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An inverse problem for fractional diffusion equation in 2-dimensional case: Stability analysis and regularization

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ABSTRACT

In this paper we investigate an inverse problem for a time-fractional diffusion equation which is highly ill-posed in the two-dimensional setting. Based on an a priori assumption, we give a conditional stability result. Some new regularization methods are constructed for solving the inverse problem and the corresponding error estimates are proved. For numerical illustration, several examples are constructed to demonstrate the feasibility and efficiency of the proposed methods.

1. Introduction

In recent years, the studies on fractional calculus and derivatives have drawn attention from various disciplines of science and engineering, for instance, mechanical engineering [1], viscoelasticity [2], Lévy motion [3], electron transport [4], dissipation [5], heat conduction [6–8] and high-frequency financial data [9]. A number of experiments have shown that, in the process of modeling real physical phenomena such as Brownian motion, fractional calculus and derivatives provide more accurate simulation than traditional calculus with integer order derivatives. Fractional derivatives have also proven to be more flexible in describing viscoelastic behavior. In particular, fractional models are believed to be more realistic in describing anomalous diffusion in heterogeneous porous media.

It is known that in solving a well-posed forward problem for partial differential equations, if the initial concentration distribution and boundary conditions are given, a complete recovery of the unknown solution is attainable. On the other hand, if the boundary data can only be measured on a portion of the boundary or at some scattered points in the solution domain, this leads to an inverse problem in the sense that the solution does not depend continuously on the given data [10]. Many kinds of inverse problems are ill-posed in the sense that a small error in the boundary data will cause an enormous error in the solution. These inverse problems are usually referred to as ill-posed backward determination problems [11] which are by nature unstable because the unknown solution and its derivatives have to be determined from indirect observable data which usually contains measurement errors. The major difficulty in establishing any numerical algorithm for approximating the solution is due to the severe ill-posedness of the inverse problem. In this paper, we investigate an inverse heat conduction problem (IHCP) for a fractional order diffusion equation in the two-dimensional (2-D) case. We remark here that the IHCP for the one-dimensional (1-D) case had been well studied in the last few decades. Due to severe ill-posedness, it is, however, much more difficult to solve IHCP in the 2-D case than in the 1-D case. To the knowledge of the authors, there are still very few results on inverse problems for fractional order diffusion equations in the 2-D case.

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For inverse problems of the fractional diffusion equation in the one-dimensional setting, many authors have investigated related topics. For example, the uniqueness of an inverse problem for one-dimensional fractional diffusion equation was given by Cheng et al. [12]. Zheng and Wei [13] gave a regularization method for a Cauchy problem of the time fractional advection–dispersion equation in a spatially unbounded domain. Numerical results using difference methods were given respectively by Bondarenko and Ivaschenko [14] and Murio [15]. In this paper, we focus on the investigation of a fractional inverse heat conduction problem (FIHCP) defined in a semi-infinite 2-D slab domain and apply the Fourier transform for stability analysis and error estimate. Many of the numerical methods for solving ill-posed problems are based on solving the corresponding direct problem iteratively. The iteration and the initial guess value are important and sensitive in these iterative computational methods. By using static and dynamic regularization methods, in this paper, we devise a one-stage direct computational method for solving the ill-posed problem. The proposed methods will improve the convergence of the iterative procedures by providing good initial guesses.

Since the 2-dimensional FIHCP is much more ill-posed than the 1-dimensional FIHCP, obtaining the convergence error estimates for the 2-dimensional case is very difficult. In this paper, we consider a revised version of the Fourier method [16] in the 2-dimensional setting. To avoid the constraint that the regularization parameter is selected statically in the Fourier method, i.e., the parameter does not depend on the location of reconstruction (see (4.4)), we construct a new dynamic regularization procedure in which the regularization parameter is selected dynamically with respect to the location of reconstruction (see (5.17)).

The paper is organized as follows. In Section 2 we first give an analysis on the ill-posedness of the 2-D FIHCP. The conditional stability analysis is then given in Section 3. In Sections 4 and 5, some regularization solutions are constructed with optimal order error estimates. In particular, we give in Section 5 some new dynamic methods for reconstructing stable solutions. We stress here that one of the proposed dynamic methods gives an optimal error bound for the 2-D problem under consideration not only in theory, but also can give a good approximation in numerical experiments. The numerical verification is demonstrated in Section 6 in which several examples are presented for the validity of the proposed dynamic regularization methods.

2. Mathematical formulation

In several engineering contexts, it is sometimes necessary to estimate the surface temperature or heat flux in a body from a measured temperature history at a fixed location inside a body. This is the so-called inverse heat conduction problem. We consider a two-dimensional inverse heat conduction problem in a semi-infinite slab made of heterogeneous material:

$$\frac{\partial^\beta u}{\partial t^\beta} - u_{xx} - u_{yy} = 0, \quad x > 0, y > 0, t > 0, \tag{2.1}$$

$$u(1, y, t) = g(y, t), \quad y \geq 0, t \geq 0, \tag{2.2}$$

with the corresponding measured data function $g_\delta(y, t)$, and the initial and boundary conditions

$$u(x, y, 0) = 0, \quad x > 0, y \geq 0, \tag{2.3}$$

$$u(x, 0, t) = 0, \quad x > 0, t \geq 0, \tag{2.4}$$

$$u_y(x, 0, t) = 0, \quad x > 0, t \geq 0, \tag{2.5}$$

$$u(x, y, t)|_{x\to\infty} \text{ bounded}, \quad y > 0, t > 0, \tag{2.6}$$

where the time fractional derivative $\frac{\partial^\beta u}{\partial t^\beta}$ is the Caputo fractional derivative of order β $(0 < \beta \leq 1)$ defined by (cf. [17])

$$\frac{\partial^\beta u}{\partial t^\beta}(x, y, t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial u(x, y, s)}{\partial s} \frac{ds}{(t-s)^\beta}, \quad 0 < \beta < 1, \tag{2.7}$$

$$\frac{\partial^\beta u}{\partial t^\beta} = \frac{\partial u(x, y, t)}{\partial t}, \quad \beta = 1. \tag{2.8}$$

Since the order β of the derivative with respect to time in the heat equation can be of arbitrary real order, the heat equation is called the fractional diffusion equation. For $\beta = 1$, the equation becomes the classical diffusion equation. For $0 < \beta < 1$, we have the ultraslow diffusion [18].

If we take the Laplace transform of both sides of (2.1) with respect to y and t, according to the properties of the Laplace transform of the Caputo derivative (cf. [17] pp 106), we get

$$s_2^\beta U(x, s_1, s_2) - u(x, y, 0) - U_{xx}(x, s_1, s_2) - (s_1^2 U(x, s_1, s_2) - u_y(x, 0, t) - s_1 u(x, 0, t)) = 0, \tag{2.9}$$

where s_1, s_2 are the variables of the Laplace transform on y, t, respectively, and U is the Laplace transform of u. Applying the conditions (2.3)–(2.5), we have

$$U_{xx}(x, s_1, s_2) = (s_2^\beta - s_1^2) U(x, s_1, s_2), \tag{2.10}$$

which is a second-order ordinary differential equation. Now using the conditions (2.2) and (2.6), we can get with the principal square root of $\sqrt{s_2^\beta - s_1^2}$

$$U(x, s_1, s_2) = \exp\left(\sqrt{s_2^\beta - s_1^2}(1-x)\right) G(s_1, s_2), \tag{2.11}$$

where G is the Laplace transform of g.

Throughout this paper, we extend all the functions to the whole plane $-\infty < t < \infty, -\infty < y < \infty$ by setting the functions to be zero for $t < 0$ and $y < 0$. Let

$$\hat{f}(\xi, \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(y, t)e^{-i(\xi y + \tau t)}dydt \tag{2.12}$$

be the Fourier transform of the function $f(y, t) \in L^2(\mathbb{R}^2)$.

For the functions $h(y, t)$ which vanish on the negative t axis and y axis, the Fourier and Laplace transforms are related via

$$H(i\xi, i\tau) = 2\pi\hat{h}(\xi, \tau).$$

Therefore, from (2.11), setting $s_1 = i\xi,\ s_2 = i\tau$, the solution of (2.1)–(2.6) can be formulated in the frequency domain:

$$\hat{u}(x, \xi, \tau) = e^{(1-x)\sqrt{(i\tau)^\beta + \xi^2}}\hat{g}(\xi, \tau), \tag{2.13}$$

where

$$(i\tau)^\beta = |\tau|^\beta\left(\cos\left(\frac{\beta\pi}{2}\right) + i\,\mathrm{sign}(\tau)\sin\left(\frac{\beta\pi}{2}\right)\right). \tag{2.14}$$

Denote η as follows

$$\eta := \sqrt{(i\tau)^\beta + \xi^2}. \tag{2.15}$$

η can be written in real and imaginary parts, i.e.

$$\eta = \sqrt{\frac{\sqrt{|\tau|^{2\beta}\sin^2\frac{\beta\pi}{2} + \left(\xi^2 + |\tau|^{2\beta}\cos\left(\frac{\beta\pi}{2}\right)\right)^2} + \left(\xi^2 + |\tau|^{2\beta}\cos\left(\frac{\beta\pi}{2}\right)\right)}{2}}$$
$$+ i\,\mathrm{sign}\left(\mathrm{sign}(\tau)|\tau|^\beta\sin\left(\frac{\beta\pi}{2}\right)\right)\sqrt{\frac{\sqrt{|\tau|^{2\beta}\sin^2\frac{\beta\pi}{2} + \left(\xi^2 + |\tau|^{2\beta}\cos\left(\frac{\beta\pi}{2}\right)\right)^2} - \left(\xi^2 + |\tau|^{2\beta}\cos\left(\frac{\beta\pi}{2}\right)\right)}{2}}.$$

The inverse Fourier transform on (2.13) yields

$$u(x, y, t) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\widehat{g}(\xi, \tau)e^{(1-x)\sqrt{(i\tau)^\beta + \xi^2}}e^{i(\xi y + \tau t)}d\xi d\tau.$$

We need to seek the solution $u(x, y, t)$ from the given data $g(y, t)$. Physically, $g(y, t)$ can only be measured, there should be measurement errors, and we would actually have some data functions $g_\delta(y, t) \in L^2(\mathbb{R}^2)$, for which

$$\|g_\delta(\cdot, \cdot) - g(\cdot, \cdot)\| \le \delta, \tag{2.16}$$

where the constant $\delta > 0$ represents a bound on the measurement errors, $\|\cdot\|$ denotes the L^2-norm. And the solution is assumed to be a priori bounded at the inaccessible surface

$$\|u(0, \cdot, \cdot)\| \le E \tag{2.17}$$

for some $E > 0$.

We note that for a fixed $0 < x \le 1$, $|e^{(1-x)\eta}|$ is unbounded as $\mathfrak{Re}(\eta) \to \infty$. There we want to seek a solution $u(x, \cdot, \cdot) \in L^2(\mathbb{R}^2)$, so that, from (2.13) and the Parseval equality, the exact data $\hat{g}(\xi, \tau)$ must decay rapidly as $|\xi|, |\tau| \to \infty$. But in practice, we can only get the noisy data $\hat{g}_\delta(\cdot, \cdot) \in L^2(\mathbb{R}^2)$. Hence for the noisy data $\hat{g}_\delta(\cdot, \cdot)$, we cannot obtain a meaningful solution. This implies that the ill-posedness of the problem (2.1)–(2.6) is caused by the high frequency components of $\hat{g}(\cdot, \cdot)$. For ill-posed problems, it is impossible to solve by using classical numerical methods and requires special techniques to be employed. A natural idea is to cut off all high frequency components from the solution. This idea was first proposed in [16,19] where the method is called the Fourier method for stabilizing a one-dimensional inverse heat conduction problem. In the forthcoming sections, we will discuss some spectral-type regularization methods. Firstly we discuss the issue on the stability estimate for the problem (2.1)–(2.6).

Throughout this paper, we denote the real part and imaginary part of η as follows

$$a := \mathfrak{Re}(\eta); \qquad b := \mathfrak{Im}(\eta). \tag{2.18}$$

3. A conditional stability estimate

A conditional stability estimate for ill-posed problems tells how much any two solutions differ from each other when a data error exists. Since the problem under consideration is linear, stability estimates can be derived by estimating the size of solutions to the corresponding homogeneous problem.

Theorem 3.1. *Suppose that $u(x, y, t)$ is the solution of the problem under consideration, and (2.16) is satisfied, then the following estimate holds for $0 < x < 1$:*

$$\|u(x,\cdot,\cdot)\| \le e^{(1-x)}\|g(\cdot,\cdot)\| + \|u(0,\cdot,\cdot)\|^{1-x}\|g(\cdot,\cdot)\|^{x}. \tag{3.1}$$

Proof. From (2.13) and (2.18), since $\hat{u}(x,\xi,\tau) = e^{(1-x)(a+bi)}\hat{g}(\xi,\tau)$, by the Parseval identity we have

$$\|u(x,\cdot,\cdot)\|^2 = \|\hat{u}(x,\cdot,\cdot)\|^2 = \int_{|\xi|\le 1}\int_{|\tau|\le 1} |e^{(1-x)(a+bi)}|^2|\hat{g}(\xi,\tau)|^2 d\xi d\tau$$
$$+ \int_{|\xi|>1}\int_{|\tau|>1} |e^{(1-x)(a+bi)}|^2|\hat{g}(\xi,\tau)|^2 d\xi d\tau := B_1 + B_2.$$

According to the inequality $|e^{(1-x)(a+bi)}|^2 \le e^{(1-x)(|\tau|^\beta+\xi^2)}$,

$$B_1 \le c_1^2\|g(\cdot,\cdot)\|^2,$$

where $c_1 = e^{(1-x)}$.

By the Hölder inequality and (2.13) for $0 < x < 1$, $\hat{g}(\xi,\tau) = \frac{1}{e^{a+bi}}\hat{u}(0,\xi,\tau)$,

$$\begin{aligned}
B_2 &= \int_{|\xi|>1}\int_{|\tau|>1} |e^{(1-x)(a+bi)}|^2|\hat{g}(\xi,\tau)|^2 d\xi d\tau \\
&= \int_{|\xi|>1}\int_{|\tau|>1} |e^{-x(a+bi)}|^2|\hat{u}(0,\xi,\tau)|^2 d\xi d\tau \\
&= \int_{|\xi|>1}\int_{|\tau|>1} (|\hat{u}(0,\xi,\tau)|^2)^{1-x}(e^{-2a}|\hat{u}(0,\xi,\tau)|^2)^{x} d\xi d\tau \\
&\le \left(\int_{|\xi|>1}\int_{|\tau|>1} |\hat{u}(0,\xi,\tau)|^2 d\xi d\tau\right)^{1-x} \cdot \left(\int_{|\xi|>1}\int_{|\tau|>1} e^{-2a}|\hat{u}(0,\xi,\tau)|^2 d\xi d\tau\right)^{x} \\
&= \left(\int_{|\xi|>1}\int_{|\tau|>1} |\hat{u}(0,\xi,\tau)|^2 d\xi d\tau\right)^{1-x} \cdot \left(\int_{|\xi|>1}\int_{|\tau|>1} |\hat{g}(\xi,\tau)|^2 d\xi d\tau\right)^{x} \\
&\le \|u(0,\cdot,\cdot)\|^{2-2x}\|g(\cdot,\cdot)\|^{2x}.
\end{aligned}$$

Finally,

$$B_2 \le \|u(0,\cdot,\cdot)\|^{2(1-x)}\|g(\cdot,\cdot)\|^{2x}. \tag{3.2}$$

Therefore, we get

$$\|u(x,\cdot,\cdot)\| = \sqrt{B_1+B_2} \le \sqrt{B_1} + \sqrt{B_2} = c_1\|g(\cdot,\cdot)\| + \|u(0,\cdot,\cdot)\|^{1-x}\|g(\cdot,\cdot)\|^{x}. \quad \square \tag{3.3}$$

Since the solution u depends linearly on the data g, from Theorem 3.1, we have the following remark.

Remark 3.1. For given two functions g_1 and $g_2(\cdot,\cdot)$, let $u_1(x,\cdot,\cdot)$ and $u_2(x,\cdot,\cdot)$ be the corresponding solutions, respectively, then

$$\|u_1(x,\cdot,\cdot) - u_2(x,\cdot,\cdot)\| \le c_1\|g_1(\cdot,\cdot) - g_2(\cdot,\cdot)\| + (\|u_1(0,\cdot,\cdot) - u_2(0,\cdot,\cdot)\|)^{1-x}\|g_1(\cdot,\cdot) - g_2(\cdot,\cdot)\|^{x}. \tag{3.4}$$

4. Fourier regularization method and error estimate

Let χ_{Ω_n} be the characteristic function defined in the domain Ω_n, and

$$\Omega_n = \left\{ (\xi,\tau) \left| \frac{\xi^2 + |\tau|^\beta\cos(\frac{\beta}{2})}{n^2} + \frac{|\tau|^{2\beta}\sin^2\left(\frac{\beta}{2}\right)}{4n^4} \le 1 \right. \right\}, \quad n = \ln\left(\frac{E}{\delta}\right), \tag{4.1}$$

where δ and E are from (2.16) and (2.17), respectively. The reason why Ω_n is selected as (4.1) can be found in the proof of Theorem 4.1.

Following the idea of truncation singular value decomposition, a regularization solution with the noisy data g_δ can be given, i.e.,

$$\widehat{v}_\delta(x,\xi,\tau)=\widehat{g}_\delta(\xi,\tau)e^{(1-x)\sqrt{(i\tau)^\beta+\xi^2}}\chi_{\Omega_n}, \tag{4.2}$$

or equivalently in the physical domain,

$$v_\delta(x,y,t)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\widehat{g}_\delta(\xi,\tau)e^{(1-x)\sqrt{(i\tau)^\beta+\xi^2}}\chi_{\Omega_n}e^{i(\xi y+\tau t)}d\xi d\tau. \tag{4.3}$$

Theorem 4.1. *. Let $u(x,y,t)$ be the solution of the problem (2.1)–(2.6) with the exact data g, and let v_δ be the regularized solution (4.3) with the noisy data g_δ which satisfies $\|g-g_\delta\|\le\delta$. If the exact boundary temperature $u(0,y,t)$ satisfies (2.17), i.e., $\|u(0,\cdot,\cdot)\|\le E$, and if we choose*

$$n=\ln\left(\frac{E}{\delta}\right), \tag{4.4}$$

then for $0<x<1$, there holds

$$\|u(x,\cdot,\cdot)-v_\delta(x,\cdot,\cdot)\|\le 2E^{1-x}\delta^x. \tag{4.5}$$

Proof. We take two steps to prove it.

Step I: Convergence. We need to prove the regularization solution v given by (4.3) approaches the exact solution u with the same exact data g.

Firstly we have

$$\begin{aligned}\widehat{u}(x,\xi,\tau)-\widehat{v}(x,\xi,\tau)&=\widehat{g}e^{(1-x)\sqrt{(i\tau)^\beta+\xi^2}}-\widehat{g}e^{(1-x)\sqrt{(i\tau)^\beta+\xi^2}}\chi_{\Omega_n}\\&=\widehat{g}e^{(1-x)\sqrt{(i\tau)^\beta+\xi^2}}\chi_{\mathbb{R}^2\setminus\Omega_n}\\&=\widehat{g}e^{\sqrt{(i\tau)^\beta+\xi^2}}e^{-x\sqrt{(i\tau)^\beta+\xi^2}}\chi_{\mathbb{R}^2\setminus\Omega_n}.\end{aligned} \tag{4.6}$$

Using Parseval's equality and the a priori bound $\|u(0,\cdot,\cdot)\|\le E$, we get

$$\|u(x,\cdot,\cdot)-v(x,\cdot,\cdot)\|=\|\widehat{u}(x,\cdot,\cdot)-\widehat{v}(x,\cdot,\cdot)\|\le E\sup_{\xi,\tau\in\mathbb{R}}A_1, \tag{4.7}$$

where

$$A_1=\left|e^{-x\sqrt{(i\tau)^\beta+\xi^2}}\chi_{\mathbb{R}^2\setminus\Omega_n}\right|. \tag{4.8}$$

Since $|e^{-x\eta}|=e^{-ax}$,

$$A_1=e^{-xa}\chi_{\mathbb{R}^2\setminus\Omega_n}. \tag{4.9}$$

According to the definition of Ω_n in (4.1),

$$(\xi,\tau)\in\mathbb{R}^2\setminus\Omega_n\Rightarrow\frac{\xi^2+|\tau|^\beta\cos\left(\frac{\beta}{2}\right)}{n^2}+\frac{|\tau|^{2\beta}\sin^2\left(\frac{\beta}{2}\right)}{4n^4}>1. \tag{4.10}$$

Because the following relations hold

$$\begin{aligned}&(\xi,\tau)\in\Omega_n\Leftrightarrow\frac{\xi^2+|\tau|^\beta\cos\left(\frac{\beta}{2}\right)}{n^2}+\frac{|\tau|^{2\beta}\sin^2\left(\frac{\beta}{2}\right)}{4n^4}\le1\Leftrightarrow\\&\quad|\tau|^{2\beta}\sin^2\left(\frac{\beta}{2}\right)+4n^2\left(\xi^2+|\tau|^\beta\cos\left(\frac{\beta}{2}\right)\right)\le4n^4\\&\Leftrightarrow a=\sqrt{\frac{\sqrt{|\tau|^{2\beta}\sin^2\frac{\beta\pi}{2}+\left(\xi^2+|\tau|^{2\beta}\cos\left(\frac{\beta\pi}{2}\right)\right)^2}+\left(\xi^2+|\tau|^{2\beta}\cos\left(\frac{\beta\pi}{2}\right)\right)}{2}}\le n,\end{aligned} \tag{4.11}$$

we have that, $(\xi,\tau)\in\mathbb{R}^2\setminus\Omega_n$ yields $a>n$, and

$$e^{-xa}<e^{-xn}. \tag{4.12}$$

With the selection of parameter n given in (4.1), it yields

$$\|u(x,\cdot,\cdot)-v(x,\cdot,\cdot)\| \le E^{1-x}\delta^{x}. \tag{4.13}$$

Step II: Stability. We now prove that the regularization solution is dependent continuously on the data. Using Parseval's equality, we obtain

$$\begin{aligned}\|v(x,\cdot,\cdot)-v_\delta(x,\cdot,\cdot)\| &= \|\widehat{v}(x,\cdot,\cdot)-\widehat{v}_\delta(x,\cdot,\cdot)\| \\ &= \|\widehat{g}e^{(1-x)\sqrt{(i\tau)^\beta+\xi^2}}\chi_{\Omega_n}-\widehat{g}_\delta e^{(1-x)\sqrt{(i\tau)^\beta+\xi^2}}\chi_{\Omega_n}\| \\ &\le \delta \sup_{\xi,\tau\in\mathbb{R}} A_2,\end{aligned} \tag{4.14}$$

where we have used the condition $\|g-g_\delta\|\le\delta$, and

$$A_2 = |e^{(1-x)\sqrt{(i\tau)^\beta+\xi^2}}\chi_{\Omega_n}|. \tag{4.15}$$

We rewritten A_2 as

$$A_2 = e^{(1-x)a}\chi_{\Omega_n}. \tag{4.16}$$

From (4.11), we have

$$A_2 \le e^{(1-x)n}. \tag{4.17}$$

Combining Step I and Step II, with the choice of parameter n given in (4.1), we get

$$\|v(x,\cdot,\cdot)-v_\delta(x,\cdot,\cdot)\| \le E^{1-x}\delta^{x}. \tag{4.18}$$

The conclusion of the theorem now follows immediately by using the triangle inequality, (4.13) and (4.18). □

Remark 4.1. We find that the error estimate (4.5) is not convergent to 0 for the location at $x=0$. This is common in ill-posed problems. If a stronger a priori condition is added, for example, $\|u(0,\cdot,\cdot)\|_p \le \tilde{E}$, with $p>0$, $\tilde{E}>0$, where $\|\cdot\|_p$ denotes the norm of the Sobolev space $H^p(\mathbb{R}^2)$, then the convergence rate is logarithmic. Please refer to [20,21].

5. Dynamic regularization methods and error estimates

By previous analysis, the cause of ill-posedness for the problem (2.1)–(2.6) lies in the amplified factor $e^{(1-x)\sqrt{(i\tau)^\beta+\xi^2}}$ of the data. To stabilize the problem, a natural idea is to construct a function that approaches the amplified factor. The constructed function must satisfy the following properties:

(I) The modulus of the constructed function should be less than or equal to that of the amplified factor.
(II) We need to preserve the information of low-frequency components and eliminate partially or completely the information of high-frequency components as well.

The Fourier method mentioned in Section 4 for solving (2.1)–(2.6) is given by

$$\widehat{v}_n(x,\xi,\tau) = \widehat{g}(\xi,\tau)e^{(1-x)\sqrt{(i\tau)^\beta+\xi^2}}\chi_{\Omega_n},$$

where n is the regularization parameter. The method stabilizes the ill-posed problems by cutting off the high frequency components completely. For the problem (2.1)–(2.6), the Fourier regularization solution in the frequency domain $\widehat{v}_n$ can be rewritten as

$$\hat{v}_\alpha(x,\xi,\tau) = \begin{cases} \widehat{g}(\xi,\tau)e^{(1-x)\sqrt{(i\tau)^\beta+\xi^2}}, & |e^{(1-x)\sqrt{(i\tau)^\beta+\xi^2}}| \le 1/\sqrt{\alpha}, \\ 0, & |e^{(1-x)\sqrt{(i\tau)^\beta+\xi^2}}| > 1/\sqrt{\alpha}, \end{cases}$$

where α is the new regularization parameter.

However, in image processing, the original images usually contain important high frequency information in the form of finely detailed features, and much of this high frequency information cannot be discerned in the images. On one hand, the high frequency components of data induce severe ill-posedness of the problem. On the other hand, the high frequency components cannot be discerned arbitrarily. In other words, in the process of constructing approximate solutions we have to preserve the major portion of the high-frequency components and at the same time we should not let the solutions blast. Fortunately, we can achieve this goal for the problem (2.1)–(2.6) in the frequency domain. Follow the idea mentioned at the beginning of this section, the regularized solutions in the frequency domain with unperturbed data g can be constructed as follows:

Method 1:

$$\hat{u}_\alpha^\delta(x,\xi,\tau)=\begin{cases} e^{(1-x)(a+bi)}\hat{g}_\delta(\xi,\tau), & e^{-a(1-x)}\geq\sqrt{\alpha},\\ \dfrac{e^{-2a(1-x)}}{\alpha}e^{(1-x)(a+bi)}\hat{g}_\delta(\xi,\tau), & e^{-a(1-x)}<\sqrt{\alpha}.\end{cases} \tag{5.1}$$

Method 2:

$$\hat{v}_\alpha^\delta(x,\xi,\tau)=\begin{cases} e^{(1-x)(a+bi)}\hat{g}_\delta(\xi,\tau), & e^{-a(1-x)}\geq\sqrt{\alpha},\\ \dfrac{e^{-a(1-x)}}{\sqrt{\alpha}}e^{(1-x)(a+bi)}\hat{g}_\delta(\xi,\tau), & e^{-a(1-x)}<\sqrt{\alpha}.\end{cases} \tag{5.2}$$

Method 3:

$$\hat{w}_\alpha^\delta(x,\xi,\tau)=\begin{cases} e^{(1-x)(a+bi)}\hat{g}_\delta(\xi,\tau), & e^{-a(1-x)}\geq\sqrt{\alpha},\\ \dfrac{e^{-\frac{a}{2}(1-x)}}{\alpha^{\frac{1}{4}}}e^{(1-x)(a+bi)}\hat{g}_\delta(\xi,\tau), & e^{-a(1-x)}<\sqrt{\alpha}.\end{cases} \tag{5.3}$$

In general,

$$\hat{z}_\alpha^\delta(x,\xi,\tau)=\begin{cases} e^{(1-x)(a+bi)}\hat{g}_\delta(\xi,\tau), & e^{-a(1-x)}\geq\sqrt{\alpha},\\ \dfrac{e^{-\nu a(1-x)}}{\sqrt{\alpha}^{\nu}}e^{(1-x)(a+bi)}\hat{g}_\delta(\xi,\tau), & e^{-a(1-x)}<\sqrt{\alpha},\end{cases} \tag{5.4}$$

where ν is a positive real number. In (5.4), we note that the factor $\frac{e^{-\nu a(1-x)}}{\sqrt{\alpha}^{\nu}}<1\ (\nu>0)$ under the restriction $e^{-a(1-x)}<\sqrt{\alpha}$. Obviously the blasting components of high frequency in the data have been suppressed by introducing the factor $\frac{e^{-\nu a(1-x)}}{\sqrt{\alpha}^{\nu}}$. Thus it guarantees the stability even if there exist some high frequency components of the data.

According to the convolution theorem in Fourier analysis, we have

$$\left(\frac{e^{-\nu a(1-x)}}{\sqrt{\alpha}^{\nu}}e^{(1-x)(a+bi)}\hat{g}_\delta(\xi,\tau)\right)^{\vee}=\frac{1}{2\pi}\left(\frac{e^{-\nu a(1-x)}}{\sqrt{\alpha}^{\nu}}e^{(1-x)(a+bi)}\right)^{\vee}*g(y,t),$$

where $*$ represents convolution operation, $\vee$ denotes inverse Fourier transform. Unfortunately, in the physical domain, we cannot obtain the expression of $\left(\frac{e^{-\nu a(1-x)}}{\sqrt{\alpha}^{\nu}}e^{(1-x)(a+bi)}\right)^{\vee}$ in a simple form explicitly.

Remark 5.1. Obviously, if $\alpha\to 0$ as $\delta\to 0$, the regularization solutions approach the exact solution. The regularization parameter α depends on the fractional order β implicitly via the relationship that $e^{-a(1-x)}<\sqrt{\alpha}$ (or $e^{-a(1-x)}\geq\sqrt{\alpha}$). In fact, we recall that

$$a=\sqrt{\frac{\sqrt{|\tau|^{2\beta}\sin^2\frac{\beta\pi}{2}+\left(\xi^2+|\tau|^{2\beta}\cos\left(\frac{\beta\pi}{2}\right)\right)^2}+\left(\xi^2+|\tau|^{2\beta}\cos\left(\frac{\beta\pi}{2}\right)\right)}{2}}.$$

We now only deal with the Methods 1 and 2. According to (5.1) and (5.2), we have:

Method 1:

$$\hat{u}_\alpha^\delta(x,\xi,\tau)=\begin{cases} e^{(1-x)(a+bi)}\hat{g}_\delta(\xi,\tau), & e^{-2a(1-x)}\geq\alpha,\\ \dfrac{1}{\alpha}e^{-(1-x)(a-bi)}\hat{g}_\delta(\xi,\tau), & e^{-2a(1-x)}<\alpha.\end{cases} \tag{5.5}$$

Method 2:

$$\hat{v}_\alpha^\delta(x,\xi,\tau)=\begin{cases} e^{(1-x)(a+bi)}\hat{g}_\delta(\xi,\tau), & e^{-2a(1-x)}\geq\alpha,\\ \dfrac{1}{\sqrt{\alpha e^{-2a(1-x)}}}e^{-(1-x)(a-bi)}\hat{g}_\delta(\xi,\tau), & e^{-2a(1-x)}<\alpha.\end{cases} \tag{5.6}$$

For the sake of clarity, we rewrite these two methods.

Method 1:

$$\hat{u}_\alpha^\delta(x,\xi,\tau)=\begin{cases} e^{(1-x)(a+bi)}\hat{g}_\delta(\xi,\tau), & e^{-2a(1-x)}\geq\alpha,\\ \dfrac{e^{-2a(1-x)}}{\alpha}e^{(1-x)(a+bi)}\hat{g}_\delta(\xi,\tau), & e^{-2a(1-x)}<\alpha.\end{cases} \tag{5.7}$$

Method 2:

$$\hat{v}_\alpha^\delta(x,\xi,\tau) = \begin{cases} e^{(1-x)(a+bi)}\hat{g}_\delta(\xi,\tau), & e^{-2a(1-x)} \geq \alpha, \\ \dfrac{1}{\sqrt{\alpha}} e^{(1-x)bi}\hat{g}_\delta(\xi,\tau), & e^{-2a(1-x)} < \alpha. \end{cases} \tag{5.8}$$

By the inverse Fourier transform, we can get the corresponding regularized solutions $u_\alpha^\delta(x,y,t)$ and $v_\alpha^\delta(x,y,t)$ in the physical domain.

Lemma 5.1. *For $0 < x < 1$, suppose that $u_\alpha(x,y,t)$ is the solution by Method 1 with exact data $g(y,t)$, $u_\alpha^\delta(x,y,t)$ is the solution by Method 1 with noisy data $g_\delta(y,t)$, (2.16) holds, we have*

$$\|u_\alpha^\delta(x,\cdot,\cdot) - u_\alpha(x,\cdot,\cdot)\| \leq \alpha^{-\frac{1}{2}}\delta. \tag{5.9}$$

Proof. By Parseval's equality and (5.7), then

$$\begin{aligned} \|u_\alpha(x,,\cdot,\cdot) - u_\alpha^\delta(x,\cdot,\cdot)\| &= \|\hat{u}_\alpha(x,\cdot,\cdot) - \hat{u}_\alpha^\delta(x,\cdot,\cdot)\| \\ &= \left\| \min\left\{1, \frac{e^{-2a(1-x)}}{\alpha}\right\} e^{(1-x)(a+bi)}(\hat{g} - \hat{g}_\delta) \right\| \leq \alpha^{-\frac{1}{2}}\delta. \quad \square \end{aligned}$$

Lemma 5.2. *For $0 < x < 1$, suppose that $u(x,y,t)$ is the exact solution with exact data $g(y,t)$, $u_\alpha(x,y,t)$ is the solution by Method 1 with exact data $g(y,t)$, (2.17) holds, we have*

$$\|u_\alpha(x,\cdot,\cdot) - u(x,\cdot,\cdot)\| \leq \frac{2(1-x)}{2-x}\left(\frac{x}{2-x}\right)^{\frac{x}{2(1-x)}} \alpha^{\frac{x}{2(1-x)}} E. \tag{5.10}$$

Proof. By Parseval's equality and using $\hat{g}(\xi,\tau) = e^{-(a+bi)}\hat{u}(0,\xi,\tau)$, then

$$\begin{aligned} \|u_\alpha(x,\cdot,\cdot) - u(x,\cdot,\cdot)\| &= \|\hat{u}_\alpha(x,\cdot,\cdot) - \hat{u}(x,\cdot,\cdot)\| \\ &= \left\| \min\left\{1, \frac{e^{-2a(1-x)}}{\alpha}\right\} e^{(1-x)(a+bi)}\hat{g} - e^{(1-x)(a+bi)}\hat{g} \right\| \\ &= \left\| \left(1 - \min\left\{1, \frac{e^{-2a(1-x)}}{\alpha}\right\}\right) e^{(1-x)(a+bi)}\hat{g} \right\| \\ &= \left\| \left(1 - \min\left\{1, \frac{e^{-2a(1-x)}}{\alpha}\right\}\right) e^{(1-x)(a+bi)}e^{-(a+bi)}\hat{u}(0,\cdot,\cdot) \right\| \\ &\leq \sup_{e^{-2a(1-x)} \leq \alpha} \left(1 - \frac{e^{-2a(1-x)}}{\alpha}\right) e^{-ax}\|\hat{u}(0,\cdot,\cdot)\|. \end{aligned}$$

Now we seek the maximum of $F(a) := \left(1 - \frac{e^{-2a(1-x)}}{\alpha}\right)e^{-ax}$ under the restriction $e^{-2a(1-x)} \leq \alpha$. It is easy to find the zero point a^* of $F'(a)$, i.e., a^* satisfies

$$\frac{x\alpha}{2-x} = e^{-2a^*(1-x)}, \tag{5.11}$$

and a^* maximize the function $F(a)$, hence $F(a) \leq F(a^*)$. Finally,

$$\|u_\alpha(x,\cdot,\cdot) - u(x,\cdot,\cdot)\| \leq F(a^*)E = \left(1 - \tfrac{x}{2-x}\right)\left(\tfrac{x}{2-x}\alpha\right)^{\frac{x}{2(1-x)}} E. \quad \square$$

Theorem 5.1. *For $0 < x < 1$, suppose that $u(x,y,t)$ is the exact solution with the exact data $g(y,t)$, $u_\alpha^\delta(x,y,t)$ is the solution by Method 1 with noisy data $g_\delta(y,t)$, (2.16) and (2.17) holds. If α is selected dynamically*

$$\alpha(x) = 2^{2x-2}\left(\frac{x}{2-x}\right)^{x-2}\left(\frac{\delta}{E}\right)^{2(1-x)}, \tag{5.12}$$

then the following error estimate holds

$$\|u_\alpha^\delta(x,\cdot,\cdot) - u(x,\cdot,\cdot)\| \leq \frac{2^{1-x}}{2-x}\left(\frac{x}{2-x}\right)^{-\frac{x}{2}} \delta^x E^{1-x}. \tag{5.13}$$

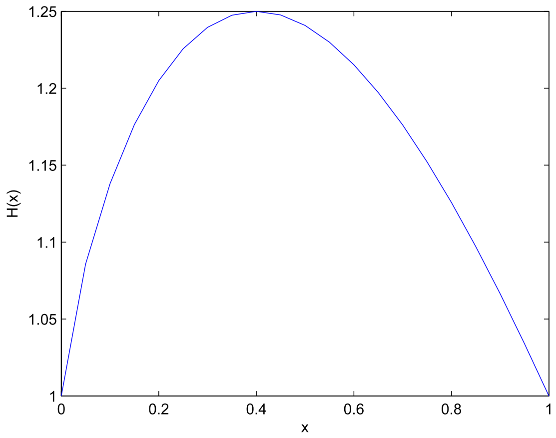

Fig. 1. The plot of $H(x)$.

Proof. According to Lemmas 5.1 and 5.2, we have

$$\|u_\alpha^\delta(x,\cdot,\cdot)-u(x,\cdot,\cdot)\| \le \alpha^{-\frac{1}{2}}\delta + \frac{2(1-x)}{2-x}\left(\frac{x}{2-x}\right)^{\frac{x}{2(1-x)}}\alpha^{\frac{x}{2(1-x)}}E.$$

Minimizing the right-hand side of above inequality with respect to α, we can get (5.12). Thus, (5.13) holds. □

Remark 5.2. The function $\frac{2^{1-x}}{2-x}\left(\frac{x}{2-x}\right)^{-\frac{x}{2}}$ in (5.13) is denoted as $H(x)$ which is plotted as Fig. 1.

Similarly, for Method 2, we have the following conclusions.

Lemma 5.3. *For $0 < x < 1$, suppose that $v_\alpha(x,y,t)$ is the solution by Method 2 with exact data $g(y,t)$, $v_\alpha^\delta(x,y,t)$ is the solution by Method 2 with noisy data $g_\delta(y,t)$, (2.16) holds, we have*

$$\|v_\alpha^\delta(x,\cdot,\cdot)-v_\alpha(x,\cdot,\cdot)\| \le \alpha^{-\frac{1}{2}}\delta. \tag{5.14}$$

Proof. By Parseval's equality and (5.8), then

$$\begin{aligned}\|v_\alpha(x,\cdot,\cdot)-v_\alpha^\delta(x,\cdot,\cdot)\| &= \|\hat{v}_\alpha(x,\cdot,\cdot)-\hat{v}_\alpha^\delta(x,\cdot,\cdot)\|\\ &= \left\|\min\left\{1,\frac{e^{-a(1-x)}}{\sqrt{\alpha}}\right\}e^{(1-x)(a+bi)}(\hat{g}-\hat{g}_\delta)\right\| \le \alpha^{-\frac{1}{2}}\delta. \quad \square\end{aligned}$$

Lemma 5.4. *For $0 < x < 1$, suppose that $u(x,y,t)$ is the exact solution with exact data $g(y,t)$, $v_\alpha(x,y,t)$ is the solution by Method 2 with exact data $g(y,t)$, (2.17) holds, we have*

$$\|v_\alpha(x,\cdot,\cdot)-u(x,\cdot,\cdot)\| \le (1-x)x^{\frac{x}{1-x}}\alpha^{\frac{x}{2(1-x)}}E. \tag{5.15}$$

Proof. By Parseval's equality and noting $\hat{g}(\xi,\tau)=e^{-(a+bi)}\hat{u}(0,\xi,\tau)$, we have

$$\begin{aligned}\|v_\alpha(x,\cdot,\cdot)-u(x,\cdot,\cdot)\| &= \|\hat{v}_\alpha(x,\cdot,\cdot)-\hat{u}(x,\cdot,\cdot)\|\\ &= \left\|\min\left\{1,\frac{e^{-a(1-x)}}{\sqrt{\alpha}}\right\}e^{(1-x)(a+bi)}\hat{g}-e^{(1-x)(a+bi)}\hat{g}\right\|\\ &= \left\|\left(1-\min\left\{1,\frac{e^{-a(1-x)}}{\sqrt{\alpha}}\right\}\right)e^{(1-x)(a+bi)}\hat{g}\right\|\\ &= \left\|\left(1-\min\left\{1,\frac{e^{-a(1-x)}}{\sqrt{\alpha}}\right\}\right)e^{(1-x)(a+bi)}e^{-(a+bi)}\hat{u}(0,\cdot,\cdot)\right\|\\ &\le \sup_{e^{-2a(1-x)}\le\alpha}\left(1-\frac{e^{-a(1-x)}}{\sqrt{\alpha}}\right)e^{-ax}\|\hat{u}(0,\cdot,\cdot)\|.\end{aligned}$$

Similarly we want to seek the maximum of $F(a) := \left(1 - \frac{e^{-a(1-x)}}{\sqrt{\alpha}}\right) e^{-ax}$ under the restriction $e^{-2a(1-x)} \le \alpha$. It is easy to find the zero point a^* of $F'(a)$, i.e., a^* satisfies

$$x\sqrt{\alpha} = e^{-a^*(1-x)}, \tag{5.16}$$

and a^* maximizes the function $F(a)$, hence $F(a) \le F(a^*)$. Finally,

$$\|v_\alpha(x,\cdot,\cdot) - u(x,\cdot,\cdot)\| \le F(a^*)E = (1-x)\left(x\sqrt{\alpha}\right)^{\frac{x}{(1-x)}} E. \quad \square$$

Theorem 5.2. *For* $0 < x < 1$, *suppose that* $u(x, y, t)$ *is the exact solution with exact data* $g(y, t)$, $v_\alpha^\delta(x, y, t)$ *is the solution by Method* 2 *with noisy data* $g_\delta(y, t)$, (2.16) *and* (2.17) *holds. If* α *is selected dynamically*

$$\alpha(x) = x^{-2}\left(\frac{\delta}{E}\right)^{2(1-x)}, \tag{5.17}$$

then we have

$$\|v_\alpha^\delta(x,\cdot,\cdot) - u(x,\cdot,\cdot)\| \le \delta^x E^{1-x}. \tag{5.18}$$

Proof. According to Lemmas 5.3 and 5.4, we have

$$\|v_\alpha^\delta(x,\cdot,\cdot) - u(x,\cdot,\cdot)\| \le \alpha^{-\frac{1}{2}}\delta + (1-x)x^{\frac{x}{1-x}}\alpha^{\frac{x}{2(1-x)}}E.$$

Minimize the right-hand side of above inequality with respect to α, we can get (5.17). Thus, (5.18) holds.

Remark 5.3. Following the definition of optimality in [22,23], similarly we can obtain the optimal error bound for the problem (2.1)–(2.6). Here we can predict that the error estimate (5.18) is optimal. $\square$

6. Numerical examples

Despite its lengthy theoretical analysis, the spectral regularization methods are relatively simple to be implemented. Here, the proposed methods will be implemented in Matlab.

In this section, we will show some numerical results to illustrate the validity of the regularization methods presented in this paper. Firstly we consider the following direct problem for the given data $f(y, t)$:

$$\begin{aligned}
&\frac{\partial^\beta u}{\partial t^\beta} - u_{xx} - u_{yy} = 0, \quad x > 0, y > 0, t > 0,\\
&u(x, y, 0) = 0, \quad x > 0, y > 0,\\
&u(x, 0, t) = 0, \quad x > 0, t > 0,\\
&u_y(x, 0, t) = 0, \quad x > 0, t > 0,\\
&u(0, y, t) = f(y, t), \quad y > 0, t > 0,\\
&u(x, y, t)|_{x\to\infty} \text{ bounded}, \quad y > 0, t > 0.
\end{aligned} \tag{6.1}$$

This problem is a well-posed problem and its solution at $x = 1$ is given by

$$g(y, t) := u(1, y, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi, \tau)e^{-\sqrt{(i\tau)^\beta+\xi^2}}e^{-i(\xi y+\tau t)}d\xi d\tau. \tag{6.2}$$

In numerical implementations, we give the data $f(y, t)$ and sample at an equidistant grid in the domain $[0, 1] \times [0, 1]$ with 100×100 grid points, then carry out a 2-dimensional discrete Fourier transformation. We obtain the data $g(y, t)$ via 2-dimensional inverse discrete Fourier transformation according to (6.2), then generate the noisy data g_δ:

$$g_\delta(\cdot,\cdot) = g(\cdot,\cdot) + g_{\max} * \delta \ rand(size(g(\cdot,\cdot))), \tag{6.3}$$

where δ indicates the error level of g, $g_{\max}$ is the maximum of sampled data g, the symbol $rand(size(\cdot))$ is a random number between $[-1, 1]$. Let RMS denotes the root mean square for a sampled function $W(\cdot,\cdot)$ which is defined by

$$RMS(W) = \sqrt{\frac{1}{(s+1)^2}\sum_{j=1}^{s+1}\sum_{k=1}^{s+1}(W(y_j, t_k))^2}, \tag{6.4}$$

where $s + 1$ is the total number of test points. Similarly, we can define the root mean square error (RMSE) between the computed data and the exact data.

Table 1
Error behavior of Fourier method.

x	0.9	0.5	0.1
n	9.81	9.81	9.81
$RMSE(u - v^{\delta})$	$1.42 * 10^{-6}$	$2.10 * 10^{-4}$	$3.26 * 10^{-2}$

Table 2
Error behavior of Method 2.

x	0.9	0.5	0.1
α	$1.70 * 10^{-2}$	$2.19 * 10^{-4}$	$2.13 * 10^{-6}$
$RMSE(u - v_{\alpha}^{\delta})$	$4.40 * 10^{-7}$	$1.82 * 10^{-4}$	$3.23 * 10^{-2}$

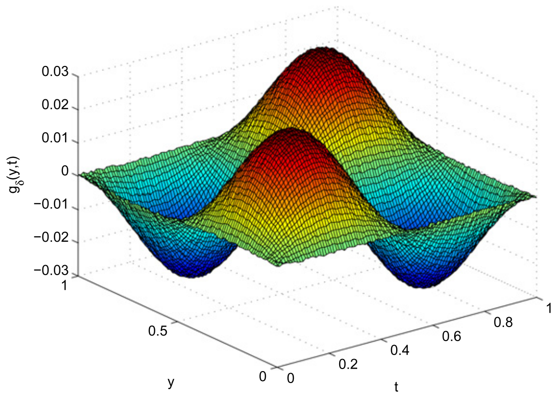

Fig. 2. The noisy input data $g_{\delta}(y, t)$ with $\beta = 0.1$.

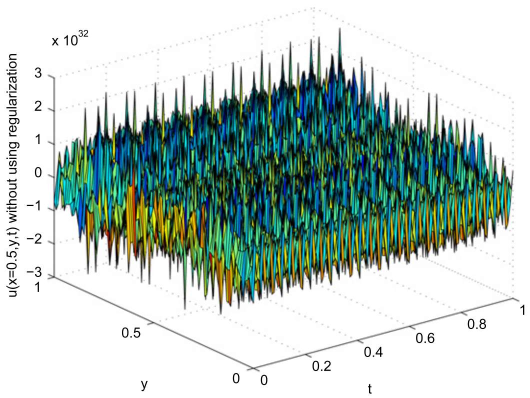

Fig. 3. The un-regularized solution with $\delta = 3\%$, $x = 0.5$, $\beta = 0.1$.

The regularized solutions were computed by the 2D discrete Fast Fourier Transform (2D FFT) and 2D inverse discrete Fast Fourier Transform (2D IFFT) according to the formulas in Sections 4 and 5. The regularization parameter α is chosen by Theorem 5.1, Theorem 5.2 where E is computed by (6.4) for the sampled $f(y, t)$ at the grids. Now we construct two examples to investigate the regularization methods.

Example 1. Consider a smooth function $u(0, y, t) := f(y, t) = 16 \sin(2\pi y) \sin(2\pi t)$.

Fig. 2 shows the input noisy data $g(y, t)$ with $\delta = 3\%$.

Fig. 3 shows the result with $\delta = 3\%$ where we do not use any regularization method at $x = 0.5$.

Fig. 4 shows the result computed by Fourier method with $\delta = 3\%$ at $x = 0.5$.

Fig. 5 shows the result computed by Method 2 with $\delta = 3\%$ at $x = 0.1$.

Now we compare Method 2 with the Fourier method for this example.

For the Fourier method, the regularization parameter n and *RMSE* are listed in Table 1.

For Method 2, the regularization parameter α and *RMSE* are listed in Table 2.

From Tables 1 and 2, we can see that Fourier method and the dynamic Method 2 are comparable. The numerical results are in excellent agreement with our theoretical analysis.

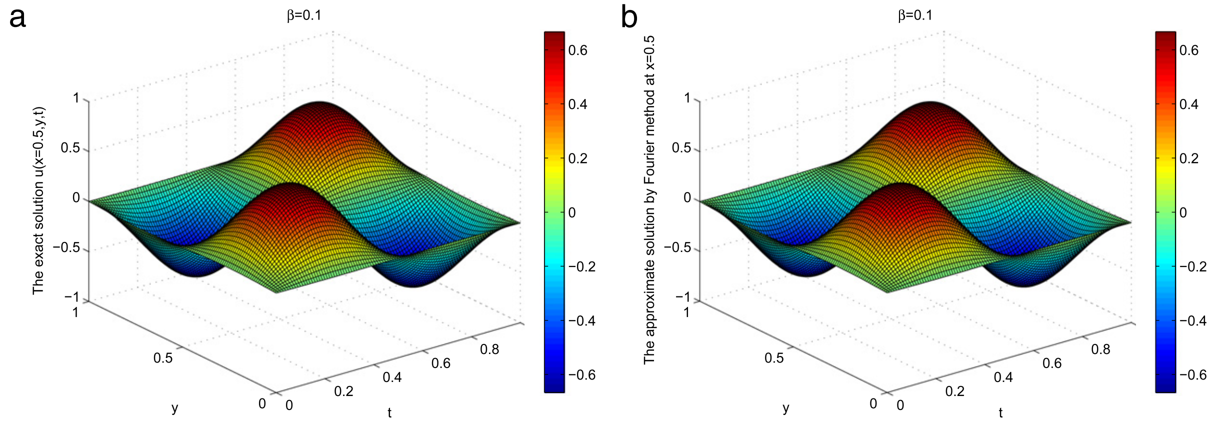

Fig. 4. (a) The exact solution at $x = 0.5$; (b) reconstruction by Fourier method at $x = 0.5$.

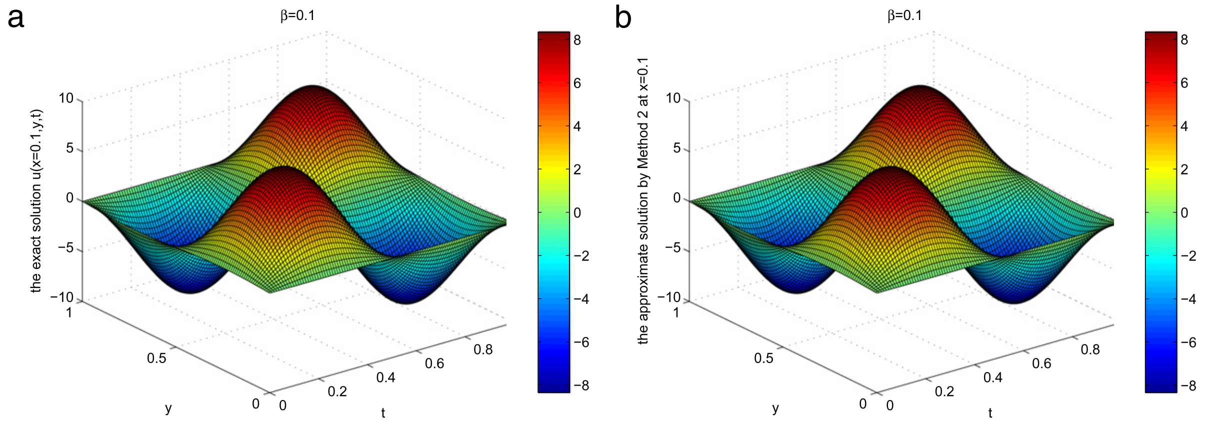

Fig. 5. (a) The exact solution at $x = 0.1$; (b) reconstruction by Method 2 at $x = 0.1$.

Example 2. Let $\Omega = \{(y, t)|0.1 \le y \le 0.3,\ 0.1 \le t \le 0.3\} \bigcup \{(y, t)|0.6 \le y \le 0.8,\ 0.6 \le t \le 0.8\}$. Consider a non-smooth function

$$f(y, t) = \begin{cases} 1, & \text{if } (y, t) \in \Omega \\ 0, & \text{if else.} \end{cases}$$

In this example, we fix $\delta = 1\%$ and $x = 0.4$.

Fig. 6 shows the noisy data $g_\delta(y, t)$ with $\delta = 1\%$.

Fig. 7 shows the result by Fourier method and Method 1 with $\delta = 1\%$, $\beta = 0.1$.

This example shows the comparison between the Fourier and the dynamic spectral regularization methods. From Fig. 7, we can see that the computed result supports the theoretical result.

Fig. 8 shows the results by Method 1 with different values of β at $x = 0.4$. From these results, it can be seen that the solution continuously depends on the time fractional derivative.

7. Concluding remarks

In this paper, we obtained a stability result for the inverse heat conduction problem of a fractional heat diffusion equation in 2-D setting. In the theoretical aspect, order optimal error estimations were proved. Especially, using a new dynamic spectral method, an optimal error estimation was obtained. Comparing with the static spectral regularization method, i.e. the Fourier method, the regularization parameters in the dynamic methods are related to the locations x of reconstruction. The proposed dynamic methods could give higher accuracy not only in theoretical analysis, but also in the numerical experiment. The numerical examples show the comparison between the static and the dynamic spectral regularization methods and the numerical results support the theoretical analysis.

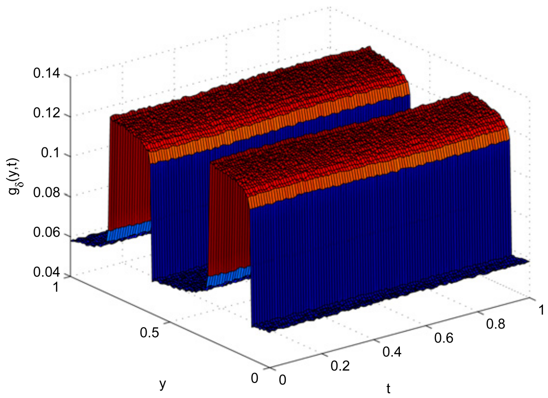

Fig. 6. The noisy input data $g_\delta(y, t)$ with $\beta = 0.1$.

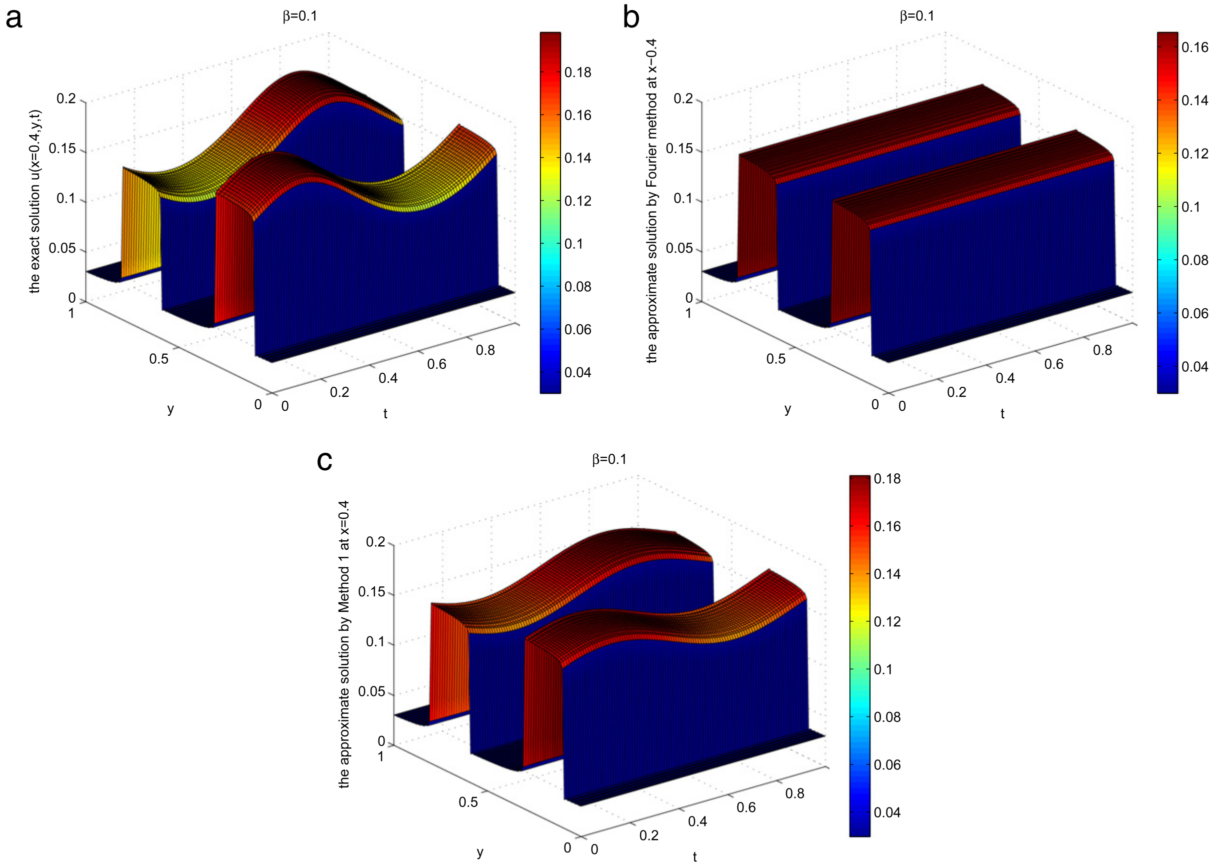

Fig. 7. (a) The exact solution at $x = 0.4$; (b) reconstruction by Fourier method at $x = 0.4$ with $RMSE = 0.014$ and $n = 5.77$; (c) reconstruction by Method 1 at $x = 0.4$ with $RMSE = 0.007$ and $\alpha = 9.8 * 10^{-4}$.

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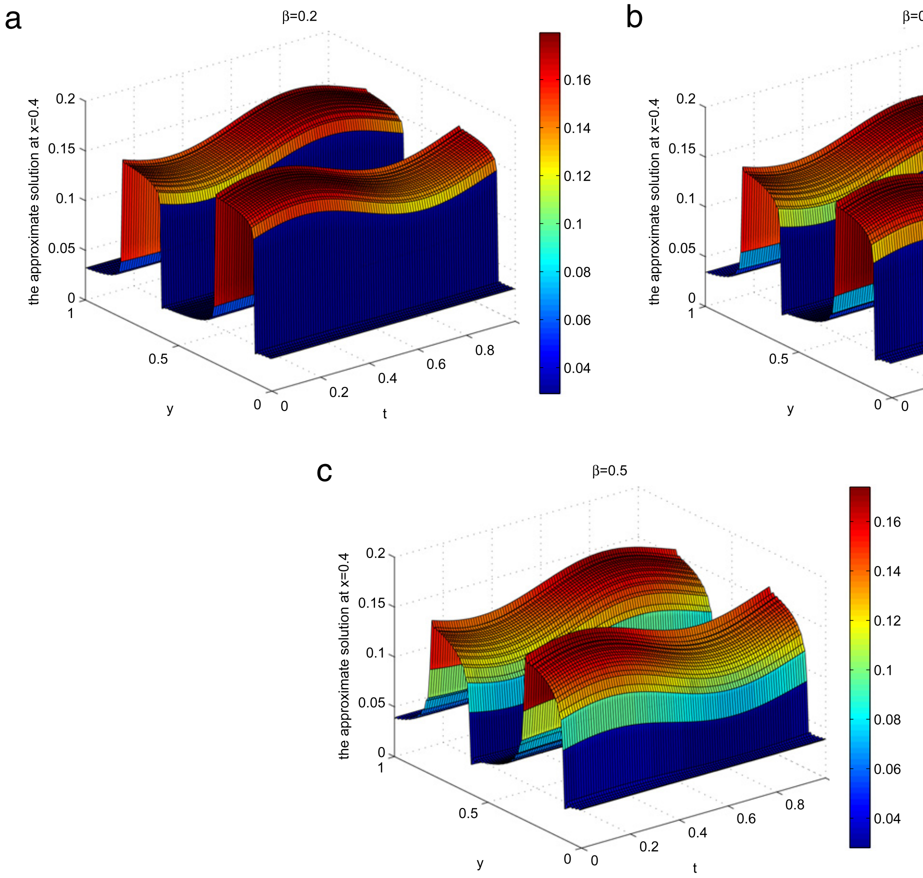

Fig. 8. (a) Reconstruction with $\beta = 0.2$; (b) reconstruction with $\beta = 0.3$; (c) reconstruction with $\beta = 0.5$.

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References

[1] W. Chen, L.J. Ye, H.G. Sun, Fractional diffusion equations by the Kansa method, Comput. Math. Appl. 59 (2010) 1614–1620.
[2] Z. Yu, J. Lin, Numerical research on the coherent structure in the viscoelastic second-order mixing layers, Appl. Math. Mech. 19 (1998) 671–677.
[3] N. Laskin, I. Lambadaris, F.C. Harmantzis, M. Devetsikiotis, Fractional Lévy motion and its application to network traffic modeling, Comput. Netw. 40 (3) (2002) 363–375.
[4] H. Scher, E.W. Montroll, Anomalous transit-time dispersion in amorphous, Phys. Rev. B 12 (6) (1975) 2455–2477.
[5] T.L. Szabo, J. Wu, A model for longitudinal and shear wave propagation in viscoelastic media, J. Acoust. Soc. Am. 107 (5) (2000) 2437–2446.
[6] R. Gorenflo, F. Mainardi, D. Moretti, G. Pagnini, P. Paradisi, Discrete random walk models for space–time fractional diffusion, Chem. Phys. 284 (2002) 521–541.
[7] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, Phys. Rep. 339 (2000) 1–77.
[8] I.M. Sokolov, J. Klafter, A. Blumen, Fractional kinetics, Phys. Today 55 (2002) 48–54.
[9] R.V. Mendes, A fractional calculus interpretation of the fractional volatility model, Nonlinear Dynam. 55 (2009) 395–399.
[10] J. Hadamard, Lectures on Cauchy's Problem in Linear Partial Differential Equations, Dover Publications, New York, 1953.
[11] Y.C. Hon, M. Li, A computational method for inverse free boundary determination problem, Int. J. Numer. Methods Eng. 73 (2008) 1291–1309.
[12] J. Cheng, J. Nakagawa, M. Yamamoto, T. Yamazaki, Uniqueness in an inverse problem for one-dimensional fractional diffusion equation, Inverse Probl. 16 (2009) 115002. (16 pp).
[13] G.H. Zheng, T. Wei, Spectral regularization method for a Cauchy problem of the time fractional advection-dispersion equation, J. Comput. Appl. Math. 233 (2010) 2631–2640.
[14] A.N. Bondarenko, D.S. Ivaschenko, Numerical methods for solving inverse problems for time fractional diffusion equation with variable coefficient, J. Inverse Ill-Posed Probl. 17 (5) (2009) 419–440.
[15] D.A. Murio, Stable numerical solution of a fractional-diffusion inverse heat conduction problem, Comput. Math. Appl. 53 (2007) 1492–1501.
[16] L. Eldén, F. Berntsson, T. Regińska, Wavelet and Fourier methods for solving the sideways heat equation, SIAM J. Sci. Comput. 21 (6) (2000) 2187–2205.
[17] I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.
[18] R. Gorenflo, R. Rutman, On ultraslow and on intermediate processes, in: P. Rusev, I. Dimovski, V. Kiryakova (Eds.), Transform Methods and Special Functions, SCT Publishers, Singapore, 1995.
[19] F. Berntsson, A spectral method for solving the sideways heat equation, Inverse Probl. 15 (1999) 891–906.
[20] Z. Qian, C.L. Fu, Regularization strategy for a two-dimensional inverse heat conduction problem, Inverse Probl. 23 (2007) 1053–1068.

[21] X.T. Xiong, Regularization theory and algorithm for some inverse problems for parabolic differential equations, Ph.D. Dissertation, Lanzhou University, 2007 (in Chinese).
[22] U. Tautenhahn, Optimal stable approximations for the sideways heat equation, J. Inverse Ill-Posed Probl. 5 (1997) 287–307.
[23] U. Tautenhahn, Optimality for ill-posed problems under general source conditions, Numer. Funct. Anal. Optim. 19 (1998) 377–398.