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On the properties of nonlinear nonlocal operators arising in neural field models

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ABSTRACT

We study the existence and continuous dependence of stationary solutions of the one-population Wilson–Cowan model on the steepness of the firing rate functions. We investigate the properties of the nonlinear nonlocal operators which arise when formulating the stationary one-population Wilson–Cowan model as a fixed point problem. The theory is used to study the existence and continuous dependence of localized stationary solutions of this model on the steepness of the firing rate functions. The present work generalizes and complements previously obtained results as we relax on the assumptions that the firing rate functions are given by smoothed Heaviside functions.

1. Introduction

The macroscopic dynamics of neural networks is often studied by means of neural field models. Here we consider a neural field model of the Wilson–Cowan type [1–5]

$$\frac{\partial}{\partial t}u(x,t) = -u(x,t) + \int_{\mathbb{R}} \omega(x,y)P(u(y,t))dy, \quad x \in \mathbb{R},\ t > 0. \tag{1.1}$$

Eq. (1.1) describes the dynamics of the spatio-temporal electrical activity in neural tissue in one spatial dimension. Here $u(x,t)$ is interpreted as a local activity of a neural population at the position $x \in \mathbb{R}$ and time $t > 0$. The second term on the right hand side of (1.1) represents the synaptic input where P is a firing rate function. Typically P is a smooth function that has sigmoidal shape (the shape of the logistic function). The spatial strength of the connectivity between the neurons is modeled by means of a connectivity function ω. We refer the reader to [1–5] for more details regarding the relevance of Eq. (1.1) in neural field theory.

The most common 'simplification' of the model consists of replacing the smooth firing rate function by the unit step function. The existence of solutions to a neural field equation with smooth firing rate functions can be studied using methods of classical fixed point theory; see e.g. [6,7]. These methods have been applied to the particular type of neural field model by various authors; see [8–11]. Dealing with the unit step function however leads to the discontinuity in the integral operator involved in (1.1), which makes it impossible to apply the classical theory.

Despite difficulties in mathematical treatment, the mentioned 'simplification' allows to obtain closed form expressions for solutions describing coherent structures like stationary localized solutions (bumps) and traveling fronts [5] as well as to assess the stability of these structures using the Evans function approach [12]. To benefit from both representation of P it is often conjectured that the 'simplified' model reproduces the essential features of the model with smooth P in the steep

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firing rate regimes. While this conjecture is supported by numerical simulations (see for example [13]) there are few and far between works addressing this problem in a rigorous mathematical way. Namely, Potthast and Beim Graben provided a rigorous approach to study global existence of solutions to the Wilson–Cowan type of the model with the smooth firing rate function as well as with the unit step function, [11]. They demonstrated that the latter case requires more restrictions on the choice of a functional space as well as some extra assumptions on ω. In [14–16] the reader can find the analysis of existence and stability of localized stationary solutions (bumps) for a special class of the firing rate functions, the functions that are 'squeezed' between two unit step functions. This class of functions is also referred to as the smoothed Heaviside functions [17]. It has been shown that if the both bump solutions to the model with the unit step functions are stable/unstable then the bump in the framework of the corresponding smoothed Heaviside firing rate function has the same stability property, [14–16]. To the best of our knowledge, no analysis has been done on the passage from a smooth to discontinuous firing rate function in the framework of neural field models.

In the present paper we study the existence and continuous dependence of stationary solutions to (1.1) under the transition from a smooth firing rate function to the unit step function. The stationary solutions of (1.1) are solutions to a fixed point problem. We describe the fixed point problem in terms of a Hammerstein operator that is represented as the superposition of a Nemytskii operator $\mathcal{N} : u \to P(u(x))$ and a linear integral operator. We study properties of the operators when the firing rate function is represented as a one-parameter family of functions that approach the unit step function with the step taking place in $x = \theta$, when the steepness parameter goes to infinity. The main challenge here is to choose function spaces and a suitable topology of the operators convergence that allow the continuous dependence properties of solutions to be fulfilled.

We introduce the notion of the θ-condition, the condition on a function, say u, to have finite number of only simple roots to $u(x) - \theta$; for details see Definition 3.4. We show that the Nemytskii operator in the limit case (when the steepness parameter goes to infinity) preserves continuity if the functions from the operator domain satisfy the θ-condition. We demonstrate that the choice of the norm is crucial here since, e.g., the θ-condition is achieved in $W^{1,\infty}$-norm but not in $W^{1,q}$-norms, $q < \infty$. Our main results are summarized in Theorems 3.14 and 3.15, which we will refer to as the continuous dependence theorem and the existence theorem, respectively. These theorems enable us to show the existence and continuous dependence of bumps on the steepness parameter when it approaches infinity. We provide two examples of assumptions on ω: one is for the inhomogeneous and one is for the homogeneous function ω, to demonstrate the applicability of our results. In particular, in the latter case we prove the existence of bumps in a steep firing rate regime where the firing rate function takes values zero on a ray $(-\infty, \theta)$. We emphasize that this result is more general than results on the existence of bumps obtained in [14–16].

The paper is organized as follows. In Section 2 we explain our notations, prove some useful theorems, and state lemmas from functional analysis, to which we refer in the subsequent sections. In Section 3 we give a detailed description of the model. Next, we study continuity and compactness of the associated operators in Sobolev spaces, formulate and prove the main theorems. In Section 4 we apply the results of Section 3 to prove continuous dependence of spatially localized stationary solutions (bumps) of (1.1) on the steepness of the firing rate function for both inhomogeneous and homogeneous connectivity functions, and show the existence of the bumps in the framework of the homogeneous ω. Section 5 contains conclusions and outlook.

2. Preliminaries

Let B be an open set of a real Banach space $\mathcal{B}$, then $\overline{B}$ denotes the closure of B in $\mathcal{B}$. We use the notation $\deg(A, B, p)$ for the degree defined for an operator $A : \overline{B} \to \mathcal{B}$, and $p \in \mathcal{B}$. We use $\mathrm{ind}(A, B)$ for the topological index of A, [18].

Let $W^{1,q}(\mathbb{R}, \mu)$, $1 \le q \le \infty$, denote a Sobolev space which consists of all functions $w \in L^q(\mathbb{R}, \mu)$ such that their generalized derivatives (with respect to the given measure μ) $dw/d\mu = \tilde{w}$ belong to $L^q(\mathbb{R}, \mu)$.

The element $w \in W^{1,q}(\mathbb{R}, \mu)$ then can be represented as

$$w(x) = w(0) + \int_0^x \widetilde{w}(\xi) d\mu(\xi). \tag{2.1}$$

We consider the following two norms in $W^{1,q}(\mathbb{R}, \mu)$

$$\|w\|_1 = \|w\|_{L^q} + \|\widetilde{w}\|_{L^q} \tag{2.2}$$

and

$$\|w\|_2 = |w(0)| + \|\widetilde{w}\|_{L^q} \tag{2.3}$$

where $\|\cdot\|_{L^q}$ is the norm in $L^q(\mathbb{R}, \mu)$, i.e.,

$$\|w\|_{L^q} = \left(\int_{\mathbb{R}} |w(x)|^q d\mu(x)\right)^{1/q}, \quad 1 \le q < \infty$$

and

$$\|w\|_{L^\infty} = \sup_{x\in\mathbb{R}} |w(x)|.$$

Theorem 2.1. *The norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent whenever μ is finite.*

Proof. From the representation (2.1) we have

$$\begin{aligned}\|w\|_{L^q} &= \|w(0)+\int_0^x \tilde{w}(y)d\mu(y)\|_{L^q} \le \||w(0)|+\int_0^x |\tilde{w}(y)|d\mu(y)\|_{L^q}\\ &\le \||w(0)|+\int_{\mathbb{R}} |\tilde{w}(y)|d\mu(y)\|_{L^q} = [\mu(\mathbb{R})]^{1/q}(|w(0)|+\|\tilde{w}\|_{L^1}).\end{aligned}$$

Using the Hölder inequality we get $\|\tilde{w}\|_{L^1} \le [\mu(\mathbb{R})]^{1/q'}\|\tilde{w}\|_{L^q}$ where q' is defined by the equality $1/q+1/q'=1$. Thus,

$$\|w\|_{L^q} \le [\mu(\mathbb{R})]^{1/q}|w(0)| + \mu(\mathbb{R})\|\tilde{w}\|_{L^q}. \tag{2.4}$$

From (2.2) and (2.4) we obtain

$$\|w\|_1 \le [\mu(\mathbb{R})]^{1/q}|w(0)| + \mu(\mathbb{R})\|\tilde{w}\|_{L^q} + \|\tilde{w}\|_{L^q}.$$

Therefore we get

$$\|w\|_1 \le C_2\|w\|_2, \quad C_2 = \max\left\{[\mu(\mathbb{R})]^{1/q}, 1+\mu(\mathbb{R})\right\}.$$

In a similar way we estimate $|w(0)|$, i.e.,

$$|w(0)| = [\mu(\mathbb{R})]^{-1/q}\|w(0)\|_{L^q} = [\mu(\mathbb{R})]^{-1/q}\|w(x)-\int_0^x \tilde{w}(y)d\mu(y)\|_{L^q} \le [\mu(\mathbb{R})]^{-1/q}\left(\|w\|_{L^q}+\|\tilde{w}\|_{L^q}\right).$$

We have

$$\|w\|_2 \le c_1\|w\|_1, \quad c_1 = [\mu(\mathbb{R})]^{-1/q}+1.$$

Hence, we get

$$C_1\|w\|_2 \le \|w\|_1 \le C_2\|w\|_2$$

with

$$C_1 = c_1^{-1} = \frac{[\mu(\mathbb{R})]^{1/q}}{[\mu(\mathbb{R})]^{1/q}+1}, \qquad C_2 = \max\left\{[\mu(\mathbb{R})]^{1/q}, 1+\mu(\mathbb{R})\right\}.$$

By definition the norms then are equivalent. □

We denote the norm in $W^{1,q}(\mathbb{R},\mu)$ by $\|\cdot\|_{W^{1,q}}$.

Lemma 2.2. *Let A be the following operator*

$$(Au)(x) = \int_{\mathbb{R}} k(x,y)u(y)d\mu(y), \quad x\in\mathbb{R}, \tag{2.5}$$

where μ is a finite complete measure on $\mathbb{R}$ and $k(x,y)$ is measurable on $\mathbb{R}^2$. Let the following conditions be satisfied

(i) *for any $x\in\mathbb{R}$, $k(x,\cdot)\in L^p(\mathbb{R},\mu)$,*
(ii) *for any $\varepsilon>0$ there exist a finite partitioning of $\mathbb{R}$ into measurable sets, say $D_1, D_2, \ldots, D_n$, such that*

$$\sup_{x_1,x_2\in D_j} \|k(x_1,y)-k(x_2,y)\|_{L^{p'}} < \varepsilon, \quad j=1,2,\ldots,n. \tag{2.6}$$

Then the integral operator A maps $L^p(\mathbb{R},\mu)$ to $L^\infty(\mathbb{R},\mu)$ and it is compact; see [19].

Theorem 2.3. *Let D be a closed bounded subset of a real Banach space $\mathcal{B}$, Λ be a closed subset of $\mathbb{R}$, and an operator $T(\lambda,u):\Lambda\times D\to\mathcal{B}$ be continuous with respect to both variables and collectively compact (i.e., $T(\Lambda\times D)$ is a pre-compact set in $\mathcal{B}$). Assume that $\lambda_n\to\lambda^*$ and $T(\lambda_n,u_n)=u_n$. Then the equation $T(\lambda^*,u)=u$ has at least one solution. Moreover, any limit point of the sequence $\{u_n\}$ is a solution of this equation, i.e., if $u_{n_k}\to u^*$ then u^* is a solution of $T(\lambda^*,u^*)=u^*$.*

Proof. The sequence $\{u_n\}$ defined by $T(\lambda_n,u_n)=u_n$ is a pre-compact set due to T is collectively compact. Thus, there exist convergent subsequences of $\{u_n\}$, i.e., $\{u_{n_k}\}\to u^*\in D$. The continuity of T yields $\lim_{n_k\to\infty}T(\lambda_{n_k},u_{n_k})=T(\lambda^*,u^*)=u^*$. □

Remark 2.4. If u^* is unique in D then u_n has only one limit point, that is, $u_n\to u^*$.

Lemma 2.5 (*Homotopy Invariance*)**.** *Let D be an open bounded subset of a real Banach space $\mathcal{B}$. Suppose that $\{h_t\}$ is a homotopy of operators $h_t:\bar{D}\to\mathcal{B}$ for $t\in[0,1]$, and assume that h_t-I is collectively compact. If $h_tf\ne p$ for any $f\in\partial D$ and $t\in[0,1]$, then $\deg(h_t,D,p)$ is independent of t; see* [20].

3. Main results

The stationary Wilson–Cowan model (1.1) is equivalent to the fixed point problem

$$u = \mathcal{H}u, \tag{3.1}$$

where $\mathcal{H}$ is the Hammerstein operator

$$(\mathcal{H}u)(x) = \int_{\mathbb{R}} \frac{\omega(x,y)}{\rho(y)} P(u(y))d\mu(y) \tag{3.2}$$

and

$$\mu(A) = \int_A \rho(y)dy$$

is an arbitrary probability measure which is absolutely continuous with respect to the Lebesgue measure (i.e., $\mu(\mathbb{R}) = 1$ and $\mu(A) \geq 0$ whenever the Lebesgue measurable set A has a positive Lebesgue measure). This can be achieved by putting some necessary properties on the function ρ.

We assume that the function $\omega(x, y)$ is a measurable function satisfying the following assumptions:

(i) for any $x \in \mathbb{R}$, $\omega(x, \cdot) \in L^1(\mathbb{R})$, i.e.,

$$\forall x \in \mathbb{R} \quad \int_{\mathbb{R}} |\omega(x,y)|dy < \infty,$$

(ii) ω is differentiable a.e. with respect to the first variable and

$$\omega'_x(x,\cdot) \in L^1_{loc}(\mathbb{R}) \quad \forall x \in \mathbb{R},$$

(iii) ω is bounded, i.e.,

$$\exists C > 0 \quad |\omega(x,y)| < C \quad \forall x, y \in \mathbb{R},$$

(iv) for any $y \in \mathbb{R}$ $\lim_{x\to\infty} \omega(x, y) = 0$.

The function P can be interpreted as a probability function of firing. Thus, P is a map from $\mathbb{R}$ to $[0, 1]$. We consider the special family of P: $P(u) = S(\beta, u)$ where β takes values from $(0, \infty]$. We assume that S satisfies the following properties:

(i) $S : (0,\infty) \times \mathbb{R} \to [0, 1]$ is a continuous function,
(ii) $S(\beta, \cdot)$ is monotonically non-decreasing,
(iii) $S(\beta, \cdot) \to S(\beta_0, \cdot)$ uniformly on $\mathbb{R}$ as $\beta \to \beta_0 \in (0, \infty)$,
(iv) as $\beta \to \infty$ $S(\beta, u)$ approaches $S(\infty, u)$ uniformly on $(-\infty, \theta - \varepsilon] \cup [\theta + \varepsilon, \infty)$ for any $\varepsilon > 0$, where $S(\infty, u)$ is the unit step function

$$S(\infty, u) = \begin{cases} 0, & u < \theta \\ 1, & u \geq \theta \end{cases}$$

with some threshold value $\theta > 0$.

The Hammerstein operator (3.2) can be represented as the superposition

$$(\mathcal{H}u)(x) = (\Omega \circ \mathcal{N}u)(x)$$

of the linear operator

$$(\Omega u)(x) = \int_{\mathbb{R}} \frac{\omega(x,y)}{\rho_1(y)} u(y)d\mu(y), \tag{3.3}$$

and the Nemytskii operator

$$(\mathcal{N}u)(x) = \frac{\rho_1(x)}{\rho(x)} P(u(x)). \tag{3.4}$$

Here ρ_1 is an auxiliary function satisfying the following properties

(i) $0 \leq \rho_1(x) \leq C_{\rho_1}$, where $C_{\rho_1} > 0$,
(ii) $\mathrm{supp}(\rho_1) \supseteq \mathrm{supp}(\rho)$,
(iii) $|\omega(0, y)| \leq C_\omega \rho_1(y)$ $\forall y \in \mathbb{R}$ and $C_\omega > 0$.

We set $\rho_1(x)/\rho(x) = 0$ and $\rho(x)/\rho_1(x) = 0$ for all $x \in \mathbb{R} \setminus \mathrm{supp}(\rho_1)$.

Remark 3.1. In particular, one can assume $\rho_1 \equiv \rho$. However, in order to keep the theory as general as possible, we allow ρ_1 to differ from ρ. In Section 4 we make use of this difference. See Example 4.5.

When we want to emphasize that some particular property is valid only for operators corresponding to $S(\beta,\cdot)$, $\beta\in(0,\infty)$ or $S(\infty,\cdot)$ we use the subindexes β and ∞, respectively. That is, we denote the Hammerstein operator (3.2) and the Nemytskii operator (3.4) as $\mathcal{H}_\beta$, $\mathcal{H}_\infty$ and $\mathcal{N}_\beta$, $\mathcal{N}_\infty$. When a property is valid for an operator with any $P:\mathbb{R}\to[0,1]$ we do not use any subindex, e.g., $\mathcal{H}$, $\mathcal{N}$.

Lemma 3.2. *Let $\widetilde{\Omega}: L^p(\mathbb{R},\mu)\to L^q(\mathbb{R},\mu)$, $1\le p,q\le\infty$ be an operator defined as*

$$(\widetilde{\Omega}v)(x)=\int_{\mathbb{R}}\frac{\omega'_x(x,y)}{\rho(x)\rho_1(y)}u(y)d\mu(y). \tag{3.5}$$

Then, the operator Ω in (3.3) is a map from $L^p(\mathbb{R},\mu)$ to $W^{1,q}(\mathbb{R},\mu)$ and (a) *it is a continuous operator if and only if $\widetilde{\Omega}$ is continuous,* (b) *it is a compact operator if and only if $\widetilde{\Omega}$ is a compact operator.*

Proof. We formally apply (2.1) to the element $(\Omega u)(x)$. We have

$$(\Omega u)(x)=(\Omega u)(0)+\int_0^x\widetilde{\Omega}u(y)d\mu(y)$$

where

$$(\Omega u)(0)=\int_{\mathbb{R}}\frac{\omega(0,y)}{\rho_1(y)}u(y)d\mu(y),$$

and

$$(\widetilde{\Omega}u)(x)=\frac{d(\Omega u)(x)}{d\mu}=\int_{\mathbb{R}}\frac{\omega'_x(x,y)}{\rho(x)\rho_1(y)}u(y)d\mu(y).$$

Using properties of ρ_1 we get

$$|(\Omega u)(0)|=\left|\int_{\mathbb{R}}\frac{\omega(0,y)}{\rho_1(y)}u(y)d\mu(y)\right|\le C_\omega\|u\|_{L^p}. \tag{3.6}$$

Further, $\widetilde{\Omega}$ is a map from $L^p(\mathbb{R},\mu)$ to $L^q(\mathbb{R},\mu)$, hence Ω maps $L^p(\mathbb{R},\mu)$ to $W^{1,q}(\mathbb{R},\mu)$.

To prove (a) continuity and (b) compactness of Ω we introduce a linear operator $\mathcal{J}: W^{1,p}(\mathbb{R},\mu)\to\mathbb{R}\times L^p(\mathbb{R},\mu)$ such that

$$\mathcal{J}=(\mathcal{J}_1,\mathcal{J}_2):\ \mathcal{J}_1w=w(0)\in\mathbb{R},\qquad \mathcal{J}_2w=\frac{d}{d\mu}w\equiv\tilde{w}\in L^p(\mu,\mathbb{R}).$$

The inverse operator $\mathcal{J}^{-1}:\mathbb{R}\times L^p(\mathbb{R},\mu)\to W^{1,p}(\mathbb{R},\mu)$ then is given as

$$\mathcal{J}^{-1}(a,u)=a+\int_0^x u(y)d\mu(y),\quad (a,u)\in\mathbb{R}\times L^p(\mathbb{R},\mu).$$

It is easy to check that $\mathcal{J}$ is a homeomorphism: Indeed $\mathcal{J}$ is an isomorphism [21] and linear continuous. Thus, $\mathcal{J}^{-1}$ is continuous by the Banach theorem [22]. We present the proof of (b). The operator $\Omega_0: L^p(\mathbb{R},\mu)\to\mathbb{R}$ given by $(\Omega_0u)(x)=(\Omega u)(0)$ is compact as soon as it is bounded, which is the case due to the estimate (3.6). Therefore, for any bounded subset $D\subset L^p(\mathbb{R},\mu)$ there is a corresponding pre-compact subset $(\Omega_0D,\tilde{\Omega}D)\subset\mathbb{R}\times L^p(\mathbb{R},\mu)$ which is homeomorphic to ΩD. Hence, ΩD is a pre-compact set in $W^{1,p}(\mathbb{R},\mu)$ and Ω is a compact operator.

Let us assume now that Ω is a compact operator, while $\tilde{\Omega}$ is not compact. Then, for any bounded D we get a pre-compact set ΩD which is homeomorphic to the non pre-compact set $(\Omega_0D,\tilde{\Omega}D)$. This contradiction completes the proof. To prove (a) one can proceed in a similar way assuming boundedness of a set D instead of pre-compactness. □

Lemma 3.3. *If $\rho_1(x)/\rho(x)$ belongs to $L^p(\mathbb{R},\mu)$ the Nemytskii operator $\mathcal{N}$ maps $W^{1,q}(\mathbb{R},\mu)$ to $L^p(\mathbb{R},\mu)$, $1\le p,q\le\infty$. Moreover $\mathcal{N}_\beta$, $\beta<\infty$, is continuous. The operator $\mathcal{N}_\infty$ is discontinuous on $W^{1,q}(\mathbb{R},\mu)$, $1\le q\le\infty$.*

Proof. First of all, we notice that due to the boundedness of P, i.e., $|P(u)|\le 1$, we have

$$\left|\frac{\rho_1(x)}{\rho(x)}P(u)\right|\le\left|\frac{\rho_1(x)}{\rho(x)}\right|\in L^p(\mathbb{R},\mu).$$

Hence, the Nemytskii operator is a map from $W^{1,q}(\mathbb{R},\mu)$ to $L^p(\mathbb{R},\mu)$. Moreover, $S(\beta,\cdot)$ satisfies to the Caratheodory conditions [19,20]. We conclude that the Nemytskii operator $\mathcal{N}_\beta$ is continuous [19,20].

To show that N_∞ is not continuous on $W^{1,q}(\mathbb{R},\mu)$ it is enough to give an example. Consider $u_n(x)=\theta+1/n$ and $u(x)=\theta$. We have $(\mathcal{N}_\infty u_n)(x)=\rho_1(x)/\rho(x)$ for all $n\in N$ and $\mathcal{N}_\infty u=0$. When $n\to\infty$ we get

$$\|u_n-u\|_{W^{1,q}}\to 0,\quad\text{and}\quad \|\mathcal{N}_\infty u_n-\mathcal{N}_\infty u\|_{L^p}=\|\rho_1/\rho\|_{L^p}\not\to 0.\quad\square$$

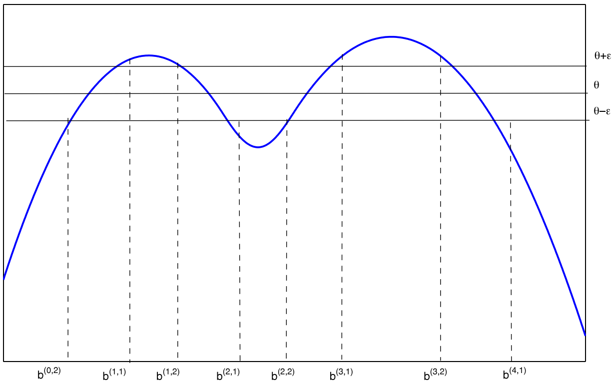

Fig. 1. The θ-condition in $W^{1,\infty}(\mathbb{R},\mu)$ for $N=4$.

Definition 3.4. Let $\theta>0$ be fixed. We say that $u\in W^{1,q}(\mathbb{R},\mu)$ satisfies the θ-condition if

- the function $u(x)-\theta$ has finitely many simple roots (i.e. $u(a)=\theta$ always implies $u'(a)\neq 0$);
- there exist $\sigma>0$ and $A>0$ such that $u(x)\le\theta-\sigma$ for all $|x|>A$.

Remark 3.5. Definition 3.4 implies that if $u\in W^{1,q}(\mathbb{R},\mu)$ satisfies the θ-condition then the number of intersections $u(x)=\theta$ is an even number.

Lemma 3.6. *Let $\theta>0$ be fixed and let $U\in W^{1,\infty}(\mathbb{R},\mu)$ satisfies the θ-condition. Assume that the equation $U(x)=\theta$ has N solutions. Then there exists $\varepsilon>0$ such that for any $u\in B(U,\varepsilon)=\{u:\ \|u-U\|_{W^{1,\infty}}<\varepsilon\}$*

- *the function u satisfies the θ-condition;*
- *the equation $u(x)=\theta$ has exactly N solutions.*

Proof. Here we are going to use $\|\cdot\|_{W^{1,\infty}}$ given by (2.2). Let U satisfy the θ-condition, and $a^{(k)}$: $a^{(k)}<a^{(k+1)}$, $k=1,\dots,N$, be all solutions of the equation $U(x)=\theta$. Due to these assumptions there exist a positive ε and the points $b^{(k,1)}$, $b^{(k,2)}$, $k=0,\dots,N$, satisfying

$$\begin{aligned}&a^{(k)}<b^{(k,1)}<b^{(k,2)}<a^{(k+1)},\quad k=1,\dots,N-1,\\&b^{(0,1)}=-\infty,\qquad b^{(0,2)}<a^{(1)},\qquad a^{(N)}<b^{(N,1)},\qquad b^{(N,2)}=\infty,\end{aligned}$$

such that

- $U(x)>\theta+2\varepsilon$ if $x\in(b^{(k,1)},b^{(k,2)})$, $k=2m-1$, $0\le k\le N$;
- $U(x)<\theta-2\varepsilon$ if $x\in(b^{(k,1)},b^{(k,2)})$, $k=2m$, $0\le k\le N$;
- $|U'(x)|>2M\varepsilon$ if $x\in(b^{(k,2)},b^{(k+1,1)})$, $0\le k\le N-1$, where $M=\sup_{x\in\mathbb{R}}\rho(x)$.

Let $u\in B(U,\varepsilon)$. Clearly,

$$|u(x)-U(x)|<\varepsilon,\qquad |u'(x)-U'(x)|<\varepsilon\rho(x)\quad \text{a.e. } x\in\mathbb{R}.$$

This implies the following estimates:

$$u(x)>\theta+\varepsilon\quad \text{if } x\in(b^{(k,1)},b^{(k,2)}),\ k=2m-1,\ 0\le k\le N;\tag{A1}$$

$$u(x)<\theta-\varepsilon\quad \text{if } x\in(b^{(k,1)},b^{(k,2)}),\ k=2m,\ 0\le k\le N;\tag{A2}$$

$$|u'(x)|>M\varepsilon\quad \text{if } x\in(b^{(k,2)},b^{(k+1,1)}),\ 0\le k\le N-1.\tag{A3}$$

Therefore, the equation $u(x)=\theta$ has a unique solution in each interval $(b^{(k,2)},b^{(k+1,1)})$, $0\le k\le N-1$, while $|u'(x)|>M\varepsilon$ within any of these intervals. Remembering that $b^{(0,2)}=-\infty$ and that $u(x)<\theta$ for $x>b^{(N,1)}$ yield exactly N solutions of the equation $u(x)=\theta$, and all of these solutions must be simple. Fig. 1 illustrates graphically Lemma 3.6 for $N=4$. Here we have plotted schematically a function $u\in B(U,\varepsilon)$, where U has $N=4$ intersections with $u=\theta$. $\square$

Lemma 3.7. *Let $\theta>0$ be fixed and let $U\in W^{1,q}(\mathbb{R},\mu)$, $1\le q<\infty$, satisfy the θ-condition. For any $\varepsilon>0$ the ball $B(U,\varepsilon)=\{u:\ \|u-U\|_{W^{1,q}}<\varepsilon\}$ contains functions which do not satisfy the θ-condition.*

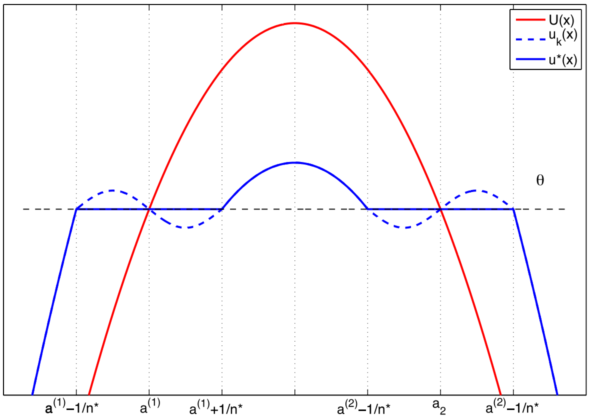

Fig. 2. The violation of the θ-condition in $W^{1,q}(\mathbb{R},\mu)$ for $q<\infty$.

Proof. For the proof we give the following example

$$u_n(x)=\begin{cases} U(x)-U(a^{(1)}-1/n)+\theta, & x\in(-\infty,a^{(1)}-1/n)\\ \theta, & x\in\bigcup_{j=1}^{2}[a^{(j)}-1/n,a^{(j)}+1/n]\\ U(x)-U(a^{(2)}+1/n)+\theta, & x\in(a^{(2)}+1/n,+\infty). \end{cases} \tag{3.7}$$

We consider the norm in $W^{1,q}(\mathbb{R},\mu)$ given by (2.3). Without loss of generality, we assume that one of $a^{(j)}$, $j=1,2$, is equal to zero. Then, it is easy to see that

$$\begin{aligned}\|u_n-U\|_{W^{1,q}} &= \|\tilde u_n-\tilde U\|_{L^q}=\left(\bigcup_{j=1}^{2}\int_{a^{(j)}-1/n}^{a^{(j)}+1/n}|\tilde U(x)|^q d\mu(x)\right)^{1/q}\\ &\le A\left(\mu[a^{(1)}-1/n,a^{(1)}+1/n]+\mu[a^{(2)}-1/n,a^{(2)}+1/n]\right)^{1/q},\end{aligned}$$

where $A=\sup_x\{\tilde U(x)\}$, for $x\in[a^{(1)}-1/n,a^{(1)}+1/n]\bigcup[a^{(2)}-1/n,a^{(2)}+1/n]$. Thus, $\|u_n-U\|_{W^{1,q}}\to 0$ as $n\to\infty$, i.e., for any $\varepsilon>0$ there exist such n_ε that $\|u_n-U\|_{W^{1,q}}\le\varepsilon$ for all $n\ge n_\varepsilon$. In Fig. 2 we have plotted the graphs of $U(x)$ (red solid line) and $u^*(x)$ (blue solid line), where $u^*(x)$ is an example of (3.7) for some $n^*\le n_\varepsilon$, together with the constant function θ. From the figure it is clear that $u^*(x)$ does not satisfy the θ-condition. $\square$

Theorem 3.8. *Let $\theta>0$ be fixed, $U(x)\in W^{1,\infty}(\mathbb{R},\mu)$ satisfies the θ-condition and $U(x)=\theta$ has, say, N solutions $a^{(1)}$, $a^{(2)},\dots,a^{(N)}$. Let $\rho_1(x)/\rho(x)$ belong to $L^p(\mathbb{R},\mu)$. (a) There exist $\varepsilon>0$ such that $\mathcal{N}_\infty: B(U,\varepsilon)=\{u:\|u-U\|_{W^{1,\infty}}<\varepsilon\}\to L^p(\mathbb{R},\mu)$ is continuous when $1\le p<\infty$. (b) The operator $\mathcal{N}_\infty: B(U,\varepsilon)\to L^\infty(\mathbb{R},\mu)$ is continuous provided that there exist some $\delta>0$ that $\operatorname{supp}(\rho_1)\bigcap(a^{(k)}-\delta,a^{(k)}+\delta)=\emptyset$ for any $k=1,2,\dots,N$. Otherwise, i.e., if for any $\delta>0$ there exist some $\hat k$ such that $\operatorname{supp}(\rho_1)\bigcap(a^{(\hat k)}-\delta,a^{(\hat k)}+\delta)\neq\emptyset$, we get discontinuity of $\mathcal{N}_\infty: B(U,\varepsilon)\to L^\infty(\mathbb{R},\mu)$.*

Proof. Let us consider $u_n,\ u\in B(U,\varepsilon)\subset W^{1,\infty}(\mathbb{R},\mu)$ such that $\|u_n-u\|_{W^{1,\infty}}\to 0$. By Lemma 3.6 it is always possible to choose ε in a such way that both u_n and u satisfy the θ-condition and the equations $u_n(x)=\theta$, $u(x)=\theta$ possess N simple roots each. We denote these roots as $a_n^{(k)}$ for the first equation, and $a_0^{(k)}$ for the second, $k=1,\dots,N$.

We derive the estimate

$$|(N_\infty u_n)(x)-(N_\infty u)(x)|=|\rho_1(x)/\rho(x)|\,\chi(x),$$

where

$$\chi(x)=\begin{cases}1, & x\in\bigcup_{k=1}^{N}[a_n^{(k)},a_0^{(k)}],\\ 0, & \text{otherwise}.\end{cases}$$

Here $[x_1,x_2]$ defines the interval $[x_1,x_2]$ when $x_2\ge x_1$ and $[x_2,x_1]$ if $x_2<x_1$.

First we consider $p<\infty$. Then, after the lemma (follows below), the case $p=\infty$ will be considered. When $1\le p<\infty$ we have the following equality

$$\|(N_\infty u_n)(x)-(N_\infty u)(x)\|_{L^p}=\left(\int_{\mathbb{R}}\left|\frac{\rho_1(x)}{\rho(x)}\right|^p\chi(x)d\mu(x)\right)^{1/p}.$$

Using now the Hölder inequality we get

$$\left(\int_{\mathbb{R}} \left|\frac{\rho_1(x)}{\rho(x)}\right|^p \chi(x)d\mu(x)\right)^{1/p} \le \|\rho_1/\rho\|_{L^\infty} \left(\mu\left(\bigcup_{k=1}^{N}[a_n^k, a_0^k]\right)\right)^{1/p}.$$

Since $|\rho_1(x)/\rho(x)| \in L^p(\mathbb{R},\mu) \subset L^\infty(\mathbb{R},\mu)$ then $\|\rho_1/\rho\|_{L^\infty} \le C_\rho$, where $C_\rho > 0$ is some constant. If now $\mu(\bigcup_{k=1}^{N}[a_n^{(k)}, a_0^{(k)}]) \to 0$ as $n \to \infty$ we get the continuity of $N_\infty : B(U,\varepsilon) \to L^p(\mathbb{R},\mu)$. To prove that we use the following lemma. □

Lemma 3.9. *For any $u_n,\ u \in B(U,\varepsilon) \subset W^{1,\infty}(\mathbb{R},\mu)$ such that $\|u_n - u\|_{W^{1,\infty}} \to 0$ we have $a_n^{(k)} \to a_0^{(k)}$.*

Proof. For our proof we use the norm (2.2) of $W^{1,\infty}(\mathbb{R},\mu)$. From $\|u_n-u\|_{W^{1,\infty}} \to 0$ follows that $\sup_{x\in\mathbb{R}} |u_n(x)-u(x)| \to 0$.

Let us assume the contrary, i.e., there is $k = k^*$ such that $a_n^{(k^*)} \not\to a_0^{(k^*)}$. This means that

$$(\exists \sigma_0 > 0)\ (\forall N \in \mathbb{N})\ (\exists \tilde{n} \ge N):\ |a_{\tilde{n}}^{(k^*)} - a_0^{(k^*)}| \ge \sigma_0.$$

Then we have

$$\sup_{x\in\mathbb{R}} |u_{\tilde{n}}(x) - u(x)| \ge |u_{\tilde{n}}(a_{\tilde{n}}^{(k^*)}) - u(a_{\tilde{n}}^{(k^*)})| = |\theta - u(a_{\tilde{n}}^{(k^*)})| = |u(a_0^{(k^*)}) - u(a_{\tilde{n}}^{(k^*)})|.$$

Due to the transversality condition on the intersection of any $u(x) \in B(U,\varepsilon)$ and θ we have $|u(a)-u(b)| \ge \kappa$ if $|a-b| \ge \sigma_0$. Therefore we get

$$(\forall N \in \mathbb{N})\ (\exists \tilde{n} \ge N):\ \sup_{x\in\mathbb{R}} |u_{\tilde{n}}(x) - u(x)| > \kappa.$$

By definition $\sup_{x\in\mathbb{R}} |u_n(x) - u(x)|$ diverges. Then $\|u_n - u\|_{W^{1,\infty}}$ diverges too. This contradiction completes the proof of the lemma. □

Next, we consider the case $p = \infty$. We get

$$\|N_\infty u_n - N_\infty u\|_{L^\infty} = \sup_{x\in\mathbb{R}} |(N_\infty u_n)(x) - (N_\infty u)(x)| = \alpha_n,$$

where α_n is a smallest value that $\mu\{x : |(N_\infty u_n)(x) - (N_\infty u)(x)| \ge \alpha_n\} = 0$, i.e.,

$$\alpha_n = \sup_{x\in Q} |\rho_1(x)/\rho(x)|, \quad Q = \bigcup_{k=1}^{N}[a_n^{(k)}, a_0^{(k)}].$$

Let us assume first that there is some $\delta > 0$ such that $\mathrm{supp}(\rho_1)\bigcap(a^{(k)} - \delta, a^{(k)} + \delta) = \emptyset$ for any $k = 1, 2, \dots, N$. This means that $\mathrm{supp}(\rho_1)\bigcap Q = \emptyset$ which implies $\alpha_n = 0$. Thus, $\mathcal{N}_\infty$ is continuous. Assume now that $\mathrm{supp}(\rho_1)\bigcap Q \ne \emptyset$. Due to $\rho_1(x) > 0$ for all $x \in \mathrm{supp}(\rho_1)$ we have $\alpha_n = 0$ if and only if $a_n^{(k)} = a_0^{(k)}$, for all $k = 1, 2, \dots, N$. That is not necessarily the case, thus, $\mathcal{N}_\infty$ discontinuous on $B(U,\varepsilon)$. □

Remark 3.10. We notice here that the assumption $\mathrm{supp}(\rho_1)\bigcap(a^{(k)}-\delta, a^{(k)}+\delta) = \emptyset$ for all $k = 1, 2, \dots, N$ is not interesting here, as it breaks properties of the model. Thus, further we exclude these types of ρ_1 from consideration.

Theorem 3.11. *Let θ be fixed, $U(x) \in W^{1,q}(\mathbb{R},\mu)$, $1 \le q < \infty$, satisfies the θ-condition. There exist no such $\varepsilon > 0$ that $\mathcal{N}_\infty : B(U,\varepsilon) \subset W^{1,q}(\mathbb{R},\mu) \to L^p(\mathbb{R},\mu)$, $1 \le p \le \infty$ is continuous operator.*

Proof. In Lemma 3.7 it has been shown that for any $\varepsilon > 0$ there exists some n_ε that u_n, given by (3.7), for all $n \ge n_\varepsilon$ belongs to the ball $B(U,\varepsilon)$. We fix $n^* = n_{\varepsilon/2}$ and denote $u^* = u_{n_{\varepsilon/2}}$. Then we consider the sequence $u_k(x)$ given as

$$u_k(x) = \begin{cases} \theta - \dfrac{1}{\gamma k}\sin(\pi n^*(x - a_1)), & x \in [a^{(1)} - 1/n^*, a^{(1)} + 1/n^*] \\ \theta + \dfrac{1}{\gamma k}\sin(\pi n^*(x - a_2)), & x \in [a^{(2)} - 1/n^*, a^{(2)} + 1/n^*] \\ u^*(x), & \text{otherwise} \end{cases}$$

where γ is a positive constant. We have plotted the graphs of $U(x)$ (red solid line), $u^*(x)$ (blue solid line), and $u_k(x)$ (blue dashed line) in Fig. 2 together with the constant θ. First, we prove that $u_k \to u^*$ and show that there exists $\gamma = \gamma^*$ such that $u^k \in B(U,\varepsilon)$. Next we prove that $\|\mathcal{N}_\infty u_k - \mathcal{N}_\infty u^*\|_{L^p}$ does not converges to zero.

Without loss of generality we assume that one of $a^{(j)}, j = 1, 2$ is equal to zero. We calculate the norm of $|u_k(x) - u^*(x)|$ using (2.3) and derive the following inequality

$$\begin{aligned}\|u_k - u^*\|_{W^{1,q}} &= \|\tilde{u}_k - \tilde{u^*}\|_{L^q} = \|(u_k' - (u^*)')\rho^{-1}(x)\|_{L^q} \\ &= \left(\sum_{j=1}^{2}\int_{a^{(j)}-1/n^*}^{a^{(j)}+1/n^*}\left|\frac{1}{\pi n^*\gamma k}\cos(\pi n^*(x-a^{(j)}))\right|^q dx\right)^{1/q} \\ &\le \frac{1}{\pi n^*\gamma k}\left(\sum_{j=1}^{2}\left|(a^{(j)}+1/n^*)-(a^{(j)}-1/n^*)\right|\right)^{1/q} = \frac{1}{\pi n^*\gamma k}\left(\frac{2}{n^*}\right)^{1/q}.\end{aligned}$$

From this inequality we see that $\|u_k - u^*\|_{W^{1,q}} \to 0$ as $k \to \infty$. Moreover, as we assign $\gamma^* = 2/(\varepsilon\pi n^*)$ we get $\|u_k - u^*\|_{W^{1,q}} \le \varepsilon/2$. We have

$$\|u_k - U\|_{W^{1,q}} \le \|u_k - u^*\|_{W^{1,q}} + \|u^* - U\|_{W^{1,q}} \le \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

for $k = 1, 2, \ldots$, i.e., $u_k \in B(U, \varepsilon)$ for all $k \in \mathbb{N}$.

Using the definition of $\mathcal{N}_\infty$ we have

$$|(\mathcal{N}_\infty u_k)(x) - (\mathcal{N}u^*)(x)| = \begin{cases} \dfrac{\rho_1(x)}{\rho(x)}, & x \in \bigcup_{j=1}^{2}[a^{(j)} - 1/n^*, a^{(j)} + /n^*] \\ 0, & \text{otherwise.}\end{cases}$$

Due to $\rho_1(x)/\rho(x) > 0$ we have $\|(\mathcal{N}_\infty u_k)(x) - (\mathcal{N}u^*)(x)\|_{L^p} = \delta > 0$ independently of k. Hence, we conclude that $(\mathcal{N}_\infty u_k)(x)$ does not converges to $(\mathcal{N}u^*)(x)$. It completes our proof. □

Now we consider the Nemytskii operator $\mathcal{N}_\beta$ when β is not fixed, but belongs to $(0, \infty]$. Then, $\mathcal{N}_\beta$ is a map $(0, \infty] \times B(U, \varepsilon) \subset W^{1,\infty}(\mathbb{R}, \mu) \to W^{1,\infty}(\mathbb{R}, \mu)$. We have the following lemma.

Lemma 3.12. *Let $\theta > 0$ be fixed, $U \in W^{1,\infty}(\mathbb{R}, \mu)$ satisfy the θ-condition, $\rho_1/\rho \in L^1(\mathbb{R}, \mu)$ and $B(U, \varepsilon) = \{u : \|u-U\|_{W^{1,\infty}} < \varepsilon\}$. The operator $\mathcal{N}_\beta : (0, \infty] \times B(U, \varepsilon) \to L^1(\mathbb{R}, \mu)$ is continuous at $\beta_0 \in (0, \infty]$ uniformly for all $u \in B(U, \varepsilon)$.*

Proof. By Lemma 3.3 and Theorem 3.8 $\mathcal{N}_\beta$ is a map from $(0, \infty] \times B(U, \varepsilon)$ to $L^1(\mathbb{R}, \mu)$. Using properties of ρ_1 we have

$$\|\mathcal{N}_\beta - \mathcal{N}_{\beta_0}\|_{L^1} \le C_{\rho_1}\int_{\mathbb{R}}|S(\beta, u(x)) - S(\infty, u(x))|dx.$$

For $\beta_0 < \infty$ from uniform convergence $S(\beta, z) \to S(\beta_0, z)$ we get pointwise convergence $S(\beta, u(x)) \to S(\beta_0, u(x))$. Boundedness of S allows us to applying the Lebesgue dominated convergence theorem. Thus, we get

$$\|\mathcal{N}_\beta - \mathcal{N}_{\beta_0}\|_{L^1} \to 0, \quad \forall u \in B(U, \varepsilon). \tag{3.8}$$

When $\beta_0 = \infty$ the proof is not so straightforward. By Lemma 3.6 there is $\varepsilon > 0$ such that for given U there are defined $b^{(k,i)}$ $(k = 1, \ldots, N, i = 1, 2)$ such that for any $u \in B(U, \varepsilon)$ the conditions (A1)–(A3) are satisfied. We have

$$\begin{aligned}\|\mathcal{N}_\beta - \mathcal{N}_\infty\|_{L^1} &\le C_{\rho_1}\int_{\mathbb{R}}|S(\beta, u(x)) - S(\infty, u(x))|dx \\ &= \sum_{k=0}^{N}\int_{b^{(k,1)}}^{b^{(k,2)}}|S(\beta, u(x)) - S(\infty, u(x))|dx = C_{\rho_1}(\Sigma_1 + \Sigma_2),\end{aligned} \tag{3.9}$$

where

$$\Sigma_1 = \sum_{k=0}^{N}\int_{b^{(k,1)}}^{b^{(k,2)}}|S(\beta, u(x)) - S(\infty, u(x))|dx$$

and

$$\Sigma_2 = \sum_{k=0}^{N-1}\int_{b^{(k,2)}}^{b^{(k+1,1)}}|S(\beta, u(x)) - S(\infty, u(x))|dx.$$

Notice, that Σ_1 contains only the integrals over such intervals that $S(\infty, u(x))$ does not have singularities; see for example Fig. 1.

Let us consider first Σ_1 and then Σ_2. Using (A1)–(A2) we have

$$\begin{aligned}\Sigma_1 &= \sum_{m=0}^{N/2} \int_{b^{(2m,1)}}^{b^{(2m,2)}} |S(\beta, u(x)) - S(\infty, u(x))| dx + \sum_{m=0}^{N/2-1} \int_{b^{2m+1,1}}^{b^{(2m+1,2)}} |S(\beta, u(x)) - S(\infty, u(x))| dx \\ &\le \sum_{m=0}^{N/2} \int_{b^{(2m,1)}}^{b^{(2m,2)}} S(\beta, \theta - \varepsilon) dx + \sum_{m=0}^{N/2-1} \int_{b^{(2m+1,1)}}^{b^{(2m+1,2)}} (1 - S(\beta, \theta + \varepsilon)) dx \\ &\le S(\beta, \theta - \varepsilon) \sum_{m=0}^{N/2} (b^{(2m,2)} - b^{(2m,1)}) + (1 - S(\beta, \theta + \varepsilon)) \sum_{m=0}^{N/2-1} (b^{(2m+1,2)} - b^{(2m+1,1)}).\end{aligned}$$

Using the property (iii) of $S(\beta, x)$ we have $\Sigma_1 \to 0$ as $\beta \to \infty$.

Consider now the second term in (3.9). On each interval $[b^{(k,2)}, b^{(k+1,1)}]$ the function $u(x)$ is monotone. Using the condition (A3) we get

$$\Sigma_2 \le \int_{U_{\min}}^{U_{\max}} |S(\beta, y) - S(\infty, y)| \frac{dy}{|u'(x)|} \le \frac{1}{M\varepsilon} \int_{U_{\min}}^{U_{\max}} |S(\beta, y) - S(\infty, y)| dy,$$

where $M = \sup_{x\in\mathbb{R}} \rho(x)$, $U_{\min} = \inf_{x\in\mathbb{R}} U(x)$, and $U_{\max} = \sup_{x\in\mathbb{R}} U(x)$. Since $S(\beta, x) \to S(\infty, x)$ as $\beta \to \infty$ almost everywhere on $\mathbb{R}$ (see property (iii) of $S(\beta, x)$) and $|S(\beta, x)| \le 1$ for all $\beta \in (0, \infty]$, then by the Lebesgue dominance convergence theorem, the integral in the last inequality converges to 0 as $\beta \to \infty$. Combining the results for Σ_1 and Σ_2 we complete the proof. □

Theorem 3.13. *Let $\theta > 0$ be fixed and let $U \in W^{1,\infty}(\mathbb{R}, \mu)$ satisfy the θ-condition. We define a set Q as $Q = \operatorname{supp}(\rho)$ and a ball $B(U, \varepsilon) = \{u : \|u - U\|_{W^{1,\infty}} < \varepsilon\}$. If the following condition is fulfilled*

$$\sup_{x\in Q} \left(\sup_{y\in Q} \left| \frac{\omega'_x(x, y)}{\rho(x)\rho_1(y)} \right| \right) < \infty, \tag{3.10}$$

then there exists $\varepsilon > 0$ that the Hammerstein operator $\mathcal{H}_\beta : (0, \infty] \times B(U, \varepsilon)$ is continuous at $\beta_0 \in (0, \infty]$ uniformly for all $u \in B(U, \varepsilon)$.

Proof. We consider the norm of $|(\mathcal{H}_\beta u)(x) - (\mathcal{H}_{\beta_0} u)(x)|$ in $W^{1,\infty}(\mathbb{R}, \mu)$ given by (2.3). Here u is an arbitrary function from the ball $B(U, \varepsilon)$. We have

$$\|\mathcal{H}_\beta u - \mathcal{H}_{\beta_0} u\|_{W^{1,\infty}} = |(\mathcal{H}_\beta u)(0) - (\mathcal{H}_{\beta_0} u)(0)| + \|(\tilde{\Omega} \circ \mathcal{N}_\beta) u - (\tilde{\Omega} \circ \mathcal{N}_{\beta_0}) u\|_{L^\infty}.$$

Here $\tilde{\Omega}$ is given as in (3.5). We consider the first and the second term separately.

$$\begin{aligned}|(\mathcal{H}_\beta u)(0) - (\mathcal{H}_{\beta_0} u)(0)| &= \left| \int_{\mathbb{R}} \frac{\omega(0, y)}{\rho(y)} (S(\beta, u(y)) - S(\beta_0, u(y))) d\mu(y) \right| \\ &\le \left| \int_{\mathbb{R}} \left| \frac{\omega(0, y)}{\rho_1(y)} \right| \left| \frac{\rho_1(y)}{\rho(y)} \right| S(\beta, u(y)) - S(\beta_0, u(y)) d\mu(y) \right| \\ &\le C_\omega \|\mathcal{N}_\beta u - \mathcal{N}_{\beta_0} u\|_{L^1}.\end{aligned}$$

By Lemma 3.12$|(\mathcal{H}_\beta u)(0) - (\mathcal{H}_{\beta_0} u)(0)|$ uniformly converges to zero.

Under the conditions of the theorem $\tilde{\Omega} : L^1(\mathbb{R}, \mu) \to L^\infty(\mathbb{R}, \mu)$ is bounded. Indeed, using the Hölder inequality we get

$$\|\tilde{\Omega} v\|_{L^\infty} \le C_{\tilde{\Omega}} \|u\|_{L^1}, \quad C_{\tilde{\Omega}} = \sup_{x\in Q} \left(\sup_{y\in Q} \left| \frac{\omega'_x(x, y)}{\rho(x)\rho_1(y)} \right| \right) < \infty.$$

Then it is easy to see that

$$\|(\tilde{\Omega} \circ \mathcal{N}_\beta) u - (\tilde{\Omega} \circ \mathcal{N}_{\beta_0}) u\|_{L^\infty} \le C_{\tilde{\Omega}} \|\mathcal{N}_\beta u - \mathcal{N}_{\beta_0} u\|_{L^1}.$$

Applying Lemma 3.12 we complete our proof. □

We formulate our main theorem.

Theorem 3.14 (*Continuous dependence*)**.** *Let $\theta > 0$ be fixed and $U \in W^{1,\infty}(\mathbb{R}, \mu)$ satisfy the θ-condition. Assume that $1 \le p < \infty$ and $\rho_1/\rho \in L^p(\mathbb{R}, \mu)$, the operator Ω in (3.3) is a compact operator from $L^p(\mathbb{R}, \mu)$ to $W^{1,\infty}(\mathbb{R}, \mu)$, and (3.10) is satisfied. If there exist solutions of the equation $\mathcal{H}_\beta u_\beta = u_\beta$ which belong to $B(U, \varepsilon) = \{u : \|u - U\|_{W^{1,\infty}} < \varepsilon\}$ for any $\beta \in [C_\beta, \infty]$, $C_\beta > 0$, then there exist a solution of $\mathcal{H}_\infty u = u$ and it is a limit point of the net $\{u_\beta\}$. Moreover, if the solution of $\mathcal{H}_\infty u = u$, say u^*, is unique then $\{u_\beta\} \to u^*$.*

Proof. We base our proof on Theorem 2.3. We choose D (in Theorem 2.3) to be a closure of $B(U,\varepsilon)=\{u:\|u-U\|_{W^{1,\infty}}<\varepsilon\}$, and $\lambda=1/\beta$ and, thus, $\Lambda=[0,1/C_\beta]$. The operator $\mathcal{H}_\beta\equiv\mathcal{H}_{1/\lambda}:\Lambda\times\overline{B(U,\varepsilon)}\to W^{1,\infty}(\mathbb{R},\mu)$ is continuous as a superposition of continuous operators, Ω and $N_{1/\lambda}$, for each $\lambda\in\Lambda$. The operator $\Omega:L^p(\mathbb{R},\mu)\to W^{1,\infty}(\mathbb{R},\mu)$ is continuous by the conditions of the theorem, and continuity of $N_{1/\lambda}:\overline{B(U,\varepsilon)}\to L^p(\mathbb{R},\mu)$ follows from Lemma 3.3 (for $\lambda>0$) and Theorem 3.8 (for $\lambda=0$). Moreover, from Theorem 3.13 we conclude that $\mathcal{H}_{1/\lambda}:\Lambda\times\overline{B(U,\varepsilon)}\to W^{1,\infty}(\mathbb{R},\mu)$ is continuous with respect to both variables.

The observation that the operators $\mathcal{H}_{1/\lambda}$ are collectively compact as a superposition of a compact operator Ω and collectively bounded operators $\mathcal{N}_{1/\lambda}$ completes the proof. □

It is usually much easier to study the existence of solutions (which satisfy the θ-condition) of the fixed point problem $\mathcal{H}_\infty u=u$ than $\mathcal{H}_\beta u=u$, $\beta<\infty$. The next theorem allows us to prove the existence of the fixed points of $\mathcal{H}_\beta$ using some knowledge about the fixed point of the limit problem. We give more details on the existence of fixed points of $\mathcal{H}_\infty$ in the next section.

Theorem 3.15 (*Existence*)**.** *Let the conditions of Theorem 3.14 be satisfied, i.e., $\theta>0$ be fixed, $U\in W^{1,\infty}(\mathbb{R},\mu)$ satisfy the θ-condition, $\rho_1/\rho\in L^p(\mathbb{R},\mu)$, $1\le p<\infty$, the operator $\Omega:L^p(\mathbb{R},\mu)\to W^{1,\infty}(\mathbb{R},\mu)$ is compact, and (3.10) is fulfilled. Let B_0 define an open subset of $B(U,\varepsilon)=\{u:\|u-U\|_{W^{1,\infty}}<\varepsilon\}$. If U is a unique fixed point of the operator $\mathcal{H}_\infty$ on $\overline{B_0}$ such that $\deg(\mathcal{H}_\infty-I,B_0,0)\neq 0$, then $\mathcal{H}_\beta$ possesses a fixed point $u_\beta\in B_0$ for any $\beta\gg 1$.*

Proof. We define $h_t(u)=(\mathcal{H}_{k/t}-I)(u)$, where I is an identity operator, $t\in[0,1]$, and $k\ge 1$. We show that h_t is a homotopy, i.e., (a) continuous with respect to t and u for all $t\in[0,1]$ and $u\in\overline{B_0}$, (b) $h_t(u)\neq 0$ for any $t\in[0,1]$ and $u\in\partial B_0$. The property (a) is satisfied. Indeed $\mathcal{H}_{k/t}$ is continuous (for details see the proof of Theorem 3.14) and thus, $\mathcal{H}_{k/t}-I$ is continuous as well.

In the proof of (b) we first observe that $\mathcal{H}_\infty u\neq u$ for all $u\in\partial B_0$ since U is a unique solution on $\bar{B}_0$ and $U\in B_0$. Assume that $\mathcal{H}_\beta u\neq u$ for all $u\in\partial B_0$ does not hold true, i.e., there exist $\{u_n\}\in\partial B_0$ such that $\mathcal{H}_{\beta_n}u_n=u_n$. From Theorem 3.14 it follows that $u_n\to u_0\in\partial B_0$ where $u_0=\mathcal{H}_\infty u_0$. This contradiction competes the proof of (b). It is easy to see that h_t satisfies the conditions of Lemma 2.5 and thus, $\deg(h_t,B_0,0)=\deg(\mathcal{H}_\infty-I,B_0,0)\neq 0$ for any $t\in[0,1]$. This implies the existence of solutions of $\mathcal{H}_{k/t}u=u$ belonging to $B_0\in B(U,\varepsilon)$. □

4. Bumps in neural field model

Definition 4.1. Let $\theta>0$ be fixed, and U be a stationary solution of (1.1) where $P(u)=S(\infty,u)$. A set $R[U]=\{x:U(x)\ge\theta\}$ is called an excited region of U, [4].

Definition 4.2. Let $\theta>0$ be fixed, and U be a stationary solution of (1.1) where $P(u)=S(\infty,u)$. If the excited region of U is such that $R[U]=\bigcup_{k=1}^{N}[a^{(2k-1)},a^{(2k)}]$ and $U'(a^{(k)})\neq 0$, $k=1,\dots,2N$ then $U(x)$ is called a bump, or more specificity, N-bump.

The existence of 1-bump solutions was studied in [4]. Later, 2-bumps and multibumps were considered in [23,24]. In all these cases the connectivity function was assumed to be translation homogeneous, i.e., $\omega(x,y)=\varpi(x-y)$ where $\varpi(z)$ is an even function. These type of solutions were linked to the mechanisms of the working memory, representations in the head-direction system, and feature selectivity in the visual cortex; see [5] and references therein.

Remark 4.3. Although the condition $U'(a^{(k)})\neq 0$, $k=1,\dots,2N$ was not postulated in [4], it was used for studying stability of these bumps.

Theorem 4.4. *A bump solution of (1.1) with $P(u)=S(\infty,u)$ belongs to $W^{1,\infty}(\mathbb{R},\mu)$ and satisfy the θ-condition.*

Proof. By Definition 4.2 and (3.1) a bump is given as

$$U(x)=\int_{R[U]}\omega(x,y)dy. \tag{4.1}$$

We use the norm (2.3) and get the following estimate

$$\begin{aligned}\|U\|_{W^{1,\infty}}&=\left|\int_{R[U]}\omega(0,y)dy\right|+\sup_{x\in\mathbb{R}}\left|\rho(x)\int_{R[U]}\omega'_x(x,y)dy\right|\\&\le\int_{R[U]}|\omega(0,y)|dy+M\sup_{x\in\mathbb{R}}\int_{R[U]}|\omega'_x(x,y)|dy,\quad M=\sup_{x\in\mathbb{R}}\rho(x).\end{aligned}$$

Applying the property (i) of ω to the first term of the sum, and the property (ii) to the second, we get $\|U\|_{W^{1,\infty}}<\infty$, i.e., $u\in W^{1,\infty}(\mathbb{R},\mu)$.

Next, we show that U satisfies the θ-condition. By the definition of bumps the first condition of Definition 3.4 is fulfilled. To show that the second one is fulfilled as well we consider the limit

$$\lim_{|x|\to\infty} U(x) = \lim_{|x|\to\infty} \int_{R[U]} w(x,y)dy.$$

The properties (iii) and (iv) of ω allows us to apply the Lebesgue dominated convergence theorem, i.e., we get $\lim_{|x|\to\infty} U(x) = 0$. This observations complete the proof. □

Below we give two examples where quite simple requirements on $\omega(x, y)$ allow us to choose ρ, ρ_1 in a such way that all conditions of Theorem 3.14 are satisfied.

Example 4.5. For any ω such that

$$\left|\frac{\partial^m \omega(x,y)}{\partial x^m}\right| \le Ce^{-a|x|}e^{-b|y|}, \quad m = 1, 2,\ C, a, b > 0, \tag{4.2}$$

the conditions of Theorem 3.14 are satisfied.

Proof. We set

$$\rho_1(x) = e^{-\alpha|x|}, \qquad \rho(x) = \frac{\beta}{2}e^{-\beta|x|}, \tag{4.3}$$

for some positive α and β. To satisfy the first condition of Theorem 3.14 ($\rho_1/\rho \in L^p(\mathbb{R}, \mu)$, $1 \le p < \infty$), it is sufficient to fulfill the following inequality $p(\alpha - \beta) + \beta > 0$ or, equivalently,

$$\alpha > 0 \quad \text{if } p = 1, \quad \text{and} \quad \beta > \alpha p' \text{ if } p > 1.$$

Let us now focus on the second condition of Theorem 3.14 (Ω is compact). By Lemma 3.2 it is sufficient to prove compactness of $\tilde{\Omega}$ given by (3.5). We use Lemma 2.2. We denote the kernel of the operator $\tilde{\Omega}$ as $k(x, y)$. Using the estimates (4.2) and (4.3) we have

$$|k(x,y)| \le 2\frac{C}{\beta}e^{-(a-\alpha)|x|}e^{-(b-\beta)|y|}, \tag{4.4}$$

and

$$\begin{aligned} |k'_x(x,y)| &\le \frac{|\omega''_{xx}(x,y)| + |\rho'(x)/\rho(x)||\omega'_x(x,y)|}{\rho_1(y)\rho(x)} \\ &\le \frac{|\omega''_{xx}(x,y)| + \alpha|\omega'_x(x,y)|}{\rho_1(y)\rho(x)} \le 2C\frac{1+\alpha}{\alpha}e^{-(a-\alpha)|x|}e^{-(b-\beta)|y|}. \end{aligned}$$

Moreover, the requirement

$$a > \alpha, \quad b > \beta, \tag{4.5}$$

implies that both conditions of Lemma 2.2 are satisfied. While it is obvious that $k(x, \cdot) \in L^{p'}(\mathbb{R}, \mu)$ for almost all $x \in \mathbb{R}$, (2.6) is needed to be explained. We notice that

$$|k(x,y)| \le C_1e^{-c|x|}, \qquad |k'_x(x,y)| \le C_2e^{-c|x|}, \tag{4.6}$$

where $c = a - \alpha > 0$, and $C_1, C_2 > 0$. First, we assign A to be some constant larger than $c^{-1}\ln(2C_1/\varepsilon)$. Then, for $|x| > A$ we have $C_1e^{-c|x|} < \varepsilon/2$ and for any $x_1, x_2 :\ |x_1| > A, |x_2| > A$ we get

$$|k(x_1,y) - k(x_2,y)| \le C_1(e^{-c|x_1|} + e^{-c|x_2|}) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \tag{4.7}$$

Next, using the mean value theorem we have

$$\begin{aligned} |k(x_1,y) - k(x_2,y)| &\le |k'_x(\tilde{x},y)(x_2 - x_1)| \\ &\le C_2e^{-c|\tilde{x}|}|x_2 - x_1| \le C_2|x_2 - x_1|, \end{aligned} \tag{4.8}$$

where $\tilde{x} = \lambda x_1 + (1 - \lambda x_2)$, $\lambda \in [0, 1]$. We define some $\Delta :\ 0 < \Delta < \varepsilon/C_2$ and set

$$D_1 = (\infty, -A),\ D_2 = (-A, -A + \Delta), \qquad D_3 = (-A + \Delta, -A + 2\Delta), \ldots, D_n = (-A + n\Delta, +\infty),$$

where n is defined in a such way that $-A + n\Delta > A$, e.g., $n = [2A/\Delta] + 1$. Therefore, (2.6) is fulfilled for $j = 1, n$ due to (4.7), and for $j = 2, 3, \ldots, n - 1$ due to (4.8). Thus, under assumptions (4.5) the operator $\tilde{\Omega}$ maps $L^p(\mathbb{R}, \mu)$ to $L^\infty(\mathbb{R}, \mu)$ and is compact.

Combining all the restrictions on α, β we have

$$\begin{array}{ll} 0 < \alpha < a,\ 0 < \beta < b & \text{if } p = 1, \\ 0 < \alpha < a,\ \alpha p' < \beta < b & \text{if } p > 1. \end{array} \tag{4.9}$$

It is clear that for any given $a, b > 0$, and $1 \leq p < \infty$ it is always possible to choose α and β satisfying (4.9).

Finally, using (4.4) it is easy to see that (3.10) is valid. □

The case when $\omega(x, y) = \varpi(x - y)$ seems to be more complicated. The main difficulty here is that the kernel of the operator Ω becomes unbounded along the line $y = x$. We have not found a general approach how to deal with this problem. However, the theory developed in previous section works very well for the family of firing rate functions, $S(\beta, u)$, which possesses the following property:

$$S(\beta, u) = 0 \quad \text{for all } u \leq \theta - \tau, \text{ and } \beta > 0. \tag{4.10}$$

Example 4.6. Let ω be given as $\omega(x, y) = \varpi(x - y)$ with $\varpi(z)$ such that

$$\varpi^{(m)}(z) \leq Ce^{-c|z|}, \quad m = 1, 2,\ c > 0. \tag{4.11}$$

In addition to the properties of S, we assume (4.10). Then there exist such $\tau > 0$ that the conditions of Theorem 3.14 are satisfied.

Proof. Let $\theta > 0$ be fixed and $U \in W^{1,\infty}(\mathbb{R}, \mu)$ satisfy the θ-condition. Moreover, assume that $U(x) = \theta$ has N solutions. Then, by Lemma 3.6 there exist such $\varepsilon > 0$, and $b^{(0,2)}$, $b^{(N,1)}$ that for any $u \in B(U, \varepsilon) = \{u : \|u - U\|_{W^{1,\infty}} < \varepsilon\}$ the following inequality $U(x) < \theta - \varepsilon$ for all $x \in (-\infty, b^{(0,2)}) \cup (b^{(N,1)}, \infty)$ is valid.

We introduce a set $D = [b^{(0,2)}, b^{(N,1)}]$. Next, we set $\tau < \varepsilon$ and define

$$\rho(x) \equiv \rho_1(x) = \begin{cases} 1/(b^{(N,1)} - b^{(0,2)}), & x \in D, \\ 0, & x \notin D. \end{cases}$$

This definition of ρ implies that $L^s(\mathbb{R}, \mu) \equiv L^s(D)$ and $W^{1,s}(\mathbb{R}, \mu) \equiv W^{1,s}(D)$ for any $1 \leq s \leq \infty$. By Lemma 3.3 and Theorem 3.8 $\mathcal{N}$ is a continuous map from $B(U, \varepsilon)$ to $L^p(D)$. Using Lemmas 2.2 and 3.2 we next show that the operator Ω is compact operator from $L^p(D)$ to $W^{1,\infty}(D)$. The operator $\tilde{\Omega}$ is given by

$$(\tilde{\Omega}u)(x) = \int_D \varpi'(y - x)u(y)dy, \quad x \in D.$$

Due to the estimate (4.11) the first condition of Lemma 2.2 is satisfied. It remains to check the second condition of Lemma 2.2. Making use of the mean value theorem and (4.11) for $m = 2$, we get

$$\|\varpi'(x_1 - y) - \varpi'(x_2 - y)\|_{L^{p'}} \leq \|\varpi''(\tilde{x} - y)\|_{L^{p'}} |x_2 - x_1| \leq C|x_2 - x_1|.$$

Here we assume $\tilde{x} = \lambda x_1 + (1 - \lambda x_2)$, $\lambda \in [0, 1]$. Similarly to Example 4.5 we choose some $\Delta :\ \Delta < \varepsilon/C$ and set

$$D_1 = (b^{(0,2)}, b^{(0,2)} + \Delta), \quad D_2 = (b^{(0,2)} + \Delta, b^{(0,2)} + 2\Delta), \dots, D_n = (b^{(0,2)} + n\Delta, b^{(N,1)}),$$

where n is defined in a such way that $b^{(0,2)} + n\Delta < b^{(N,1)}$ and $b^{(0,2)} + (n + 1)\Delta \geq b^{(N,1)}$, e.g., $n = \left[(b^{(N,1)} - b^{(0,2)})/\Delta\right]$. Thus, by Lemma 2.2 $\tilde{\Omega} : L^p(\mathbb{R}, \mu) \to L^\infty(\mathbb{R}, \mu)$ is compact. This implies $\Omega : L^p(\mathbb{R}, \mu) \to W^\infty(\mathbb{R}, \mu)$ be a compact operator; see Lemma 3.2. Finally, we remark that (3.10) is fulfilled. Hence, all the conditions of Theorem 3.14 are verified.

In neural field theory one often assumes that $\omega(x, y)$ is given as a homogeneous and distant dependent function, i.e., $\omega(x, y) = \varpi(x - y)$, where ϖ is an even function. In this case any stationary solution of (1.1) is translation invariant.[1] A typical example of a homogeneous connectivity function ϖ is

$$\varpi(x) = M_1 e^{-m_1|x|} - M_2 e^{-m_2|x|}, \quad M_1 > M_2,\ m_1 > m_2. \tag{4.12}$$

This function is called a 'Mexican-hat' function and models a neural network with local excitation and distal inhibition. This function satisfies (4.11) and thus it is a particular case of Example 4.6. The existence of 1- and 2-bumps for the model (1.1) with $P(\cdot) = S(\infty, \cdot)$ and this type of connection was shown in [4] and [23], respectively.

Next, we formulate our second theorem which rigorously shows that the bumps solutions in the steep firing rate regime approach the bumps solutions of the stationary Wilson–Cowan model in the unit step function approximation of the firing rate function.

[1] i.e., if $U(x)$ is a solution so is $U(x + c)$ for any $c \in \mathbb{R}$.

Theorem 4.7. *Assume that $\omega(x, y) = \varpi(x-y)$ with ϖ be an even function satisfying (4.11). Let $\theta > 0$ be fixed and U be a symmetric 1-bump solution of (1.1) where $P(\cdot) = S(\infty, \cdot)$. Moreover, we let $S(\beta, \cdot)$ satisfy the condition (4.10). Then, there exists $\varepsilon > 0$ such that for any $\tau < \varepsilon$ and all $\beta \gg 1$ the operator $\mathcal{H}_\beta$ has a fixed point $u_\beta \in B_e(U, \varepsilon) = \{u - \text{even function}: \|u-U\|_{W^{1,\infty}} < \varepsilon\}$. Moreover, u_β depends continuously on β, i.e., $u_\beta \to U$ as $\beta \to \infty$.*

The proof of the theorem involves some knowledge of degree theory and topological fixed point index theory. We do not recall any definitions and properties here but refer a reader to [18].

Proof. We notice that all conditions of Theorem 3.14 are fulfilled; see Example 4.6. This means that if $u = \mathcal{H}_\beta u$ possesses a solution $u_\beta \in B(U, \varepsilon)$ for all $C_\beta < \beta < \infty$ then $u_\beta \to U$. It remains to show that such u_β exist. For that we are going to use Theorem 3.15 where we choose $B_0 = B_e(U, \varepsilon)$. To be able to apply Theorem 3.15 we need to show that (a) U is a unique fixed point of $\mathcal{H}_\infty$ in $B_e(U, \varepsilon)$ and (b) $\deg(\mathcal{H}_\infty - I, B_e(U, \varepsilon, 0)) \neq 0$.

By Theorem 4.4, U satisfies the θ-condition and $U \in W^{1,\infty}(\mathbb{R}, \mu)$. By Lemma 3.6 there exists $\varepsilon > 0$ such that any $u \in B(U, \varepsilon) = \{u : \|u-U\|_{W^{1,\infty}} < \varepsilon\}$ satisfies the θ-condition and possesses exactly two intersections with the straight line θ. Obviously, the same properties are valid for $u \in B_e(U, \varepsilon)$, i.e., for any $u \in B_e(U, \varepsilon)$ there is $c_u > 0$ such that $u(\pm c_u) = \theta$, $u'(\pm c_u) \neq 0$. For $u = U$ we denote the intersections as $\pm a$.

We define an auxiliary function

$$W(x) = \int_0^x \varpi(y)dy.$$

Then, for any $u \in B_e(U, \varepsilon)$ there is defined $v(x) = (H_\infty u)(x) = W(x + c_u) - W(x - c_u)$. In particular, we have

$$U(x) = W(x + a) - W(x - a), \tag{4.13}$$

where

$$W(2a) = \theta, \qquad \varpi(2a) < 0. \tag{4.14}$$

Lemma 4.8. *A symmetric 1-bump, U, is a unique fixed point of $\mathcal{H}_\infty$ on $B_e(U, \varepsilon)$.*

Proof. Let us assume the contrary. Then there exist a sequence $\{u_n\} \in B_e(U, \varepsilon)$ such that $u_n \to U$ and $\mathcal{H}_\infty u_n = u_n$. Similarly to (4.13) and (4.14) we have

$$u_n(x) = W(x + a_n) - W(x - a_n) \tag{4.15}$$

with

$$W(2a_n) = \theta, \qquad \varpi(2a_n) < 0 \tag{4.16}$$

where we set $a_n = c_{u_n}$.

From Lemma 3.9 we have $a_n \to a$. The condition $\varpi(2a) < 0$ implies that any vicinity of a contains such a_n that $W(2a_n) \neq \theta$. This contradicts with (4.16) and thus, with u_n being a fixed point of $\mathcal{H}_\infty$. We conclude that U is an isolated fixed point of the operator $\mathcal{H}_\infty$ on $B_e(U, \varepsilon)$. Therefore, without loss of generality we assume that $B_e(U, \varepsilon)$ does not contain any other fixed points than U. We emphasize here, that U is not an isolated fixed point of $\mathcal{H}_\infty$ on $W^{1,\infty}(\mathbb{R}, \mu)$ due to the translation invariance of bumps in a homogeneous neural field. □

Due to Lemma 4.8 and definition of the topological fixed point index we have

$$\deg(\mathcal{H}_\infty - I, B_e(U, \varepsilon), 0) = \text{ind}(\mathcal{H}_\infty, B_e(U, \varepsilon)).$$

We notice that $\mathcal{H}_\infty$ maps $\overline{B_e(U, \varepsilon)}$ into a manifold $E_M \subset W^{1,\infty}(\mathbb{R}, \mu)$, where $E_M = \{v : v = W(\cdot + c) - W(\cdot - c),\ c \in [m, M]\}$. The interval $[m, M]$ is chosen in a such way that it contains c_u for all $u \in \overline{B_e(U, \varepsilon)}$. By Lemma 3.6 this is possible to achieve if one chooses, for example, $m = 0$, and $M = b^{(N,1)}$.

We define $\phi : [m, M] \to E_M$ where $\phi(c) = v(x) \equiv W(x + c) - W(x - c)$, $x \in \mathbb{R}$. Next, we show that ϕ is a homeomorphism.

Lemma 4.9. *The map ϕ is a homeomorphism from $[m, M]$ to E_M, and E_M is ANR.*[2]

Proof. First we show that ϕ is bijection. It is a surjection since E_M is defined as an image of $[m, M]$. To prove that ϕ is injection we assume the contrary: Let $c_1, c_2 \in [m, M]$ and $c_1 \neq c_2$ imply $v_1 = v_2$. From $v_1 = v_2$ and $v_1, v_2 \in W^{1,\infty}(\mathbb{R}, \mu)$ it follows that $|v_1(x) - v_2(x)| = 0$ for almost all $x \in \mathbb{R}$. Applying the mean value theorem we get

$$\begin{aligned} |v_1(x) - v_2(x)| &= |W(x + c_1) - W(x - c_1) - W(x + c_2) + W(x - c_2)| \\ &= |\varpi(x + \xi) + \varpi(x - \eta)||c_1 - c_2| = 0, \text{ a.e. on } \mathbb{R} \end{aligned} \tag{4.17}$$

[2] Absolute Neighborhood Retract, see [18].

where $\xi, \eta \in [c_1, c_2]$. As $c_1 \neq c_2$ we have

$$\varpi(x+\xi) = -\varpi(x-\eta) \text{ a.e. on } \mathbb{R},$$

or, that is equivalent,

$$\varpi(x+2(\xi+\eta)) = \varpi(x) \text{ a.e. on } \mathbb{R}.$$

The last equality contradicts with the property (iv) of ω. Thus, ϕ is a bijective map. Next, we observe that ϕ is differentiable for all $c \in [m, M]$ and $\phi'(c) \neq 0$. Indeed, as we assume the contrary we get

$$\varpi(x+c) = -\varpi(x-c), \text{ a.e. on } \mathbb{R}$$

which implies $4c$ periodicity of ϖ. This contradicts with the property (iv) of ω. Hence, we conclude that ϕ defines a homeomorphism on $[m, M]$. Moreover, since $[m, M]$ is a closed convex subset on $\mathbb{R}$ then it is ANR. By properties of homeomorphism $\phi([m, M]) = E_M$ is ANR too. □

Let $\mathcal{H}'_\infty$ be the excision of $\mathcal{H}_\infty$ on $E_M \cap \overline{B_e(U,\varepsilon)}$, i.e.,

$$\mathcal{H}'_\infty = \mathcal{H}_\infty|_{E_M \cap \overline{B_e(U,\varepsilon)}} : E_M \cap \overline{B_e(U,\varepsilon)} \to E_M. \tag{4.18}$$

The fixed point U belongs to $E_M \cap \overline{B_e(U,\varepsilon)}$ and thus, by the property of the topological fixed point index [18] $\mathcal{H}'_\infty$ is *admissible*[3] compact map and

$$\operatorname{ind}(\mathcal{H}_\infty, B_e(U,\varepsilon)) = \operatorname{ind}(\mathcal{H}'_\infty, E_M \cap B_e(U,\varepsilon)).$$

Next, we apply the topological invariance property of the index and get

$$\operatorname{ind}(\mathcal{H}'_\infty, E_M \cap B_e(U,\varepsilon)) = \operatorname{ind}(\phi^{-1} \circ \mathcal{H}'_\infty \circ \phi, \mathcal{D})$$

where $\mathcal{D}$ denotes the following set $\phi^{-1}(\mathcal{H}_\infty(E_M \cap B_e(U,\varepsilon)))$.

We prove the following lemma which enables us to compute $\operatorname{ind}(\phi^{-1} \circ \mathcal{H}'_\infty \circ \phi, \mathcal{D})$.

Lemma 4.10. *There exist such $\delta > 0$ that $\mathcal{D} = \phi^{-1}(\mathcal{H}_\infty(E_M \cap \overline{B_e(U,\varepsilon)})) \supset [a-\delta, a+\delta]$.*

Proof. The map $\bar{c} : u \mapsto c_u$ is defined for all $u \in \overline{B_e(U,\varepsilon)}$. Let $v(x) = W(x+c) - W(x-c)$, $c \in [m, M]$. Then using the norm (2.3) in $W^{1,\infty}(\mathbb{R}, \mu)$, the equality (4.13), and the mean value theorem we have

$$\|U - v\|_{W^{1,\infty}} = (\|\varpi(\xi_1) + \varpi(\eta_1)\|_{L^\infty} + \|\varpi(x+\xi_2) + \varpi(x-\eta_2)\|_{L^\infty})\, |c-a|,$$

$\xi_i, \eta_i \in [c, a]$, $i = 1, 2$. Thus, using (4.11) we get

$$\|U - v\|_{W^{1,\infty}} \le 4C|c-a| < \varepsilon,$$

for all $c \in [a-\delta, a+\delta]$, where $\delta < \varepsilon/4C$. From this observation we conclude that

$$\bar{c}(B(U,\varepsilon) \cap E_M) \supset [a-\delta, a+\delta]$$

which implies

$$\mathcal{H}_\infty(\overline{B_e(U,\varepsilon)} \cap E_M) \supset E_\delta = \{v : v = W(\cdot + c) - W(\cdot - c), c \in [a-\delta, a+\delta]\}.$$

Furthermore, it follows that

$$\phi^{-1}(\mathcal{H}_\infty(B_e(U,\varepsilon) \cap E_M)) \supset \phi^{-1}(E_\delta) = [a-\delta, a+\delta]. \quad \square$$

Finally, we have all the ingredients to calculate $\operatorname{ind}(\phi^{-1} \circ \mathcal{H}'_\infty \circ \phi, \mathcal{D})$. We define the finite dimension operator $T = \phi^{-1} \circ \mathcal{H}'_\infty \circ \phi$ which, as we have shown above, maps $[a-\delta, a+\delta] \to [m, M]$. It is easy to check that a is a fixed point of T, i.e., $T(a) = a$. Moreover a is an isolated fixed point of T. The latter statement follows from U being the isolated fixed point of $\mathcal{H}_\infty$ and topological invariance property of the index. The topological index of a finite dimensional map can be calculated as

$$\operatorname{ind}(T, \mathcal{D}) = \operatorname{sgn}(T'(a) - 1);$$

see [25].

The following equality holds true for all $c \in [a-\delta, a+\delta]$

$$W(T(c) + c) - W(T(c) - c) = \theta.$$

[3] A continuous map $g : B \to \mathcal{B}$ is called admissible provided B is an open subset of $\mathcal{B}$ and the fixed point set of g is compact; see [18].

Using the implicit function theorem and the chain rule for differentiation we find

$$T'(a) = \frac{\varpi(0) + \varpi(2a)}{\varpi(0) + \varpi(2a)}.$$

Thus, we have

$$\deg(\mathcal{H}_\infty, B_e(U, \varepsilon), 0) = \operatorname{ind}(T, \mathcal{D}) = \operatorname{sgn}(\varpi(2a)) = 1.$$

Combining all the results, we get that there exists $C_\beta \gg 1$ that $u = \mathcal{H}_\beta u$ possesses a solution $u_\beta \in B_e(U, \varepsilon)$ for all $\beta > C_\beta$ and $u_\beta \to U$. We also notice here that u_β is a symmetric function which satisfy the θ-condition and has two intersection points with straight line θ. □

5. Conclusions and outlook

In the present paper we have studied the properties of the one-parameter family of Hammerstein operators $\mathcal{H}_\beta$, $0 < \beta \leq \infty$ given by (3.2). Fixed points of an operator belonging to this family are stationary solutions of (1.1). For functions in $W^{1,q}(\mathbb{R}, \mu)$, $1 \leq q \leq \infty$ we have introduced the definition of the θ-condition, which means that we consider functions with a finite number of intersection points with the line $u = \theta$. We have shown that the continuous dependence theorem (Theorem 3.14) holds in a vicinity of a function $U \in W^{1,\infty}(\mathbb{R}, \mu)$ satisfying the θ-condition, while for the case $U \in W^{1,q}(\mathbb{R}, \mu)$, $1 \leq q < \infty$ the conditions of the theorem are not satisfied. Next, with Theorem 3.15 we show that if $\mathcal{H}_\infty$ possess an unique fixed point with some additional assumptions, then the solutions of the fixed point problem $\mathcal{H}_\beta u = u$ exist for $\beta > C_\beta$. This theorem allows us to prove the existence of multibump solutions of (1.1) with sigmoidal firing rate functions.

We believe that these results can be very useful in neural-field theory. We have given two examples of restrictions on the connectivity functions (one with inhomogeneous connectivity and the second one for homogeneous connectivity) where all the conditions of the continuous dependence theorem are satisfied. Moreover, for a homogeneous type of connectivity we have proved the existence of 1-bump solutions for (1.1) with steep gradient continuous firing rate function, $S(\beta, \cdot)$, satisfying the condition (4.10). Although the latter condition imposes restrictions on the choice of $S(\beta, \cdot)$, we would like to emphasize that this result is more general than one obtained in [14,15]. Here we would like to point out that the results of this paper can be useful for studying not only continuous dependence and existence of bumps but also stability of these solutions. The methods for studying stability of bumps, Evans function technique and Amari approach, assume that small perturbation of a bump solution possess the same number of intersection with a straight line θ as a bump itself. As we have shown, this is the case only if we work in the Sobolev space $W^{1,\infty}(\mathbb{R}, \mu)$. However, if one studies stability of bump in $W^{1,q}(\mathbb{R}, \mu)$, $1 \leq q < \infty$, then any vicinity of a bump contains functions which do not satisfy the θ-condition, and thus, these stability approaches do not work.

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References

[1] H.R. Wilson, J.D. Cowan, Excitatory and inhibitory interactions in localized populations of model neurons, Biophysical Journal 12 (1972) 1–24.
[2] H.R. Wilson, J.D. Cowan, A mathematical theory of the functional dynamics of cortical and thalamic nervous tissue, Kybernetik 13 (1973) 55–80.
[3] S. Amari, Homogeneous nets of neuron-like elements, Biol. Cybernet. 17 (1975) 211–220.
[4] S. Amari, Dynamics of pattern formation in literal-inhibition type neural fields, Biol. Cybernet. 27 (1977) 77–87.
[5] S. Coombes, Waves, bumps, and patterns in neural field theories, Biol. Cybernet. 93 (2005).
[6] J.K. Hale, S.M.V. Lunel, Introduction to Functional Differentail Equations, Springer Verlag, 1993, p. 447.
[7] E. Zeidler, Nonlinear Functional Analysis, vol.1: Fixed-Point Theorems, Springer, 1986, p. 909.
[8] O. Faugeras, F. Grimbert, J.-J. Slotine, Absolute stability and complete synchronization in a class of neural field models, SIAM J. Appl. Math. 63 (2008) 205–250.
[9] G.B. Ermentrout, J.B. McLeod, Existence and uniqueness of travelling waves for a neural network in: Proceedings of the Royal Society of Edinburgh 123A, 1993, pp. 461–478.
[10] G. Faye, O. Faugeras, Some theoretical and numerical results for delayed neural field equations, Physica D 239 (2010) 561–579.
[11] R. Potthast, P. Beim Graben, Existence and properties of solutions for neural field equations, Math. Methods Appl. Sci. 33 (8) (2010) 935–949.
[12] S. Coombes, M.R. Owen, Evans functios for integral field equations with Heavisite firing rate function, SIAM Journal on Applied Dynamical Systems 34 (2004) 574–600.
[13] J. Wyller, P. Blomquist, G.T. Einevoll, Turing instability and pattern formation in a two-population neuronal network model, Physica D 225 (2007) 7593.

[14] K. Kishimoto, S. Amari, Existence and stability of local excitations in homogeneous neural fields, J. Math. Biol. 7 (1979) 303–318.
[15] A. Oleynik, A. Ponosov, J. Wyller, Iterative schemes for bump solutions in a neural field model, In Anna Oleynik, Ph.D. Thesis: Mathematical aspects of localized activity in neural models, ISBN 978-82-575-1000-8, ISSN 1503-1667, 2011, Aas, Norway, pp. 15–32.
[16] V. Kostrykin, A. Oleynik, On the existence of unstable bumps in neural networks, 2011, Preprint arXiv:1112.2941 [math.DS].
[17] S. Coombes, H. Schmidt, Neural fields with sigmoidal firing rates: Approximate solutions, Discrete Contin. Dyn. Syst. 28 (2010) 1369–1379.
[18] A. Granas, The Leray–Schuder index and the fixed point theorey for arbitrary ANRs, Bull. Soc. Math. France 100 (1972) 209–228.
[19] M.A. Krasnosel'skii, A.I. Koshelev, S.G. Mikhlin, L.S. Rakovshchik, V.Ya. Stet'senko, P.P. Zabreiko, Integral Equations, Noordhoff International Publishing, Leyden, 1975, p. 443.
[20] V. Hutson, J.S. Pym, M.J. Cloud, Applications of Functional Analysis and Operator Theory, 2nd Edition, vol. 200, Elsevier, 2005.
[21] N. Azbelev, V. Maksimov, L. Rakhmatullina, Introduction to the Theory of Functional Differential Equations: Methods and Applications, Hindawi Publishing Corporation, New York, Cairo, 2007, p. 314.
[22] A.N. Kolmogorov, S.V. Fomin, Introductory Real Analasis, Dover Publications Inc, 1975, p. 403.
[23] C.R. Laing, W.C. Troy, Two-bump solutions of Amari-type models pf neuronal patter formation, Physica D 178 (2003) 190–218.
[24] C. Laing, W.C. Troy, B. Gutkin, G.B. Ermentrout, Multiple bumps in a neural model of working memory, SIAM J. Appl. Math. 63 (2005) 62–97. 1.
[25] M.A. Krasnosel'skii, P.P. Zabreiko, Geometrical methods of nonlinear analysis, in: Grundlehren Der Mathematischen Wissenschaften, in: A Series of Comprehensive Studies in Mathematics, vol. 263, Springer Verlag, Berlin-Heidelberg-New York-Tokyo, 1984, p. 409.