J. Math. Anal. Appl. 395 (2012) 643–653

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Convergence rates of limit distribution of maxima of lognormal samples

Xin Liao, Zuoxiang Peng *

School of Mathematics and Statistics, Southwest University, 400715 Chongqing, China

ARTICLE INFO

Article history:
Received 31 January 2012
Available online 7 June 2012
Submitted by Vladimir Pozdnyakov

Keywords:
Extreme value distribution
Logarithmic normal distribution
Maximum
Uniform convergence rate

ABSTRACT

Let M_n denote the partial maximum of an independent and identically distributed lognormal random sequence. In this paper, we derive the exact uniform convergence rate of the distribution of the normalized maximum $(M_n - b_n)/a_n$ to its extreme value distribution, where the constants a_n and b_n are chosen by an optimal way.

1. Introduction and main results

Let $(\xi_n, n \geq 1)$ be a sequence of independent and identically distributed standardized normal random variables with distribution function $\Phi(x)$. Let $M_n = \max(\xi_k, 1 \leq k \leq n)$ denote the partial maximum. It is well-known that

$$\lim_{n\to\infty} P\left(M_n \leq a_n x + b_n\right) = \lim_{n\to\infty} \Phi^n\left(a_n x + b_n\right) = \exp\left(-e^{-x}\right) = \Lambda(x) \tag{1.1}$$

with the normalized constants a_n and b_n given by

$$a_n = (2\log n)^{-1/2}, \qquad b_n = a_n^{-1} - \frac{a_n}{2}\left(\log\log n + \log 4\pi\right).$$

The pointwise convergence rate of (1.1) is

$$\Phi^n\left(a_n x + b_n\right) - \Lambda(x) \sim \frac{e^{-x}\exp\left(-e^{-x}\right)}{16}\frac{(\log\log n)^2}{\log n} \tag{1.2}$$

for large n. For more details, see [1,2]. Hall [1] investigated this problem further and proved

$$\frac{\mathbb{C}_1}{\log n} < \sup_{x\in\mathbb{R}}\left|P\left(M_n \leq a_n x + b_n\right) - \Lambda(x)\right| < \frac{\mathbb{C}_2}{\log n} \tag{1.3}$$

for some absolute constants $0 < \mathbb{C}_1 < \mathbb{C}_2$ if we choose a_n and b_n by the following equations:

$$2\pi b_n^2 \exp(b_n^2) = n^2, \qquad a_n = b_n^{-1}. \tag{1.4}$$

For the extreme value distributions and their associated uniform convergence rates of some given distributions, see [3,4] respectively for exponential and mixed exponential distributions, [5,6] for the general error distribution, [7,8] for the short-tailed symmetric distribution.

* Corresponding author.
E-mail address: pzx@swu.edu.cn (Z. Peng).

0022-247X/$ – see front matter © 2012 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2012.05.077

It is popular in theoretical analysis and wide applications that the normal distribution is carried over to logarithmic normal one. Let $\xi \sim N(0, 1)$, we say that X follows the logarithmic normal distribution if X is defined by $X = e^{\xi}$, abbreviated as *lognormal distribution*. In this paper, we are interested in the uniform convergence rate of extremes from samples with common distribution F following the lognormal distribution. The probability density function of the lognormal distribution is given by

$$F'(x) = \frac{x^{-1}}{\sqrt{2\pi}} \exp\left(-\frac{(\log x)^2}{2}\right), \quad x > 0.$$

In the sequel, let $(X_n, n \geq 1)$ be a sequence of independent random variables with common distribution F following the lognormal distribution, and let M_n denote its partial maximum. For the limiting distribution of the maximum of lognormal, by Lemma 2 in Section 2 and the arguments similar to the proof of Theorem 1.5.3 in [2], we can derive

$$\lim_{n\to\infty} P\left(M_n \leq \alpha_n x + \beta_n\right) = \lim_{n\to\infty} F^n\left(\alpha_n x + \beta_n\right) = \Lambda(x)$$

with normalized constants α_n and β_n given by

$$\alpha_n = \frac{\exp\left((2\log n)^{1/2}\right)}{(2\log n)^{1/2}}, \tag{1.5}$$

and

$$\beta_n = \left(\exp\left((2\log n)^{1/2}\right)\right)\left(1 - \frac{\log 4\pi + \log\log n}{2\,(2\log n)^{1/2}}\right). \tag{1.6}$$

As mentioned by Hall [1] and Leadbetter et al. [2] that the choices of normalized constants may influence the convergence rate of extremes, and the distribution tail representation may help us find the optimal normalized constants. For the distributional tail representation of lognormal distribution, Lemma 2(ii) in Section 2 shows that

$$1 - F(x) = c(x) \exp\left(-\int_e^x \frac{g(t)}{f(t)} dt\right)$$

for sufficiently large x, where $c(x) \to (2\pi e)^{-1/2}$ as $x \to \infty$, $f(x) = x/\log x$ and $g(x) = 1 + (\log x)^{-2}$. Noting that $f'(x) \to 0$ and $g(x) \to 1$ as $x \to \infty$. By Proposition 1.1(a) and Corollary 1.7 of [9], we can choose the norming constants a_n and b_n in such a way that b_n is the solution of the equation

$$2\pi\,(\log b_n)^2 \exp\left((\log b_n)^2\right) = n^2 \tag{1.7}$$

and a_n satisfies

$$a_n = f(b_n) = \frac{b_n}{\log b_n}, \tag{1.8}$$

then

$$\lim_{n\to\infty} P\left(M_n \leq a_n x + b_n\right) = \lim_{n\to\infty} F^n\left(a_n x + b_n\right) = \Lambda(x). \tag{1.9}$$

The aim of this paper is to prove that the uniform convergence rate of (1.9) is proportional to $1/(\log n)^{1/2}$. However, for $F^n(\alpha_n x + \beta_n)$ the convergence rate is no better than $(\log\log n)^2/(\log n)^{1/2}$ even though $\alpha_n/a_n \to 1$ and $(\beta_n - b_n)/a_n \to 0$ as $n \to \infty$. The main results are stated as follows:

Theorem 1. *Let $\{X_n, n \geq 1\}$ denote a sequence of independent identically distributed random variables with common distribution F which is lognormal distribution.*

(i) *For norming constants a_n and b_n given respectively by (1.7) and (1.8), then there exist absolute constants $0 < C_1 < C_2$ such that*

$$\frac{C_1}{(\log n)^{1/2}} < \sup_{x\in\mathbb{R}} \left|F^n\left(a_n x + b_n\right) - \Lambda(x)\right| < \frac{C_2}{(\log n)^{1/2}}$$

for $n \geq 2$.

(ii) *For norming constants α_n and β_n respectively given by (1.4) and (1.6), we have*

$$F^n\left(\alpha_n x + \beta_n\right) - \Lambda(x) \sim -\frac{e^{-x}\exp\left(-e^{-x}\right)}{8}\frac{(\log\log n)^2}{(2\log n)^{1/2}}$$

for large n.

2. Auxiliary lemmas

In this section we provide several key properties of lognormal distribution which are needed for the proofs of our main results. The first one is the distributional tail decomposition of lognormal distribution, which is stated as follows.

Lemma 1. *Let F denote the lognormal distribution function. For $x > 1$, we have*

$$1 - F(x) = \frac{1}{\sqrt{2\pi}}(\log x)^{-1}\exp\left(-\frac{(\log x)^2}{2}\right) - \gamma(x) \tag{2.1}$$

$$= \frac{1}{\sqrt{2\pi}}(\log x)^{-1}\exp\left(-\frac{(\log x)^2}{2}\right)\left(1-(\log x)^{-2}\right) + s(x), \tag{2.2}$$

where

$$0 < \gamma(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty (\log t)^{-2}\exp\left(-\frac{(\log t)^2}{2}\right)\frac{1}{t}dt < \frac{1}{\sqrt{2\pi}}(\log x)^{-3}\exp\left(-\frac{(\log x)^2}{2}\right) \tag{2.3}$$

and

$$0 < s(x) = \frac{3}{\sqrt{2\pi}}\int_x^\infty (\log t)^{-4}\exp\left(-\frac{(\log t)^2}{2}\right)\frac{1}{t}dt < \frac{3}{\sqrt{2\pi}}(\log x)^{-5}\exp\left(-\frac{(\log x)^2}{2}\right). \tag{2.4}$$

Proof. By integration by parts we have

$$\begin{aligned}1 - F(x) &= \frac{1}{\sqrt{2\pi}}(\log x)^{-1}\exp\left(-\frac{(\log x)^2}{2}\right) - \frac{1}{\sqrt{2\pi}}\int_x^\infty (\log t)^{-2}\exp\left(-\frac{(\log t)^2}{2}\right)\frac{1}{t}\,dt\\ &= \frac{1}{\sqrt{2\pi}}(\log x)^{-1}\exp\left(-\frac{(\log x)^2}{2}\right) - \gamma(x),\end{aligned}$$

which is (2.1). Similarly,

$$\gamma(x) = \frac{1}{\sqrt{2\pi}}(\log x)^{-3}\exp\left(-\frac{(\log x)^2}{2}\right) - s(x).$$

Putting it into (2.1), we obtain (2.2), where

$$\begin{aligned}s(x) &= \frac{3}{\sqrt{2\pi}}\int_x^\infty (\log t)^{-4}\exp\left(-\frac{(\log t)^2}{2}\right)\frac{1}{t}\,dt\\ &= \frac{3}{\sqrt{2\pi}}(\log x)^{-5}\exp\left(-\frac{(\log x)^2}{2}\right) - \frac{15}{\sqrt{2\pi}}\int_x^\infty (\log t)^{-6}\exp\left(-\frac{(\log t)^2}{2}\right)\frac{1}{t}\,dt\\ &< \frac{3}{\sqrt{2\pi}}(\log x)^{-5}\exp\left(-\frac{(\log x)^2}{2}\right).\end{aligned}$$

The proof is complete. □

By Lemma 1, we can derive the Mills ratio and the distributional tail representation of lognormal distribution, which are helpful, just as mentioned in Section 1, to find the optimal normalized constants a_n and b_n given by (1.7) and (1.8). The following result is about the Mills-type ratio and distributional tail representation of lognormal distribution, which is a special case of tail behavior and Mills ratio of logarithmic general error distribution studied by Liao and Peng [10].

Lemma 2. *Let $F(x)$ denote the distribution of lognormal random variable with density $F'(x)$, then*

(i) *for $x > 1$ we have*

$$\frac{xF'(x)}{\log x}\left(1 - \frac{1}{(\log x)^2}\right) < 1 - F(x) < \frac{x}{\log x}F'(x)$$

and

$$\frac{1 - F(x)}{F'(x)} \sim \frac{x}{\log x} \tag{2.5}$$

as $x \to \infty$;

(ii) *for large x we have*

$$1 - F(x) = c(x)\exp\left(-\int_e^x \frac{g(t)}{f(t)}dt\right) \tag{2.6}$$

with $c(x) \to (2\pi e)^{-1/2}$, $g(x) = 1 + (\log x)^{-2} \to 1$ as $x \to \infty$ and the auxiliary function $f(x) = x(\log x)^{-1}$ satisfies $f'(x) \to 0$ as $x \to \infty$.

Let the norming constants a_n and b_n are defined by (1.7) and (1.8) respectively. Let

$$a_n^* = a_n r_n, \qquad b_n^* = b_n + \delta_n a_n \tag{2.7}$$

with $r_n \to 1$ and $\delta_n \to 0$ as $n \to \infty$. Then $a_n^*/a_n \to 1$, and $\left(b_n^* - b_n\right)/a_n \to 0$ which implies $F^n(a_n^* x + b_n^*) \to \Lambda(x)$. Since $a_n^* x + b_n^* \to \infty$, for large n we have the following expansion.

Lemma 3. *Let a_n^* and b_n^* be defined by (2.7). For fixed $x \in R$ and sufficiently large n,*

$$\begin{aligned} F^n\left(a_n^* x + b_n^*\right) - \Lambda(x) = \Lambda(x) e^{-x}\Bigg(&(r_n - 1)x + \delta_n - \frac{1}{2} a_n b_n^{-1}(r_n x + \delta_n)^2 \\ &+ O\left((r_n - 1)^2 + \delta_n^2 + \left(a_n b_n^{-1}\right)^2\right)\Bigg). \end{aligned} \tag{2.8}$$

Proof. Note by (1.7) that $\log b_n \sim (2\log n)^{1/2}$, which implies that

$$a_n b_n^{-1} = \frac{1}{\log b_n} \sim (2\log n)^{-1/2} \to 0$$

as $n \to \infty$ by virtue of (1.8). Notice

$$\log\left(1 + a_n b_n^{-1}(r_n x + \delta_n)\right) = a_n b_n^{-1}(r_n x + \delta_n) - \frac{1}{2}\left(a_n b_n^{-1}\right)^2 (r_n x + \delta_n)^2 + O\left(\left(a_n b_n^{-1}\right)^3\right),$$

implying

$$\left(a_n b_n^{-1}\right)^{-1}\log\left(1 + a_n b_n^{-1}(r_n x + \delta_n)\right) = r_n x + \delta_n - \frac{1}{2} a_n b_n^{-1}(r_n x + \delta_n)^2 + O\left(\left(a_n b_n^{-1}\right)^2\right)$$

and

$$\left(\log\left(1 + a_n b_n^{-1}(r_n x + \delta_n)\right)\right)^2 = \left(a_n b_n^{-1}\right)^2 (r_n x + \delta_n)^2 + O\left(\left(a_n b_n^{-1}\right)^3\right),$$

so we have

$$\begin{aligned} &\exp\left(-\left(a_n b_n^{-1}\right)^{-1}\log\left(1 + a_n b_n^{-1}(r_n x + \delta_n)\right) - \frac{1}{2}\left(\log\left(1 + a_n b_n^{-1}(r_n x + \delta_n)\right)\right)^2\right) \\ &\quad = e^{-x}\left(1 - (r_n - 1)x - \delta_n + \frac{1}{2} a_n b_n^{-1}(r_n x + \delta_n)^2 + O\left(\left(a_n b_n^{-1}\right)^2 + \delta_n^2 + (r_n - 1)^2\right)\right) \end{aligned}$$

and

$$\left(1 + a_n b_n^{-1}\log\left(1 + a_n b_n^{-1}(r_n x + \delta_n)\right)\right)^{-1} = 1 - \left(a_n b_n^{-1}\right)^2 (r_n x + \delta_n) + O\left(\left(a_n b_n^{-1}\right)^3\right).$$

Hence,

$$\begin{aligned} &\left(1 + a_n b_n^{-1}\log\left(1 + a_n b_n^{-1}(r_n x + \delta_n)\right)\right)^{-1} \\ &\qquad \times \exp\left(-\left(a_n b_n^{-1}\right)^{-1}\log\left(1 + a_n b_n^{-1}(r_n x + \delta_n)\right) - \frac{1}{2}\left(\log\left(1 + a_n b_n^{-1}(r_n x + \delta_n)\right)\right)^2\right) \\ &\quad = \left(1 - \left(a_n b_n^{-1}\right)^2 (r_n x + \delta_n) + O\left(\left(a_n b_n^{-1}\right)^3\right)\right) e^{-x} \\ &\qquad \times \left(1 - (r_n - 1)x - \delta_n + \frac{1}{2} a_n b_n^{-1}(r_n x + \delta_n)^2 + O\left(\left(a_n b_n^{-1}\right)^2 + \delta_n^2 + (r_n - 1)^2\right)\right) \\ &\quad = e^{-x}\left(1 - (r_n - 1)x - \delta_n + \frac{1}{2} a_n b_n^{-1}(r_n x + \delta_n)^2 + O\left(\left(a_n b_n^{-1}\right)^2 + \delta_n^2 + (r_n - 1)^2\right)\right). \end{aligned}$$

Thus for large n we have

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\left(\log\left(a_n^*x+b_n^*\right)\right)^{-1}\exp\left(-\frac{1}{2}\left(\log\left(a_n^*x+b_n^*\right)\right)^2\right)\\
&=\frac{1}{\sqrt{2\pi}}\left(\log b_n\left(1+a_nb_n^{-1}\left(r_nx+\delta_n\right)\right)\right)^{-1}\exp\left(-\frac{1}{2}\left(\log b_n\left(1+a_nb_n^{-1}\left(r_nx+\delta_n\right)\right)\right)^2\right)\\
&=n^{-1}\left(1+a_nb_n^{-1}\log\left(1+a_nb_n^{-1}\left(r_nx+\delta_n\right)\right)\right)^{-1}\\
&\quad\times\exp\left(-\left(a_nb_n^{-1}\right)^{-1}\log\left(1+a_nb_n^{-1}\left(r_nx+\delta_n\right)\right)-\frac{1}{2}\left(\log\left(1+a_nb_n^{-1}\left(r_nx+\delta_n\right)\right)\right)^2\right)\\
&=n^{-1}e^{-x}\left(1-\left(r_n-1\right)x-\delta_n+\frac{1}{2}a_nb_n^{-1}\left(r_nx+\delta_n\right)^2+O\left(\left(a_nb_n^{-1}\right)^2+\delta_n^2+\left(r_n-1\right)^2\right)\right).
\end{aligned} \tag{2.9}$$

Similarly, one can get

$$\left(\log\left(a_n^*x+b_n^*\right)\right)^{-2}=\left(a_nb_n^{-1}\right)^2+O\left(\left(a_nb_n^{-1}\right)^4\right) \tag{2.10}$$

and

$$s(x)=O\left(n^{-1}\left(a_nb_n^{-1}\right)^4\right) \tag{2.11}$$

by virtue of (2.4).

Using (2.9)–(2.11) together with (2.2) and (2.4) we have

$$\begin{aligned}
&F^n\left(a_n^*x+b_n^*\right)-\Lambda(x)\\
&=\left(1-n^{-1}e^{-x}\left(1-\left(r_n-1\right)x-\delta_n+\frac{1}{2}a_nb_n^{-1}\left(r_nx+\delta_n\right)^2+O\left(\left(r_n-1\right)^2+\delta_n^2+\left(a_nb_n^{-1}\right)^2\right)\right)\right.\\
&\quad\left.\times\left(1-\left(a_nb_n^{-1}\right)^2+O\left(\left(a_nb_n^{-1}\right)^4\right)\right)+O\left(n^{-1}\left(a_nb_n^{-1}\right)^4\right)\right)^n-\Lambda(x)\\
&=\Lambda(x)e^{-x}\left(\left(r_n-1\right)x+\delta_n-\frac{1}{2}a_nb_n^{-1}\left(r_nx+\delta_n\right)^2+O\left(\left(r_n-1\right)^2+\delta_n^{-1}+\left(a_nb_n^{-1}\right)^2\right)\right),
\end{aligned}$$

which completes the proof. □

3. Proofs of the main results

The aim of this section is to prove the main results.

Proof of Theorem 1. (i) Letting $r_n=1,\delta_n=0$ in (2.7) and noting $a_nb_n^{-1}\sim 1/\left(2\log n\right)^{1/2}$, by Lemma 3 we can prove that there exists an absolute constant $C_1>0$ such that

$$\sup_{x\in\mathbb{R}}\left|F^n(a_nx+b_n)-\Lambda(x)\right|>\frac{C_1}{\left(\log n\right)^{1/2}}$$

for $n\geq 2$. In order to obtain the upper bound, we need to prove the following.

$$\sup_{d_n\leq x<\infty}\left|F^n(a_nx+b_n)-\Lambda(x)\right|<\mathbb{D}_1\,a_nb_n^{-1}, \tag{3.1}$$

$$\sup_{0\leq x<d_n}\left|F^n(a_nx+b_n)-\Lambda(x)\right|<\mathbb{D}_2\,a_nb_n^{-1}, \tag{3.2}$$

$$\sup_{-c_n\leq x<0}\left|F^n(a_n+b_n)-\Lambda(x)\right|<\mathbb{D}_3\,a_nb_n^{-1} \tag{3.3}$$

$$\sup_{-\infty<x<-c_n}\left|F^n(a_nx+b_n)-\Lambda(x)\right|<\mathbb{D}_4\,a_nb_n^{-1} \tag{3.4}$$

for $n\geq n_0$ as there must exist absolute constant $C_2>C_1$ such that $n_0=\sup(k:C_2/(\log k)^{1/2}\geq 1)$, where $\mathbb{D}_i>0$ for $i=1,2,3,4$ are absolute constants, and c_n and d_n are respectively defined by

$$c_n=:\log\log\log b_n>0,\qquad d_n=:-\log\log\frac{\log b_n}{\log b_n-1}>0$$

for $n \geq n_0$. Note that $x \geq -c_n$ implies

$$b_n + a_n x \geq b_n - a_n c_n = b_n \left(1 - \frac{\log\log\log b_n}{\log b_n}\right) > 0, \quad n \geq n_0.$$

For the rest of the proof, let $\mathbb{C}_j, j = 1, 2, \ldots, 12$ stand for the absolute positive constants.

For $x \geq -c_n$, let $\Psi_n(x) = 1 - F(a_n x + b_n)$, then

$$n \log F(a_n x + b_n) = -n\Psi_n(x) - R_n(x), \tag{3.5}$$

where

$$0 < R_n(x) = -n\left(\Psi_n(x) + \log\left(1 - \Psi_n(x)\right)\right) < \frac{n\Psi_n^2(x)}{2\left(1 - \Psi_n(x)\right)}$$

since

$$-x - \frac{x^2}{2(1-x)} < \log(1-x) < -x \quad \text{for } 0 < x < 1. \tag{3.6}$$

We deduce from (2.1) that for $x \geq -c_n$,

$$\begin{aligned}
\Psi_n(x) &\leq \Psi_n(-c_n) = 1 - F(b_n - a_n c_n) \\
&< n^{-1}\left(1 + a_n b_n^{-1} \log\left(1 - a_n b_n^{-1} c_n\right)\right)^{-1} \exp\left(-\left(a_n b_n^{-1}\right)^{-1} \log\left(1 - a_n b_n^{-1} c_n\right)\right) \\
&< n^{-1}\left(1 + a_n b_n^{-1} \log\left(1 - a_n b_n^{-1} c_n\right)\right)^{-1} \exp\left(c_n + \frac{a_n b_n^{-1} c_n^2}{2(1 - a_n b_n^{-1} c_n)}\right) \\
&< \frac{\log(2\log n)}{2n}\left(1 + \frac{\log\left(1 - \frac{\log\log\log b_n}{\log b_n}\right)}{\log b_n}\right)^{-1} \exp\left(\frac{\frac{(\log\log\log b_n)^2}{\log b_n}}{2\left(1 - \frac{\log\log\log b_n}{\log b_n}\right)}\right) \\
&< \mathbb{C}_1 < 1
\end{aligned} \tag{3.7}$$

for $n \geq n_0$, implying

$$\inf_{x \geq -c_n} \left(1 - \Psi_n(x)\right) > 1 - \mathbb{C}_1 > 0,$$

so,

$$\begin{aligned}
0 &< R_n(x) \leq \frac{n\Psi_n^2(x)}{2\left(1 - \Psi_n(x)\right)} \\
&< \frac{a_n b_n^{-1}}{2\sqrt{2\pi}(1 - \mathbb{C}_1)}\left(1 + \frac{\log\left(1 - \frac{\log\log\log b_n}{\log b_n}\right)}{\log b_n}\right)^{-2} \frac{(\log\log b_n)^2}{\exp\left(\frac{1}{2}(\log b_n)^2\right)} \exp\left(\frac{\frac{(\log\log\log b_n)^2}{\log b_n}}{\left(1 - \frac{\log\log\log b_n}{\log b_n}\right)}\right) \\
&< \mathbb{C}_2 a_n b_n^{-1}
\end{aligned}$$

for $n \geq n_0$. Thus

$$\left|\exp\left(-R_n(x)\right) - 1\right| < R_n(x) < \mathbb{C}_2 a_n b_n^{-1} \tag{3.8}$$

as $1 - e^{-x} < x$ for $x > 0$.

Let $A_n(x) = \exp\left(-n\Psi_n(x) + e^{-x}\right)$ and $B_n(x) = \exp\left(-R_n(x)\right)$, by (3.8) we have

$$\begin{aligned}
\left|F^n(a_n x + b_n) - \Lambda(x)\right| &\leq \Lambda(x)|A_n(x) - 1| + |B_n(x) - 1| \\
&< \Lambda(x)|A_n(x) - 1| + \mathbb{C}_2 a_n b_n^{-1}.
\end{aligned} \tag{3.9}$$

From (2.1) and (2.3) it follows that

$$-n\Psi_n(x) + e^{-x} = \left(1 + a_n b_n^{-1} \log\left(1 + a_n b_n^{-1} x\right)\right)^{-1} e^{-x} C_n(x) \tag{3.10}$$

with

$$C_n(x) = -D_n(x)E_n(x) + F_n(x), \tag{3.11}$$

where $D_n(x)$, $E_n(x)$ and $F_n(x)$ are respectively defined by

$$\begin{aligned}
D_n(x) &= \exp\left(x - \frac{1}{2}\left(\log\left(1 + a_n b_n^{-1} x\right)\right)^2 - \left(a_n b_n^{-1}\right)^{-1} \log\left(1 + a_n b_n^{-1} x\right)\right),\\
E_n(x) &= 1 - \left(a_n b_n^{-1}\right)^2 \left(1 + a_n b_n^{-1} \log\left(1 + a_n b_n^{-1} x\right)\right)^{-2} \kappa_n,\\
F_n(x) &= 1 + a_n b_n^{-1} \log\left(1 + a_n b_n^{-1} x\right),
\end{aligned}$$

where $0 < \kappa_n < 1$.

Noting that $1 + x < e^x$, $x \in \mathbb{R}$ and $\log(1 + x) < x$, $x > -1$, for $x \geq -c_n$ we have

$$\begin{aligned}
D_n(x) &> 1 + x - \frac{1}{2}\left(\log\left(1 + a_n b_n^{-1} x\right)\right)^2 - \left(a_n b_n^{-1}\right)^{-1} \log\left(1 + a_n b_n^{-1} x\right)\\
&> 1 - \frac{1}{2}\left(\log\left(1 + a_n b_n^{-1} x\right)\right)^2. \qquad (3.12)
\end{aligned}$$

First we prove (3.3). Noting that $\log\left(1 + a_n b_n^{-1} x\right) < 0$ in the case of $-c_n \leq x < 0$ and $(\log(1 + x))^2 < \left(x - \frac{x^2}{2(1+x)}\right)^2$ as $-1 < x < 0$, by (3.12) we have

$$C_n(x) < a_n b_n^{-1} \left(\frac{1}{2} a_n b_n^{-1} \left(x - \frac{a_n b_n^{-1} x^2}{2\left(1 + a_n b_n^{-1} x\right)}\right)^2 + a_n b_n^{-1} \left(1 + a_n b_n^{-1} \log\left(1 + a_n b_n^{-1} x\right)\right)^{-2}\right) \tag{3.13}$$

for $-c_n \leq x < 0$. On the other hand, since $e^x - 1 < xe^x$, $x \in \mathbb{R}$ and by (3.6) we have

$$D_n(x) < 1 + \frac{a_n b_n^{-1} x^2}{2\left(1 + a_n b_n^{-1} x\right)} \exp\left(\frac{a_n b_n^{-1} x^2}{2\left(1 + a_n b_n^{-1} x\right)}\right),$$

so,

$$C_n(x) > a_n b_n^{-1} \left(-\frac{x^2}{2\left(1 + a_n b_n^{-1} x\right)} \exp\left(\frac{a_n b_n^{-1} x^2}{2\left(1 + a_n b_n^{-1} x\right)}\right) + a_n b_n^{-1} x - \frac{\left(a_n b_n^{-1} x\right)^2}{2\left(1 + a_n b_n^{-1} x\right)}\right). \tag{3.14}$$

Combining (3.13) with (3.14), for $-c_n \leq x < 0$ we have

$$\begin{aligned}
|C_n(x)| &< a_n b_n^{-1} \left(\frac{1}{2} a_n b_n^{-1} \left(x - \frac{a_n b_n^{-1} x^2}{2\left(1 + a_n b_n^{-1} x\right)}\right)^2 + a_n b_n^{-1} \left(1 + a_n b_n^{-1} \log\left(1 + a_n b_n^{-1} x\right)\right)^{-2}\right.\\
&\quad \left. + \frac{x^2}{2\left(1 + a_n b_n^{-1} x\right)} \exp\left(\frac{a_n b_n^{-1} x^2}{2\left(1 + a_n b_n^{-1} x\right)}\right) - a_n b_n^{-1} x + \frac{\left(a_n b_n^{-1} x\right)^2}{2\left(1 + a_n b_n^{-1} x\right)}\right)\\
&< a_n b_n^{-1} \left(\frac{1}{8} a_n b_n^{-1} c_n^2 \left(1 + \frac{1}{1 - a_n b_n^{-1} c_n}\right)^2 + a_n b_n^{-1} \left(1 + a_n b_n^{-1} \log\left(1 - a_n b_n^{-1} c_n\right)\right)^{-2}\right.\\
&\quad \left. + \frac{x^2}{2\left(1 - a_n b_n^{-1} c_n\right)} \exp\left(\frac{a_n b_n^{-1} c_n^2}{2\left(1 - a_n b_n^{-1} c_n\right)}\right) + a_n b_n^{-1} c_n + \frac{\left(a_n b_n^{-1} c_n\right)^2}{2\left(1 - a_n b_n^{-1} c_n\right)}\right)\\
&< a_n b_n^{-1} \left(\mathbb{C}_3 + \mathbb{C}_4 x^2\right)
\end{aligned}$$

for $n \geq n_0$. So for $-c_n \leq x < 0$ we have

$$\begin{aligned}
|-n\Psi_n(x) + e^{-x}| &= \left(1 + a_n b_n^{-1} \log\left(1 + a_n b_n^{-1} x\right)\right)^{-1} e^{-x} |C_n(x)|\\
&< \left(1 + a_n b_n^{-1} \log\left(1 + a_n b_n^{-1} x\right)\right)^{-1} e^{-x} a_n b_n^{-1} \left(\mathbb{C}_3 + \mathbb{C}_4 x^2\right)\\
&< \left(1 + a_n b_n^{-1} \log\left(1 - a_n b_n^{-1} c_n\right)\right)^{-1} e^{-x} a_n b_n^{-1} \left(\mathbb{C}_3 + \mathbb{C}_4 x^2\right)\\
&< \left(1 + a_n b_n^{-1} \log\left(1 - a_n b_n^{-1} c_n\right)\right)^{-1} e^{c_n} a_n b_n^{-1} \left(\mathbb{C}_3 + \mathbb{C}_4 c_n^2\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left(1+a_nb_n^{-1}\log\left(1-a_nb_n^{-1}c_n\right)\right)^{-1}\left(\mathbb{C}_3\frac{\log\log b_n}{\log b_n}+\mathbb{C}_4\frac{\log\log b_n\,(\log\log\log b_n)^2}{\log b_n}\right)\\
&< \left(1+\frac{\log\left(1-\frac{\log\log\log b_n}{\log b_n}\right)}{\log b_n}\right)^{-1}\left(\mathbb{C}_3\frac{\log\log b_n}{\log b_n}+\mathbb{C}_4\frac{(\log\log b_n)^3}{\log b_n}\right)\\
&< \mathbb{C}_5
\end{aligned}
$$

for $n\ge n_0$. Hence noting $|e^x-1|<|x|\,(e^x+1)\,,\,x\in\mathbb{R}$, we have

$$
\begin{aligned}
\Lambda(x)|A_n(x)-1| &= \Lambda(x)|\exp\left(-n\Psi_n(x)+e^{-x}\right)-1|\\
&< \Lambda(x)|-n\Psi_n(x)+e^{-x}|\left(\exp\left(-n\Psi_n(x)+e^{-x}\right)+1\right)\\
&< \left(e^{\mathbb{C}_5}+1\right)\Lambda(x)\left(1+a_nb_n^{-1}\log\left(1-a_nb_n^{-1}c_n\right)\right)^{-1}e^{-x}a_nb_n^{-1}\left(\mathbb{C}_3+\mathbb{C}_4x^2\right)\\
&< \left(e^{\mathbb{C}_5}+1\right)e^{-1}a_nb_n^{-1}\left(1+a_nb_n^{-1}\log\left(1-a_nb_n^{-1}c_n\right)\right)^{-1}\left(\mathbb{C}_3+\mathbb{C}_4x^2\right)\exp\left(-\frac{x^2}{2}\right)\\
&< \mathbb{C}_6a_nb_n^{-1}
\end{aligned}
$$

for $n\ge n_0$, since $\max_{-c_n\le x\le 0}\left(\mathbb{C}_3+\mathbb{C}_4x^2\right)e^{-x^2/2}\le\mathbb{C}_3+\mathbb{C}_4$. Combining above inequality with (3.9) we have

$$
\sup_{-c_n\le x<0}\left|F^n\left(a_nx+b_n\right)-\Lambda(x)\right|<\left(\mathbb{C}_2+\mathbb{C}_6\right)a_nb_n^{-1}
$$

for $n>n_0$.

Second, consider the case in which $0\le x<d_n$. By (3.6), (3.10) and (3.12) we can derive

$$
C_n(x)<a_nb_n^{-1}\left(\frac{1}{2}a_nb_n^{-1}x^2+a_nb_n^{-1}\left(1+a_nb_n^{-1}\log\left(1+a_nb_n^{-1}x\right)\right)^{-2}+a_nb_n^{-1}x\right). \tag{3.15}
$$

On the other hand, note that $x-x^2/2<\log(1+x)$, $x>0$ and $e^x-1<xe^x$, $x\in\mathbb{R}$, implies

$$
\begin{aligned}
D_n(x) &< \exp\left(x-\left(a_nb_n^{-1}\right)^{-1}\log\left(1+a_nb_n^{-1}x\right)\right)\\
&< 1+\frac{a_nb_n^{-1}x^2}{2}\exp\left(\frac{a_nb_n^{-1}x^2}{2}\right),
\end{aligned}
$$

so,

$$
\begin{aligned}
C_n(x) &> -1-\frac{a_nb_n^{-1}x^2}{2}\exp\left(\frac{a_nb_n^{-1}x^2}{2}\right)+1+a_nb_n^{-1}\log\left(1+a_nb_n^{-1}x\right)\\
&> -\frac{a_nb_n^{-1}x^2}{2}\exp\left(\frac{a_nb_n^{-1}x^2}{2}\right)
\end{aligned} \tag{3.16}
$$

as $x\ge 0$. Combining (3.15) with (3.16), for $0\le x<d_n$ we can get

$$
|C_n(x)|<a_nb_n^{-1}\left(\mathbb{C}_7+\mathbb{C}_8x^2\right)
$$

for $n\ge n_0$, implying

$$
\begin{aligned}
|-n\Psi_n(x)+e^{-x}| &< \left(1+a_nb_n^{-1}\log\left(1+a_nb_n^{-1}x\right)\right)^{-1}e^{-x}a_nb_n^{-1}\left(\mathbb{C}_7+\mathbb{C}_8x^2\right)\\
&< e^{-x}a_nb_n^{-1}\left(\mathbb{C}_7+\mathbb{C}_8x^2\right)\\
&< a_nb_n^{-1}\left(\mathbb{C}_7+4\mathbb{C}_8e^{-2}\right)\\
&< \mathbb{C}_9.
\end{aligned}
$$

Hence by $|e^x-1|<|x|\,(e^x+1)\,,\,x\in\mathbb{R}$,

$$
\begin{aligned}
\Lambda(x)|A_n(x)-1| &= \Lambda(x)\left|\exp\left(-n\Psi_n(x)+e^{-x}\right)-1\right|\\
&< \Lambda(x)\left|-n\Psi_n(x)+e^{-x}\right|\left(\exp\left(-n\Psi_n(x)+e^{-x}\right)+1\right)\\
&< \left(e^{\mathbb{C}_9}+1\right)\Lambda(x)e^{-x}a_nb_n^{-1}\left(\mathbb{C}_7+\mathbb{C}_8x^2\right)\\
&= \left(e^{\mathbb{C}_9}+1\right)a_nb_n^{-1}\left(\mathbb{C}_7+\mathbb{C}_8x^2\right)\exp\left(-e^{-x}-x\right)
\end{aligned}
$$

$$
\begin{aligned}
&< \left(e^{\mathbb{C}_9}+1\right) a_n b_n^{-1}\left(\mathbb{C}_7+\mathbb{C}_8 x^2\right)\exp\left(-1-\frac{1}{2}x^2\right)\\
&\le \left(e^{\mathbb{C}_9}+1\right) e^{-1}\left(\mathbb{C}_7+2\mathbb{C}_8 e^{-1}\right) a_n b_n^{-1}\\
&= \mathbb{C}_{10} a_n b_n^{-1}
\end{aligned}
$$

for $n \ge n_0$ since $0 \le x < d_n$. Combining with (3.9) to get (3.2).

Third we prove (3.1). Obviously,

$$
\sup_{x\ge d_n}(1-\Lambda(x)) \le 1-\Lambda(d_n) = a_n b_n^{-1}, \tag{3.17}
$$

and from (3.5) we deduce that

$$
1-F^n(a_n d_n+b_n) < n\Psi_n(d_n)+R_n(d_n). \tag{3.18}
$$

By (2.1) we have

$$
\begin{aligned}
n\Psi_n(d_n) &< \frac{n}{\sqrt{2\pi}}(\log(a_n d_n+b_n))^{-1}\exp\left(-\frac{1}{2}(\log(a_n d_n+b_n))^2\right)\\
&< \exp\left(-\left(a_n b_n^{-1}\right)^{-1}\log\left(1+a_n b_n^{-1} d_n\right)\right)\\
&< \exp\left(-d_n+\frac{a_n b_n^{-1} d_n^2}{2}\right)\\
&= a_n b_n^{-1}(\log b_n)\left(\log\frac{\log b_n}{\log b_n-1}\right)\exp\left(\frac{\left(\log\log\frac{\log b_n}{\log b_n-1}\right)^2}{2\log b_n}\right)\\
&< \mathbb{C}_{11} a_n b_n^{-1}
\end{aligned} \tag{3.19}
$$

for $n \ge n_0$ since the continuous function $x\log(x/(x-1))\exp\left(\frac{\left(\log\log\frac{x}{x-1}\right)^2}{2x}\right)$ is decreasing as $x > 2$. Note that $R_n(x) < \mathbb{C}_2 a_n b_n^{-1}$ by (3.7). Combining with (3.17), (3.18) and (3.19), we derive that

$$
\begin{aligned}
\sup_{x\ge d_n}\left|F^n(a_n x+b_n)-\Lambda(x)\right| &\le 1-F^n(a_n d_n+b_n)+1-\Lambda(d_n)\\
&< (\mathbb{C}_{11}+\mathbb{C}_2+1)\, a_n b_n^{-1}
\end{aligned}
$$

for $n \ge n_0$ which completes the proof of (3.1).

Finally, consider the case in which $-\infty < x \le -c_n$. In this case,

$$
\begin{aligned}
\sup_{x\le -c_n} F^n(a_n x+b_n) &\le F^n(b_n-a_n c_n)\\
&\le \exp\left(-e^{c_n}\left(1-\frac{1}{2}\left(\log\left(1-a_n b_n^{-1}c_n\right)\right)^2-\left(a_n b_n^{-1}\right)^2\left(1+a_n b_n^{-1}\log\left(1-a_n b_n^{-1}c_n\right)\right)^{-2}\right)\right)\\
&\le a_n b_n^{-1}\exp\left(\frac{(\log\log b_n)\,c_n^2}{2(\log b_n)^2}+\frac{1}{2\left(1-a_n b_n^{-1}c_n\right)}\frac{(\log\log b_n)\,c_n^3}{(\log b_n)^3}\right.\\
&\qquad\left.+\frac{1}{8\left(1-a_n b_n^{-1}c_n\right)^2}\frac{(\log\log b_n)\,c_n^4}{(\log b_n)^4}+\frac{\log\log b_n}{(\log b_n)^2}\left(1+a_n b_n^{-1}\log\left(1-a_n b_n^{-1}c_n\right)\right)^{-2}\right)\\
&< \mathbb{C}_{12} a_n b_n^{-1}
\end{aligned}
$$

for $n \ge n_0$. Thus we have

$$
\begin{aligned}
\sup_{x< -c_n}\left|F^n(a_n x+b_n)-\Lambda(x)\right| &\le F^n(b_n-a_n c_n)+\Lambda(-c_n)\\
&< (\mathbb{C}_{12}+1)\, a_n b_n^{-1}
\end{aligned}
$$

for $n \ge n_0$. This completes the proof of (3.4). The proof of Theorem 1(i) is complete.

(ii) Setting

$$\begin{aligned} b_n &= \beta_n + \theta_n \\ &= \left(\exp\left((2\log n)^{1/2}\right)\right)\left(1 - \frac{B_n}{2\,(2\log n)^{1/2}} + \frac{\theta_n}{\exp\left((2\log n)^{1/2}\right)}\right), \end{aligned} \tag{3.20}$$

where β_n is defined by (1.6), $B_n = \log 4\pi + \log\log n$ and $\theta_n = \exp\left((2\log n)^{1/2}\right) o\left(1/\left(\log n\right)^{1/2}\right)$. Note that

$$\log(1+x) = x - \frac{x^2}{2} + O\left(x^3\right) \quad \text{as } x \to 0.$$

Taking logarithms of (3.20), we have

$$\begin{aligned} \log b_n &= (2\log n)^{1/2}\left(1 - \frac{B_n}{4\log n} + \frac{\theta_n}{(2\log n)^{1/2}\exp\left((2\log n)^{1/2}\right)}\right. \\ &\quad \left. - \frac{1}{2\,(2\log n)^{1/2}}\left(\frac{B_n}{2\,(2\log n)^{1/2}} - \frac{\theta_n}{\exp\left((2\log n)^{1/2}\right)}\right)^2 + O\left(\frac{B_n^3}{(\log n)^2}\right)\right). \end{aligned}$$

Hence,

$$\begin{aligned} (\log b_n)^2 &= 2\log n - B_n + \frac{B_n^2}{8\log n} - \frac{B_n\theta_n}{(2\log n)^{1/2}\exp\left((2\log n)^{1/2}\right)} + \frac{\left(1 - (2\log n)^{1/2}\right)\theta_n^2}{\exp\left(2\,(2\log n)^{1/2}\right)} \\ &\quad + \frac{\left(2\,(2\log n)^{1/2} + B_n\right)\theta_n}{\exp\left((2\log n)^{1/2}\right)} - \frac{B_n^2}{4\,(2\log n)^{1/2}} + O\left(\frac{B_n^3}{\log n}\right) \end{aligned}$$

and

$$\begin{aligned} \log\log b_n &= \frac{1}{2}\log 2 + \frac{1}{2}\log\log n - \frac{B_n}{4\log n} + \frac{\theta_n}{(2\log n)^{1/2}\exp\left((2\log n)^{1/2}\right)} - \frac{B_n^2}{32\,(\log n)^2} \\ &\quad - \frac{\theta_n^2}{4\,(\log n)\exp\left(2\,(2\log n)^{1/2}\right)} + \frac{B_n\theta_n}{2\,(2\log n)^{3/2}\exp\left((2\log n)^{1/2}\right)} + O\left(\frac{B_n^2}{(\log n)^{3/2}}\right). \end{aligned}$$

Putting above expansions of $(\log b_n)^2$ and $\log\log b_n$ into

$$\log 2\pi + 2\log\log b_n + (\log b_n)^2 = 2\log n, \tag{3.21}$$

we have

$$\theta_n \sim \frac{\exp\left((2\log n)^{1/2}\right) B_n^2}{16\log n}.$$

Hence,

$$b_n = \exp\left((2\log n)^{1/2}\right)\left(1 - \frac{B_n}{2\,(2\log n)^{1/2}} + \frac{B_n^2}{16\log n} + o\left(\frac{(\log\log n)^2}{\log n}\right)\right).$$

Now for α_n and β_n defined by (1.4) and (1.6), one can check that

$$r_n - 1 = \frac{\alpha_n}{a_n} - 1 \sim \frac{\log\log n}{2\,(2\log n)^{1/2}},$$

and

$$\delta_n = \frac{\beta_n - b_n}{a_n} \sim -\frac{(\log\log n)^2}{8\,(2\log n)^{1/2}}$$

for large n. Hence by Lemma 3 we can derive

$$F^n\left(\alpha_n x + \beta_n\right) - \Lambda(x) \sim -\frac{e^{-x}\Lambda(x)}{8}\frac{(\log\log n)^2}{(2\log n)^{1/2}}$$

for large n, which completes the proof of Theorem 1(ii). □

Acknowledgments

The authors would like to thank the Editor-in-Chief (Professor Dr. Goong Chen) and the referee for carefully reading the paper and for their comments which greatly improved the paper. This work was supported by the National Natural Science Foundation of China Grant No. 11171275 and the SWU Grant for Statistics PH.D.

References

[1] P. Hall, On the rate of convergence of normal extremes, J. Appl. Probab. 16 (1979) 433–439.
[2] M.R. Leadbetter, G. Lindgren, H. Rootzén, Extremes and Related Properties of Random Sequences and Processes, Springer-Verlag, New York, 1983.
[3] W.J. Hall, J.A. Wellner, The rate of convergence in law of the maximum of an exponential sample, Stat. Neerl. 33 (1979) 151–154.
[4] Z. Peng, Z. Weng, S. Nadarajah, Rates of convergence of extremes for mixed exponential distributions, Math. Comput. Simul. 81 (2010) 92–99.
[5] Z. Peng, B. Tong, S. Nadarajah, Tail behavior of the general error distribution, Comm. Statist. Theory Methods 38 (2009) 1884–1892.
[6] Z. Peng, S. Nadarajah, F. Lin, Convergence rate of extremes for the general error distribution, J. Appl. Probab. 47 (2010) 668–679.
[7] F. Lin, Z. Peng, Tail behavior and extremes of short-tailed symmetric distribution, Comm. Statist. Theory Methods 39 (2010) 2811–2817.
[8] F. Lin, X. Zhang, Z. Peng, Y. Jiang, On the rate of convergence of STSD extremes, Comm. Statist. Theory Methods 40 (2011) 1795–1806.
[9] S.I. Resnick, Extreme Values, Regular Variation and Point Processes, Springer-Verlag, New York, 1987.
[10] X. Liao, Z. Peng, 2012, Tail behavior and extremes of logarithmic general error distribution, (submitted for publication).