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Periodic solution generated by impulses for singular differential equations[☆]

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ABSTRACT

In this paper, we study the existence of a positive periodic solution for second-order singular differential equations with impulsive conditions. The proof is based on the mountain-pass theorem. We show that this positive periodic solution is generated by impulses.

1. Introduction

In this paper, we discuss the T-periodic solution for second-order non-autonomous singular problems

$$u''(t) - \frac{1}{u^{\alpha}(t)} = e(t) \tag{1.1}$$

under impulsive conditions

$$\Delta u'(t_j) = I_j(u(t_j)), \quad j = 1, 2, \dots, p-1, \tag{1.2}$$

where $\alpha \geq 1$, $e \in L^1([0, T], \mathbb{R})$ is T-periodic, $\Delta u'(t_j) = u'(t_j^+) - u'(t_j^-)$ with $u'(t_j^{\pm}) = \lim_{t \to t_j^{\pm}} u'(t)$; $t_j, j = 1, 2, \dots, p-1$, are the instants where the impulses occur and $0 = t_0 < t_1 < t_2 < \cdots < t_{p-1} < t_p = T$, $t_{j+p} = t_j + T$; $I_j : \mathbb{R} \to \mathbb{R} (j = 1, 2, \dots, p-1)$ are continuous and $I_{j+p} \equiv I_j$.

Impulsive differential equations have been studied by many authors [4,5,16,17,19,18,20]. Some classical tools have been used to study such problems. These classical techniques include the coincidence degree theory of Mawhin [17], the method of upper and lower solutions [4], some fixed point theorems [5], and variational methods [16,19,18,20,22,23,25]. In 2008, Tian and Ge [22] first studied the existence of solutions for impulsive differential equations by using a variational

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method. Later, Nieto and O'Regan [16] further developed the variational framework for impulsive problems and established existence results for a class of impulsive differential equations with Dirichlet boundary conditions. From then on, the variational method has been a powerful tool in many research fields, including the above-concerned impulsive problems and homoclinic solutions for Hamiltonian systems [12–14,25].

On the other hand, during the last three decades, singular differential equations with different kinds of boundary conditions have also been investigated extensively in the literature by using either topological methods or variational methods; see [1–3,6,8–11,24] and the references therein. Here, we recall one famous result proved by Lazer and Solimini [11] in 1987.

Theorem 1.1 ([11]). *Assume that $e \in L^1([0,T],\mathbb{R})$ is T-periodic. Then problem (1.1) has a positive T-periodic weak solution if and only if $\int_0^T e(t)dt < 0$.*

Compared with the classical impulsive problems or singular problems, singular problems with impulsive effects have been scarcely studied; see [7,21]. Therefore at this stage it is important to point out the dynamical differences between both models. For example, from Theorem 1.1, if $\int_0^T e(t)dt \geq 0$, then problem (1.1) does not have a positive T-periodic weak solution. However, if the impulses happen, for this singular problem there may exist a positive T-periodic weak solution.

Inspired by the above facts, the aim of this paper is to reveal a new existence result on a positive T-periodic solution for singular problem (1.1) when impulsive effects are considered, i.e., problem (1.1)–(1.2). Indeed, this periodic solution is generated by impulses. Here, we say that a solution is generated by impulses if this solution is non-trivial when $I_j \not\equiv 0$ for some $1 < j < p-1$, but it is trivial when $I_j \equiv 0$ for all $1 < j < p-1$. For example, if problem (1.1)–(1.2) does not possess a positive periodic solution when $I_j \equiv 0$ for all $1 < j < p-1$, then a positive periodic solution u of problem (1.1)–(1.2) with $I_j \not\equiv 0$ for some $1 < j < p-1$ is called a positive periodic solution generated by impulses; see [25].

Our result is presented as follows.

Theorem 1.2. *Assume that the following hold.*

(S_1) *$e \in L^1([0,T],\mathbb{R})$ is T-periodic and $\int_0^T e(t)dt \geq 0$.*

(S_2) *There exist two constants m, M such that, for any $s \in \mathbb{R}$,*

$$m \leq I_j(s) \leq M, \quad j = 1, 2, \ldots, p-1,$$

where $m \leq M < -\frac{1}{p-1}\int_0^T e(t)dt \leq 0$.

Then problem (1.1)–(1.2) has at least a positive T-periodic solution.

2. Preliminaries

Set

$$H_T^1 = \{u : \mathbb{R} \to \mathbb{R} \mid u \text{ is absolutely continuous}, u' \in L^2((0,T),\mathbb{R}) \text{ and } u(t) = u(t+T) \text{ for } t \in \mathbb{R}\}$$

with the inner product

$$(u, v) = \int_0^T u(t)v(t)dt + \int_0^T u'(t)v'(t)dt, \quad \forall u, v \in H_T^1.$$

The corresponding norm is defined by

$$\|u\|_{H_T^1} = \left(\int_0^T |u(t)|^2 dt + \int_0^T |u'(t)|^2 dt\right)^{\frac{1}{2}}, \quad \forall u \in H_T^1.$$

Then H_T^1 is a Banach space (in fact it is a Hilbert space).

In order to study problem (1.1)–(1.2), for any $\lambda \in (0, 1)$, we consider the following modified problem:

$$\begin{cases} u''(t) + f_\lambda(u(t)) = e(t), \quad a.e.\ t \in (0,T), \\ \Delta u'(t_j) = I_j(u(t_j)), \quad j = 1, 2, \ldots, p-1, \end{cases} \tag{2.1}$$

where $f_\lambda : \mathbb{R} \to \mathbb{R}$ is defined by

$$f_\lambda(s) = \begin{cases} -\dfrac{1}{s^\alpha}, & s \geq \lambda, \\ -\dfrac{1}{\lambda^\alpha}, & s < \lambda. \end{cases}$$

Following the ideas of [16,19,20], we introduce the following concept of a weak solution for problem (2.1).

Definition 2.1. We say that a function $u \in H_T^1$ is a weak solution of problem (2.1) if

$$\int_0^T u'(t)v'(t)dt + \sum_{j=1}^{p-1} I_j(u(t_j))v(t_j) - \int_0^T f_\lambda(u(t))v(t)dt + \int_0^T e(t)v(t)dt = 0$$

holds for any $v \in H_T^1$.

Let $F_\lambda \in C^1(\mathbb{R}, \mathbb{R})$ be defined by

$$F_\lambda(s) = \int_1^s f_\lambda(t)dt,$$

and consider the functional

$$\Phi_\lambda : H_T^1 \to \mathbb{R}$$

defined by

$$\Phi_\lambda(u) := \frac{1}{2}\int_0^T |u'(t)|^2 dt + \sum_{j=1}^{p-1}\int_0^{u(t_j)} I_j(s)ds - \int_0^T F_\lambda(u(t))dt + \int_0^T e(t)u(t)dt. \tag{2.2}$$

Clearly, Φ_λ is well defined on H_T^1, and is a continuously Gáteaux differentiable functional whose Gáteaux derivative is the functional $\Phi'_\lambda(u)$, given by

$$\Phi'_\lambda(u)v = \int_0^T u'(t)v'(t)dt + \sum_{j=1}^{p-1} I_j(u(t_j))v(t_j) - \int_0^T f_\lambda(u(t))v(t)dt + \int_0^T e(t)v(t)dt, \tag{2.3}$$

for any $v \in H_T^1$. Moreover, it is easy to verify that Φ_λ is weakly lower semi-continuous. Indeed, if $\{u_n\} \subset H_T^1$, $u \in H_T^1$, and $u_n \rightharpoonup u$, then $\{u_n\}$ converges uniformly to u on $[0, T]$ and $u_n \to u$ on $L^2([0, T])$, and combining the fact that $\liminf_{n\to\infty} \|u_n\|_{H_T^1} \geq \|u\|_{H_T^1}$, we have

$$\begin{aligned}
\liminf_{n\to\infty} \Phi_\lambda(u_n) &= \liminf_{n\to\infty}\left(\frac{1}{2}\|u_n\|_{H_T^1}^2 - \frac{1}{2}\int_0^T |u_n(t)|^2 dt + \sum_{j=1}^{p-1}\int_0^{u_n(t_j)} I_j(s)ds \right.\\
&\quad \left. - \int_0^T F_\lambda(u_n(t))dt + \int_0^T e(t)u_n(t)dt\right)\\
&\geq \frac{1}{2}\int_0^T |u'(t)|^2 dt + \sum_{j=1}^{p-1}\int_0^{u(t_j)} I_j(s)ds - \int_0^T F_\lambda(u(t))dt + \int_0^T e(t)u(t)dt\\
&= \Phi_\lambda(u).
\end{aligned}$$

By the standard discussion, the critical points of Φ_λ are the weak solutions of problem (2.1); see [16,19].

3. Proof of the main result

Now, we give the proof of Theorem 1.2 by using the mountain-pass theorem; see [15].

Step 1. We verify that the functional Φ_λ satisfies the Palais–Smale condition.

Let a sequence $\{u_n\}$ in H_T^1 satisfy that $\Phi_\lambda(u_n)$ is bounded and $\Phi'_\lambda(u_n) \to 0$, i.e., there exist a constant $c_1 > 0$ and a sequence $\{\epsilon_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$ with $\epsilon_n \to 0$ as $n \to +\infty$ such that, for all n,

$$\left|\int_0^T \left[\frac{1}{2}|u_n'(t)|^2 - F_\lambda(u_n(t)) + e(t)u_n(t)\right]dt + \sum_{j=1}^{p-1}\int_0^{u_n(t_j)} I_j(s)ds\right| \leq c_1, \tag{3.1}$$

and, for every $v \in H_T^1$,

$$\left|\int_0^T [u_n'(t)v'(t) - f_\lambda(u_n(t))v(t) + e(t)v(t)]dt + \sum_{j=1}^{p-1} I_j(u_n(t_j))v(t_j)\right| \leq \epsilon_n \|v\|_{H_T^1}. \tag{3.2}$$

Now we show that $\{u_n\}$ is bounded in H_T^1. Taking $v(t) \equiv -1$ in (3.2), one has

$$\left|\int_0^T [f_\lambda(u_n(t)) - e(t)]dt - \sum_{j=1}^{p-1} I_j(u_n(t_j))\right| \leq \epsilon_n\sqrt{T} \quad \text{for all } n.$$

By (S_1)–(S_2), we have

$$\left|\int_0^T f_\lambda(u_n(t))dt\right| \le \epsilon_n\sqrt{T} + \int_0^T e(t)dt + \sum_{j=1}^{p-1}|I_j(u_n(t_j))|$$
$$\le \epsilon_n\sqrt{T} + \int_0^T e(t)dt + (p-1)|m| := c_2.$$

Note that, for any $t \in [0, T]$, $f_\lambda(u_n(t)) < 0$. Thus

$$\int_0^T |f_\lambda(u_n(t))|dt = \left|\int_0^T f_\lambda(u_n(t))dt\right| \le c_2.$$

On the other hand, take, in (3.2), $v(t) \equiv w_n(t) := u_n(t) - \bar{u}_n$, where $\bar{u}_n = \frac{1}{T}\int_0^T u_n(t)dt$. By Proposition 1.1 of [15], we have

$$\begin{aligned}
c_3\|w_n\|_{H_T^1} &\ge \int_0^T [w_n'(t)^2 - f_\lambda(u_n(t))w_n(t) + e(t)w_n(t)]dt + \sum_{j=1}^{p-1} I_j(u_n(t_j))w_n(t_j)\\
&\ge \|w_n'\|_{L^2}^2 - (c_2 + \|e\|_{L^1})\|w_n\|_{L^\infty} + (p-1)m\|w_n\|_{L^\infty}\\
&= \|w_n'\|_{L^2}^2 - (c_2 + \|e\|_{L^1} - (p-1)m)\|w_n\|_{L^\infty}\\
&\ge \|w_n'\|_{L^2}^2 - c_4\|w_n\|_{H_T^1},
\end{aligned}$$

where c_3 and c_4 are two positive constants. Thus,

$$\|w_n'\|_{L^2}^2 \le (c_3 + c_4)\|w_n\|_{H_T^1}.$$

Consequently, using the Wirtinger inequality, we get the existence of $c_5 > 0$ such that

$$\|u_n'\|_{L^2}^2 \le c_5. \tag{3.3}$$

Now, suppose that

$$\|u_n\|_{H_T^1} \to +\infty \quad \text{as } n \to +\infty.$$

Since (3.3) holds, we have, passing to a subsequence if necessary, that either

$$M_n := \max u_n \to +\infty \quad \text{as } n \to +\infty, \quad \text{or}$$
$$m_n := \min u_n \to -\infty \quad \text{as } n \to +\infty.$$

(i) Assume that the first possibility occurs. By (S_2) and the fact that $f_\lambda < 0$, one has

$$\begin{aligned}
&\int_0^T [F_\lambda(u_n(t)) - e(t)u_n(t)]\,dt - \sum_{j=1}^{p-1}\int_0^{u_n(t_j)} I_j(s)ds\\
&= \int_0^T \left[\left(\int_1^{M_n} f_\lambda(s)ds - \int_{u_n(t)}^{M_n} f_\lambda(s)ds\right) - e(t)u_n(t)\right]dt - \sum_{j=1}^{p-1}\int_0^{|M_n|} I_j(s)ds - \sum_{j=1}^{p-1}\int_{|M_n|}^{u_n(t_j)} I_j(s)ds\\
&\ge \int_0^T F_\lambda(M_n)dt - \int_0^T M_n e(t)dt - \max_{t\in[0,T]}|M_n - u_n(t)|\int_0^T |e(t)|dt - (p-1)M|M_n|\\
&\quad + (p-1)m\max_{t\in[0,T]}\|M_n| - u_n(t)|\\
&\ge TF_\lambda(M_n) - M_n\int_0^T e(t)dt - (p-1)MM_n - \|e\|_{L^1}|M_n - m_n| + (p-1)m|M_n - m_n|\\
&= TF_\lambda(M_n) - M_n\left(\int_0^T e(t)dt + (p-1)M\right) - \|e\|_{L^1}\left|\int_{\bar{t}_n}^{\hat{t}_n} u_n'(t)dt\right| + (p-1)m\left|\int_{\bar{t}_n}^{\hat{t}_n} u_n'(t)dt\right|\\
&\ge TF_\lambda(M_n) - M_n\left(\int_0^T e(t)dt + (p-1)M\right) - \big(\|e\|_{L^1} - (p-1)m\big)\int_0^T |u_n'(t)|dt,
\end{aligned}$$

where $u_n(\hat{t}_n) = M_n$ and $u_n(\bar{t}_n) = m_n$. Thus, using the Hölder inequality, one has

$$\begin{aligned}
&-M_n\left(\int_0^T e(t)dt + (p-1)M\right) + TF_\lambda(M_n) \le \int_0^T [F_\lambda(u_n(t)) - e(t)u_n(t)]\,dt\\
&\quad - \sum_{j=1}^{p-1}\int_0^{u_n(t_j)} I_j(s)ds + \sqrt{T}\big(\|e\|_{L^1} - (p-1)m\big)\|u_n'\|_{L^2}.
\end{aligned} \tag{3.4}$$

If $\alpha = 1$, then $F_\lambda(M_n) = -\ln M_n$. By ($S_2$), one has

$$-M_n\left(\int_0^T e(t)dt + (p-1)M\right) - T\ln M_n \to +\infty \quad \text{as } n \to +\infty.$$

If $\alpha > 1$, then $F_\lambda(M_n) = -\frac{1}{\alpha-1}\left(\frac{1}{M_n^{\alpha-1}} - 1\right)$. By ($S_2$), one has

$$-M_n\left(\int_0^T e(t)dt + (p-1)M\right) - \frac{1}{\alpha-1}\left(\frac{1}{M_n^{\alpha-1}} - 1\right) \to +\infty \quad \text{as } n \to +\infty.$$

From (3.1) and (3.3), we see that the right-hand side of (3.4) is bounded, which is a contradiction.

(ii) Assume that the second possibility occurs, i.e., $m_n \to -\infty$ as $n \to +\infty$. We replace M_n by $-m_n$ in the preceding arguments, and we also get a contradiction. So $\{u_n\}$ is bounded in H_T^1.

Since H_T^1 is a reflexive Banach space, there exists a subsequence of $\{u_n\}$, denoted again by $\{u_n\}$ for simplicity, and $u \in H_T^1$ such that $u_n \rightharpoonup u$ in H_T^1; then, by the Sobolev embedding theorem, we get $u_n \to u$ in $C([0, T])$ and $u_n \to u$ in $L^2([0, T])$. So

$$\begin{cases} \displaystyle\int_0^T (f_\lambda(u_n(t)) - f_\lambda(u(t)))(u_n(t) - u(t))dt \to 0, \\ \displaystyle\sum_{j=1}^{p-1}(I_j(u_n(t_j)) - I_j(u(t_j)))(u_n(t_j) - u(t_j)) \to 0, \\ \displaystyle\int_0^T e(t)(u_n(t) - u(t))dt \to 0, \\ (\Phi_\lambda'(u_n) - \Phi_\lambda'(u))(u_n - u) \to 0, \quad \text{as } n \to \infty. \end{cases} \tag{3.5}$$

By (2.3), we have

$$\begin{aligned}(\Phi_\lambda'(u_n) - \Phi_\lambda'(u))(u_n - u) = {} & \int_0^T |u_n' - u|^2 dt + \int_0^T e(t)(u_n(t) - u(t))dt \\ & + \sum_{j=1}^{p-1}(I_j(u_n(t_j)) - I_j(u(t_j)))(u_n(t_j) - u(t_j)) \\ & - \int_0^T (f_\lambda(u_n(t)) - f_\lambda(u(t)))(u_n(t) - u(t))dt. \end{aligned} \tag{3.6}$$

By (3.5) and (3.6), and since $u_n \to u$ in $L^2([0, T])$, we have $\|u_n - u\|_{H_T^1} \to 0$ as $n \to \infty$. That is, $\{u_n\}$ strongly converges to u in H_T^1, which means that the Palais–Smale condition holds for Φ_λ.

Step 2. Let

$$\Omega = \left\{u \in H_T^1 \mid \min_{t\in[0,T]} u(t) > 1\right\},$$

and

$$\partial\Omega = \{u \in H_T^1 | u(t) \geq 1 \text{ for all } t \in (0, T),\ \exists t_u \in (0, T) : u(t_u) = 1\}.$$

We show that there exists $d > 0$ such that $\inf_{u\in\partial\Omega} \Phi_\lambda(u) \geq -d$ whenever $\lambda \in (0, 1)$.

For any $u \in \partial\Omega$, there exists some $t_u \in (0, T)$ such that $\min_{t\in[0,T]} u(t) = u(t_u) = 1$. By (2.2), ($S_2$), and extending the functions by T-periodicity, we have

$$\begin{aligned}\Phi_\lambda(u) = {} & \int_{t_u}^{t_u+T}\left[\frac{1}{2}|u'(t)|^2 - F_\lambda(u(t)) + e(t)u(t)\right]dt + \sum_{j=1}^{p-1}\int_0^{u(t_j)} I_j(s)ds \\ \geq {} & \frac{1}{2}\int_{t_u}^{t_u+T}|u'(t)|^2 dt + \int_{t_u}^{t_u+T} e(t)(u(t) - 1)dt + \int_{t_u}^{t_u+T} e(t)dt \\ & + \sum_{j=1}^{p-1}\int_0^1 I_j(s)ds + \sum_{j=1}^{p-1}\int_1^{u(t_j)} I_j(s)ds \\ \geq {} & \frac{1}{2}\|u'\|_{L^2}^2 - \|e\|_{L^1}\max_{t\in[0,T]}(u(t) - 1) - \|e\|_{L^1} + (p-1)m\max_{t\in[0,T]}(u(t) - 1) + (p-1)m \\ = {} & \frac{1}{2}\|u'\|_{L^2}^2 - \|e\|_{L^1}\int_{t_u}^{\check{t}_u} u'(t)dt - \|e\|_{L^1} + (p-1)m\int_{t_u}^{\check{t}_u} u'(t)dt + (p-1)m\end{aligned}$$

$$\geq \frac{1}{2}\|u'\|_{L^2}^2 - \left(\|e\|_{L^1} - (p-1)m\right)\int_{t_u}^{t_u+T} |u'(t)|dt - \|e\|_{L^1} + (p-1)m,$$

where $\check{t}_u \in [0, T]$ and $\max_{t\in[0,T]} u(t) = u(\check{t}_u)$. Applying the Hölder inequality, we get

$$\Phi_\lambda(u) \geq \frac{1}{2}\|u'\|_{L^2}^2 - \sqrt{T}\left(\|e\|_{L^1} - (p-1)m\right)\|u'\|_{L^2} - \|e\|_{L^1} + (p-1)m.$$

The above inequality shows that

$$\Phi_\lambda(u) \to +\infty \quad \text{as } \|u'\|_{L^2} \to +\infty.$$

For any $u \in \partial\Omega$, it is easy to verify the fact that $\|u\|_{H_T^1} \to +\infty$ is equivalent to $\|u'\|_{L^2} \to +\infty$. Indeed, when $\|u'\|_{L^2} \to +\infty$, it is clear that $\|u\|_{H_T^1} \to +\infty$. When $\|u\|_{H_T^1} \to +\infty$, if not, we assume that $\|u'\|_{L^2}$ is bounded; then $\|u\|_{L^2} \to +\infty$. Since $\min_{t\in[0,T]} u(t) = 1$, we have

$$u(t) - 1 = \int_{t_u}^{t} u'(s)ds \leq \int_0^T |u'(s)|ds \leq \sqrt{T}\left(\int_0^T |u'(t)|^2 dt\right)^{\frac{1}{2}}.$$

Therefore, u is bounded in $L^2(0, T)$, which is a contradiction. Hence

$$\Phi_\lambda(u) \to +\infty \quad \text{as } \|u\|_{H_T^1} \to +\infty,\ \forall u \in \partial\Omega,$$

which shows that Φ_λ is coercive. Thus it has a minimizing sequence. The weak lower semi-continuity of Φ_λ yields

$$\inf_{u\in\partial\Omega} \Phi_\lambda(u) > -\infty.$$

It follows that there exists $d > 0$ such that $\inf_{u\in\partial\Omega} \Phi_\lambda(u) > -d$ for all $\lambda \in (0, 1)$.

Step 3. We prove that there exists $\lambda_0 \in (0, 1)$ with the property that, for every $\lambda \in (0, \lambda_0)$, any solution u of problem (2.1) satisfying $\Phi_\lambda(u) > -d$ is such that $\min_{u\in[0,T]} u(t) \geq \lambda_0$, and hence u is a solution of problem (1.1)–(1.2).

Assume on the contrary that there are sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ such that

(i) $\lambda_n \leq \frac{1}{n}$;
(ii) u_n is a solution of (2.1) with $\lambda = \lambda_n$;
(iii) $\Phi_{\lambda_n}(u_n) \geq -d$;
(iv) $\min_{t\in[0,T]} u_n(t) < \frac{1}{n}$.

Since $f_{\lambda_n} < 0$ and $\int_0^T [f_{\lambda_n}(u_n(t)) - e(t)]dt = 0$, one has

$$\|f_{\lambda_n}(u_n(\cdot))\|_{L^1} \leq c_7, \quad \text{for some constant } c_7 > 0.$$

Hence

$$\|u_n'\|_{L^\infty} \leq c_8, \quad \text{for some constant } c_8 > 0. \tag{3.7}$$

From $\Phi_{\lambda_n}(u_n) \geq -d$ it follows that there must exist two constants l_1 and l_2, with $0 < l_1 < l_2$, such that

$$\max\{u_n(t);\ t \in [0, T]\} \subset [l_1, l_2].$$

If not, u_n would tend uniformly to 0 or $+\infty$. In both cases, by (S_1)–(S_2) and (3.7), we have

$$\Phi_{\lambda_n}(u_n) \to -\infty \quad \text{as } n \to +\infty,$$

which contradicts $\Phi_{\lambda_n}(u_n) \geq -d$.

Let τ_n^1, τ_n^2 be such that, for n sufficiently large,

$$u_n(\tau_n^1) = \frac{1}{n} < l_1 = u_n(\tau_n^2).$$

Multiplying the equation $u_n''(t) + f_\lambda(u_n(t)) = e(t)$ by u_n', and integrating it on $[\tau_n^1, \tau_n^2]$, or on $[\tau_n^2, \tau_n^1]$, we get

$$\begin{aligned}\Psi &:= \int_{\tau_n^1}^{\tau_n^2} u_n''(t)u_n'(t)dt + \int_{\tau_n^1}^{\tau_n^2} f_{\lambda_n}(u_n(t))u_n'(t)dt \\ &= \int_{\tau_n^1}^{\tau_n^2} e(t)u_n'(t)dt.\end{aligned} \tag{3.8}$$

It is easy to verify that

$$\Psi = \Psi_1 + \frac{1}{2}[u_n'^2(\tau_n^2) - u_n'^2(\tau_n^1)],$$

where

$$\Psi_1 = \int_{\tau_n^1}^{\tau_n^2} f_{\lambda_n}(u_n(t))u_n'(t)dt.$$

From (S_1), (3.4) and (3.8), it follows that Ψ is bounded, and consequently Ψ_1 is bounded.

On the other hand, it is easy to see that

$$f_{\lambda_n}(u_n(t))u_n'(t) = \frac{d}{dt}[F_{\lambda_n}(u_n(t))].$$

Thus, we have

$$\Psi_1 = F_{\lambda_n}(l_1) - F_{\lambda_n}\left(\frac{1}{n}\right).$$

From the fact that $F_{\lambda_n}\left(\frac{1}{n}\right) \to +\infty$ as $n \to +\infty$, we obtain $\Psi_1 \to -\infty$, i.e., Ψ_1 is unbounded. This is a contradiction.

Step 4. Φ has a mountain-pass geometry for $\lambda \le \lambda_0$.

Fixing $\lambda \in (0, \lambda_0]$, one has

$$\begin{aligned} F_\lambda(0) &= \int_1^0 f_\lambda(s)ds = -\int_0^1 f_\lambda(s)ds \\ &= -\int_0^\lambda f_\lambda(s)ds - \int_\lambda^1 f_\lambda(s)ds \\ &= \frac{1}{\lambda^{\alpha-1}} - \int_\lambda^1 f_\lambda(s)ds, \end{aligned}$$

which implies that

$$F_\lambda(0) > -\int_\lambda^1 f_\lambda(s)ds = \int_1^\lambda f_\lambda(s)ds = F_\lambda(\lambda).$$

Thus we have

$$\begin{aligned} \Phi_\lambda(0) &= -TF_\lambda(0) < -TF_\lambda(\lambda) \\ &= \begin{cases} T\ln\lambda, & \text{if } \alpha = 1, \\ -\dfrac{T}{\alpha-1}\left(\dfrac{1}{\lambda^{\alpha-1}} - 1\right), & \text{if } \alpha > 1. \end{cases} \end{aligned} \tag{3.9}$$

Consider $\lambda \in (0, \lambda_0] \cap (0, e^{-d}) \cap \left(0, \left[\frac{T}{T+d(\alpha-1)}\right]^{1/(\alpha-1)}\right)$. Thus it follows from (3.9) that $\Phi_\lambda(0) < -d$.

Also, using (S_2), we can choose a constant $R > 1$ sufficiently large such that

$$-\left(M(p-1) + \int_0^T e(t)dt\right)R - \frac{T}{\alpha-1}\left(1 - \frac{1}{R^{\alpha-1}}\right) > d,$$

and

$$-\left(M(p-1) + \int_0^T e(t)dt\right)R - T\ln R > d.$$

Thus, $R \in H_T^1$, and

$$\begin{aligned} \Phi_\lambda(R) &= \sum_{j=1}^{p-1}\int_0^R I_j(s)ds - TF_\lambda(R) + R\int_0^T e(t)dt \\ &\le \begin{cases} M(p-1)R + T\ln R + R\displaystyle\int_0^T e(t)dt, & \text{if } \alpha = 1 \\ M(p-1)R + \dfrac{T}{\alpha-1}\left(1 - \dfrac{1}{R^{\alpha-1}}\right) + R\displaystyle\int_0^T e(t)dt, & \text{if } \alpha > 1 \end{cases} \\ &= \begin{cases} \left(M(p-1) + \displaystyle\int_0^T e(t)dt\right)R + T\ln R, & \text{if } \alpha = 1 \\ \left(M(p-1) + \displaystyle\int_0^T e(t)dt\right)R + \dfrac{T}{\alpha-1}\left(1 - \dfrac{1}{R^{\alpha-1}}\right), & \text{if } \alpha > 1 \end{cases} \\ &< -d. \end{aligned}$$

Since Ω is a neighborhood of R, $0 \notin \Omega$, and

$$\max\{\Phi_\lambda(0), \Phi_\lambda(R)\} < \inf_{x\in\partial\Omega} \Phi_\lambda(u),$$

Step 1 and Step 2 imply that Φ_λ has a critical point u_λ such that

$$\Phi_\lambda(u_\lambda) = \inf_{h\in\Gamma} \max_{s\in[0,1]} \Phi_\lambda(h(s)) \geq \inf_{x\in\partial\Omega} \Phi_\lambda(u),$$

where

$$\Gamma = \{h \in C([0, 1], H_T^1) : h(0) = 0, h(1) = R\}.$$

Since $\inf_{u\in\partial\Omega} \Phi_\lambda(u_\lambda) \geq -d$, it follows from Step 3 that u_λ is a positive solution of problem (1.1)–(1.2). The proof of the main result is complete.

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