^2\xi^2 + 648(1+k)\big(13k^3 - k - 15\big)\eta\xi \\ &+ 1728(1+k)\big(48k^5 + 8k^3 - 36k^2 - 25k - 4\big)\eta - 2304(1+k)^2\big(32k^4 - 40k^2 + 15k - 4\big)\xi \\ &+ 1024(1+k)^2(1+2k)^2\big(64k^4 - 63k^3 + 36k^2 - 44k + 5\big)\big]. \end{aligned}$$

The conics $s_2 = 0$ is tangent to the conics $e_2 = 0$ at the point that $L_2 = L_3 = 0$. Denoted by M_s (resp. M_e) the intersect point of s_2 and L_4 (resp. e_2 and L_4). The abscissa of M_s and M_e are

$$x_s = -\frac{64(k+1)(1+2k)^2(36k^3 - 22k^2 - k - 13)}{9(52k^3 - 39k - 94)},$$

and

$$x_e = \frac{64(k+1)(1+2k)^2(36k^3 - 16k^2 - 13k - 7)}{9(188k^3 - 141k + 34)}$$

respectively. A simple calculation yields that

$$x_s - x_e = -\frac{384(k-1)^2(k+1)(1+2k)^2(4+5k)(32k^3 + 20k^2 + 16k + 1)}{(52k^3 - 39k - 94)(188k^3 - 141k + 34)} > 0.$$

By this information, it is not hard to check that in the region $\{(\xi,\eta) \in D\colon\ e_2 < 0,\ s_2 > 0\}$ is an empty set. See Fig. 3(ii).

Therefore, there is not $(\xi,\eta) \in \mathbf{R}^2$ such that $C_{\bar r}$ intersect $C_{\bar w}$ at three points. The proof is finished. □

4. Proof of the main theorem

In this section we are going to obtain the exact upper bound for the number of zeros of $I(h)$, counting the multiplicity.

We first use the Picard–Fuchs system (2.11) to obtain the asymptotic expansions of $P(h)$ and $Q(h)$ in a neighborhood of $h = h_1$.

Lemma 4.1. *If $h \to h_1^+$, then we have*

$$\begin{aligned} P(h) &= 1 + \frac{2k+1}{16}(h-h_1) + \frac{(2k+1)(110k^2 + 91k - 55)}{9216}(h-h_1)^2 \\ &\quad + \frac{(2k+1)(14\,740k^4 + 20\,020k^3 - 7551k^2 - 12\,530k + 1165)}{3\,538\,944}(h-h_1)^3 + \cdots, \\ Q(h) &= 1 + \frac{2k+3}{8}(h-h_1) + \frac{(k+1)(220k^2 + 292k + 231)}{4608}(h-h_1)^2 \\ &\quad + \frac{(k+1)(29\,480k^4 + 54\,780k^3 + 38\,418k^2 + 11\,789k + 1155)}{1\,769\,472}(h-h_1)^3 + \cdots. \end{aligned}$$

Proof. Recall that $I_\mu(h)(\mu = -\frac{1}{2}, 0, \frac{1}{2}, 1)$ are analytic at $h = h_1$ since the Hamiltonian value h_1 corresponds to the center of system (2.1). An easy manipulation shows that

$$I_{-\frac{1}{2}}(h_1) = I_0(h_1) = I_{\frac{1}{2}}(h_1) = I_1(h_1) = 0, \qquad I'_{-\frac{1}{2}}(h_1) = I'_0(h_1) = I'_{\frac{1}{2}}(h_1) = I'_1(h_1).$$

Hence we can set

$$I_\mu(h) = I'_1(h_1)(h-h_1) + b_\mu(h-h_1)^2 + c_\mu(h-h_1)^3 + d_\mu(h-h_1)^4 + o\big(|h-h_1|^4\big). \tag{4.1}$$

Substituting (4.1) into (2.11), we get the following expressions after a long but straightforward manipulations:

$$\begin{aligned} b_{-\frac{1}{2}} &= \frac{1}{192}\big(10k^2 + 29k + 31\big)I'_1(h_1), \qquad b_0 = \frac{1}{192}\big(10k^2 + 17k + 7\big)I'_1(h_1), \\ b_{\frac{1}{2}} &= \frac{5}{192}(k+1)(2k-1)I'_1(h_1), \qquad b_1 = \frac{1}{192}\big(10k^2 - 7k - 5\big)I'_1(h_1); \\ c_{-\frac{1}{2}} &= \frac{35}{110592}(1+k)^2\big(44k^2 + 52k + 11\big)I'_1(h_1), \\ c_0 &= \frac{35}{110592}(1+k)^2(2k-1)(22k+13)I'_1(h_1), \end{aligned}$$

$$
\begin{aligned}
c_{\frac{1}{2}} &= \frac{1}{110592}\big(308k^4+308k^3-231k^2-226k+5\big)I_1'(h_1),\\
c_1 &= \frac{35}{110592}(2k-1)\big(22k^3+9k^2-24k-11\big)I_1'(h_1),\\
d_{-\frac{1}{2}} &= \frac{5005}{127401984}(1+k)^3(2k-1)\big(68k^2+76k+17\big)I_1'(h_1),\\
d_0 &= \frac{35}{127401984}(1+k)^2\big(19448k^4+10868k^3-14586k^2-5665k+1313\big)I_1'(h_1),\\
d_{\frac{1}{2}} &= \frac{385}{127401984}(1+k)^2(2k-1)\big(884k^3-663k-103\big)I_1'(h_1),\\
d_1 &= \frac{35}{127401984}(k+1)\big(19448k^5-10868k^4-24310k^3+10637k^2+5980k-1243\big)I_1'(h_1).
\end{aligned}
$$

Using these expressions and (2.5) we obtain the required asymptotic expansion of $P(h)$ and $Q(h)$ by direct calculation. We also omit the course for sake of brevity. □

By Lemma 4.1, $P(h)$ is strict monotonous decreasing in a neighborhood of $h = h_1$. Moreover, we have the following result.

Lemma 4.2. $P'(h) < 0$, $h \in (h_1, h_2)$.

Proof. It is suffices to prove that the equation $P(h) = -1/\nu$ has at most one solution in (h_1, h_2) for any $\nu \in \mathbf{R}\setminus\{0\}$. To this end, we verify that $\nu I_0(h) + I_1(h)$ has at most one zero in (h_1, h_2). Noting that the number of the zeros of $\nu I_0(h) + I_1(h)$ equals to the number of the limit cycles of system

$$
\begin{aligned}
\dot{x} &= -2xy,\\
\dot{y} &= k-1-2kx+(k+1)x^2-\frac{1}{2}y^2+\epsilon(\nu x+1)y, \quad 0<|\epsilon|\ll 1,
\end{aligned}
$$

counting multiplicities. By the results of [29], any planar quadratic differential system with an invariant line has at most one limit cycle. Thus $\nu I_0(h) + I_1(h)$ has at most one zero. This completes the proof of the result. □

Since $P'(h) < 0$, $h \in (h_1, h_2)$, the inverse function of $P = P(h)$, say $h = h(P)$, exists and we can treat Q as a function of P: $Q = Q(h(P)) := \tilde{Q}(P)$. We will determine the geometric behavior of Σ: $Q = \tilde{Q}(P)$ in its two endpoints, namely $C = (P(h_1), P(h_1)) = (1, 1)$ and $S = (P(h_2), P(h_2))$.

Lemma 4.3. $\frac{dQ}{dP}|_{h=h_1} = \frac{2(2k+3)}{2k+1}$, $\frac{d^2Q}{dP^2}|_{h=h_1} = \frac{4(10k+11)}{(2k+1)^2}$, $\frac{d^3Q}{dP^3}|_{h=h_1} = \frac{-380k^3-320k^2+2213k+2225}{6(2k+1)^3}$.

Proof. Applying the result of Lemma 4.1 and the formulas

$$
\begin{aligned}
\frac{dQ}{dP}\bigg|_{h=h_1} &= \frac{Q'(h_1)}{P'(h_1)},\\
\frac{d^2Q}{dP^2}\bigg|_{h=h_1} &= \frac{Q''(h_1)P'(h_1)-Q'(h_1)P''(h_1)}{[P'(h_1)]^3},\\
\frac{d^3Q}{dP^3}\bigg|_{h=h_1} &= \frac{Q'''(h)(P'(h))^2-3Q''(h)P''(h)P'(h)+Q'(h)[3(P''(h))^2-P'(h)P'''(h)]}{[P'(h)]^5}\bigg|_{h=h_1},
\end{aligned}
$$

we obtain the required results by direct calculation. □

Our next goal will be to obtain the geometric behavior of Σ near the endpoint S. Since the Hamiltonian value h_2 corresponds to the homoclinic orbit of system (2.1), we can write

$$
I_\mu(h) = \tilde{a}_\mu + \tilde{b}_\mu(h_2-h)\ln(h_2-h) + \tilde{c}_\mu(h_2-h) + \tilde{d}_\mu(h_2-h)^2\ln(h_2-h) + \cdots \tag{4.2}
$$

in a neighborhood of $h = h_2$, where $\tilde{a}_\mu, \tilde{b}_\mu, \tilde{c}_\mu\tilde{d}_\mu \in \mathbf{R}$, $\mu = -\frac{1}{2}, 0, \frac{1}{2}, 1$. Clearly, $\tilde{a}_\mu < 0, \tilde{b}_\mu < 0$ since $I_\mu(h) < 0, I_\mu'(h) < 0$.

Substituting (4.2) into (2.11), some tedious computations show that

$$
\begin{aligned}
&3h_2\tilde{b}_{-\frac{1}{2}} + 24k\tilde{b}_0 - 8(k+1)\tilde{b}_1 = 0,\\
&8(k-1)\tilde{b}_{-\frac{1}{2}} + 3h_2\tilde{b}_0 + 8k\tilde{b}_{\frac{1}{2}} = 0,\\
&8(k-1)\tilde{b}_0 + 3h_2\tilde{b}_{\frac{1}{2}} + 8k\tilde{b}_1 = 0,\\
&-24k(k-1)\tilde{b}_{-\frac{1}{2}} - 6kh_2\tilde{b}_0 + 8\big(k^2-1\big)\tilde{b}_{\frac{1}{2}} + 3(k+1)h_2\tilde{b}_1 = 0.
\end{aligned}
$$

From the above equalities and (2.5) we obtain the relations between the coefficients:

$$\tilde{b}_{\frac{1}{2}} = \lambda\tilde{b}_1, \qquad \tilde{b}_{-\frac{1}{2}} = \lambda^3\tilde{b}_1, \qquad \tilde{b}_0 = \lambda^2\tilde{b}_1, \qquad \tilde{b}_{-1} = \lambda^4\tilde{b}_1, \quad \text{with } \lambda = \sqrt{\frac{k+1}{k-1}}. \tag{4.3}$$

Define $J_{-1}(h) = I_{-1}(h) - \lambda^4 I_1(h)$, $J_0(h) = I_0(h) - \lambda^2 I_1(h)$. By (4.2) and (4.3) it follows that $|J'_{-1}(h)| < +\infty$, $|J'_0(h)| < +\infty$. Let

$$\tilde{P}(h) = J_0(h)/I_1(h) = P(h) - \lambda^2, \qquad \tilde{Q}(h) = J_{-1}(h)/I_1(h) = Q(h) - \lambda^4$$

and define the corresponding curve

$$\tilde{\Sigma} := \big\{(\tilde{P}, \tilde{Q}) = \big(\tilde{P}(h), \tilde{Q}(h)\big)\colon h \in (h_1, h_2)\big\}.$$

The $\tilde{\Sigma}$ is just a translation of Σ. Denoted respectively by L_S and $L_{\tilde{S}}$ the tangent line to Σ at S and the tangent line to $\tilde{\Sigma}$ at $\tilde{S} := (\tilde{P}(h_2), \tilde{Q}(h_2)) = (P(h_2) - \lambda^2, Q(h_2) - \lambda^4)$.

Lemma 4.4.

(i) *The curve Σ lies below L_S.*
(ii) *The curve Σ is increasing near S.*

Proof. (i) Obviously, it suffices to prove that $\tilde{\Sigma}$ lies below $L_{\tilde{S}}$.

From (3.5) ($c_2 \neq 0$) and $\tilde{b}_{\frac{1}{2}} = \lambda\tilde{b}_1$ it follows that $|I'_1(h)| \to +\infty$ as $h \to h_2^-$. Since $|J'_{-1}(h)| < +\infty$, $|J'_0(h)| < +\infty$, we obtain

$$
\begin{aligned}
\lim_{h\to h_2^-}\frac{\tilde{Q}'(h)}{\tilde{P}'(h)} &= \lim_{h\to h_2^-}\frac{J'_{-1}(h)I_1(h) - J_{-1}(h)I'_1(h)}{J'_0(h)I_1(h) - J_0(h)I'_1(h)}\\
&= \lim_{h\to h_2^-}\frac{J'_{-1}(h)/I'_1(h) - J_{-1}(h)/I_1(h)}{J'_0(h)/I'_1(h) - J_0(h)/I_1(h)}\\
&= \lim_{h\to h_2^-}\frac{J_{-1}(h)/I_1(h)}{J_0(h)/I_1(h)} = \frac{\tilde{Q}(h_2)}{\tilde{P}(h_2)}.
\end{aligned}
$$

Thus the equation of $L_{\tilde{S}}$ is $Q = \frac{\tilde{Q}(h_2)}{\tilde{P}(h_2)}P$.

On the other hand, we find on the Γ_h that $x < \frac{k-1}{k+1} = \lambda^{-2}$, hence $J_0(h) = \oint_{\Gamma_h} x^{-\frac{1}{2}}(x^{-1} - \lambda^2)y\,dx < 0$. Since $I_1(h) < 0$, it turns out that $\tilde{P}(h) > 0$. Noting that

$$\tilde{Q}(h) - \frac{\tilde{Q}(h_2)}{\tilde{P}(h_2)}\tilde{P}(h) = \tilde{P}(h)\left(\frac{\tilde{Q}(h)}{\tilde{P}(h)} - \frac{\tilde{Q}(h_2)}{\tilde{P}(h_2)}\right),$$

the proof will be complete if we show that $\tilde{Q}(h)/\tilde{P}(h)$ is strictly increasing.

It is easy to see that $\tilde{Q}(h)/\tilde{P}(h)$ can be wrote as the ratio of two Abelian integrals $\frac{\oint_{H=h} f_2(x)y\,dx}{\oint_{H=h} f_1(x)y\,dx}$. In Theorem 2 of [19], a method to determine the monotonicity of such a ratio of two Abelian integrals is given. Before applying the result of that paper, we perform a change of variable: $\bar{x} = \sqrt{x}$ and change the first integral $H(x,y) = h$ of system (2.1) to $\bar{H}(\bar{x}, y) = h$, where (for the simplicity in notation we still write $\bar{x}$ as x): $\bar{H}(x,y) = \phi(x)y^2 + \Phi(x)$,

$$\phi(x) = x^{-1}, \qquad \Phi(x) = \frac{2}{3}(k+1)x^3 - 4kx - 2(k-1)x^{-1},$$

and corresponding, $I_i(h) = 2\oint_{\bar{H}=h} x^{2i-2}y\,dx$. Consequently

$$\frac{\tilde{Q}(h)}{\tilde{P}(h)}=\frac{I_{-1}(h)-\lambda^4 I_1(h)}{I_0(h)-\lambda^2 I_1(h)}=\frac{\oint_{\bar{H}=h} f_2(x)y\,dx}{\oint_{\bar{H}=h} f_1(x)y\,dx},$$

where $f_1(x)=x^{-2}-\lambda^2$, $f_2(x)=x^{-4}-\lambda^4$. Obviously, $f_1(x)$ verifies the hypothesis (H_2) of [19].

The homoclinic orbit $\bar{H}(x,y)=h_2$ cut the x-axis in two points: $(\sqrt{\frac{4k+2}{k+1}}-\sqrt{\frac{k-1}{k+1}},0)$ and $(\sqrt{\frac{k-1}{k+1}},0)$. Since $\Phi'(x)=x^{-2}(x^2-1)(x^2+kx^2-k+1)$, the hypothesis $(H1)(i)$ of [19] is satisfied. By using the formula (19) of [19], we define the function

$$\zeta(x)=\frac{f_2(x)\sqrt{\phi(\tilde{x})}\Phi'(\tilde{x})-f_2(\tilde{x})\sqrt{\phi(x)}\Phi'(x)}{f_1(x)\sqrt{\phi(\tilde{x})}\Phi'(\tilde{x})-f_1(\tilde{x})\sqrt{\phi(x)}\Phi'(x)},\quad x\in\left(\sqrt{\frac{4k+2}{k+1}}-\sqrt{\frac{k-1}{k+1}},1\right),$$

where $\tilde{x}=\tilde{x}(x)$ is an analytic involution defined by $\Phi(\tilde{x})=\Phi(x)$.

According to Theorem 2 of [19], to prove that $\tilde{Q}(h)/\tilde{P}(h)$ is strictly monotonically increasing, it is enough to prove that $\zeta'(x)<0$. It turns out after a straightforward calculation that

$$\zeta(x)=\frac{\tilde{x}^{\frac{3}{2}}(1+\lambda^2x^2)(\tilde{x}^2-1)-x^{\frac{3}{2}}(1+\lambda^2\tilde{x}^2)(x^2-1)}{(\tilde{x}x)^{\frac{3}{2}}[\sqrt{x}(\tilde{x}^2-1)-\sqrt{\tilde{x}}(x^2-1)]}:=\frac{A(x,\tilde{x})}{B(x,\tilde{x})}.$$

Then $\zeta'(x)=\frac{\kappa(x,\tilde{x})+\kappa(\tilde{x},x)\tilde{x}'(x)}{B^2(x,\tilde{x})}$, where $\kappa(x,\tilde{x})=A'_x(x,\tilde{x})B(x,\tilde{x})-A(x,\tilde{x})B'_x(x,\tilde{x})$.

Since $\tilde{x}'(x)=\frac{\Phi'(x)}{\Phi'(\tilde{x})}<0$, and

$$\kappa(x,\tilde{x})=\frac{1}{2}\sqrt{x}(\sqrt{x}-\sqrt{\tilde{x}})^2\big(1-\tilde{x}^2\big)\tilde{x}^{\frac{3}{2}}\big[x+3x^3+2\sqrt{x\tilde{x}}+6x^{\frac{5}{2}}\sqrt{\tilde{x}}+3\tilde{x}+9x^2\tilde{x}+12(\tilde{x}x)^{\frac{3}{2}}+8x\tilde{x}^2+4x\tilde{x}^{\frac{5}{2}}\big],$$

which yields that $\kappa(x,\tilde{x})<0$, $\kappa(\tilde{x},x)>0$, we conclude that $\zeta'(x)<0$. This prove that $\tilde{Q}(h)/\tilde{P}(h)$ is strictly monotonically increasing.

(ii) By (4.2) and (4.3), we obtain for $h\to h_2^-$ that

$$Q(h)=\frac{\tilde{a}_{-1}}{\tilde{a}_1}+\frac{(\lambda^4\tilde{a}_1-\tilde{a}_{-1})\tilde{b}_1}{\tilde{a}_1^2}(h_2-h)\ln(h_2-h)+\cdots.$$

Since

$$\lambda^4\tilde{a}_1-\tilde{a}_{-1}=\oint\limits_{H=h_2} x^{-\frac{1}{2}}\big(\lambda^4-x^{-2}\big)y\,dx>0$$

and $\tilde{b}_1<0$, it follows that $Q(h)$ is decreasing for $h\to h_2^-$. Hence, by Lemma 4.2, near the endpoint S we find that $dQ/dP=Q'(h)/P'(h)>0$.

The proof is finished. □

By Lemma 4.3 and Lemma 4.4, the convex properties of the centroid curve Σ at the two endpoints are different for the case that $k\in(-1.1,1)$. Thus there are at least one inflexion. One can easily find a straight line cutting Σ at least three points. Consequently, we obtain the following conclusion by using Lemma 3.6 immediately.

Theorem 4.1. *If $k\in(-1.1,1)$, then $I(h)$ has at most three isolated zeros, counting the multiplicity. This bound is exact.*

Next we consider the cases that $k\leqslant -1.1$. First of all, we prove the following lemma.

Lemma 4.5. *For each $k\in(-\infty,-1.1]$, the centroid curve Σ lies below L_C.*

Proof. By Lemma 4.3, if $k<-1.1$, then $\frac{d^2Q}{dP^2}|_{h_1}<0$. If $k=-1.1$, then $\frac{d^2Q}{dP^2}|_{h_1}=0$, $\frac{d^3Q}{dP^3}|_{h_1}=\frac{35}{4}$. Noting that $P'(h)<0$, hence it follows that for both cases, the centroid curve is concave near the endpoint C. Since L_C go through C and its slope is $\frac{2(2k+3)}{2k+1}$, the equation of L_C is

$$L_C:\quad (2k+1)Q-2(2k+3)P+2k+5=0.$$

Substituting $\alpha=2k+5$, $\beta=2k+1$, $\gamma=-2(2k+3)$ into (3.1), it follows that

$$3(k-1)\Psi(h)=-\frac{128}{3}(2k+1)\big(k^2-1\big)I'_{\frac{1}{2}}(h)+\frac{16}{3}(2k+1)\big(4k^2+4k-9\big)I'_{\frac{3}{2}}(h)+\frac{3}{4}(2k+1)h^2I'_{-\frac{1}{2}}(h).$$

Multiplied by $\frac{4}{3(2k+1)}$ both sides of the above equation yields

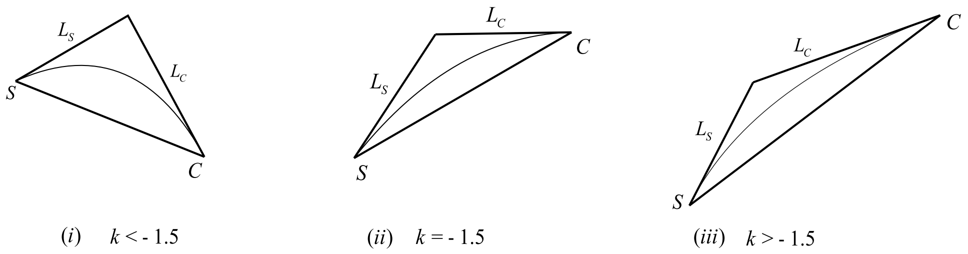

Fig. 4. Near the endpoints Σ is placed inside the triangle T.

$$\frac{4(k-1)}{2k+1}\Psi(h)=-\frac{512}{9}\big(k^2-1\big)I'_{\frac{1}{2}}(h)+\frac{64}{9}\big(4k^2+4k-9\big)I'_{\frac{3}{2}}(h)+h^2I'_{-\frac{1}{2}}(h). \tag{4.4}$$

Substituting $\xi=-\frac{512}{9}(k^2-1)$, $\eta=\frac{64}{9}(4k^2+4k-9)$ into

$$\frac{4(k-1)}{2k+1}\Psi(h)=\eta I'_{\frac{3}{2}}+\xi I'_{\frac{1}{2}}+h^2I'_{-\frac{1}{2}},$$

it turns out that

$$\frac{4(k-1)}{2k+1}\tilde{\Psi}(h)=\big(\hat{a}h^2+\hat{b}\big)I''_{\frac{1}{2}}-\big(\hat{c}h^2+\hat{d}\big)I''_{\frac{3}{2}}, \tag{4.5}$$

where

$$\begin{aligned}&\hat{a}=-2592(k-1), \qquad \hat{b}=18432(k-1)(2k-1)^2,\\ &\hat{c}=1152(k^2-1)(k+2), \qquad \hat{d}=-8192\big(k^2-1\big)(k+2)(2k-1)^2.\end{aligned} \tag{4.6}$$

If $k=-2$, then $\hat{c}=\hat{d}=0$ and

$$36\tilde{\Psi}(h)=8\big(97h^2-172800\big)I''_{\frac{1}{2}}\neq 0, \quad \text{for } h\in(h_1,h_2)=(172800/97,192).$$

Suppose that $k\neq -2$. Then by (4.5) and (4.6) we obtain

$$\frac{4(k-1)}{2k+1}\tilde{\Psi}(h)=32\big(k^2-1\big)(k+2)\big[36h^2-256(2k-1)^2\big]I''_{\frac{1}{2}}\bigg[\frac{9}{4(k+1)(k+2)}-w(h)\bigg].$$

As the proof of Lemma 3.5 has shown, $w(h)\in(w_{21},w_1)$. Computation shows

$$\bigg(\frac{9}{4(k+1)(k+2)}-w_{21}\bigg)\bigg(\frac{9}{4(k+1)(k+2)}-w_1\bigg)=\frac{(2k-1)^2(2k+1)^2(11+10k)}{80(k+1)^2(k+2)^2}\geqslant 0.$$

Since $k\leqslant -1.1$, it follows that $\frac{9}{4(k+1)(k+2)}-w(h)\neq 0$, which implies that $\tilde{\Psi}(h)\neq 0$ for $h\in(h_1,h_2)$.

Finally, noting that $\Psi(h_1)=I(h_1)-h_1I'(h_1)=-h_1I'(h_1)(\alpha+\beta+\gamma)=0$, the conclusion $\tilde{\Psi}(h)\neq 0$ yields that $\Psi(h)\neq 0$. Hence $I(h)\neq 0$. This means that L_C has not intersection point with Σ. □

Now it is ready to finish the proof of Theorem 2.1 for $k\leqslant -1.1$.

Theorem 4.2. *If $k\leqslant -1.1$, then $I(h)$ has at most two isolated zeros, counting the multiplicity.*

Proof. By Lemma 4.3 and Lemma 4.4, Σ is concave in a neighborhood of its two endpoints. Furthermore, near its endpoints it is placed inside the triangle T formed by the lines L_C, L_S and the line going through C and S (Lemma 4.4(ii)). See Fig. 4. If Σ leaves T, then there would exist a straight line L going through a point of Σ sufficiently close to some of its endpoints and intersecting Σ at least at four points (taking the multiplicities into account). This yields a contradiction. If Σ is placed inside T but not globally concave, then we can also find a straight ling L intersecting Σ at four points, also a contradiction. Therefore, Σ is a concave curve, which implies that $I(h)$ has at most two zeros. □

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