J. Math. Anal. Appl. 396 (2012) 323–344

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

journal homepage: www.elsevier.com/locate/jmaa

Global existence of the scalar field in de Sitter spacetime

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ARTICLE INFO

Article history:
Received 13 March 2012
Available online 23 June 2012
Submitted by Kenji Nishihara

Keywords:
de Sitter spacetime
Global solution
$L^p - L^q$ estimates
Higgs boson equation

ABSTRACT

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0. Introduction and statement of results

In this article we prove the global existence of the small data solutions of the Cauchy problem for the semilinear Klein–Gordon equation in the de Sitter spacetime. Unlike the same problem in the Minkowski spacetime, we have no restriction on the order of nonlinearity and structure of the nonlinear term, provided that a physical mass of the field is outside of some interval.

A large amount of work has been devoted to the Cauchy problem for the semilinear Klein–Gordon equations in the Minkowski spacetime. The existence of global weak solutions has been obtained by Jörgens [1], Segal [2,3], Pecher [4], Brenner [5], Strauss [6], Ginibre and Velo [7,8] for the equation

$$u_{tt} - \Delta u + m^2 u = |u|^\alpha u.$$

For global solvability, the exact relation between n and $\alpha > 0$ was finally established. More precisely, consider the Cauchy problem for the nonlinear Klein–Gordon equation

$$u_{tt} - \Delta u = -V'(u),$$

where Δ is the Laplace operator in $\mathbb{R}^n$ and $V' = V'(u)$ is a nonlinear function, a typical form of which is the sum of two powers

$$V'(u) = \lambda_0 u + \lambda |u|^\alpha u$$

with $\alpha \geq 0$ and $\lambda \geq 0$. For this equation, a conservation of energy is valid. For finite energy solutions the scaling arguments suggest the assumption $\alpha < 4/(n-1)$. In [7], by a contraction method, the existence and the uniqueness of strong global solutions in energy space $H_{(1)} \oplus L^2$ are proved for arbitrary space dimension n under assumptions on V' that cover the case of sum of powers $\lambda |u|^\alpha u$ with $0 \leq \alpha < 4/(n-1)$, $n \geq 2$, and $\lambda > 0$ for the highest α. Some of the results can be extended to the critical case $\alpha = 4/(n-1)$ (see, e.g. [7, Section 4]).

The Klein–Gordon equation arising in relativistic physics and, in particular, general relativity and cosmology, as well as, in more recent quantum field theories, is a covariant equation that is considered in the curved pseudo-Riemannian

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0022-247X/$ – see front matter. Published by Elsevier Inc.
doi:10.1016/j.jmaa.2012.06.020

manifolds. (See, e.g., [9–12].) Moreover, the latest astronomical observational discovery that the expansion of the universe is speeding supports the model of the expanding universe that is mathematically described by the manifold with metric tensor depending on time and spatial variables. In this paper we restrict ourselves to the manifold arising in the so-called de Sitter model of the universe, which is the curved manifold due to the cosmological constant. Thus, there is a need to study partial differential equations related to such models and, in particular, to investigate the question of the global solvability of the semilinear hyperbolic equations with variable coefficients. The lack of results for the global solvability of such semilinear hyperbolic equations can be explained, among other reasons, by the fact that there are only very few known examples of linearized equations with explicit formulas for the fundamental solutions.

The line element in the de Sitter spacetime has the form

$$ds^2 = -\left(1-\frac{r^2}{R^2}\right)c^2\,dt^2 + \left(1-\frac{r^2}{R^2}\right)^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta\,d\phi^2).$$

The Lamaître–Robertson transformation leads to the following form for the line element [13, Section 134], [14, Section 142]:

$$ds^2 = -c^2\,dt'^2 + e^{2ct'/R}(dx'^2+dy'^2+dz'^2).$$

Here R is the "radius" of the universe. In fact, the de Sitter model belongs to the family of the Friedmann–Lemaître–Robertson–Walker spacetimes (FLRW spacetimes). In the FLRW spacetime [15], one can choose coordinates so that the metric has the form $ds^2 = -dt^2 + S^2(t)d\sigma^2$. In particular, the metric in the de Sitter spacetime in the Lamaître–Robertson coordinates [13] has this form with the cosmic scale factor $S(t) = e^t$.

The homogeneous and isotropic cosmological models possess the highest degree of symmetry that makes them more amenable to rigorous study. Among them we mention FLRW models. The simplest class of cosmological models can be obtained if we assume, additionally, that the metric of the slices of constant time is flat and that the spacetime metric can be written in the form $ds^2 = -dt^2 + a^2(t)(dx^2+dy^2+dz^2)$ with an appropriate scale factor $a(t)$. The assumption that the universe is expanding leads to the positivity of the time derivative $\frac{d}{dt}a(t)$. A further assumption that the universe obeys the accelerated expansion suggests that the second derivative $\frac{d^2}{dt^2}a(t)$ is positive. Under the assumption of FLRW symmetry the equation of motion in the case of positive cosmological constant Λ leads to the solution

$$a(t) = a(0)e^{\sqrt{\frac{\Lambda}{3}}t},$$

which produces models with exponentially accelerated expansion, which is referred to as the *de Sitter model*.

In general the matter fields described by the function ϕ must satisfy equations of motion and, in the case of the massive scalar field, the equation of motion is that ϕ should satisfy the Klein–Gordon equation generated by the metric g,

$$\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ik}\frac{\partial\psi}{\partial x^k}\right) = m^2\psi + V'(\psi).$$

In physical terms this equation describes a local self-interaction for a scalar particle. In the de Sitter universe the equation for the scalar field with mass m and potential function V written out explicitly in coordinates is (See, e.g., [16, Section 5.4] and [17].)

$$\phi_{tt} + nH\phi_t - e^{-2Ht}\Delta\phi + m^2\phi = -V'(\phi). \tag{0.1}$$

Here $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and Δ is the Laplace operator on the flat metric, $\Delta := \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$, while $H = \sqrt{\Lambda/3}$ is the Hubble constant. For the sake of simplicity, from now on, we set $H = 1$. A typical example of a potential function would be $V(\phi) = \phi^4$. If we introduce a new unknown function $u = e^{\frac{n}{2}t}\phi$, then the semilinear Klein–Gordon equation for u in the de Sitter spacetime takes the form

$$u_{tt} - e^{-2t}\Delta u + \left(m^2 - \frac{n^2}{4}\right)u = -e^{\frac{n}{2}t}V'\left(e^{-\frac{n}{2}t}u\right). \tag{0.2}$$

The quantity $\mathcal{M}$, with nonnegative real part $\Re\mathcal{M} \geq 0$, defined by

$$\mathcal{M}^2 := m^2 - \frac{n^2}{4},$$

will be called the "curved mass" of the particle, which is also sometimes referred to as the "effective mass". It is convenient to use $M = |\mathcal{M}|$. We distinguish the following three cases: the case of large mass $m^2 > \frac{n^2}{4}$, the case of dimensional mass $m^2 = \frac{n^2}{4}$, and the case of small mass $m^2 < \frac{n^2}{4}$. They lead to three different equations: the Klein–Gordon equation with the "real curved mass" $\mathcal{M}$,

$$u_{tt} - e^{-2t}\Delta u + M^2u = -e^{\frac{n}{2}t}V'\left(e^{-\frac{n}{2}t}u\right),$$

the wave equation with the "zero curved mass"

$$u_{tt} - e^{-2t}\Delta u = -e^{\frac{n}{2}t} V'\left(e^{-\frac{n}{2}t}u\right),$$

and the Klein–Gordon equation with the "imaginary curved mass" $\mathcal{M}$,

$$u_{tt} - e^{-2t}\Delta u - M^2 u = -e^{\frac{n}{2}t} V'\left(e^{-\frac{n}{2}t}u\right), \tag{0.3}$$

respectively.

Let $W^{l,p}(\mathbb{R}^n)$ be the Sobolev space and $H_{(s)}(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$. To estimate the nonlinear term $V = V(u)$ we use the following Lipschitz condition.

Condition ($\mathcal{L}$). *The function F is said to be Lipschitz continuous in the space $H_{(s)}(\mathbb{R}^n)$ with the norm $\|\cdot\|_{H_{(s)}(\mathbb{R}^n)}$ if*

$$\|F(u) - F(v)\|_{H_{(s)}(\mathbb{R}^n)} \le C\|u - v\|_{H_{(s)}(\mathbb{R}^n)}\left(\|u\|^{\alpha}_{H_{(s)}(\mathbb{R}^n)} + \|v\|^{\alpha}_{H_{(s)}(\mathbb{R}^n)}\right) \quad \text{for all } u, v \in H_{(s)}(\mathbb{R}^n), \tag{0.4}$$

where $\alpha \ge 0$.

Define the complete metric space

$$X(R, s, \gamma) := \{\Phi \in C([0,\infty); H_{(s)}(\mathbb{R}^n)) \mid \|\Phi\|_X := \sup_{t\in[0,\infty)} e^{\gamma t}\|\Phi(t)\|_{H_{(s)}(\mathbb{R}^n)} \le R\}$$

with the metric

$$d(\Phi_1, \Phi_2) := \sup_{t\in[0,\infty)} e^{\gamma t}\|\Phi_1(t) - \Phi_2(t)\|_{H_{(s)}(\mathbb{R})}.$$

The main result of this paper is the following theorem.

Theorem 0.1. *Assume that the nonlinear term $F(u)$ is Lipschitz continuous in the space $H_{(s)}(\mathbb{R}^n)$, $s > n/2 \ge 1$, $F(0) = 0$, and $\alpha > 0$. Assume also that $m \in (0, \sqrt{n^2-1}/2) \cup [n/2, \infty)$. Then, there exists $\varepsilon_0 > 0$ such that, for every given functions $\varphi_0, \varphi_1 \in H_{(s)}(\mathbb{R}^n)$, such that*

$$\|\varphi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\varphi_1\|_{H_{(s)}(\mathbb{R}^n)} \le \varepsilon, \quad \varepsilon < \varepsilon_0,$$

there exists a global solution $\Phi \in C^1([0,\infty); H_{(s)}(\mathbb{R}^n))$ of the Cauchy problem

$$\Phi_{tt} + n\Phi_t - e^{-2t}\Delta\Phi + m^2\Phi = F(\Phi), \tag{0.5}$$

$$\Phi(x, 0) = \varphi_0(x), \qquad \Phi_t(x, 0) = \varphi_1(x). \tag{0.6}$$

That solution $\Phi(x, t)$ belongs to the space $X(2\varepsilon, s, \gamma)$, that is,

$$\sup_{t\in[0,\infty)} e^{\gamma t}\|\Phi(\cdot, t)\|_{H_{(s)}(\mathbb{R}^n)} < 2\varepsilon,$$

with γ such that either $0 < \gamma < (\frac{n}{2} - \sqrt{\frac{n^2}{4} - m^2})/(\alpha+1)$ if $\sqrt{n^2-1}/2 > m > 0$, or $\gamma = 0$ if $\frac{n}{2} \le m$.

In particular, if

$$F(\Phi) = \pm|\Phi|^{\alpha}\Phi \quad \text{or} \quad F(\Phi) = \pm|\Phi|^{\alpha+1},$$

and ($\mathcal{L}$), then the small data Cauchy problem is globally solvable for every $\alpha \in (0,\infty)$ if $m \in (0, \sqrt{n^2-1}/2) \cup [n/2, \infty)$. We can only conjecture that $(\sqrt{n^2-1}/2, n/2)$ is a forbidden mass interval for the small data global solvability of the Cauchy problem for all $\alpha \in (0,\infty)$ with ($\mathcal{L}$). The case of $m = \sqrt{n^2-1}/2$ will be considered in a forthcoming paper.

We note here that, due to the time dependence of the coefficient, there is no conservation of energy, and that, for the general nonlinearity $F(\Phi)$, the decay of the energy cannot be established even though the equation contains a dissipative term.

Baskin [18] discussed the small data global solutions for the scalar Klein–Gordon equation on asymptotically de Sitter spaces, which is a compact manifold with a boundary. More precisely, in [18] the Cauchy problem is considered for the semilinear equation

$$\Box_g u + m^2 u = f(u), \qquad u(x, t_0) = \varphi_0(x) \in H_{(1)}(\mathbb{R}^n), \qquad u_t(x, t_0) = \varphi_1(x) \in L^2(\mathbb{R}^n),$$

where mass is large, $m^2 > n^2/4$, F is a smooth function and satisfies conditions

$$|f(u)| \le c|u|^{\alpha+1}, \qquad |u|\cdot|f'(u)| \sim |f(u)|, \quad f(u) - f'(u)\cdot u \le 0,$$

$\int_0^u f(v)dv \geq 0$, and $\int_0^u f(v)dv \sim |u|^{\alpha+2}$ for large $|u|$. It is also assumed that $\alpha = \frac{4}{n-1}$. In Theorem 1.3 [18] the existence of the global solution for small energy data is stated. (For more references on the asymptotically de Sitter spaces, see the bibliography in [19,20].)

D'Ancona [21] considered the Cauchy problem for the equation

$$u_{tt} - a(t)\Delta u = -V'(u), \quad t \in [0, T], \; x \in \mathbb{R}^3,$$

with the nonnegative real-analytic function $a(t)$, which has a locally finite number of zeros and those zeros are of finite order only. It was supposed in [21] that the nonlinear term obeys conditions $V'(u)u \geq 0$, $V(0) = 0$,

$$|V'(s)|^{1+1/p} \leq c(1 + V(s)), \qquad |V^{(2)}(s)| \leq c(1 + |s|)^{p-1}, \qquad V^{(4)}(s)V^{(2)}(s) + \beta(V^{(3)}(s))^2 \geq 0,$$

with some $p \geq 1$ and $\beta < 1$. Then, assuming that the possible zeros of $a(t)$ are of order not greater than 2λ, $\lambda = 1, 2, \ldots$, the existence of solution $u \in C^\infty([0, T] \times \mathbb{R}^3)$ without restriction on the size of the initial data is proved, provided that $p < (3\lambda + 5)/(3\lambda + 1)$. In [22] this result was extended to the case of 1 and 2 space dimensions.

The remaining part of this paper is organized as follows. In Section 1 we give integral representations for the solutions of the Cauchy problem for the linear equation with large physical mass. Then, we quote from [23], the $L^p - L^q$ estimate for the solutions of that equation with and without a source term. In Section 2 we introduce similar representations for the case of small real mass and of the imaginary mass. These representations are used for the derivation in Sections 2.1 and 2.2 the $L^p - L^q$ estimate for the linear equation with and without a source term. The last section, Section 3, is devoted to the solvability of the associated integral equation and to the proof of Theorem 0.1.

1. The case of large mass, $m \geq n/2$

The nonlinear equations (0.1) and (0.2) are those we would like to solve, but the linear problem is a natural first step. An exceptionally efficient tool for studying nonlinear equations is the fundamental solution of the associated linear operator. We extract a linear part of Eq. (0.2) as an initial model that must be treated first:

$$u_{tt} - e^{-2t}\Delta u + \mathcal{M}^2 u = f. \tag{1.1}$$

In this section we list the explicit formulas for the solution to the Cauchy problem for Eq. (1.1).

Eq. (1.1) is strictly hyperbolic. That implies the well-posedness of the Cauchy problem for (1.1) in the different function spaces. The coefficient of the equation is an analytic function and, consequently, the Holmgren theorem implies local uniqueness in the space of distributions. Moreover, the speed of propagation is finite, namely, it is equal to e^{-t} for every $t \in \mathbb{R}$. The second-order strictly hyperbolic equation (1.1) possesses two fundamental solutions resolving the Cauchy problem. They can be written microlocally in terms of the Fourier integral operators [24], which give a complete description of the wave front sets of the solutions. The distance between two characteristic roots $\lambda_1(t, \xi)$ and $\lambda_2(t, \xi)$ of Eq. (1.1) is $|\lambda_1(t, \xi) - \lambda_2(t, \xi)| = e^{-t}|\xi|$, $t \in \mathbb{R}$, $\xi \in \mathbb{R}^n$. It tends to zero as t approaches ∞. Thus, the operator is not uniformly strictly hyperbolic. Moreover, the finite integrability of the characteristic roots, $\int_0^\infty |\lambda_i(t, \xi)|dt < \infty$, leads to the existence of so-called "horizon" for that equation. More precisely, any signal emitted from the spatial point $x_0 \in \mathbb{R}^n$ at time $t_0 \in \mathbb{R}$ remains inside the ball $|x - x_0| < e^{-t_0}$ for all time $t \in (t_0, \infty)$. Eq. (1.1) is neither Lorentz invariant nor invariant with respect to usual scaling and that brings additional difficulties.

In this section we introduce some necessary notations, definitions, formulas, and results from [23], where the case of the large mass, that is, $m^2 \geq n^2/4$, is discussed. First, we define "forward light cone" $D_+(x_0, t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, and the "backward light cone" $D_-(x_0, t_0)$, $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$, as follows:

$$D_\pm(x_0, t_0) := \left\{(x, t) \in \mathbb{R}^{n+1};\, |x - x_0| \leq \pm(e^{-t_0} - e^{-t})\right\}. \tag{1.2}$$

In fact, any intersection of $D_-(x_0, t_0)$ with the hyperplane $t = \text{const} < t_0$ determines the so-called dependence domain for the point (x_0, t_0), while the intersection of $D_+(x_0, t_0)$ with the hyperplane $t = \text{const} > t_0$ is the so-called domain of influence of the point (x_0, t_0). Eq. (1.1) is non-invariant with respect to time inversion. Moreover, the dependence domain is wider than any given ball if time const $> t_0$ is sufficiently large, while the domain of influence is permanently, for all time *const* $< t_0$, in the ball of the radius e^{t_0}.

Define for $t_0 \in \mathbb{R}$ in the domain $D_+(x_0, t_0) \cup D_-(x_0, t_0)$ the function

$$\begin{aligned} E(x, t; x_0, t_0) = {} & (4e^{-t_0-t})^{iM} \left((e^{-t} + e^{-t_0})^2 - (x - x_0)^2\right)^{-\frac{1}{2}-iM} F \\ & \times \left(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(e^{-t_0} - e^{-t})^2 - (x - x_0)^2}{(e^{-t_0} + e^{-t})^2 - (x - x_0)^2}\right), \end{aligned} \tag{1.3}$$

where $F(a, b; c; \zeta)$ is the hypergeometric function. (For the definition of $F(a, b; c; \zeta)$ see, e.g., [25].) Here the notation $(x - x_0)^2 = (x - x_0) \cdot (x - x_0)$ for the points $x, x_0 \in \mathbb{R}^n$ has been used. The kernels $K_0(z, t)$ and $K_1(z, t)$ are defined by

$$K_0(z, t) := -\left[\frac{\partial}{\partial b} E(z, t; 0, b)\right]_{b=0}$$

$$= (4e^{-t})^{iM} \left((1+e^{-t})^2 - z^2\right)^{-iM} \frac{1}{[(1-e^{-t})^2 - z^2]\sqrt{(1+e^{-t})^2 - z^2}}$$
$$\times \left[\left(e^{-t} - 1 - iM(e^{-2t} - 1 - z^2)\right) F\left(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(1-e^{-t})^2 - z^2}{(1+e^{-t})^2 - z^2}\right)\right.$$
$$\left. + \left(1 - e^{-2t} + z^2\right)\left(\frac{1}{2} - iM\right) F\left(-\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(1-e^{-t})^2 - z^2}{(1+e^{-t})^2 - z^2}\right)\right]$$

and $K_1(z,t) := E(z,t;0,0)$, that is,

$$K_1(z,t) = (4e^{-t})^{iM} \left((1+e^{-t})^2 - z^2\right)^{-\frac{1}{2} - iM} F\left(\frac{1}{2} + iM, \frac{1}{2} + iM; 1; \frac{(1-e^{-t})^2 - z^2}{(1+e^{-t})^2 - z^2}\right), \quad 0 \le z \le 1 - e^{-t},$$

respectively. The kernels $K_0(z,t)$ and $K_1(z,t)$ play leading roles in the derivation of $L^p - L^q$ estimates. Their main properties of $K_0(z,t)$ and $K_1(z,t)$ are listed and proved in Section 3 [23].

We consider the equation with $n \ge 2$. The solution $u = u(x,t)$ to the Cauchy problem

$$u_{tt} - e^{-2t}\Delta u + M^2 u = f, \qquad u(x,0) = 0, \qquad u_t(x,0) = 0, \tag{1.4}$$

with $f \in C^\infty(\mathbb{R}^{n+1})$ and with vanishing initial data is given by the next expression

$$u(x,t) = 2\int_0^t db \int_0^{e^{-b} - e^{-t}} dr\, v(x,r;b) E(r,t;0,b),$$

where the function $v(x,t;b)$ is a solution to the Cauchy problem for the wave equation

$$v_{tt} - \Delta v = 0, \qquad v(x,0;b) = f(x,b), \qquad v_t(x,0;b) = 0. \tag{1.5}$$

Thus, for the solution Φ of the equation

$$\Phi_{tt} + n\Phi_t - e^{-2t}\Delta\Phi + m^2\Phi = f, \tag{1.6}$$

due to the relation $u = e^{\frac{n}{2}t}\Phi$, we obtain

$$\Phi(x,t) = 2e^{-\frac{n}{2}t}\int_0^t db \int_0^{e^{-b} - e^{-t}} dr\, e^{\frac{n}{2}b} v(x,r;b) E(r,t;0,b), \tag{1.7}$$

where the function $v(x,t;b)$ is defined by (1.5).

The solution $u = u(x,t)$ to the Cauchy problem

$$u_{tt} - e^{-2t}\Delta u + M^2 u = 0, \qquad u(x,0) = \varphi_0(x), \qquad u_t(x,0) = \varphi_1(x), \tag{1.8}$$

with $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$, $n \ge 2$, can be represented as follows:

$$u(x,t) = e^{\frac{t}{2}} v_{\varphi_0}(x, \phi(t)) + 2\int_0^1 v_{\varphi_0}(x, \phi(t)s) K_0(\phi(t)s, t)\phi(t)\, ds$$
$$+ 2\int_0^1 v_{\varphi_1}(x, \phi(t)s) K_1(\phi(t)s, t)\phi(t)\, ds, \quad x \in \mathbb{R}^n, \ t > 0,$$

where $\phi(t) := 1 - e^{-t}$. Here, for $\varphi \in C_0^\infty(\mathbb{R}^n)$ and for $x \in \mathbb{R}^n$, the function $v_\varphi(x, \phi(t)s)$ coincides with the value $v(x, \phi(t)s)$ of the solution $v(x,t)$ of the Cauchy problem

$$v_{tt} - \Delta v = 0, \qquad v(x,0) = \varphi(x), \qquad v_t(x,0) = 0. \tag{1.9}$$

Thus, for the solution Φ of the Cauchy problem

$$\Phi_{tt} + n\Phi_t - e^{-2t}\Delta\Phi + m^2\Phi = 0, \qquad \Phi(x,0) = \varphi_0(x), \qquad \Phi_t(x,0) = \varphi_1(x), \tag{1.10}$$

due to the relation $u = e^{\frac{n}{2}t}\Phi$, we obtain

$$\Phi(x,t) = e^{-\frac{n-1}{2}t} v_{\varphi_0}(x, \phi(t)) + e^{-\frac{n}{2}t}\int_0^1 v_{\varphi_0}(x, \phi(t)s)\left(2K_0(\phi(t)s, t) + nK_1(\phi(t)s, t)\right)\phi(t)\, ds$$
$$+ 2e^{-\frac{n}{2}t}\int_0^1 v_{\varphi_1}(x, \phi(t)s) K_1(\phi(t)s, t)\phi(t)\, ds, \quad x \in \mathbb{R}^n, \ t > 0.$$

1.1. $L^p - L^q$ estimates for equation with source, $n \geq 2$

The Cauchy problem

$$v_{tt} - \Delta v = 0, \qquad v(x,0) = \psi_0(x), \qquad v_t(x,0) = \psi_1(x),$$

with $\psi_0, \psi_1 \in C_0^\infty(\mathbb{R}^n)$ for the linear wave equation has a unique solution that can be written as follows:

$$u_0(x,t) = V_1(t,D_x)\psi_0(x) + V_2(t,D_x)\psi_1(x).$$

The operators $V_1(t,D_x)$ and $V_2(t,D_x)$ are chosen in accordance with

$$V_1(0,D_x) = I \quad \text{(identity operator)}, \qquad \partial_t V_1(0,D_x) = 0,$$
$$V_2(0,D_x) = 0, \qquad \partial_t V_2(0,D_x) = I \quad \text{(identity operator)}.$$

The microlocal description of those operators by means of the Fourier integral operators is known (see, e.g. [26]). Let $W^{l,p}(\mathbb{R}^n)$, $B^{l,p}(\mathbb{R}^n)$, and $\dot{B}^{l,p}(\mathbb{R}^n)$ be Sobolev, Besov, and homogeneous Besov spaces, respectively. In what follows the space $\mathcal{M}^{s,q}$ can be each of the next spaces $L^q(\mathbb{R}^n)$, $W^{s,q}(\mathbb{R}^n)$, $\dot{W}^{s,q}(\mathbb{R}^n)$, $B^{s,q}(\mathbb{R}^n)$, or $\dot{B}^{s,q}(\mathbb{R}^n)$. The following decay estimates for the linear operators $V_1(t,D_x)$ and $V_2(t,D_x)$ can be found, e.g., in [5,4].

For all $\psi \in C_0^\infty(\mathbb{R}^n)$, $n > 1$, one has the estimates

$$\|(-\Delta)^{-s}V_1(t,D_x)\psi(x)\|_{\mathcal{M}^{l,q}} \leq C\, t^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)}\|\psi\|_{\mathcal{M}^{l,p}}, \quad t \in (0,\infty),$$

under the conditions $s \geq 0$, $1 < p \leq 2$, $\frac{1}{p}+\frac{1}{q}=1$, and $\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q}) \leq 2s \leq n(\frac{1}{p}-\frac{1}{q})$. Then, for all $g \in C_0^\infty(\mathbb{R}^n)$ one has the estimate

$$\|(-\Delta)^{-s}V_2(t,D_x)g(x)\|_{\mathcal{M}^{l,q}} \leq C\, t^{1+2s-n\left(\frac{1}{p}-\frac{1}{q}\right)}\|g\|_{\mathcal{M}^{l,p}}, \quad t \in (0,\infty),$$

under the conditions $s \geq 0$, $1 < p \leq 2$, $\frac{1}{p}+\frac{1}{q}=1$, $k \geq 0$, and $\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q})-1 \leq 2s \leq n(\frac{1}{p}-\frac{1}{q})$. Moreover, a standard interpolation implies that these estimates hold for s and r in some range (see for details [27]). The scaling arguments show that the time dependent factors are exact. The Duhamel principle gives corresponding estimate for the equation with the source term.

Let $u = u(x,t)$ be a solution of the Cauchy problem (1.4). Then according to Corollary 9.3 [23][1] for $n \geq 2$ one has the following estimate

$$\|(-\Delta)^{-s}u(x,t)\|_{L^q(\mathbb{R}^n)} \leq C_M \int_0^t \|f(x,b)\|_{L^p(\mathbb{R}^n)} e^b \left(e^{-b}-e^{-t}\right)^{1+2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} (1+t-b)^{1-\operatorname{sgn}M}\, db,$$

provided that $1 < p \leq 2$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{2}(n+1)\left(\frac{1}{p}-\frac{1}{q}\right) \leq 2s \leq n\left(\frac{1}{p}-\frac{1}{q}\right) < 2s+1$. Thus, for the solution Φ of Eq. (1.6), due to the relation $u = e^{\frac{n}{2}t}\Phi$, we obtain

$$\|(-\Delta)^{-s}\Phi(x,t)\|_{L^q(\mathbb{R}^n)} \leq C_M e^{-\frac{n}{2}t}\int_0^t e^{\frac{n}{2}b}\|f(x,b)\|_{L^p(\mathbb{R}^n)} e^b \left(e^{-b}-e^{-t}\right)^{1+2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} (1+t-b)^{1-\operatorname{sgn}M}\, db.$$

For $M > 0$ we obtain

$$\|(-\Delta)^{-s}\Phi(x,t)\|_{L^q(\mathbb{R}^n)} \leq C_M e^{-\frac{n}{2}t}\int_0^t e^{\frac{n}{2}b}\|f(x,b)\|_{L^p(\mathbb{R}^n)} e^b \left(e^{-b}-e^{-t}\right)^{1+2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} db.$$

For $M = 0$ we obtain

$$\|(-\Delta)^{-s}\Phi(x,t)\|_{L^q(\mathbb{R}^n)} \leq C_M e^{-\frac{n}{2}t}\int_0^t e^{\frac{n}{2}b}\|f(x,b)\|_{L^p(\mathbb{R}^n)} e^b \left(e^{-b}-e^{-t}\right)^{1+2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} (1+t-b)\, db.$$

In particular, for $s = 0$ and $p = q = 2$, we have

$$\|\Phi(x,t)\|_{L^2(\mathbb{R}^n)} \leq C_M e^{-\frac{n}{2}t}\int_0^t e^{\frac{n}{2}b}\|f(x,b)\|_{L^2(\mathbb{R}^n)} (1+t-b)^{1-\operatorname{sgn}M}\, db.$$

Here the rates of exponential factors are independent of the curved mass $\mathcal{M}$ and, consequently, of the mass m.

[1] There is a misprint in [23].

1.2. $L^p - L^q$ estimates for equation without source, $n \geq 2$

According to Theorem 10.1 [23] the solution $u = u(x,t)$ of the Cauchy problem (1.8) satisfies the following $L^p - L^q$ estimate

$$\|(-\Delta)^{-s}u(x,t)\|_{L^q(\mathbb{R}^n)} \leq C_M(1+t)^{1-\mathrm{sgn}M}(1-e^{-t})^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{e^{\frac{t}{2}}\|\varphi_0(x)\|_{L^p(\mathbb{R}^n)}+(1-e^{-t})\|\varphi_1\|_{L^p(\mathbb{R}^n)}\right\}$$

for all $t \in (0,\infty)$, provided that $1 < p \leq 2$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{2}(n+1)\left(\frac{1}{p}-\frac{1}{q}\right) \leq 2s \leq n\left(\frac{1}{p}-\frac{1}{q}\right) < 2s+1$.

In particular, for large t we obtain "no decay" estimate

$$\|(-\Delta)^{-s}u(x,t)\|_{L^q(\mathbb{R}^n)} \leq C_M(1+t)^{1-\mathrm{sgn}M}\left\{e^{\frac{t}{2}}\|\varphi_0(x)\|_{L^p(\mathbb{R}^n)}+\|\varphi_1\|_{L^p(\mathbb{R}^n)}\right\}.$$

Thus, for the solution Φ of the Cauchy problem (1.10), due to the relation $u = e^{\frac{n}{2}t}\Phi$, we obtain the decay estimate

$$\|(-\Delta)^{-s}\Phi(x,t)\|_{L^q(\mathbb{R}^n)} \leq C_M e^{-\frac{n}{2}t}(1+t)^{1-\mathrm{sgn}M}(1-e^{-t})^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{e^{\frac{t}{2}}\|\varphi_0(x)\|_{L^p(\mathbb{R}^n)}+(1-e^{-t})\|\varphi_1\|_{L^p(\mathbb{R}^n)}\right\} \quad (1.11)$$

for all $t > 0$, while

$$\|(-\Delta)^{-s}\Phi(x,t)\|_{L^q(\mathbb{R}^n)} \leq C_M e^{-\frac{n}{2}t}(1+t)^{1-\mathrm{sgn}M}\left\{e^{\frac{t}{2}}\|\varphi_0(x)\|_{L^p(\mathbb{R}^n)}+\|\varphi_1\|_{L^p(\mathbb{R}^n)}\right\}$$

for large t. Here the rate of decay is independent of the curved mass $\mathcal{M}$ and, consequently, of the mass m.

2. The equation with the imaginary curved mass

In this section we consider the linear part of the equation

$$u_{tt} - e^{-2t}\Delta u - M^2u = -e^{\frac{n}{2}t}V'\left(e^{-\frac{n}{2}t}u\right), \quad (2.1)$$

with $M \geq 0$. Eq. (2.1) covers two important cases. The first one is the Higgs boson equation, which has $V'(\phi) = \lambda\phi^3$ and $M^2 = \mu m^2 + n^2/4$ with $\lambda > 0$ and $\mu > 0$, while $n = 3$. The second case is the case of the small physical mass, that is $0 \leq m \leq \frac{n}{2}$. For the last case $M^2 = \frac{n^2}{4} - m^2$.

We introduce new functions $E(x,t;x_0,t_0;M)$, $K_0(z,t;M)$, and $K_1(z,t;M)$, which can be obtained by continuation in complex domain, the ones introduced in [23] and which have been used in Section 1. First we define the function

$$\begin{aligned}E(x,t;x_0,t_0;M) &= 4^{-M}e^{M(t_0+t)}\left((e^{-t}+e^{-t_0})^2-(x-x_0)^2\right)^{-\frac{1}{2}+M}\\ &\quad\times F\left(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(e^{-t_0}-e^{-t})^2-(x-x_0)^2}{(e^{-t_0}+e^{-t})^2-(x-x_0)^2}\right).\end{aligned}$$

Hence, it is related to the function $E(x,t;x_0,t_0)$ of (1.3) as follows:

$$E(x,t;x_0,t_0) = E(x,t;x_0,t_0;-iM).$$

Next we define also new kernels $K_0(z,t;M)$ and $K_1(z,t;M)$ by

$$\begin{aligned}K_0(z,t;M) &:= -\left[\frac{\partial}{\partial b}E(z,t;0,b;M)\right]_{b=0}\\ &= 4^{-M}e^{tM}\left((1+e^{-t})^2-z^2\right)^M\frac{1}{[(1-e^{-t})^2-z^2]\sqrt{(1+e^{-t})^2-z^2}}\\ &\quad\times\left[\left(e^{-t}-1+M(e^{-2t}-1-z^2)\right)F\left(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(1-e^{-t})^2-z^2}{(1+e^{-t})^2-z^2}\right)\right.\\ &\quad\left.+\left(1-e^{-2t}+z^2\right)\left(\frac{1}{2}+M\right)F\left(-\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(1-e^{-t})^2-z^2}{(1+e^{-t})^2-z^2}\right)\right]\end{aligned}$$

and $K_1(z,t;M) := E(z,t;0,0;M)$, that is,

$$K_1(z,t;M) = 4^{-M}e^{Mt}\left((1+e^{-t})^2-z^2\right)^{-\frac{1}{2}+M}F\left(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(1-e^{-t})^2-z^2}{(1+e^{-t})^2-z^2}\right), \quad 0 \leq z \leq 1-e^{-t},$$

respectively. These kernels will be used in the representation of the solutions of the Cauchy problem.

The solution $u = u(x,t)$ to the Cauchy problem

$$u_{tt} - e^{-2t}\Delta u - M^2u = f, \qquad u(x,0) = 0, \qquad u_t(x,0) = 0,$$

with $f \in C^\infty(\mathbb{R}^{n+1})$ and with vanishing initial data is given [28] by the next expression

$$u(x,t) = 2\int_0^t db \int_0^{e^{-b}-e^{-t}} dr\, v(x,r;b)E(r,t;0,b;M), \tag{2.2}$$

where the function $v(x,t;b)$ is a solution to the Cauchy problem for the wave equation (1.5).

The solution $u = u(x,t)$ to the Cauchy problem

$$u_{tt} - e^{-2t}\Delta u - M^2 u = 0, \qquad u(x,0) = \varphi_0(x), \qquad u_t(x,0) = \varphi_1(x),$$

with $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$, $n \geq 2$, can be represented [28] as follows:

$$\begin{aligned} u(x,t) \,=\, & e^{\frac{t}{2}} v_{\varphi_0}(x,\phi(t)) + 2\int_0^1 v_{\varphi_0}(x,\phi(t)s)K_0(\phi(t)s,t;M)\phi(t)\,ds \\ & + 2\int_0^1 v_{\varphi_1}(x,\phi(t)s)K_1(\phi(t)s,t;M)\phi(t)\,ds, \quad x \in \mathbb{R}^n,\ t > 0, \end{aligned}$$

where $\phi(t) := 1 - e^{-t}$. Here, for $\varphi \in C_0^\infty(\mathbb{R}^n)$ and for $x \in \mathbb{R}^n$, the function $v_\varphi(x,\phi(t)s)$ coincides with the value $v(x,\phi(t)s)$ of the solution $v(x,t)$ of the Cauchy problem (1.9).

Thus, for the solution Φ of the Cauchy problem

$$\Phi_{tt} + n\Phi_t - e^{-2t}\Delta\Phi + m^2\Phi = f, \qquad \Phi(x,0) = 0, \qquad \Phi_t(x,0) = 0,$$

due to the relation $u = e^{\frac{n}{2}t}\Phi$, we obtain with $f \in C^\infty(\mathbb{R}^{n+1})$ and with vanishing initial data the next expression

$$\Phi(x,t) = 2e^{-\frac{n}{2}t}\int_0^t db \int_0^{e^{-b}-e^{-t}} dr\, e^{\frac{n}{2}b} v(x,r;b)E(r,t;0,b;M),$$

where the function $v(x,t;b)$ is a solution to the Cauchy problem for the wave equation (1.5).

Thus, for the solution Φ of the Cauchy problem (1.10), due to the relation $u = e^{\frac{n}{2}t}\Phi$, we obtain

$$\begin{aligned} \Phi(x,t) \,=\, & e^{-\frac{n-1}{2}t} v_{\varphi_0}(x,\phi(t)) + e^{-\frac{n}{2}t}\int_0^1 v_{\varphi_0}(x,\phi(t)s)\left(2K_0(\phi(t)s,t;M) + nK_1(\phi(t)s,t;M)\right)\phi(t)\,ds \\ & + 2e^{-\frac{n}{2}t}\int_0^1 v_{\varphi_1}(x,\phi(t)s)K_1(\phi(t)s,t;M)\phi(t)\,ds, \quad x \in \mathbb{R}^n,\ t > 0. \end{aligned}$$

Here for $\varphi \in C_0^\infty(\mathbb{R}^n)$ and for $x \in \mathbb{R}^n$, the function $v_\varphi(x,\phi(t)s)$ coincides with the value $v(x,\phi(t)s)$ of the solution $v(x,t)$ of the Cauchy problem (1.9).

In fact, the representation formulas of this section have been used to establish in [29] some qualitative properties of the solutions of the Higgs boson equation.

2.1. $L^p - L^q$ estimates for equation without source, $n \geq 2$

Consider the solution Φ of the problem (1.10), which is generated by the smooth initial functions $\varphi_0(x)$ and $\varphi_1(x)$ with compact supports, $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$. Then $\Phi \in C^\infty([0,\infty) \times \mathbb{R}^n)$ and the support of Φ is contained in some cylinder $B_R \times [0,\infty)$, where $B_R \subset \mathbb{R}^n$ is a ball of radius R centered at the origin and is depending on the supports of φ_0 and φ_1. We may say that the support of the solution is permanently bounded. That is a consequence of the finite propagation speed property of the hyperbolic equation and due to the existence of horizon for the de Sitter spacetime. Next, we integrate the equation of (1.10) with respect to x and obtain the following initial value problem for the second-order ordinary differential equation,

$$I_{tt} + nI_t + m^2 I = 0, \qquad I(0) = C_0, \qquad I_t(x,0) = C_1,$$

with the solution $I(t) := \int_{\mathbb{R}^n}\Phi(x,t)\,dx$, where $C_0 = \int_{\mathbb{R}^n}\varphi_0(x)\,dx$ and $C_1 = \int_{\mathbb{R}^n}\varphi_1(x)\,dx$. For the case of small mass m, $m \in (0, n/2)$, the last problem implies

$$I(t) = \frac{C_0\lambda_2 - C_1}{\lambda_2 - \lambda_1}e^{-(\frac{n}{2}-M)t} + \frac{C_1 - C_0\lambda_1}{\lambda_2 - \lambda_1}e^{-(\frac{n}{2}+M)t},$$

where $M = \sqrt{n^2/4 - m^2}$, $\lambda_1 := -\frac{n}{2} + M < 0$, and $\lambda_2 := -\frac{n}{2} - M < 0$ since $0 < M < \frac{n}{2}$. Hence, we have

$$\left|\int_{\mathbb{R}^n}\Phi(x,t)\,dx\right| \leq C(\varphi_0,\varphi_1)e^{-(\frac{n}{2}-M)t} \quad \text{for all } t > 0.$$

The last estimate is optimal in the sense that there are $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$ and $C(\varphi_0, \varphi_1) > 0$ such that

$$\left| \int_{\mathbb{R}^n} \Phi(x,t)\, dx \right| \ge C(\varphi_0, \varphi_1) e^{-(\frac{n}{2}-M)t}.$$

Moreover, since the support of Φ is permanently bounded, we have

$$\left| \int_{\mathbb{R}^n} \Phi(x,t)\, dx \right| \le C(\varphi_0, \varphi_1, q) \|\Phi(\cdot, t)\|_{L^q(\mathbb{R}^n)}, \qquad q \in [1, \infty].$$

In the case of the dimensional mass, $n^2/4 = m^2$, the curved mass vanishes, $M = 0$, and we have

$$I(t) = C_0 e^{-\frac{n}{2}t} + \left(C_0 \frac{n}{2} + C_1 \right) t e^{-\frac{n}{2}t}.$$

Hence, there exist $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$ and $C(\varphi_0, \varphi_1) > 0$ such that

$$C(\varphi_0, \varphi_1, q) \|\Phi(\cdot, t)\|_{L^q(\mathbb{R}^n)} \ge \left| \int_{\mathbb{R}^n} \Phi(x,t)\, dx \right| \ge C(\varphi_0, \varphi_1) t e^{-\frac{n}{2}t}, \quad q \in [1, \infty].$$

In the case of the imaginary mass the corresponding equation is

$$I_{tt} + nI_t - m^2 I = 0, \qquad I(0) = C_0, \qquad I_t(x, 0) = C_1,$$

where $M = \sqrt{\frac{n^2}{4} + m^2} > 0$ and $\lambda_1 := -\frac{n}{2} + M > 0$ while $\lambda_2 := -\frac{n}{2} - M < 0$. Consequently, there are $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$ and $C(\varphi_0, \varphi_1) > 0$ such that

$$\left| \int_{\mathbb{R}^n} \Phi(x,t)\, dx \right| \ge C(\varphi_0, \varphi_1) \exp\left(\sqrt{\frac{n^2}{4} + m^2} - \frac{n}{2} \right) t,$$

as well as,

$$\|\Phi(\cdot, t)\|_{L^q(\mathbb{R}^n)} \ge \delta \exp\left(\sqrt{\frac{n^2}{4} + m^2} - \frac{n}{2} \right) t,$$

and the norms of the solution are increasing in time. Thus, we have proven the following statement.

Lemma 2.1. *If $q \in [1, \infty]$, then for both equations, with the real small mass ($M = \sqrt{\frac{n^2}{4} - m^2} \ge 0$, $0 \le m \le \frac{n}{2}$) and with the imaginary mass ($M = \sqrt{\frac{n^2}{4} + m^2} > \frac{n}{2}$, $m > 0$), there exist $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$ and $\delta > 0$ such that*

$$\|\Phi(\cdot, t)\|_{L^q(\mathbb{R}^n)} \ge \delta t^{1-\mathrm{sgn} M} e^{-(\frac{n}{2}-M)t} \quad \textit{for all } t \in (0, \infty).$$

The lemma shows that the estimate of the next theorem is optimal.

Theorem 2.2. *The solution $\Phi = \Phi(x,t)$ of the Cauchy problem*

$$\Phi_{tt} + n\Phi_t - e^{-2t}\Delta\Phi \pm m^2\Phi = 0, \qquad \Phi(x, 0) = \varphi_0(x), \qquad \Phi_t(x, 0) = \varphi_1(x),$$

with either $M = \sqrt{\frac{n^2}{4} - m^2}$ and $m < \sqrt{n^2 - 1}/2$ for the case of "plus", or $M = \sqrt{\frac{n^2}{4} + m^2}$ for the case of "minus", satisfies the following $L^p - L^q$ estimate

$$\|(-\Delta)^{-s}\Phi(x,t)\|_{L^q(\mathbb{R}^n)} \le C_{M,n,p,q,s}(1 - e^{-t})^{2s - n\left(\frac{1}{p} - \frac{1}{q}\right)} e^{(M - \frac{n}{2})t} \left\{ \|\varphi_0\|_{L^p(\mathbb{R}^n)} + (1 - e^{-t})\|\varphi_1\|_{L^p(\mathbb{R}^n)} \right\}$$

for all $t \in (0, \infty)$, provided that $1 < p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{2}(n+1)\left(\frac{1}{p} - \frac{1}{q}\right) \le 2s \le n\left(\frac{1}{p} - \frac{1}{q}\right) < 2s + 1$.

Proof. First we consider the case of $\varphi_1 = 0$. Then

$$\Phi(x,t) = e^{-\frac{n-1}{2}t} v_{\varphi_0}(x, \phi(t)) + e^{-\frac{n}{2}t} \int_0^1 v_{\varphi_0}(x, \phi(t)s)\, (2K_0(\phi(t)s, t; M) + nK_1(\phi(t)s, t; M))\, \phi(t)\, ds$$

and, consequently,

$$\begin{aligned} \|(-\Delta)^{-s}\Phi(x,t)\|_{L^q(\mathbb{R}^n)} \le\ & e^{-\frac{n-1}{2}t} \|(-\Delta)^{-s} v_{\varphi_0}(x, \phi(t))\|_{L^q(\mathbb{R}^n)} + e^{-\frac{n}{2}t} \int_0^1 \|(-\Delta)^{-s} v_{\varphi_0}(x, \phi(t)s)\|_{L^q(\mathbb{R}^n)} \\ & \times \left| 2K_0(\phi(t)s, t; M) + nK_1(\phi(t)s, t; M) \right| \phi(t)\, ds. \end{aligned} \tag{2.3}$$

If $n \ge 2$, then for the solution $v = v(x,t)$ of the Cauchy problem (1.9) for the wave equation in the Minkowski spacetime with $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ one has (see, e.g., [5,4]) the following so-called $L^p - L^q$ decay estimate

$$\|(-\Delta)^{-s} v(\cdot,t)\|_{L^q(\mathbb{R}^n)} \le Ct^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} \|\varphi\|_{L^p(\mathbb{R}^n)} \quad \text{for all } t>0, \tag{2.4}$$

provided that $s \ge 0$, $1 < p \le 2$, $\frac{1}{p}+\frac{1}{q}=1$, and $\frac{1}{2}(n+1)(\frac{1}{p}-\frac{1}{q}) \le 2s \le n(\frac{1}{p}-\frac{1}{q})$. Hence,

$$\|(-\Delta)^{-s} v_{\varphi_0}(x,\phi(t))\|_{L^q(\mathbb{R}^n)} \le C\phi(t)^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} \|\varphi_0\|_{L^p(\mathbb{R}^n)} \quad \text{for all } t>0,$$

where $\phi(t) = 1-e^{-t}$. Consequently, for the first term of the right-hand side of (2.3) we have

$$e^{-\frac{n-1}{2}t}\|(-\Delta)^{-s} v_{\varphi_0}(x,\phi(t))\|_{L^q(\mathbb{R}^n)} \le Ce^{-\frac{n-1}{2}t}(1-e^{-t})^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} \|\varphi_0\|_{L^p(\mathbb{R}^n)} \quad \text{for all } t>0,$$

while for the second term we obtain

$$\begin{aligned} &e^{-\frac{n}{2}t}\int_0^1 \|(-\Delta)^{-s} v_{\varphi_0}(x,\phi(t)s)\|_{L^q(\mathbb{R}^n)} \left|2K_0(\phi(t)s,t;M)+nK_1(\phi(t)s,t;M)\right|\phi(t)\,ds \\ &\quad \le \|\varphi_0\|_{L^p(\mathbb{R}^n)} e^{-\frac{n}{2}t}\int_0^1 \phi(t)^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} s^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\left|2K_0(\phi(t)s,t;M)\right| + n\left|K_1(\phi(t)s,t;M)\right|\right)\phi(t)\,ds. \end{aligned}$$

We have to estimate the following two integrals of the last inequality:

$$\int_0^1 \phi(t)^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} s^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} \left|K_0(\phi(t)s,t;M)\right|\phi(t)\,ds$$

and

$$\int_0^1 \phi(t)^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} s^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} \left|K_1(\phi(t)s,t;M)\right|\phi(t)\,ds,$$

where $\phi(t) = 1-e^{-t}$ and $t>0$. We are going to apply the next two lemmas in the case of $a = 2s - n(\frac{1}{p}-\frac{1}{q})$ and to prove the following estimates

$$\begin{aligned} &\int_0^1 \phi(t)^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} s^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} \left|K_0(\phi(t)s,t;M)\right|\phi(t)\,ds \\ &\quad \le C_{M,n,p,q,s} e^{-Mt-\left[2s-n\left(\frac{1}{p}-\frac{1}{q}\right)\right]t}(e^t-1)^{1+2s-n\left(\frac{1}{p}-\frac{1}{q}\right)}(e^t+1)^{2M-1} \quad \text{for all } t>0, \end{aligned}$$

and

$$\begin{aligned} &\int_0^1 \phi(t)^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} s^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} \left|K_1(\phi(t)s,t;M)\right|\phi(t)\,ds \\ &\quad \le C_{M,n,p,q,s} e^{-Mt-t\left(2s-n\left(\frac{1}{p}-\frac{1}{q}\right)\right)}(e^t-1)^{1+2s-n\left(\frac{1}{p}-\frac{1}{q}\right)}(e^t+1)^{2M-1} \quad \text{for all } t>0. \end{aligned}$$

In particular,

$$\int_0^1 \phi(t)^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} s^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} \left|K_0(\phi(t)s,t;M)\right|\phi(t)\,ds \le C_{M,n,p,q,s} t^{1+2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} \quad \text{for small } t>0,$$

$$\int_0^1 \phi(t)^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} s^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} \left|K_0(\phi(t)s,t;M)\right|\phi(t)\,ds \le C_{M,n,p,q,s} e^{Mt} \quad \text{for large } t,$$

and

$$\int_0^1 \phi(t)^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} s^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} \left|K_1(\phi(t)s,t;M)\right|\phi(t)\,ds \le C_{M,n,p,q,s} t^{1+2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} \quad \text{for small } t,$$

$$\int_0^1 \phi(t)^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} s^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} \left|K_1(\phi(t)s,t;M)\right|\phi(t)\,ds \le C_{M,n,p,q,s} e^{Mt} \quad \text{for large } t. \quad \square$$

Lemma 2.3. *Let $a > -1$ and $\phi(t) = 1-e^{-t}$. Then*

$$\int_0^1 \phi(t)^a s^a \left|K_1(\phi(t)s,t;M)\right|\phi(t)\,ds \le C_M e^{-Mt-at}(e^t-1)^{a+1}(e^t+1)^{2M-1} \quad \textit{for all } t>0.$$

In particular,

$$\int_0^1 \phi(t)^a s^a \left|K_1(\phi(t)s,t;M)\right|\phi(t)\,ds \le C_{M,a} t^{1+a} \quad \textit{for small } t,$$
$$\int_0^1 \phi(t)^a s^a \left|K_1(\phi(t)s,t;M)\right|\phi(t)\,ds \le C_{M,a} e^{Mt} \quad \textit{for large } t.$$

Proof. By the definition of the kernel K_1, we obtain

$$\begin{aligned}
\int_0^1 \phi(t)^a s^a \left|K_1(\phi(t)s,t;M)\right|\phi(t)\,ds &= \int_0^{1-e^{-t}} r^a \left|K_1(r,t;M)\right| dr \\
&\le 4^{-M} e^{Mt} \int_0^{1-e^{-t}} r^a \left((1+e^{-t})^2 - r^2\right)^{-\frac{1}{2}+M} \\
&\quad \times F\left(\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(1-e^{-t})^2-r^2}{(1+e^{-t})^2-r^2}\right) dr \\
&\le 4^{-M} e^{Mt} \int_0^{e^t-1} e^{t-2Mt} e^{-at} y^a \left((e^t+1)^2-y^2\right)^{-\frac{1}{2}+M} \\
&\quad \times F\left(\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(e^t-1)^2-y^2}{(e^t+1)^2-y^2}\right) e^{-t} dy,
\end{aligned}$$

where the substitution $e^t r = y$ has been used. Thus,

$$\begin{aligned}
\int_0^1 \phi(t)^a s^a \left|K_1(\phi(t)s,t;M)\right|\phi(t)\,ds &\le 4^{-M} e^{-Mt-at} \int_0^{e^t-1} y^a \left((e^t+1)^2-y^2\right)^{-\frac{1}{2}+M} \\
&\quad \times F\left(\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(e^t-1)^2-y^2}{(e^t+1)^2-y^2}\right) dy.
\end{aligned}$$

On the other hand, for $M > 0$ we have

$$F\left(\frac{1}{2}-M, \frac{1}{2}-M; 1; z\right) \le C_M \quad \text{for all } z \in [0,1],$$

where

$$z := \frac{(e^t-1)^2-y^2}{(e^t+1)^2-y^2} \in [0,1) \quad \text{for all } y \in [0, e^t-1] \text{ and all } t > 0.$$

Hence,

$$\int_0^1 \phi(t)^a s^a \left|K_1(\phi(t)s,t;M)\right|\phi(t)\,ds \le C_M e^{-Mt-at} \int_0^{e^t-1} y^a \left((e^t+1)^2-y^2\right)^{-\frac{1}{2}+M} dy.$$

On the other hand, for $M > 0$ we have

$$\int_0^{e^t-1} y^a \left((e^t+1)^2-y^2\right)^{-\frac{1}{2}+M} dy = \frac{1}{1+a}(z-1)^{1+a}(z+1)^{2M-1} F\left(\frac{1+a}{2}, \frac{1}{2}-M; \frac{3+a}{2}; \frac{(z-1)^2}{(z+1)^2}\right),$$

where $a > -1$. Hence, for $M > 0$ we have

$$\int_0^1 \phi(t)^a s^a \left|K_1(\phi(t)s,t;M)\right|\phi(t)\,ds \le C_M e^{-Mt-at}(e^t-1)^{a+1}(e^t+1)^{2M-1} \quad \text{for all } t > 0.$$

Thus the lemma is proven. □

Lemma 2.4. *Let $a > -1$, $M > 1/2$, and $\phi(t) = 1 - e^{-t}$. Then*

$$\int_0^1 \phi(t)^a s^a \left|K_0(\phi(t)s,t;M)\right|\phi(t)\,ds \le C_{M,a} e^{-Mt-at}(e^t-1)^{a+1}(e^t+1)^{2M-1} \quad \textit{for all } t > 0.$$

In particular,

$$\int_0^1 \phi(t)^a s^a \left|K_0(\phi(t)s, t; M)\right| \phi(t)\, ds \le C_{M,a} t^{a+1} \quad \text{for small } t > 0,$$
$$\int_0^1 \phi(t)^a s^a \left|K_0(\phi(t)s, t; M)\right| \phi(t)\, ds \le C_{M,a} e^{Mt} \quad \text{for large } t.$$

Proof. By substituting K_0 into integral, we obtain

$$\begin{aligned}
&\int_0^1 \phi(t)^a s^a \left|K_0(\phi(t)s, t; M)\right| \phi(t)\, ds \\
&= 4^{-M} e^{tM} \int_0^{1-e^{-t}} r^a \left((1+e^{-t})^2 - r^2\right)^M \frac{1}{[(1-e^{-t})^2 - r^2]\sqrt{(1+e^{-t})^2 - r^2}} \\
&\quad \times \left|\left[\left(e^{-t} - 1 + M(e^{-2t} - 1 - r^2)\right) F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1-e^{-t})^2 - r^2}{(1+e^{-t})^2 - r^2}\right)\right.\right. \\
&\quad \left.\left. + \left(1 - e^{-2t} + r^2\right)\left(\frac{1}{2} + M\right) F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1-e^{-t})^2 - r^2}{(1+e^{-t})^2 - r^2}\right)\right]\right| dr.
\end{aligned}$$

Now we make the change $r = e^{-t}y$ in the last integral and obtain the following estimate:

$$\begin{aligned}
&\int_0^{1-e^{-t}} r^a \left((1+e^{-t})^2 - r^2\right)^M \frac{1}{[(1-e^{-t})^2 - r^2]\sqrt{(1+e^{-t})^2 - r^2}} \\
&\quad \times \left|\left[\left(e^{-t} - 1 + M(e^{-2t} - 1 - r^2)\right) F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1-e^{-t})^2 - r^2}{(1+e^{-t})^2 - r^2}\right)\right.\right. \\
&\quad \left.\left. + \left(1 - e^{-2t} + r^2\right)\left(\frac{1}{2} + M\right) F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(1-e^{-t})^2 - r^2}{(1+e^{-t})^2 - r^2}\right)\right]\right| dr \\
&\le e^{-2Mt} e^{-at} \int_0^{e^t - 1} y^a \left((e^t+1)^2 - y^2\right)^M \frac{1}{\left((e^t-1)^2 - y^2\right)\sqrt{(e^t+1)^2 - y^2}} \\
&\quad \times \left|\left[\left(e^t - e^{2t} + M(1 - e^{2t} - y^2)\right) F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^t-1)^2 - y^2}{(e^t+1)^2 - y^2}\right)\right.\right. \\
&\quad \left.\left. + \left(e^{2t} - 1 + y^2\right)\left(\frac{1}{2} + M\right) F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(e^t-1)^2 - y^2}{(e^t+1)^2 - y^2}\right)\right]\right| dy.
\end{aligned}$$

Then we denote $z = e^t$ and obtain

$$\begin{aligned}
&\int_0^1 \phi(t)^{2s - n\left(\frac{1}{p} - \frac{1}{q}\right)} s^{2s - n\left(\frac{1}{p} - \frac{1}{q}\right)} \left|K_0(\phi(t)s, t; M)\right| \phi(t)\, ds \\
&\le z^{-[2M+a]} \int_0^{z-1} y^a \left((z+1)^2 - y^2\right)^M \frac{1}{\left((z-1)^2 - y^2\right)\sqrt{(z+1)^2 - y^2}} \\
&\quad \times \left|\left[\left(z - z^2 + M(1 - z^2 - y^2)\right) F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right)\right.\right. \\
&\quad \left.\left. + \left(z^2 - 1 + y^2\right)\left(\frac{1}{2} + M\right) F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right)\right]\right| dy.
\end{aligned}$$

To complete the proof of lemma we need the estimate given by the following proposition. □

Proposition 2.5. *If $a > -1$ and $M > 1/2$, then*

$$\int_0^{z-1} y^a \left((z+1)^2 - y^2\right)^M \frac{1}{\left((z-1)^2 - y^2\right)\sqrt{(z+1)^2 - y^2}}$$

$$\times \left| \left(z - z^2 + M(1 - z^2 - y^2)\right) F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right.$$
$$\left. + \left(z^2 - 1 + y^2\right)\left(\frac{1}{2} + M\right) F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right| dy$$
$$\le C_{M,n,p,q,s}(z-1)^{1+a}(1+z)^{2M-1}.$$

Proof. We follow the arguments have been used in the proof of Lemma 4.7 [23]. If $M > 0$, then the both involved hypergeometric functions are bounded and the integral can be estimated as follows. We divide the domain of integration into two zones:

$$Z_1(\varepsilon, z) := \left\{(z, r) \mid \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2} \le \varepsilon,\ 0 \le r \le z - 1\right\},$$
$$Z_2(\varepsilon, z) := \left\{(z, r) \mid \varepsilon \le \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2},\ 0 \le r \le z - 1\right\},$$

and then split the integral into two parts,

$$\int_0^{z-1} \star\, dr = \int_{(z,r)\in Z_1(\varepsilon,z)} \star\, dr + \int_{(z,r)\in Z_2(\varepsilon,z)} \star\, dr.$$

In the first zone $Z_1(\varepsilon, z)$ we have

$$F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) = 1 + \left(\frac{1}{2} - M\right)^2 \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2} + O\left(\left(\frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right)^2\right),$$
$$F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) = 1 - \left(\frac{1}{4} - M^2\right) \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2} + O\left(\left(\frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right)^2\right).$$

We use the last formulas to estimate the term containing the hypergeometric functions:

$$\left| \left(z - z^2 + M(1 - z^2 - y^2)\right) F\left(\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right.$$
$$\left. + \left(z^2 - 1 + y^2\right)\left(\frac{1}{2} + M\right) F\left(-\frac{1}{2} - M, \frac{1}{2} - M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right) \right|$$
$$\le \frac{1}{2}\left((z-1)^2 - y^2\right) + \frac{1}{8}|2M - 1| \left|y^2 + 2z(z-1) + z^2 - 1 + 2M\left(3y^2 + 2z(z-1) + z^2 - 1\right)\right|$$
$$\times \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2} + \frac{1}{2}\left((z-1)^2 - y^2\right) O\left(\left(\frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right)^2\right).$$

Hence, we have to consider the following two integrals, which can be easily estimated,

$$A_1 := \int_{(z,y)\in Z_1(\varepsilon,z)} y^a \left((z+1)^2 - y^2\right)^{M-\frac{1}{2}}\, dy,$$
$$A_2 := z^2 \int_{(z,y)\in Z_1(\varepsilon,z)} y^a \left((z+1)^2 - y^2\right)^{M-\frac{3}{2}}\, dy,$$

for all $z \in [1, \infty)$. Indeed, for A_1 we obtain

$$A_1 \le \int_0^{z-1} y^a \left((z+1)^2 - y^2\right)^{M-\frac{1}{2}}\, dy$$
$$= \frac{1}{1+a}(z-1)^{1+a}(z+1)^{2M-1} F\left(\frac{1+a}{2}, \frac{1}{2} - M; \frac{3+a}{2}; \frac{(z-1)^2}{(z+1)^2}\right)$$
$$\le C_{M,n,p,q,s}(z-1)^{1+a}(z+1)^{2M-1}.$$

Similarly, if $M > 0$, then

$$A_2 \le z^2 \int_0^{z-1} y^a \left((z+1)^2 - y^2\right)^{M-\frac{3}{2}}\, dy$$

$$= z^2 \frac{1}{1+a}(z-1)^{1+a}(z+1)^{2M-3} F\left(\frac{1+a}{2}, \frac{3}{2}-M; \frac{3+a}{2}; \frac{(z-1)^2}{(z+1)^2}\right).$$

On the other hand, for $M > 1/2$, and $a > -1$, the inequality

$$\left|F\left(\frac{1+a}{2}, \frac{3}{2}-M; \frac{3+a}{2}; \zeta\right)\right| \le C \quad \text{for all } \zeta \in [0,1),$$

implies

$$A_2 \le C_{M,n,p,q,s}(z-1)^{1+a}(z+1)^{2M-1}.$$

Finally, for the integral over the first zone $Z_1(\varepsilon, z)$ we have obtained

$$\int_{(z,r)\in Z_1(\varepsilon,z)} \star\, dr \le C_{M,n,p,q,s}(z-1)^{1+a}(z+1)^{2M-1}.$$

In the second zone we have

$$0 < \varepsilon \le \frac{(z-1)^2 - r^2}{(z+1)^2 - r^2} < 1 \quad \text{and} \quad \frac{1}{(z-1)^2 - r^2} \le \frac{1}{\varepsilon[(z+1)^2 - r^2]}.$$

Then, the hypergeometric functions obey the estimates

$$\left|F\left(-\frac{1}{2}-M, \frac{1}{2}-M; 1; \zeta\right)\right| \le C \quad \text{and} \quad \left|F\left(\frac{1}{2}-M, \frac{1}{2}-M; 1; \zeta\right)\right| \le C_M \quad \text{for all } \zeta \in [\varepsilon, 1).$$

This allows us to estimate the integral over the second zone:

$$\begin{aligned}
&\int_{(z,y)\in Z_2(\varepsilon,z)} y^a \left((z+1)^2 - y^2\right)^M \frac{1}{\left((z-1)^2 - y^2\right)\sqrt{(z+1)^2 - y^2}} \\
&\quad \times \left|\left(z - z^2 + M(1 - z^2 - y^2)\right) F\left(\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right)\right. \\
&\quad \left. + \left(z^2 - 1 + y^2\right)\left(\frac{1}{2}+M\right) F\left(-\frac{1}{2}-M, \frac{1}{2}-M; 1; \frac{(z-1)^2 - y^2}{(z+1)^2 - y^2}\right)\right| dy \\
&\le C_{M,n,p,q,s} z^2 \int_{(z,y)\in Z_2(\varepsilon,z)} y^a \left((z+1)^2 - y^2\right)^{M-\frac{3}{2}}\, dy \\
&\le C_{M,n,p,q,s} z^2 \int_0^{z-1} y^a \left((z+1)^2 - y^2\right)^{M-\frac{3}{2}}\, dy \\
&\le C_{M,n,p,q,s}(z-1)^{1+a}(z+1)^{2M-1} \quad \text{for all } z \in [1,\infty).
\end{aligned}$$

The rest of the proof is a repetition of the above used arguments. Thus, the proposition is proven. □

2.2. $L^p - L^q$ estimates for equation with source, $n \ge 2$

Theorem 2.6. *Let $\Phi = \Phi(x,t)$ be a solution of the Cauchy problem*

$$\Phi_{tt} + n\Phi_t - e^{-2t}\Delta\Phi \pm m^2\Phi = f, \qquad \Phi(x,0) = 0, \qquad \Phi_t(x,0) = 0, \tag{2.5}$$

with either $M = \sqrt{\frac{n^2}{4} - m^2}$ and $m < \sqrt{n^2-1}/2$ for the case of "plus", or $M = \sqrt{\frac{n^2}{4} + m^2}$ for the case of "minus". Then $\Phi = \Phi(x,t)$ satisfies the following $L^p - L^q$ estimate:

$$\begin{aligned}
\|(-\Delta)^{-s}\Phi(x,t)\|_{L^q(\mathbb{R}^n)} \le\ & C_M e^{-Mt} e^{-\frac{n}{2}t} e^{-t\left[2s - n\left(\frac{1}{p}-\frac{1}{q}\right)\right]} \int_0^t e^{\frac{n}{2}b} e^{Mb} \\
& \times (e^{t-b} - 1)^{1+2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} (e^{t-b}+1)^{2M-1} \|f(x,b)\|_{L^p(\mathbb{R}^n)}\, db,
\end{aligned}$$

for all $t > 0$, provided that $1 < p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{2}(n+1)\left(\frac{1}{p} - \frac{1}{q}\right) \le 2s \le n\left(\frac{1}{p} - \frac{1}{q}\right) < 2s + 1$.

Proof. From (2.2) we have

$$\Phi(x,t) = 2e^{-\frac{n}{2}t}\int_0^t db\int_0^{e^{-b}-e^{-t}} dr\, e^{\frac{n}{2}b}v(x,r;b)4^{-M}e^{M(b+t)}\left((e^{-t}+e^{-b})^2-r^2\right)^{-\frac{1}{2}+M}$$
$$\times F\left(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(e^{-b}-e^{-t})^2-r^2}{(e^{-b}+e^{-t})^2-r^2}\right),$$

where according to (2.4) we can write

$$\|(-\Delta)^{-s}v(x,r;b)\|_{L^q(\mathbb{R}^n)} \le Cr^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)}\|f(x,b)\|_{L^p(\mathbb{R}^n)} \quad \text{for all } r>0.$$

Hence,

$$\begin{aligned}\|(-\Delta)^{-s}\Phi(x,t)\|_{L^q(\mathbb{R}^n)} &\le 2e^{-\frac{n}{2}t}\int_0^t db\int_0^{e^{-b}-e^{-t}} dr\, e^{\frac{n}{2}b}\|(-\Delta)^{-s}v(x,r;b)\|_{L^q(\mathbb{R}^n)}4^{-M}\\
&\quad\times e^{M(b+t)}\left((e^{-t}+e^{-b})^2-r^2\right)^{-\frac{1}{2}+M}\\
&\quad\times\left|F\left(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(e^{-b}-e^{-t})^2-r^2}{(e^{-b}+e^{-t})^2-r^2}\right)\right|\\
&\le C_Me^{Mt}e^{-\frac{n}{2}t}\int_0^t e^{\frac{n}{2}b}e^{Mb}\|f(x,b)\|_{L^p(\mathbb{R}^n)}\,db\\
&\quad\times\int_0^{e^{-b}-e^{-t}} r^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)}\left((e^{-t}+e^{-b})^2-r^2\right)^{-\frac{1}{2}+M}\\
&\quad\times\left|F\left(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(e^{-b}-e^{-t})^2-r^2}{(e^{-b}+e^{-t})^2-r^2}\right)\right|dr.\end{aligned}$$

Following the outline of the proof of Lemma 2.3 we set $r=ye^{-t}$ and obtain

$$\begin{aligned}\|(-\Delta)^{-s}\Phi(x,t)\|_{L^q(\mathbb{R}^n)} &\le C_Me^{-Mt}e^{-\frac{n}{2}t}e^{-t\left[2s-n\left(\frac{1}{p}-\frac{1}{q}\right)\right]}\int_0^t e^{\frac{n}{2}b}e^{Mb}\|f(x,b)\|_{L^p(\mathbb{R}^n)}\,db\int_0^{e^{t-b}-1} y^{2s-n\left(\frac{1}{p}-\frac{1}{q}\right)}\\
&\quad\times\left((e^{t-b}+1)^2-y^2\right)^{-\frac{1}{2}+M}\left|F\left(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(e^{t-b}-1)^2-y^2}{(e^{t-b}+1)^2-y^2}\right)\right|dy.\end{aligned}$$

Hence, we have to estimate the integral

$$\int_0^{z-1} y^a\left((z+1)^2-y^2\right)^{-\frac{1}{2}+M}\left|F\left(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(z-1)^2-y^2}{(z+1)^2-y^2}\right)\right|dy,$$

where $z=e^{t-b}>1$ and $a=2s-n(\frac{1}{p}-\frac{1}{q})>-1$. On the other hand, if $M>1/2$, then we have

$$\begin{aligned}&\int_0^{z-1} y^a\left((z+1)^2-y^2\right)^{-\frac{1}{2}+M}\left|F\left(\frac{1}{2}-M,\frac{1}{2}-M;1;\frac{(z-1)^2-y^2}{(z+1)^2-y^2}\right)\right|dy\\
&\le C_M\int_0^{z-1} y^a\left((z+1)^2-y^2\right)^{-\frac{1}{2}+M}dy\\
&= C_M\frac{1}{1+a}(e^{t-b}-1)^{1+a}(e^{t-b}+1)^{-1+2M}F\left(\frac{1+a}{2},\frac{1}{2}-M;\frac{3+a}{2};\frac{(e^{t-b}-1)^2}{(e^{t-b}+1)^2}\right)\\
&\le C_M\frac{1}{1+a}(e^{t-b}-1)^{1+a}(e^{t-b}+1)^{-1+2M}.\end{aligned}$$

Thus,

$$\begin{aligned}\|(-\Delta)^{-s}\Phi(x,t)\|_{L^q(\mathbb{R}^n)} &\le C_Me^{-Mt}e^{-\frac{n}{2}t}e^{-t\left[2s-n\left(\frac{1}{p}-\frac{1}{q}\right)\right]}\int_0^t e^{\frac{n}{2}b}e^{Mb}(e^{t-b}-1)^{1+2s-n\left(\frac{1}{p}-\frac{1}{q}\right)}\\
&\quad\times(e^{t-b}+1)^{2M-1}\|f(x,b)\|_{L^p(\mathbb{R}^n)}\,db.\end{aligned}$$

The theorem is proven. □

Corollary 2.7. *Let $\Phi = \Phi(x,t)$ be a solution of the Cauchy problem considered in Theorem 2.6. Then for $n \geq 2$ and $M > 1/2$ one has the following estimate*

$$\|(-\Delta)^{-s}\Phi(x,t)\|_{L^q(\mathbb{R}^n)} \leq C_M e^{-(\frac{n}{2}-M)t}\int_0^t e^{(\frac{n}{2}-M)b}e^{-b\left(2s-n\left(\frac{1}{p}-\frac{1}{q}\right)\right)}\|f(x,b)\|_{L^p(\mathbb{R}^n)}\,db,$$

provided that $1 < p \leq 2$, $\frac{1}{p}+\frac{1}{q}=1$, $\frac{1}{2}(n+1)\left(\frac{1}{p}-\frac{1}{q}\right) \leq 2s \leq n\left(\frac{1}{p}-\frac{1}{q}\right) < 2s+1$.

Proof. Since $M > 1/2$ and $1+2s-n(\frac{1}{p}-\frac{1}{q}) \geq 0$ we have

$$\begin{aligned}\|(-\Delta)^{-s}\Phi(x,t)\|_{L^q(\mathbb{R}^n)} &\leq C_M e^{-Mt}e^{-\frac{n}{2}t}e^{-t\left[2s-n\left(\frac{1}{p}-\frac{1}{q}\right)\right]}\int_0^t e^{\frac{n}{2}b}e^{Mb}(e^{t-b}-1)^{1+2s-n\left(\frac{1}{p}-\frac{1}{q}\right)}\\ &\quad\times(e^{t-b}+1)^{2M-1}\|f(x,b)\|_{L^p(\mathbb{R}^n)}\,db\\ &\leq C_M e^{Mt}e^{-\frac{n}{2}t}\int_0^t e^{\frac{n}{2}b}e^{Mb}(e^{-b}-e^{-t})^{1+2s-n\left(\frac{1}{p}-\frac{1}{q}\right)}e^{-b(2M-1)}\|f(x,b)\|_{L^p(\mathbb{R}^n)}\,db\\ &\leq C_M e^{Mt}e^{-\frac{n}{2}t}\int_0^t e^{\frac{n}{2}b}e^{-Mb}e^{-b\left(2s-n\left(\frac{1}{p}-\frac{1}{q}\right)\right)}\|f(x,b)\|_{L^p(\mathbb{R}^n)}\,db.\end{aligned}$$

The corollary is proven. □

3. Global existence. Small data solutions

The Cauchy problem for Eq. (0.3) was studied in [28]. For the case of nonlinearity $F(\Phi) = c|\Phi|^{\alpha+1}$, $c \neq 0$, Theorem 1.1 [28], implies the nonexistence of the global solution even for arbitrary small initial functions $\varphi_0(x)$ and $\varphi_1(x)$ under some conditions on n, α, and M. By means of the evident transformation one can apply the conclusion of Theorem 1.1 [28] to the equation with imaginary physical mass (see (3.1)) and derive the following blowup result.

Theorem 3.1. *Suppose that $F(\Phi) = c|\Phi|^{\alpha+1}$, $c \neq 0$, and $\alpha > 0$. Then, for every $\alpha > 0$, N, and ε, there exist $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$ such that*

$$\|\varphi_0\|_{C^N(\mathbb{R}^n)} + \|\varphi_1\|_{C^N(\mathbb{R}^n)} < \varepsilon$$

but a global in time solution $\Phi \in C^2([0,\infty); L^q(\mathbb{R}^n))$ of the equation

$$\Phi_{tt} + n\Phi_t - e^{-2t}\Delta\Phi - m^2\Phi = c|\Phi|^{\alpha+1}, \tag{3.1}$$

with permanently bounded support does not exist for all $q \in [2,\infty)$. More precisely, there is $T > 0$ such that

$$\lim_{t \nearrow T}\int_{\mathbb{R}^n}\Phi(x,t)dx = \infty.$$

This theorem shows that instability of the trivial solution occurs in a very strong sense, that is, an arbitrarily small perturbation of the initial data can make the perturbed solution blowing up in finite time.

If we allow large initial data, then, according to Theorem 1.2 [28], the concentration of the mass, due to the non-dispersion property of the de Sitter spacetime, leads to the nonexistence of the global solution, which cannot be recovered even by adding an exponentially decaying factor in the nonlinear term. More precisely, the next theorem states that the solution blows up in finite time.

Theorem 3.2. *Suppose that $F(\Phi) = ce^{\gamma t}|\Phi|^{\alpha+1}$, $c \neq 0$, $\alpha > 0$, and $\gamma \in \mathbb{R}$. Then, for every $\alpha > 0$ and n there exist $\varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n)$ such that a global in time solution $\Phi \in C^2([0,\infty); L^q(\mathbb{R}^n))$ of Eq. (3.1) with permanently bounded support does not exist for all $q \in [2,\infty)$. More precisely, there is $T > 0$ such that*

$$\lim_{t \nearrow T}\int_{\mathbb{R}^n}\Phi(x,t)\,dx = \infty.$$

Thus, for every $\alpha > 0$ the large energy classical solution of the Cauchy for Eq. (3.1) blows up.

It is evident that, if solution is real-valued and either α is odd or the nonlinear term is $\Phi^{\alpha+1}$ with an integer nonnegative α, then the support of the solution with such initial data is permanently bounded.

In fact, the results of the previous sections are valid also in more general spaces of functions. In what follows, the space $\mathcal{M}^{s,q}$ can be each of the following spaces $L^q(\mathbb{R}^n)$, Sobolev spaces $W^{s,q}(\mathbb{R}^n)$, $\dot{W}^{s,q}(\mathbb{R}^n)$, or Besov spaces $B^{s,q}(\mathbb{R}^n)$, $\dot{B}^{s,q}(\mathbb{R}^n)$.

Lemma 3.3. *Let $\Phi = \Phi(x,t)$ be a solution of the Cauchy problem (2.5) with either $M = \sqrt{\frac{n^2}{4} - m^2}$ and $m < \sqrt{n^2-1}/2$ for the case of "plus", or $M = \sqrt{\frac{n^2}{4} + m^2}$ for the case of "minus". Then for $n \ge 2$ one has the following estimate*

$$\|(-\Delta)^{l-s}\Phi(x,t)\|_{L^q(\mathbb{R}^n)} \le C_M e^{-(\frac{n}{2}-M)t} \int_0^t e^{(\frac{n}{2}-M)b} e^{-b\left(2s-n\left(\frac{1}{p}-\frac{1}{q}\right)\right)} \|(-\Delta)^l f(x,b)\|_{L^p(\mathbb{R}^n)}\, db,$$

for all $t > 0$, provided that $1 < p \le 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{2}(n+1)\left(\frac{1}{p} - \frac{1}{q}\right) \le 2s \le n\left(\frac{1}{p} - \frac{1}{q}\right) < 2s+1$. Moreover,

$$\|(-\Delta)^{-s}\Phi(x,t)\|_{\mathcal{M}^{l,q}} \le C_M e^{-(\frac{n}{2}-M)t} \int_0^t e^{(\frac{n}{2}-M)b} e^{-b\left(2s-n\left(\frac{1}{p}-\frac{1}{q}\right)\right)} \|f(x,b)\|_{\mathcal{M}^{l,p}}\, db.$$

In particular,

$$\|\Phi(x,t)\|_{H_{(l)}(\mathbb{R}^n)} \le C_M e^{-(\frac{n}{2}-M)t} \int_0^t e^{(\frac{n}{2}-M)b} \|f(x,b)\|_{H_{(l)}(\mathbb{R}^n)}\, db.$$

For the equation with "plus" and large mass, $m \ge n/2$, and with the curved mass $M = \sqrt{m^2 - n^2/4}$, one has the following estimate

$$\|(-\Delta)^{l-s}\Phi(x,t)\|_{L^q(\mathbb{R}^n)} \le C_M e^{-\frac{n}{2}t} \int_0^t e^{\frac{n}{2}b} e^b \left(e^{-b} - e^{-t}\right)^{1+2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} (1+t-b)^{1-\operatorname{sgn} M} \|(-\Delta)^l f(x,b)\|_{L^p(\mathbb{R}^n)}\, db.$$

Moreover,

$$\|(-\Delta)^{-s}\Phi(x,t)\|_{\mathcal{M}^{l,q}} \le C_M e^{-\frac{n}{2}t} \int_0^t e^{\frac{n}{2}b} e^b \left(e^{-b} - e^{-t}\right)^{1+2s-n\left(\frac{1}{p}-\frac{1}{q}\right)} (1+t-b)^{1-\operatorname{sgn} M} \|f(x,b)\|_{\mathcal{M}^{l,p}}\, db.$$

In particular,

$$\|\Phi(x,t)\|_{H_{(l)}(\mathbb{R}^n)} \le C_M e^{-\frac{n}{2}t} \int_0^t e^{\frac{n}{2}b} (1+t-b)^{1-\operatorname{sgn} M} \|f(x,b)\|_{H_{(l)}(\mathbb{R}^n)}\, db.$$

Here the rate of exponential factors is independent of the curved mass $\mathcal{M}$ and, consequently, of the mass m.

Proof. The statements of this lemma follow immediately from Corollary 2.7 and Section 1. □

The last lemma and the fixed point theorem allow us to prove global existence in the Cauchy problem for semilinear equations. For the simplicity we consider the potential function

$$V'(\phi) = m^2\phi - F(\Phi). \tag{3.2}$$

For instance, it can be

$$V'(\phi) = m^2\phi - \lambda|\phi|^\alpha\phi, \qquad \alpha > 0,$$

where $m > 0$ and $\lambda \neq 0$. In the de Sitter universe the equation for the scalar field with potential function V is the covariant wave equation

$$\square_g \phi = V'(\phi) \quad \text{or} \quad \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i}\left(\sqrt{|g|} g^{ik} \frac{\partial \phi}{\partial x^k}\right) = V'(\phi),$$

with the usual summation convention. Written explicitly in coordinates in the de Sitter spacetime it, in particular, for (3.2), has the form (0.5):

$$\Phi_{tt} + n\Phi_t - e^{-2t}\Delta\Phi + m^2\Phi = F(\Phi).$$

Scalar fields play a fundamental role in the standard model of particle physics, as well as, its possible extensions. In particular, scalar fields generate spontaneous symmetry breaking and provide masses to gauge bosons and chiral fermions by the Brout–Englert–Higgs mechanism [30] using a Higgs-type potential [31].

We study the Cauchy problem (0.5), (0.6) through the integral equation. To determine that integral equation we appeal to the operator

$$G := \mathcal{K} \circ \mathcal{W}\mathcal{E},$$

where

$$\mathcal{W}\mathcal{E}[f](x,t;b) = v(x,t;b)$$

and the function $v(x,t;b)$ is a solution to the Cauchy problem for the wave equation (1.5), while $\mathcal{K}$ is introduced either by (1.7),

$$\mathcal{K}[v](x,t) := 2e^{-\frac{n}{2}t}\int_0^t db \int_0^{e^{-b}-e^{-t}} dr\, e^{\frac{n}{2}b} v(x,r;b)E(r,t;0,b),$$

for the large mass m, or by (2.2),

$$\mathcal{K}[v](x,t) := 2e^{-\frac{n}{2}t}\int_0^t db \int_0^{e^{-b}-e^{-t}} dr\, e^{\frac{n}{2}b} v(x,r;b)E(r,t;0,b;M), \tag{3.3}$$

for the small mass m. Hence,

$$G[f](x,t) = 2e^{-\frac{n}{2}t}\int_0^t db \int_0^{e^{-b}-e^{-t}} dr\, e^{\frac{n}{2}b}\, \mathcal{WE}[f](x,r;b)E(r,t;0,b;M).$$

Thus, the Cauchy problem (0.5), (0.6) leads to the following integral equation

$$\Phi(x,t) = \Phi_0(x,t) + G[F(\Phi)](x,t). \tag{3.4}$$

Every solution to Eq. (0.5) solves also the last integral equation with some function $\Phi_0(x,t)$, which, in fact, is a solution of the Cauchy problem (1.10).

3.1. Solvability of the integral equation associated with Klein–Gordon equation

Consider the integral equation (3.4) where $\Phi_0 = \Phi_0(x,t)$ is a given function. Every solution to Eq. (0.5) solves also the last integral equation with some function $\Phi_0 = \Phi_0(x,t)$. We are going to apply the Banach fixed-point theorem. To estimate nonlinear term we use the Lipschitz Condition ($\mathcal{L}$). Evidently, the Condition ($\mathcal{L}$) imposes some restrictions on n, α, s. Now we consider the integral equation (3.4), where the function $\Phi_0 \in C([0,\infty); L^q(\mathbb{R}^n))$ is given. We note here that any classical solution to Eq. (0.5) solves also the integral equation (3.4) with some function $\Phi_0(t,x)$, which is classical solution to the Cauchy problem for the linear equation (1.10).

Solvability of the integral equation (3.4) depends on the operator G. For the operator G generated by the linear part of Eq. (3.1) the global solvability of the integral equation (3.4) was studied in [28]. For the case of nonlinearity $F(\Phi) = c|\Phi|^{\alpha+1}$, $c \neq 0$, the results of [28] imply the nonexistence of the global solution even for arbitrary small function $\Phi_0(x,0)$ under some conditions on n, α, and M.

We start with the case of Sobolev space $H_{(s)}(\mathbb{R}^n)$ with $s > n/2$, which is an algebra. In the next theorem operator $\mathcal{K}$ is generated by linear part of Eq. (0.5).

Theorem 3.4. *Assume that $F(u)$ is Lipschitz continuous in the space $H_{(s)}(\mathbb{R}^n)$, $s > n/2$, $F(0) = 0$, and also that $\alpha > 0$. Then for every given function $\Phi_0(x,t) \in X(R,s,\gamma_0)$ such that*

$$\sup_{t\in[0,\infty)} e^{\gamma_0 t}\|\Phi_0(\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} < \varepsilon,$$

$$\gamma_0 \le \frac{n}{2} - \sqrt{\frac{n^2}{4}-m^2} \quad \text{if } 0 < m < \sqrt{n^2-1}/2, \text{ while } \gamma_0 = 0 \text{ if } \frac{n}{2} \le m,$$

and for sufficiently small ε the integral equation (3.4) has a unique solution $\Phi(x,t) \in X(R,s,\gamma)$ with $0 < \gamma < \gamma_0/(\alpha+1)$ if $0 < m < \sqrt{n^2-1}/2$ and $\gamma_0 \le \frac{n}{2} - \sqrt{\frac{n^2}{4}-m^2}$, while $\gamma = 0$ if $\frac{n}{2} \le m$. For the solution one has

$$\sup_{t\in[0,\infty)} e^{\gamma t}\|\Phi(\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} < 2\varepsilon.$$

Proof. Consider the mapping

$$S[\Phi](x,t) := \Phi_0(x,t) + G[F(\Phi)](x,t).$$

We are going to prove that S maps $X(R,s,\gamma)$ into itself and is a contraction provided that ε and R are sufficiently small.

The case of the small physical mass $0 < m < \sqrt{n^2-1}/2$. In this case the operator $\mathcal{K}$ is given by (2.2) and $M = \sqrt{\frac{n^2}{4}-m^2}$. Lemma 3.3 with $\gamma = \frac{1}{\alpha+1}(\frac{n}{2}-M-\delta) > 0$ and $\delta > 0$ implies

$$\begin{aligned}
\|S[\Phi](\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} &\le \|\Phi_0(\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} + \|G[F(\Phi)](\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)}\\
&\le \|\Phi_0(\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} + C_M e^{-(\frac{n}{2}-M)t}\int_0^t e^{(\frac{n}{2}-M)b}\|F(\Phi)(\cdot,b)\|_{H_{(s)}(\mathbb{R}^n)}\, db\\
&\le \|\Phi_0(\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} + C_M e^{-\gamma(\alpha+1)t-\delta t}\int_0^t e^{\gamma(\alpha+1)b+\delta b}\|F(\Phi)(\cdot,b)\|_{H_{(s)}(\mathbb{R}^n)}\, db.
\end{aligned}$$

Taking into account the Condition $(\mathcal{L})$ we arrive at

$$\begin{aligned}\|S[\Phi](\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} &\le \|\Phi_0(\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} + C_M e^{-\gamma(\alpha+1)t-\delta t}\int_0^t e^{\gamma(\alpha+1)b+\delta b}\|\Phi(\cdot,b)\|^{\alpha+1}_{H_{(s)}(\mathbb{R}^n)}\,db\\ &\le \|\Phi_0(\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} + C_M e^{-\gamma(\alpha+1)t-\delta t}\int_0^t e^{\delta b}\left(e^{\gamma b}\|\Phi(\cdot,b)\|_{H_{(s)}(\mathbb{R}^n)}\right)^{\alpha+1}\,db.\end{aligned}$$

Then

$$\begin{aligned}e^{\gamma t}\|S[\Phi](x,t)\|_{H_{(s)}(\mathbb{R}^n)} &\le e^{\gamma(\alpha+1)t}\|S[\Phi](x,t)\|_{H_{(s)}(\mathbb{R}^n)}\\ &\le e^{\gamma(\alpha+1)t}\|\Phi_0(\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} + C_M\left(\sup_{\tau\in[0,\infty)} e^{\gamma\tau}\|\Phi(\cdot,\tau)\|_{H_{(s)}(\mathbb{R}^n)}\right)^{\alpha+1} e^{-\delta t}\int_0^t e^{\delta b}\,db\\ &\le e^{\gamma_0 t}\|\Phi_0(\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} + C_M\delta^{-1}\left(\sup_{\tau\in[0,\infty)} e^{\gamma\tau}\|\Phi(\cdot,\tau)\|_{H_{(s)}(\mathbb{R}^n)}\right)^{\alpha+1},\end{aligned}$$

and

$$\begin{aligned}\sup_{t\in[0,\infty)} e^{\gamma t}\|S[\Phi](x,t)\|_{H_{(s)}(\mathbb{R}^n)} &\le \sup_{t\in[0,\infty)} e^{\gamma_0 t}\|\Phi_0(\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} + C_M\delta^{-1}\\ &\quad\times\left(\sup_{t\in[0,\infty)} e^{\gamma t}\|\Phi(\cdot,\tau)\|_{H_{(s)}(\mathbb{R}^n)}\right)^{\alpha+1}.\end{aligned}\tag{3.5}$$

In particular, if $\gamma_0 = \frac{n}{2} - M > 0$, then, with $\delta > 0$ such that $\gamma(\alpha+1) = \frac{n}{2} - M - \delta < \gamma_0$, we have

$$\sup_{t\in[0,\infty)} e^{\gamma t}\|S[\Phi](\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} \le \sup_{t\in[0,\infty)} e^{(\frac{n}{2}-M)t}\|\Phi_0(\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} + C\left(\sup_{t\in[0,\infty)} e^{\gamma t}\|\Phi(\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)}\right)^{\alpha+1}.$$

Thus, the last inequality proves that the operator S maps $X(R,s,\gamma)$ into itself if ε and R are sufficiently small, namely, if $\varepsilon + CR^{\alpha+1} < R$.

It remains to prove that S is a contraction mapping. As a matter of fact, we just need to apply estimate (0.4) and get the contraction property from

$$e^{\gamma t}\|S[\Phi](\cdot,t) - S[\Psi](\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} \le CR(t)^\alpha d(\Phi,\Psi),$$

where $R(t) := \max\{\sup_{0\le\tau\le t} e^{\gamma\tau}\|\Phi(\cdot,\tau)\|_{H_{(s)}(\mathbb{R}^n)}, \sup_{0\le\tau\le t} e^{\gamma\tau}\|\Psi(\cdot,\tau)\|_{H_{(s)}(\mathbb{R}^n)}\} \le R$. Indeed, we have

$$\begin{aligned}\|S[\Phi](\cdot,t) - S[\Psi](\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} &= \|G[\,(F(\Phi)-F(\Psi))](\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)}\\ &\le C_M e^{-(\frac{n}{2}-M)t}\int_0^t e^{(\frac{n}{2}-M)b}\|(F(\Phi)-F(\Psi))(\cdot,b)\|_{H_{(s)}(\mathbb{R}^n)}\,db\\ &\le C_M e^{-\gamma(\alpha+1)t-\delta t}\int_0^t e^{\gamma(\alpha+1)b+\delta b}\|(F(\Phi)-F(\Psi))(\cdot,b)\|_{H_{(s)}(\mathbb{R}^n)}\,db\\ &\le C_M e^{-\gamma(\alpha+1)t-\delta t}\int_0^t e^{\gamma(\alpha+1)b+\delta b}\|\Phi(\cdot,b)-\Psi(\cdot,b)\|_{H_{(s)}(\mathbb{R}^n)}\\ &\quad\times\left(\|\Phi(\cdot,b)\|^\alpha_{H_{(s)}(\mathbb{R}^n)} + \|\Psi(\cdot,b)\|^\alpha_{H_{(s)}(\mathbb{R}^n)}\right)db.\end{aligned}$$

Thus, taking into account the last estimate and the definition of the metric $d(\Phi,\Psi)$, we obtain

$$\begin{aligned}e^{\gamma(\alpha+1)t}\|S[\Phi](\cdot,t) - S[\Psi](\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} &\le C_M e^{-\delta t}\int_0^t e^{\gamma(\alpha+1)b+\delta b}\|\Phi(\cdot,b)-\Psi(\cdot,b)\|_{H_{(s)}(\mathbb{R}^n)}\\ &\quad\times\left(\|\Phi(\cdot,b)\|^\alpha_{H_{(s)}(\mathbb{R}^n)} + \|\Psi(\cdot,b)\|^\alpha_{H_{(s)}(\mathbb{R}^n)}\right)db\\ &\le C_M e^{-\delta t}\int_0^t e^{\delta b}\left(\max_{0\le\tau\le b} e^{\gamma\tau}\|\Phi(\cdot,\tau)-\Psi(\cdot,\tau)\|_{H_{(s)}(\mathbb{R}^n)}\right)\\ &\quad\times\left(\left(\max_{0\le\tau\le b} e^{\gamma\tau}\|\Phi(\cdot,\tau)\|_{H_{(s)}(\mathbb{R}^n)}\right)^\alpha + \left(\max_{0\le\tau\le b} e^{\gamma\tau}\|\Psi(\cdot,\tau)\|_{H_{(s)}(\mathbb{R}^n)}\right)^\alpha\right)db\\ &\le C_{M,\alpha} d(\Phi,\Psi)R(t)^\alpha e^{-\delta t}\int_0^t e^{\delta b}\,db\\ &\le C_{M,\alpha}\delta^{-1} d(\Phi,\Psi)R(t)^\alpha.\end{aligned}$$

Consequently,

$$e^{\gamma t}\|S[\Phi](\cdot,t)-S[\Psi](\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} \le C_{M,\alpha}\delta^{-1}R(t)^\alpha d(\Phi,\Psi).$$

Then we choose ε and R such that $C_{M,\alpha}\delta^{-1}R^\alpha < 1$. The Banach fixed point theorem completes the proof for the case of small physical mass.

The case of the large physical mass $m \ge n/2$. In this case the operator $\mathcal{K}$ is given by (1.7). We set $\gamma = 0$ in the definition of metric of the space $X(R,s,\gamma)$. Then we have

$$\begin{aligned}
\|S[\Phi](\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} &\le \|\Phi_0(\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} + \|G[F(\Phi)](\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} \\
&\le \|\Phi_0(\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} + C_M e^{-\frac{n}{2}t}\int_0^t e^{\frac{n}{2}b}(1+t-b)^{1-\mathrm{sgn}M}\|F(\Phi)(\cdot,b)\|_{H_{(s)}(\mathbb{R}^n)}\,db \\
&\le \|\Phi_0(\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} + C_{M,\alpha} e^{-\frac{n}{2}t}\int_0^t e^{\frac{n}{2}b}(1+t-b)^{1-\mathrm{sgn}M}\|\Phi(\cdot,b)\|^{\alpha+1}_{H_{(s)}(\mathbb{R}^n)}\,db.
\end{aligned}$$

Hence

$$\begin{aligned}
\|S[\Phi](\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} &\le \|\Phi_0(\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} + C_{M,\alpha}\left(\sup_{\tau\in[0,\infty)}\|\Phi(\cdot,\tau)\|_{H_{(s)}(\mathbb{R}^n)}\right)^{\alpha+1} e^{-\frac{n}{2}t}\int_0^t e^{\frac{n}{2}b}(1+t-b)^{1-\mathrm{sgn}M}\,db \\
&\le \|\Phi_0(\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} + C_{M,\alpha}\frac{4}{n}\left(\sup_{\tau\in[0,\infty)}\|\Phi(\cdot,\tau)\|_{H_{(s)}(\mathbb{R}^n)}\right)^{\alpha+1}.
\end{aligned}$$

Then we choose ε and R such that $\varepsilon + 4C_{M,\alpha}R^{\alpha+1}/n < R$.

To prove that S is a contraction mapping, we just need to apply estimate (3.5) and get the contraction property from

$$\|S[\Phi](\cdot,t)-S[\Psi](\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} \le CR(t)^\alpha d(\Phi,\Psi),$$

where $R(t) := \max\{\sup_{0\le\tau\le t}\|\Phi(\cdot,\tau)\|_{H_{(s)}(\mathbb{R}^n)}, \sup_{0\le\tau\le t}\|\Psi(\cdot,\tau)\|_{H_{(s)}(\mathbb{R}^n)}\} \le R$. Indeed, we have

$$\begin{aligned}
\|S[\Phi](\cdot,t)-S[\Psi](\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} &= \|G[\,(F(\Phi)-F(\Psi))](\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} \\
&\le C_M e^{-\frac{n}{2}t}\int_0^t e^{\frac{n}{2}b}(1+t-b)^{1-\mathrm{sgn}M}\|(F(\Phi)-F(\Psi))(\cdot,b)\|_{H_{(s)}(\mathbb{R}^n)}\,db \\
&\le C_{M,\alpha} e^{-\frac{n}{2}t}\int_0^t e^{\frac{n}{2}b}(1+t-b)^{1-\mathrm{sgn}M}\|\Phi(\cdot,b)-\Psi(\cdot,b)\|_{H_{(s)}(\mathbb{R}^n)} \\
&\quad\times\left(\|\Phi(\cdot,b)\|^\alpha_{H_{(s)}(\mathbb{R}^n)} + \|\Psi(\cdot,b)\|^\alpha_{H_{(s)}(\mathbb{R}^n)}\right)db.
\end{aligned}$$

Thus, taking into account the last estimate and a definition of the metric, we obtain

$$\begin{aligned}
\|S[\Phi](\cdot,t)-S[\Psi](\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} &\le C_{M,\alpha} e^{-\frac{n}{2}t}\int_0^t e^{\frac{n}{2}b}(1+t-b)^{1-\mathrm{sgn}M}\|\Phi(\cdot,b)-\Psi(\cdot,b)\|_{H_{(s)}(\mathbb{R}^n)} \\
&\quad\times\left(\|\Phi(\cdot,b)\|^\alpha_{H_{(s)}(\mathbb{R}^n)} + \|\Psi(\cdot,b)\|^\alpha_{H_{(s)}(\mathbb{R}^n)}\right)db \\
&\le C_{M,\alpha} e^{-\frac{n}{2}t}\int_0^t e^{\frac{n}{2}b}(1+t-b)^{1-\mathrm{sgn}M}\left(\max_{0\le\tau\le b}\|\Phi(\cdot,\tau)-\Psi(\cdot,\tau)\|_{H_{(s)}(\mathbb{R}^n)}\right) \\
&\quad\times\left(\left(\max_{0\le\tau\le b}\|\Phi(\cdot,\tau)\|_{H_{(s)}(\mathbb{R}^n)}\right)^\alpha + \left(\max_{0\le\tau\le b}\|\Psi(\cdot,\tau)\|_{H_{(s)}(\mathbb{R}^n)}\right)^\alpha\right)db \\
&\le C_{M,\alpha} d(\Phi,\Psi)R(t)^\alpha e^{-\frac{n}{2}t}\int_0^t e^{\frac{n}{2}b}(1+t-b)^{1-\mathrm{sgn}M}\,db \\
&\le C_{M,\alpha}\frac{4}{n}d(\Phi,\Psi)R(t)^\alpha,
\end{aligned}$$

and, consequently,

$$\|S[\Phi](\cdot,t)-S[\Psi](\cdot,t)\|_{H_{(s)}(\mathbb{R}^n)} \le C_{M,\alpha}\frac{4}{n}\delta^{-1}R(t)^\alpha d(\Phi,\Psi).$$

Then we choose ε and R such that $4C_{M,\alpha}\delta^{-1}R^\alpha/n < 1$. The Banach fixed point theorem completes the proof of theorem. □

3.2. Proof of Theorem 0.1

The case of the small physical mass $m < \sqrt{n^2-1}/2$. In this case the operator $\mathcal{K}$ is given by (2.2) and $M = \sqrt{\frac{n^2}{4} - m^2}$. Then for the function Φ_0, that is, for the solution of the Cauchy problem (1.10) and for $s > \frac{n}{2}$, $p = q = 2$, $n \geq 2$, according to Theorem 2.2 we have the estimate

$$\|\Phi_0(x,t)\|_{H_{(s)}(\mathbb{R}^n)} \leq C_{M,n,p,q,s} e^{\left(M-\frac{n}{2}\right)t} \left\{ \|\varphi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\varphi_1\|_{H_{(s)}(\mathbb{R}^n)} \right\}.$$

According to Theorem 3.4, for every initial functions φ_0 and φ_1 the function Φ_0 belongs to the space $X(R,s,\gamma)$, where the operator S is a contraction. In the case of $n = 3$ that means $m^2 < 2$, that is, the physical mass must be inside of the Higuchi bound [32]. The consideration done in Section 3.1 completes the proof of the existence of the global solution.

The case of the large physical mass $m \geq n/2$. In this case the operator $\mathcal{K}$ is given by (1.7). We set $\gamma = 0$ in the definition of metric of the space $X(R,s,\gamma)$. Then for the function Φ_0, that is for the solution of the Cauchy problem (1.10) and for $s > \frac{n}{2}$, $p = q = 2$, $n \geq 2$ we have the estimate (1.11),

$$\begin{aligned}\|\Phi_0(x,t)\|_{H_{(s)}(\mathbb{R}^n)} &\leq C_M e^{-\frac{n}{2}t}(1+t)^{1-\operatorname{sgn} M} \left\{ e^{\frac{t}{2}} \|\varphi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\varphi_1\|_{H_{(s)}(\mathbb{R}^n)} \right\} \\ &\leq C_M \left\{ \|\varphi_0\|_{H_{(s)}(\mathbb{R}^n)} + \|\varphi_1\|_{H_{(s)}(\mathbb{R}^n)} \right\}.\end{aligned}$$

Thus, $\Phi_0 \in X(R,s,0)$. According to Section 3.1, the Banach fixed point theorem implies the existence of the solution $\Phi \in X(R,s,0)$ of the integral equation (3.5) provided that R is sufficiently small. The theorem is proven. □

One can consider the problem with the initial functions φ_0, φ_1 in the Sobolev spaces with a higher index and in the Besov spaces. The necessary modifications we leave to the reader.

Acknowledgments

This paper was started during my stay in the summer of 2011 at the Technical University Bergakademie Freiberg. I am grateful to Michael Reissig for the invitation to Freiberg and for the warm hospitality. I express my gratitude to the Deutsche Forschungsgemeinschaft for the financial support under grant "Bilateral Cooperation" GZ RE 961/18-1 AOBJ 584929.

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