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Weak sharpness for gap functions in vector variational inequalities[☆]

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ABSTRACT

In this paper, characterizations of the set of solutions for VVI are presented by using scalarization approaches. The set of solutions of VVI is shown to be the set of weak sharpness for gap functions of some scalarization of VVI and for gap functions of VVI under semi-strong monotonicity. Some examples are given to illustrate these results.

1. Introduction

Let X be a real Banach space with its dual X^*, and K be a nonempty, closed and convex subset of X. Denote by $\langle l, x\rangle$ the value of $l \in X^*$ at $x \in X$. Let $A \subseteq X$. For $x \in X$, denote by $d(x, A)$ the distance from x to A, i.e., $d(x, A) = \inf_{a\in A} \|x - a\|$. Let $F_i : X \to X^*$ $(i = 1, \dots, n)$ and $F = (F_1, \dots, F_n)$. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and denote $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) | x_i \geq 0, i = 1, \dots, n\}$ and

$$S_0^n = \left\{ \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n_+ \,\middle|\, \sum_{i=1}^n \xi_i = 1 \right\}.$$

In this paper, we consider the following vector variational inequality problem (VVI) of finding $x^* \in K$ such that

$$F(x^*)(y - x^*) \notin -\mathrm{int}\, \mathbb{R}^n_+, \quad \forall y \in K.$$

Clearly, VVI can be rewritten as follows: finding $x^* \in K$ such that

$$(\langle F_1(x^*), y - x^*\rangle, \dots, \langle F_n(x^*), y - x^*\rangle) \notin -\mathrm{int}\mathbb{R}^n_+, \quad \forall y \in K.$$

Denote by S_{VVI} the set of solutions of VVI.

In this paper, we always consider the max-norm in Euclidean space $\mathbb{R}^l$, i.e.,

$$\|x - y\| = \max_{1\leq i\leq l} |x_i - y_i|,$$

where $x = (x_1, \dots, x_l)$ and $y = (y_1, \dots, y_l) \in \mathbb{R}^l$ with $l = m$ or n.

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The notion of VVI was first introduced by Giannessi [1] in finite-dimensional spaces. Recently, extensive study of VVI has been done by many authors (see, for example, [2,3] and the references therein). Among solution approaches for VVI, scalarization is one of the most analyzed topics at least from the computational point of view (see, for example, [2–5]).

The concept of a sharp minimum for real-valued functions was introduced in [6]. Weak sharp minima for real-valued functions, as a generalization of sharp minima, were introduced and investigated by Ferris [7]. Weak sharp minima play important roles in mathematical programming. It is well known that weak sharp minima are closely related to error bounds in convex programming, the sensitivity analysis of optimization problems and the convergence analysis of some algorithms (see, for example, [8–14]). Recently, Marcotte and Zhu [15] have introduced the notion of weak sharpness for a variational inequality problem (VI) and derived a necessary and sufficient condition for the solution set of VI to be weakly sharp. Deng and Yang [16] studied the existence of weak sharp minima in multicriteria linear programming problems (MCLP) and proved that weak sharp minimality holds for certain distance functions and gap functions. Bednarczuk [17] and Studniarski [18] investigated global/local weak sharp minima in vector optimization problems in terms of some distance functions, respectively. For related works on piecewise linear multiobjective problems, we refer [19,20].

Compared with the weak sharpness for variational inequalities, the investigation of that for VVI is very limited. In order to characterize the set of solutions of VVI we will employ gap functions for VVI similar to the ones in [21] and investigate the weak sharpness property of the sets of solutions for VVI via gap functions by introducing a semi-strong monotonicity assumption. We first present characterizations of the set of solutions for VVI by using scalarization approaches, which extend corresponding results of Lee et al. [5]. Compared with the proof by Lee et al. [5], gap functions instead of a separation theorem are employed in this paper. We prove that the set of solutions of VVI is the set of weak sharpness for gap functions of some scalarization of VVI and for gap functions of VVI under semi-strong monotonicity of each component mapping. We will give some examples to illustrate these results.

2. Equivalent characterizations for VVI via scalarization approaches

This section is devoted to some preliminary results on characterizations of the set of solutions for VVI by using scalarization approaches, which will be used in the sequel.

For each $\xi \in \mathbb{R}^n_+ \setminus \{0\}$, we consider the following scalar variational inequality problem ($(\mathrm{VI})_\xi$) of finding $x^* \in K$ such that

$$\langle \xi, F(x^*)(y - x^*)\rangle \geq 0, \quad \forall y \in K.$$

Denote by S^{ξ}_{VI} the set of solutions of $(\mathrm{VI})_\xi$. Notice that when $X = \mathbb{R}^m$, $(\mathrm{VI})_\xi$ has been investigated by Lee et al. [5].

Let $\xi \in \mathbb{R}^n_+ \setminus \{0\}$ be given. Define functions $\varphi, \phi_\xi : K \to \mathbb{R} \cup \{+\infty\}$, respectively, by

$$\varphi(x) = \sup_{y\in K} \min_{1\leq i\leq n} \langle F_i(x), x - y\rangle, \quad x \in K, \tag{2.1}$$

and

$$\phi_\xi(x) = \sup_{y\in K} \left\langle \sum_{i=1}^{n} \xi_i F_i(x), x - y \right\rangle, \quad x \in K. \tag{2.2}$$

The concept of a gap function for a scalar variational inequality problem was introduced in [22]. Some important algorithms can be provided based on gap function to solve variational inequalities and optimization problems (see, for example, [23,24]).

Definition 2.1 (*[2]*). A function $p : K \to \mathbb{R} \cup \{+\infty\}$ is said to be a gap function for VVI if,

(i) $p(x) \geq 0, \forall x \in K$;
(ii) $p(x^*) = 0$ if and only if $x^* \in S_{\mathrm{VVI}}$.

We begin with the following useful proposition.

Proposition 2.1 (*[22,4,21,25,26]*). *The function φ given by (2.1) is a gap function of* VVI, *and the function ϕ_ξ given by (2.2) is a gap function of* $(\mathrm{VI})_\xi$, *where $\xi \in S^n_0$.*

We now turn to the investigation of the relationship between the functions ϕ_ξ and φ.

Proposition 2.2. *The following equality holds:*

$$\varphi(x) = \min_{\xi\in S^n_0} \phi_\xi(x), \quad \forall x \in K.$$

Proof. The proof follows from [21]. □

The result below follows from Propositions 2.1 and 2.2.

Proposition 2.3. *The following equality holds:*

$$\bigcup_{\xi\in S_0^n} S_{\mathrm{VI}}^{\xi} = S_{\mathrm{VVI}}.$$

Proof. From Proposition 2.1, we have $S_{\mathrm{VVI}} = \{x \in K | \varphi(x) = 0\}$ and $S_{\mathrm{VI}}^{\xi} = \{x \in K | \phi_\xi(x) = 0\}$, where $\xi \in S_0^n$. Let $x \in S_{\mathrm{VVI}}$. Then $\varphi(x) = 0$ and from Proposition 2.2, there is $\xi_0 \in S_0^n$ such that $\phi_{\xi_0}(x) = 0$, i.e., $x \in S_{\mathrm{VI}}^{\xi_0}$. Therefore, $x \in \cup_{\xi\in S_0^n} S_{\mathrm{VI}}^{\xi}$.

Let $x \in \cup_{\xi\in S_0^n} S_{\mathrm{VI}}^{\xi}$. Then there is $\xi_0 \in S_0^n$ such that $x \in S_{\mathrm{VI}}^{\xi_0}$, i.e., $\phi_{\xi_0}(x) = 0$. From Proposition 2.1, one has $\phi_\xi(x) \geq 0$ for each $\xi \in S_0^n$. It thus follows from Proposition 2.2 that $\varphi(x) = 0$, i.e., $x \in S_{\mathrm{VVI}}$. □

Remark 2.1. Let $B^n = \{\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}_+^n | \, \|\xi\| = \max_{1\leq i\leq n} |\xi_i| = 1\}$ and $S^n = \{\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}_+^n | \, \|\xi\| = \max_{1\leq i\leq n} |\xi_i| \leq 1$ and $\sum_{i=1}^n \xi_i \geq 1\}$. It is obvious that $B^n \cup S_0^n \subseteq S^n$, B^n is compact, S_0^n and S^n are compact and convex. Thus, by Proposition 2.3, the following equalities hold:

$$\bigcup_{\xi\in S_0^n} S_{\mathrm{VI}}^{\xi} = \bigcup_{\xi\in S^n} S_{\mathrm{VI}}^{\xi} = \bigcup_{\xi\in B^n} S_{\mathrm{VI}}^{\xi} = \bigcup_{\xi\in \mathbb{R}_+^n\setminus\{0\}} S_{\mathrm{VI}}^{\xi} = S_{\mathrm{VVI}}.$$

The relations presented above extend the corresponding results of Lee et al. [5] in finite dimensional spaces. We would like to point out that Lee et al. [5] investigated the inclusion $\cup_{\xi\in B^n} S_{\mathrm{VI}}^{\xi} \supseteq S_{\mathrm{VVI}}$ by using a separation theorem.

3. Weak sharpness for VVI

In this section, we shall investigate the weak sharpness property of S_{VVI} for gap functions of some scalar VI and gap functions of VVI. We first recall some basic definitions.

Definition 3.1. Let $q : K \to \mathbb{R} \cup \{+\infty\}$ be a gap function for some scalar VI of VVI or for VVI. We say that S_{VVI} is the set of weak sharpness with respect to the function q on K if there exists $\mu > 0$ such that

$$d(x, S_{\mathrm{VVI}}) \leq \mu q(x), \quad \forall x \in K.$$

Definition 3.2 (*[15]*)**.** A mapping $T : X \to X^*$ is said to be

(i) strongly pseudomonotone (SPM) at $y \in K$ on K with modulus $\lambda > 0$ if, for any $x \in K$,

$$\langle T(y), x - y\rangle \geq 0 \Rightarrow \langle T(x), x - y\rangle \geq \lambda\|x - y\|^2;$$

(ii) strongly monotone (SM) at $y \in K$ on K with modulus $\lambda > 0$ if, for any $x \in K$,

$$\langle T(x) - T(y), x - y\rangle \geq \lambda\|x - y\|^2.$$

Next we introduce the following new concept on monotonicity of T.

Definition 3.3. A mapping $T : X \to X^*$ is said to be semi-strongly monotone (semi-SM) at $y \in K$ on K with modulus $\lambda > 0$ if, for any $x \in K$,

$$\langle T(y), x - y\rangle \geq 0 \Rightarrow \langle T(x) - T(y), x - y\rangle \geq \lambda\|x - y\|^2.$$

Remark 3.1. Recall that $T : X \to X^*$ is SPM (respectively, SM, semi-SM) on K if, the relation in (i) of Definition 3.2 (respectively, (ii) of Definitions 3.2 and 3.3) holds for all $x, y \in K$. It is easy to see that the following relations hold:

$$\boxed{\text{SM at } y \in K} \Longrightarrow \boxed{\text{semi-SM at } y \in K} \Longrightarrow \boxed{\text{SPM at } y \in K}.$$

Let $X = K = \mathbb{R}$ and $T : K \to \mathbb{R}_+$ be given by $T(x) = e^x$ for all $x \in X$. Then T is semi-strongly monotone at any $y \in K$ but not strongly monotone at $y \in K$. That a mapping is strongly pseudomonotone at $y \in K$ but not semi-strongly monotone at $y \in K$ can be illustrated by the following example. Let $X = \mathbb{R}$, $K = \mathbb{R}_+$ and $T : X \to X$ be given by

$$T(x) = \begin{cases} 2x + 1, & \text{if } x \in (-\infty, 1], \\ x + 1, & \text{if } x \in (1, +\infty). \end{cases}$$

(Choose $y = 1$).

In the following, we investigate the weak sharpness of VVI under strong monotonicity or semi-strong monotonicity assumptions, none of which implies the nonemptiness of S_{VVI}, as shown below.

Example 3.1. Let $X = K = \mathbb{R}$,

$$F_1(x) = \begin{cases} x+1, & \text{if } x \in \mathbb{R}_+, \\ x-1, & \text{if } x \in (-\infty, 0), \end{cases}$$

and

$$F_2(x) = \begin{cases} x+2, & \text{if } x \in \mathbb{R}_+, \\ x-2, & \text{if } x \in (-\infty, 0). \end{cases}$$

Consider the following VVI: finding $x^* \in K$ such that

$$(\langle F_1(x^*), y - x^*\rangle, \langle F_2(x^*), y - x^*\rangle) \notin -\mathrm{int}\mathbb{R}^2_+, \quad \forall y \in K.$$

It is easy to check that both F_1 and F_2 are strongly monotone on K with modulus 1. However, $S_{\mathrm{VVI}} = \emptyset$.

It is worth noting that the nonemptiness of S_{VVI} can only be guaranteed by the monotonicity plus certain coercivity condition; see [2]. But the coercivity is not needed in our study in this paper. Thus we always suppose that S_{VVI} is nonempty in this paper.

3.1. Weak sharpness for VVI via ϕ_ξ

In this subsection, we prove that the set of solutions of VVI has the weak sharp property for the function $\sqrt{\phi_\xi(x)}$ under the assumption that each component mapping involved in VVI is strongly monotone, where ξ is some vector in S_0^n.

Theorem 3.1. *Assume that for each $i = 1, \ldots, n$, F_i is strongly monotone on K with modulus $\lambda_i > 0$. Then there is $\xi \in S_0^n$ such that S_{VVI} is of the weak sharpness property for the function $\sqrt{\phi_\xi(x)}$.*

Proof. Let $x^* \in S_{\mathrm{VVI}}$ be given. Then from Proposition 2.3, there exists $\xi \in S_0^n$ such that $x^* \in S_{\mathrm{VI}}^{\xi}$ and hence

$$\left\langle \sum_{i=1}^{n} \xi_i F_i(x^*), x - x^* \right\rangle \geq 0, \quad \forall x \in K.$$

Since F_i is strongly monotone on K with modulus $\lambda_i > 0$ for each $i = 1, \ldots, n$, we have

$$\langle F_i(x), x - x^*\rangle \geq \langle F_i(x^*), x - x^*\rangle + \lambda_i \|x - x^*\|^2, \quad \forall x \in K.$$

It follows that

$$\begin{aligned} \left\langle \sum_{i=1}^{n} \xi_i F_i(x), x - x^* \right\rangle &\geq \left\langle \sum_{i=1}^{n} \xi_i F_i(x^*), x - x^* \right\rangle + \sum_{i=1}^{n} \xi_i \lambda_i \|x - x^*\|^2 \\ &\geq \sum_{i=1}^{n} \xi_i \lambda_i \|x - x^*\|^2, \quad \forall x \in K. \end{aligned}$$

Consequently,

$$\begin{aligned} \sqrt{\phi_\xi(x)} &= \sqrt{\sup_{y \in K} \left\langle \sum_{i=1}^{n} \xi_i F_i(x), x - y \right\rangle} \\ &\geq \sqrt{\left\langle \sum_{i=1}^{n} \xi_i F_i(x), x - x^* \right\rangle} \\ &\geq \sqrt{\sum_{i=1}^{n} \xi_i \lambda_i} \|x - x^*\| \\ &\geq \sqrt{\sum_{i=1}^{n} \xi_i \lambda_i}\, d(x, S_{\mathrm{VVI}}), \quad \forall x \in K, \end{aligned}$$

and so

$$\frac{1}{\sqrt{\sum_{i=1}^{n}\xi_i\lambda_i}}\sqrt{\phi_\xi(x)} \geq d(x, S_{\mathrm{VVI}}), \quad \forall x \in K.$$

This completes the proof. $\square$

Remark 3.2. (i) It seems that it is very difficult to derive the weak sharpness property of S_{VVI} for the function $\sqrt{\varphi(x)}$.
(ii) If the assumption "for each $i = 1, \ldots, n$, F_i is strongly monotone on K with modulus $\lambda_i > 0$" in Theorem 3.1 is replaced by "for each $i = 1, \ldots, n$, F_i is semi-strongly monotone or strongly pseudomonotone on K with modulus $\lambda_i > 0$", then the conclusion of Theorem 3.1 may not be true.

3.2. Weak sharpness for VVI via gap functions

In this subsection, we define some gap functions for scalar VI and apply them to establish the weak sharpness of VVI. Let $x, y \in X$ and $\xi \in \mathbb{R}^n_+ \setminus \{0\}$. Define the following functions:

$$\begin{aligned} d(\langle\xi, F(x)\rangle, \langle\xi, F(y)\rangle) &= \left\|\sum_{i=1}^{n}\xi_i(F_i(x) - F_i(y))\right\| \\ &= \sup_{\substack{z\in X\\ z\neq 0}} \frac{\left|\left\langle\sum_{i=1}^{n}\xi_i(F_i(x) - F_i(y)), z\right\rangle\right|}{\|z\|}, \end{aligned}$$

$$\begin{aligned} d(F_i(x), F_i(y)) &= \|F_i(x) - F_i(y)\| \\ &= \sup_{\substack{z\in X\\ z\neq 0}} \frac{|\langle F_i(x) - F_i(y), z\rangle|}{\|z\|}, \quad i = 1, \ldots, n, \end{aligned}$$

and

$$\begin{aligned} d(F(x), F(y)) &= \|F(x) - F(y)\| \\ &= \sup_{\substack{z\in X\\ z\neq 0}} \frac{\|\langle F(x) - F(y), z\rangle\|}{\|z\|} \\ &= \sup_{\substack{z\in X\\ z\neq 0}} \frac{\max\limits_{1\leq i\leq n} |\langle F_i(x) - F_i(y), z\rangle|}{\|z\|}. \end{aligned}$$

The following result is useful in the proof of main results of this paper.

Proposition 3.1. *For $x, y \in X$, we have*

$$\max_{1\leq i\leq n} d(F_i(x), F_i(y)) = \max_{\xi\in S_0^n} d(\langle\xi, F(x)\rangle, \langle\xi, F(y)\rangle) = d(F(x), F(y)).$$

Proof. Let $\xi = (\xi_1, \ldots, \xi_n) \in S_0^n$. Then $\xi_i \geq 0$ $(i = 1, \ldots, n)$ with $\sum_{i=1}^{n}\xi_i = 1$. For any $z \in X$ with $z \neq 0$, we get

$$\begin{aligned} \left|\left\langle\sum_{i=1}^{n}\xi_i(F_i(x) - F_i(y)), z\right\rangle\right| &\leq \sum_{i=1}^{n}\xi_i|\langle F_i(x) - F_i(y), z\rangle| \\ &\leq \max_{1\leq i\leq n} |\langle F_i(x) - F_i(y), z\rangle| \sum_{i=1}^{n}\xi_i \\ &= \max_{1\leq i\leq n} |\langle F_i(x) - F_i(y), z\rangle| \end{aligned}$$

and so

$$\max_{\xi\in S_0^n} d(\langle\xi, F(x)\rangle, \langle\xi, F(y)\rangle) = \max_{\xi\in S_0^n}\sup_{\substack{z\in X\\ z\neq 0}} \frac{\left|\left\langle\sum_{i=1}^{n}\xi_i(F_i(x) - F_i(y)), z\right\rangle\right|}{\|z\|}$$

$$\begin{aligned}
&\le \sup_{\substack{z\in X\\ z\neq 0}} \frac{\max_{1\le i\le n} |\langle F_i(x)-F_i(y), z\rangle|}{\|z\|}\\
&= d(F(x),F(y))\\
&= \max_{1\le i\le n}\sup_{\substack{z\in X\\ z\neq 0}} \frac{|\langle F_i(x)-F_i(y), z\rangle|}{\|z\|}\\
&= \max_{1\le i\le n} d(F_i(x),F_i(y)).
\end{aligned}$$

For any $x, y \in K$, there exists i_0 ($1 \le i_0 \le n$) such that

$$\max_{1\le i\le n} d(F_i(x),F_i(y)) = d(F_{i_0}(x),F_{i_0}(y)).$$

Since $\xi^0 = (0,\ldots,\overset{(i_0)}{1},\ldots,0) \in S_0^n$, it follows that

$$\begin{aligned}
\max_{1\le i\le n} d(F_i(x),F_i(y)) &= d(F_{i_0}(x),F_{i_0}(y))\\
&= d(\langle \xi^0, F(x)\rangle, \langle \xi^0, F(y)\rangle)\\
&\le \max_{\xi\in S_0^n} d(\langle \xi, F(x)\rangle, \langle \xi, F(y)\rangle).
\end{aligned}$$

Now above arguments lead to

$$\max_{1\le i\le n} d(F_i(x),F_i(y)) = \max_{\xi\in S_0^n} d(\langle \xi, F(x)\rangle, \langle \xi, F(y)\rangle) = d(F(x),F(y)).$$

This completes the proof. □

Under certain mild conditions, we will show the closedness of the set of solutions of VVI.

Proposition 3.2. *If for each $i = 1,\ldots,n$, F_i is continuous in the weak* topology of X^*, then S_{VVI} is closed.*

Proof. Let $\{x_m\} \subseteq S_{\mathrm{VVI}}$ with $x_m \to x_0$ as $m\to\infty$. Then

$$(\langle F_1(x_m), y-x_m\rangle,\ldots,\langle F_n(x_m), y-x_m\rangle) \notin -\mathrm{int}\mathbb{R}^n_+, \quad \forall y\in K$$

or equivalently,

$$(\langle F_1(x_m), y-x_m\rangle,\ldots,\langle F_n(x_m), y-x_m\rangle) \in W = \mathbb{R}^n \setminus (-\mathrm{int}\mathbb{R}^n_+), \quad \forall y\in K.$$

Let $y\in K$ and $i = 1,\ldots,n$. Since F_i is continuous in the weak* topology of X^*, one has that $F_i(x_m)$ converges weak* to $F_i(x_0)$ and so $\{\|F_i(x_m)\|\}$ is bounded. Consequently,

$$|\langle F_i(x_m)-F_i(x_0), y-x_0\rangle| \to 0$$

and

$$|\langle F_i(x_m), x_0-x_m\rangle| \le \|F_i(x_m)\|\,\|x_m-x_0\| \to 0$$

as $m\to\infty$. It follows that for each $i = 1,\ldots,n$,

$$\begin{aligned}
\langle F_i(x_m), y-x_m\rangle &= \langle F_i(x_0), y-x_0\rangle + \langle F_i(x_m)-F_i(x_0), y-x_0\rangle + \langle F_i(x_m), x_0-x_m\rangle\\
&\to \langle F_i(x_0), y-x_0\rangle, \quad \text{as } m\to\infty,
\end{aligned}$$

which implies that $x_0\in S_{\mathrm{VVI}}$ since W is closed. Thus, S_{VVI} is closed. □

For $x\in X$ and $\xi \in \mathbb{R}^n_+\setminus\{0\}$, let

$$D(\xi, x) = d(\langle \xi, F(x)\rangle, \langle \xi, F(S_{\mathrm{VVI}})\rangle).$$

Now, we discuss the relationship among the functions related to VVI defined above.

Proposition 3.3. *For $x\in X$, we have*

$$\max_{\xi\in S_0^n} D(\xi, x) \le d(F(x), F(S_{\mathrm{VVI}}))$$

and

$$\max_{1\le i\le n} d(F_i(x), F_i(S_{\mathrm{VVI}})) \le d(F(x), F(S_{\mathrm{VVI}})).$$

Proof. Let $x \in X$. It is clear that

$$\begin{aligned}\max_{\xi\in S_0^n} D(\xi, x) &= \max_{\xi\in S_0^n} \inf_{y\in S_{\mathrm{VVI}}} d(\langle \xi, F(x)\rangle, \langle \xi, F(y)\rangle)\\ &\le \inf_{y\in S_{\mathrm{VVI}}} \max_{\xi\in S_0^n} d(\langle \xi, F(x)\rangle, \langle \xi, F(y)\rangle)\\ &= \inf_{y\in S_{\mathrm{VVI}}} d(F(x), F(y)) \quad (\text{From Proposition 3.1})\\ &= d(F(x), F(S_{\mathrm{VVI}})).\end{aligned}$$

Notice that

$$\begin{aligned}\max_{1\le i\le n} d(F_i(x), F_i(S_{\mathrm{VVI}})) &= \max_{1\le i\le n} \inf_{y\in S_{\mathrm{VVI}}} d(F_i(x), F_i(y))\\ &\le \inf_{y\in S_{\mathrm{VVI}}} \max_{1\le i\le n} d(F_i(x), F_i(y))\\ &= \inf_{y\in S_{\mathrm{VVI}}} d(F(x), F(y)) \quad (\text{From Proposition 3.1})\\ &= d(F(x), F(S_{\mathrm{VVI}})).\end{aligned}$$

This completes the proof. □

Denote by SF the system of the following functions:

$$\max_{1\le i\le n} d(F_i(x), F_i(S_{\mathrm{VVI}})), \qquad \max_{\xi\in S_0^n} D(\xi, x), \qquad d(F(x), F(S_{\mathrm{VVI}})).$$

Based on Proposition 3.3, we next prove the weak sharpness property of the set of solutions of VVI for each gap function of VVI in SF under the assumption that each component mapping $F_i (i = 1, \dots, n)$ of F is semi-strongly monotone.

Theorem 3.2. *Let F_i be continuous in the weak* topology of X^* for each $i = 1, \dots, n$. Assume that for each $i = 1, \dots, n$, F_i is semi-strongly monotone at each point of S_{VVI} on K with modulus $\lambda_i > 0$. Then S_{VVI} is weakly sharp for each function of* SF. *In this case, each function of* SF *serves as a gap function of* VVI *for S_{VVI}.*

Proof. It is obvious that each function of SF restricted to K is nonnegative, and if $x^* \in S_{\mathrm{VVI}}$, then $d(F(x^*), F(S_{\mathrm{VVI}})) = 0$. It follows from Proposition 3.3 that each function of SF is zero at $x^* \in S_{\mathrm{VVI}}$. Since $F_i (i = 1, \dots, n)$ are continuous in the weak* topology of X^*, from Proposition 3.2, S_{VVI} is closed.

From Proposition 3.3, it suffices to prove that S_{VVI} is weakly sharp for the functions $\max_{1\le i\le n} d(F_i(x), F_i(S_{\mathrm{VVI}}))$ and $\max_{\xi\in S_0^n} D(\xi, x)$, respectively.

Let $x \in K \setminus S_{\mathrm{VVI}}$ and $x^* \in S_{\mathrm{VVI}}$. Then there exists $i_0 (1 \le i_0 \le n)$ such that

$$\langle F_{i_0}(x^*), x - x^*\rangle \ge 0.$$

Since F_{i_0} is semi-strongly monotone at x^* on K with modulus $\lambda_{i_0} > 0$, we have

$$\begin{aligned}\left\langle \sum_{i=1}^n \xi_i^0 (F_i(x) - F_i(x^*)), x - x^* \right\rangle &= \langle F_{i_0}(x) - F_{i_0}(x^*), x - x^*\rangle\\ &\ge \lambda_{i_0} \|x - x^*\|^2,\end{aligned}$$

where $\xi^0 = (0, \dots, \overset{(i_0)}{1}, \dots, 0) \in S_0^n$. As a consequence,

$$\begin{aligned}d(F_{i_0}(x), F_{i_0}(x^*)) &= \sup_{\substack{z\in X\\ z\neq 0}} \frac{|\langle F_{i_0}(x) - F_{i_0}(x^*), z\rangle|}{\|z\|}\\ &\ge \frac{|\langle F_{i_0}(x) - F_{i_0}(x^*), x - x^*\rangle|}{\|x - x^*\|}\\ &\ge \lambda_{i_0} \|x - x^*\|\end{aligned}$$

and

$$d(\langle \xi^0, F(x)\rangle, \langle \xi^0, F(x^*)\rangle) = \sup_{\substack{z\in X\\ z\neq 0}} \frac{\left|\left\langle \sum_{i=1}^n \xi_i^0 (F_i(x) - F_i(x^*)), z \right\rangle\right|}{\|z\|}$$

$$\geq \frac{\left\langle \sum_{i=1}^{n} \xi_i^0 (F_i(x) - F_i(x^*)), x - x^* \right\rangle}{\|x - x^*\|}$$
$$\geq \lambda_{i_0} \|x - x^*\|.$$

It follows that

$$\max_{1 \leq i \leq n} d(F_i(x), F_i(S_{\mathrm{VVI}})) \geq \min_{1 \leq i \leq n} \lambda_i d(x, S_{\mathrm{VVI}})$$

and

$$\max_{\xi \in S_0^n} D(\xi, x) \geq \min_{1 \leq i \leq n} \lambda_i d(x, S_{\mathrm{VVI}}),$$

i.e.,

$$d(x, S_{\mathrm{VVI}}) \leq \frac{1}{\min\limits_{1 \leq i \leq n} \lambda_i} \max_{1 \leq i \leq n} d(F_i(x), F_i(S_{\mathrm{VVI}})) \tag{3.1}$$

and

$$d(x, S_{\mathrm{VVI}}) \leq \frac{1}{\min\limits_{1 \leq i \leq n} \lambda_i} \max_{\xi \in S_0^n} D(\xi, x). \tag{3.2}$$

If $x \in S_{\mathrm{VVI}}$, then inequalities (3.1) and (3.2) also hold.

Since S_{VVI} is weakly sharp for each function of SF, it follows that if each function of SF is zero at $x^* \in K$, then $x^* \in S_{\mathrm{VVI}}$ since S_{VVI} is closed. Consequently, each function of SF is a gap function of VVI. This completes the proof. □

Remark 3.3. Let $n > 1$. If F_i is strongly monotone on K with modulus $\lambda_i > 0$ for each $i = 1, \ldots, n$, then S_{VVI} is not necessarily a singleton. Let $X = \mathbb{R}^2$, $K = [-1, 0] \times [-1, 0]$ and $F = (F_1, F_2) : X \to \mathbb{R}^2$ be given by $F_1(x) = x + (1, 1)$ and $F_2(x) = 2x$ for all $x \in X$. It is easy to verify that F_1 and F_2 are strongly monotone on K with moduli $\lambda_1 = 1$ and $\lambda_2 = 2$, respectively. However, $S_{\mathrm{VVI}} \supseteq \{(0, 0), (-1, -1)\}$.

Remark 3.4. (i) From the proof of Theorem 3.2, the assumption of semi-strong monotonicity of F_i, $i = 1, \ldots, n$, plays a key role in obtaining the weak sharpness property of S_{VVI} for each function of SF.

(ii) If the assumption "for each $i = 1, \ldots, n$, F_i is semi-strongly monotone at each point of S_{VVI} on K with modulus $\lambda_i > 0$" in Theorem 3.2 is replaced by "there exists $i_0 (1 \leq i_0 \leq n)$ such that F_{i_0} is strongly monotone on K with modulus $\lambda_{i_0} > 0$" and other assumptions in Theorem 3.2 are satisfied, then one can easily verify that any nonempty subset of K (certainly, the set of solutions of VVI) is weakly sharp for each function of SF.

(iii) If the assumption "for each $i = 1, \ldots, n$, F_i is semi-strongly monotone at each point of S_{VVI} on K with modulus $\lambda_i > 0$" in Theorem 3.2 is replaced by "for each $i = 1, \ldots, n$, F_i is strongly pseudomonotone on K with modulus $\lambda_{i_0} > 0$" and other assumptions in Theorem 3.2 are satisfied, then one cannot obtain that the set of solutions of VVI is weakly sharp for any function of SF.

The following example illustrates that the semi-strong monotonicity cannot be replaced by the strong pseudomonotonicity in Theorem 3.2.

Example 3.2. Let $X = \mathbb{R}$, $K = \mathbb{R}_+$,

$$F_1(x) = \begin{cases} -x + 1, & \text{if } x \in \left(-\infty, \frac{1}{2}\right], \\ x, & \text{if } x \in \left(\frac{1}{2}, +\infty\right), \end{cases}$$

and

$$F_2(x) = 2F_1(x) = \begin{cases} -2x + 2, & \text{if } x \in \left(-\infty, \frac{1}{2}\right], \\ 2x, & \text{if } x \in \left(\frac{1}{2}, +\infty\right). \end{cases}$$

Consider the following VVI of finding $x^* \in K$ such that

$$(\langle F_1(x^*), y - x^* \rangle, \langle F_2(x^*), y - x^* \rangle) \notin -\mathrm{int}\mathbb{R}_+^2, \quad \forall y \in K.$$

Clearly, $S_{\mathrm{VVI}} = \{0\}$, F_1 and F_2 are continuous and strongly pseudomonotone at 0 on K with moduli 1 and 2, respectively, but not semi-strongly monotone at 0 on K. Moreover,

$$d(x, S_{\mathrm{VVI}}) = x, \quad \forall x \in K,$$

and

$$\begin{aligned}\max_{i=1,2} d(F_i(x), F_i(S_{\mathrm{VVI}})) &= d(F(x), F(S_{\mathrm{VVI}})) \\ &= \max_{\xi \in S_0^2} D(\xi, x) \\ &= \begin{cases} 2x, & \text{if } x \in \left[0, \frac{1}{2}\right], \\ 2|x-1|, & \text{if } x \in \left(\frac{1}{2}, +\infty\right). \end{cases}\end{aligned}$$

For any $t > 0$ large enough, let $x = 1 + \frac{1}{4t-2} \in (\frac{1}{2}, +\infty) \subseteq K$. Then

$$\begin{aligned} d(x, S_{\mathrm{VVI}}) &= x = 1 + \frac{1}{4t-2} \\ &> \frac{1}{2 - \frac{1}{t}} = \frac{t}{2t-1} = 2t|x-1| \\ &= t \max_{i=1,2} d(F_i(x), F_i(S_{\mathrm{VVI}})) \\ &= t d(F(x), F(S_{\mathrm{VVI}})) \\ &= t \max_{\xi \in S_0^2} D(\xi, x), \end{aligned}$$

which implies that S_{VVI} is not weakly sharp for any function of SF.

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