



# On an open problem of Chen and Mortici concerning the Euler–Mascheroni constant

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## ABSTRACT

Chen and Mortici [Ch.-P. Chen, C. Mortici, New sequence converging towards the Euler–Mascheroni constant, *Comput. Math. Appl.* (2011); <http://dx.doi.org/10.1016/j.camwa.2011.03.099>] proposed an open problem: for a given positive integer  $s$ , find the constants  $a_i$  ( $i = 0, 1, 2, \dots, s$ ) such that

$$\sum_{k=1}^n \frac{1}{k} - \log \left( n + \sum_{i=0}^s a_i n^{-i} \right)$$

is the fastest sequence which would converge to the Euler–Mascheroni constant  $\gamma$ . Using logarithmic type Bell polynomials, we solve this problem. The main result shows that the  $a_i$ 's can be recursively determined.

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## 1. Introduction

The Euler–Mascheroni constant  $\gamma = 0.57721566490153286 \dots$  is usually defined by the limit

$$\gamma = \lim_{n \rightarrow \infty} \gamma_n,$$

where

$$\gamma_n = \sum_{k=1}^n \frac{1}{k} - \log n. \quad (1.1)$$

The sequence  $\{\gamma_n\}_{n \geq 1}$  and the constant  $\gamma$  have wide applications in many areas of mathematics, such as analysis, probability theory and special functions. Despite the existence of the limit of the sequence  $\{\gamma_n\}_{n \geq 1}$ , the rate of convergence is extremely slow, since

$$\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2n}.$$

An elementary proof of this inequality was given in Young [1]. It is interesting to see, however, what an unexpected large effect has on the rate of convergence if the logarithmic term in (1.1) is replaced by  $\log(n + \frac{1}{2})$ . This remarkable result was proved in DeTemple [2]. More precisely, he showed

$$\frac{1}{24(n+1)^2} < \sum_{k=1}^n \frac{1}{k} - \log \left( n + \frac{1}{2} \right) < \frac{1}{24n^2}. \quad (1.2)$$

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This opened a direction for improving the speed of convergence of  $\gamma_n$  to  $\gamma$ . Later, some faster approximations to the Euler–Mascheroni constant were established in [3–6]. For example, Negoï [6] proved that the sequence

$$\sum_{k=1}^n \frac{1}{k} - \log \left( n + \frac{1}{2} + \frac{1}{24n} \right)$$

is strictly increasing and convergent to  $\gamma$ . Chen and Mortici [7] considered

$$w_n := w_n(a, b, c, d) = \sum_{k=1}^n \frac{1}{k} - \log \left( n + a + \frac{b}{n} + \frac{c}{n^2} + \frac{d}{n^3} \right)$$

and found, among others, the values of  $a, b, c, d$ , providing the new fastest sequence  $\{w_n\}_{n \geq 1}$  approximating the Euler–Mascheroni constant. They further proposed the following open problem.

*Open problem.* For a given positive integer  $s$ , please find the constants  $a_i$  ( $i = 0, 1, 2, \dots, s$ ) such that

$$\sum_{k=1}^n \frac{1}{k} - \log \left( n + \sum_{i=0}^s a_i n^{-i} \right) \quad (1.3)$$

is the fastest sequence which would converge to  $\gamma$ . The aim of this paper is to give an answer to this kind of question. Our approach is based on the complete asymptotic expansion of  $\gamma_n$  and some basic facts from combinatorics.

## 2. Preliminaries

It is well known that Bell polynomials are key ingredients in enumerative combinatorics, see e.g. [8]. Bell polynomials are important in our derivation, so we begin with the definition of Bell polynomials and related polynomials and collect some properties of them. The *exponential partial Bell polynomials* [8, pp. 133 and 134], named in honor of Bell [9], are the polynomials

$$B_{n,k} := B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$$

in an infinite number of variables  $x_1, x_2, \dots$ , defined by the formal power series expansion

$$\frac{1}{k!} \left( \sum_{m=1}^{\infty} x_m \frac{t^m}{m!} \right)^k = \sum_{n=k}^{\infty} B_{n,k} \frac{t^n}{n!}, \quad k = 0, 1, 2, \dots \quad (2.1)$$

An alternative representation is

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum \frac{n!}{j_1! j_2! \dots j_{n-k+1}!} \left( \frac{x_1}{1!} \right)^{j_1} \left( \frac{x_2}{2!} \right)^{j_2} \dots \left( \frac{x_{n-k+1}}{(n-k+1)!} \right)^{j_{n-k+1}}, \quad (2.2)$$

the sum extending over all sequences  $j_1, j_2, \dots, j_{n-k+1}$  of non-negative integers such that

$$j_1 + j_2 + \dots = k, \quad \text{and} \quad j_1 + j_2 + \dots = n.$$

These polynomials are quite general and have numerous applications in combinatorics and other fields. The interested reader is referred to, e.g., [8, Chapter 3].

Related to Bell polynomials are *logarithmic type Bell polynomials*, or *logarithmic polynomials* in short,

$$L_n := L_n(x_1, x_2, \dots, x_n),$$

defined by

$$\log \left( 1 + \sum_{m=1}^{\infty} x_m \frac{t^m}{m!} \right) = \sum_{n=1}^{\infty} L_n \frac{t^n}{n!}. \quad (2.3)$$

The logarithmic polynomials can be expressed in Bell polynomials:

$$L_n(x_1, x_2, \dots, x_n) = \sum_{k=1}^n (-1)^{k-1} (k-1)! B_{n,k}(x_1, x_2, \dots, x_{n-k+1}), \quad (2.4)$$

where  $L_0 = 1$ .

The Bernoulli numbers  $B_k$  are defined by

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

The reader is warned in advance not to confuse the symbols for the Bernoulli numbers and Bell polynomials.

The following well known fact can be found e.g. in [10, p. 896].

**Lemma 2.1.** *We have the complete asymptotic expansion of  $\gamma_n$ :*

$$\gamma_n \sim \gamma + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} \frac{1}{n^{2k}}. \quad (2.5)$$

It is equally well known that

$$B_1 = -\frac{1}{2}, \quad B_{2k+1} = 0, \quad k = 1, 2, \dots$$

With this in mind and for later use, we reformulate Lemma 2.1 as follows.

**Lemma 2.2.** *We have the complete asymptotic expansion of  $\gamma_n$ :*

$$\gamma_n \sim \gamma - \sum_{k=1}^{\infty} \frac{B_k}{k} \frac{1}{n^k}. \quad (2.6)$$

### 3. Main result

For convenience, set

$$w_n(p) := \sum_{k=1}^n \frac{1}{k} - \log \left( n + \sum_{i=0}^s a_i n^{-i} \right). \quad (3.1)$$

First, note the following simple but important relation:

$$\begin{aligned} w_n(p) &= \sum_{k=1}^n \frac{1}{k} - \log \left( n + \sum_{i=0}^s a_i n^{-i} \right) \\ &= \sum_{k=1}^n \frac{1}{k} - \log n - \log \left( 1 + \sum_{i=0}^s a_i n^{-i-1} \right) \\ &= \gamma_n - \log(p(n^{-1})), \end{aligned} \quad (3.2)$$

where  $p(t)$  is a polynomial of degree  $s+1$  with  $p(0) = 1$ . In order to facilitate our statement and derivation, we will allow  $p(t)$  to be any formal power series of the form

$$p(t) = 1 + \sum_{k=1}^{\infty} p_k \frac{t^k}{k!}. \quad (3.3)$$

Now the above-mentioned open problem can be reformulated as follows.

*Modified open problem.* For a given formal power series  $p$  of the form (3.3), find its coefficients  $p_k$  ( $k = 1, 2, \dots$ ) such that

$$w_n(p) = \gamma_n - \log(p(n^{-1})) \quad (3.4)$$

is the fastest sequence which would converge to  $\gamma$ .

We are now at a stage to state our main result.

**Theorem 3.1.** *For a given formal power series  $p$  of the form (3.3), then we have*

$$\gamma_n - \log(p(n^{-1})) \sim \gamma - \sum_{k=1}^{\infty} \frac{L_k(p_1, p_2, \dots, p_k) + (k-1)!B_k}{k!} \frac{1}{n^k}. \quad (3.5)$$

**Proof.** From (2.3) and (2.6), it follows

$$\begin{aligned} \gamma_n - \log(p(n^{-1})) &\sim \gamma - \sum_{k=1}^{\infty} \frac{B_k}{k} \frac{1}{n^k} - \sum_{k=1}^{\infty} \frac{L_k(p_1, p_2, \dots, p_k)}{k!} \frac{1}{n^k} \\ &= \gamma - \sum_{k=1}^{\infty} \frac{L_k(p_1, p_2, \dots, p_k) + (k-1)!B_k}{k!} \frac{1}{n^k}. \end{aligned}$$

This finishes the proof.  $\square$

**Corollary 3.2.** For a given formal power series  $p$  of the form (3.3), let its coefficients  $p_k$  ( $k = 1, 2, \dots$ ) satisfy the following system of equations

$$L_k(p_1, p_2, \dots, p_k) = -(k-1)!B_k, \quad k = 1, 2, \dots, m, \quad (3.6)$$

then we have

$$\gamma_n - \log(p(n^{-1})) \sim \gamma + O\left(\frac{1}{n^{m+1}}\right), \quad n \rightarrow \infty. \quad (3.7)$$

**Proof.** If the system (3.6) of equations is fulfilled, then it follows from (3.5)

$$\gamma_n - \log(p(n^{-1})) \sim \gamma + O\left(\frac{1}{n^{m+1}}\right), \quad n \rightarrow \infty. \quad (3.8)$$

This completes the proof.  $\square$

From Corollary 3.2, the modified open problem can be settled down through solving the system (3.6) of equations. This system, although a nonlinear polynomial one, can be recursively solved. More precisely, the solving procedure can be described as follows. First, it is well known

$$L_1(p_1) = p_1$$

and

$$B_1 = -\frac{1}{2}.$$

So we must have from (3.6)

$$p_1 = \frac{1}{2}. \quad (3.9)$$

Second, for any positive integer  $m$ , we have from (2.4) and (3.6)

$$\sum_{i=1}^k (-1)^{i-1} (i-1)! B_{k,i}(p_1, p_2, \dots, p_{k-i+1}) = -(k-1)! B_k, \quad k = 1, 2, \dots, m,$$

which, together with the well known equality  $p_k = B_{k,1}(p_1, p_2, \dots, p_k)$ , yields

$$p_k = \sum_{i=2}^k (-1)^i (i-1)! B_{k,i}(p_1, p_2, \dots, p_{k-i+1}) - (k-1)! B_k, \quad k = 1, 2, \dots, m. \quad (3.10)$$

Note that  $p_k$  does not appear in the right-hand side of (3.10). Note also both the Bell polynomials  $B_{k,i}$ 's and the Bernoulli number  $B_k$ 's can be recursively calculated. This means that the system (3.6) of equations can be computed via symbolic software such as Mathematica. Having previously computed  $p_1, p_2, \dots, p_{k-1}$ , we can then compute  $p_k$  using (3.10).

**Remark 1.** Eq. (3.9) shows that  $a = p_1 = 1/2$  is the only possible value such that  $\sum_{k=1}^n \frac{1}{k} - \log(n+a)$  converges to  $\gamma$  in  $O(n^{-2})$ . This shows that it is remarkable to replace  $\log n$  by  $\log(n + \frac{1}{2})$  in  $\gamma_n$  to accelerate its convergence to  $\gamma$  in DeTemple [2].

We end with some computations of  $p_k$ 's. Note  $B_{2k+1} = 0$ ,  $k \geq 1$ . For the reader's reference, we record the first few of  $B_{2k}$  and  $L_k$  in the Appendix:

**Example 3.3.**

$$p_1 = \frac{1}{2},$$

$$p_2 = p_1^2 - B_2 = \frac{1}{12},$$

$$p_3 = 3p_1p_2 - 2p_1^3 = -\frac{1}{8},$$

$$p_4 = 4p_1p_3 - 12p_1^2p_2 + 6p_1^4 + 3p_2^2 - 3!B_4 = \frac{23}{240},$$

$$p_5 = 5p_1p_4 + 10p_2p_3 - 20p_1^2p_3 - 30p_1p_2^2 + 60p_1^3p_2 - 24p_1^5 = \frac{17}{32}.$$

For  $m = 4$ , let

$$a = p_1 = \frac{1}{2}, \quad b = \frac{p_2}{2!} = \frac{1}{24}, \quad c = \frac{p_3}{3!} = -\frac{1}{48}, \quad d = \frac{p_4}{4!} = \frac{23}{5760},$$

which, combined with Corollary 3.2, yields

$$\sum_{k=1}^n \frac{1}{k} - \log \left( n + \frac{1}{2} + \frac{1}{24n} - \frac{1}{48n^2} + \frac{23}{5760n^3} \right) \sim \gamma + O\left(\frac{1}{n^5}\right), \quad n \rightarrow \infty.$$

This recaptures one of the main results in Chen and Mortici [7]. Moreover, by (3.3), we set

$$p(t) = 1 + \frac{1}{2}t + \frac{1}{24}t^2 - \frac{1}{48}t^3 + \frac{23}{5760}t^4 + \frac{17}{3840}t^5.$$

From the above computation and Corollary 3.2, we obtain

$$\sum_{k=1}^n \frac{1}{k} - \log n - \log(p(n^{-1})) \sim \gamma + O\left(\frac{1}{n^6}\right), \quad n \rightarrow \infty.$$

## Appendix

For the reader's convenience, here we record  $B_1$  and the first few of  $B_{2k}$  and  $L_k$ . One can find more in [8,10]:

$$B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30},$$

$$B_6 = \frac{1}{42}, \quad B_8 = -\frac{1}{30}, \quad B_{10} = -\frac{5}{66}.$$

$$L_1(p_1) = p_1,$$

$$L_2(p_1, p_2) = p_2 - p_1^2,$$

$$L_4(p_1, p_2, p_3, p_4) = p_4 - 4p_1p_3 + 12p_1^2p_2 - 6p_1^4 - 3p_2^2,$$

$$L_5(p_1, p_2, p_3, p_4, p_5) = p_5 - 5p_1p_4 - 10p_2p_3 + 20p_1^2p_3 + 30p_1p_2^2 - 60p_1^3p_2 + 24p_1^5.$$

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