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A blow-up criterion of strong solutions to the 3D compressible MHD equations with vacuum

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ABSTRACT

In this paper, we prove a blow-up criterion of strong solutions to 3D viscous isentropic compressible magnetohydrodynamic equations. It is shown that if ρ and H satisfy $\|\rho\|_{L^\infty(0,T;L^\infty)} + \|H\|_{L^\infty(0,T;L^r)} < \infty$, for any $24/5 \le r \le \infty$ and $3\mu > \lambda$, then the strong solutions to the Cauchy problem of the compressible magnetohydrodynamic equations can exist globally on $[0, T]$. In addition, initial vacuum is allowed.

1. Introduction

This paper is mainly concerned with the three-dimensional compressible magnetohydrodynamic (MHD) system as follows:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla \left(P + \dfrac{1}{2}|H|^2 \right) = H \cdot \nabla H + \mu \Delta u + (\mu + \lambda)\nabla \operatorname{div} u, \\ H_t + u \cdot \nabla H - H \cdot \nabla u + H \operatorname{div} u = \nu \Delta H, \quad \operatorname{div} H = 0, \end{cases} \tag{1.1}$$

for $x \in \mathbb{R}^3$ and $t \ge 0$. The unknowns ρ, u, P and H represent the density, velocity, pressure and magnetic field respectively. The constants $\mu > 0$ and λ are viscosity constants, satisfying $3\lambda + 2\mu \ge 0$, the constant $\nu > 0$ is the magnetic diffusivity acting as the magnetic diffusion coefficient of the magnetic field. Here we only consider the isentropic MHD flows in which the equation of state has the form $P(\rho) = a\rho^\gamma$ where $\gamma > 1$ and $a > 0$ are physical constants.

There have been numerous studies on the MHD problem by many physicists and mathematicians due to its physical importance, complexity, rich phenomena and mathematical challenges; see, for example, [1,2,4–8,11–13,15,17,19,20,22,24,25] and the references therein. Briefly, for the two-dimensional case, Kawashima [11] obtained the global existence of smooth solutions to the general electro–magneto-fluid equations when the initial data are small perturbations of some given constant states. Zhou–Fan in [25] established a regularity criterion for the 2D incompressible MHD system with zero magnetic diffusivity. For the linearized 3D compressible MHD equations, Umeda, Kawashima and Shizuta [19] showed the global existence and the time decay of smooth solutions. When the initial density is strictly positive, the local strong solution to the compressible MHD with large initial data was obtained by Vol'pert and Khudiaev [20]. In the case that the initial density need not be positive and may vanish in some open sets, the local well-posedness of strong solutions to the full compressible MHD equations in three dimensions was investigated by Fan and Yu [6].

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On the other hand, many fundamental problems for MHD are still unsolved, even for the one-dimensional case, that the global existence of classical solutions to the full perfect compressible MHD with large data remains unknown when all the viscosity, heat conductivity, and magnetic diffusivity coefficients are constant, although the corresponding problem for the Navier–Stokes equations was solved in [14] many years ago. This is mainly because of the presence of the magnetic field and its interaction with the hydrodynamic motion in the MHD flow of large oscillation.

The main purpose in this paper is to give a blow-up criterion of strong solutions for compressible MHD (1.1) with initial and boundary conditions:

$$\begin{cases}(\rho, u, H)(x, 0) = (\rho_0, u_0, H_0)(x) & x \in \mathbb{R}^3,\\ (\rho, u, H)(x, t) \to 0 & |x| \to \infty.\end{cases} \tag{1.2}$$

Before stating the main results, we explain the notations and conventions used throughout this paper. We denote

$$\int f dx = \int_\Omega f dx, \qquad \partial_i f = \frac{\partial f}{\partial x_i}, \qquad L^q = L^q(\mathbb{R}^3).$$

For $1 < r < \infty$, the standard homogeneous and inhomogeneous Sobolev spaces are denoted as follows:

$$\begin{cases}D^{k,r} = \{u \in L^1_{loc} \mid \|\nabla^k u\|_{L^r} < \infty\}, \qquad \|u\|_{D^{k,r}} = \|\nabla^k u\|_{L^r},\\ W^{k,r} = L^r \cap D^{k,r}, \qquad D^k = D^{k,2}, \qquad D^1 = \{u \in L^6 \mid \|\nabla u\|_{L^2} < \infty\}.\end{cases}$$

Definition 1.1. A pair of functions (ρ, u, H) is called a strong solution to the problem (1.1)–(1.2), if for some $3 < q \le 6$,

$$\begin{cases}\rho \ge 0, \quad \rho \in C(0, T; L^1 \cap H^1 \cap W^{1,q}),\\ u \in C(0, T; D^1 \cap D^2) \cap L^2(0, T; D^{2,q}), \quad H \in C(0, T; H^2) \cap L^2(0, T; W^{2,q})\\ u_t, H_t \in L^2(0, T; D^1), \quad \sqrt{\rho} u_t, H_t \in L^\infty(0, T; L^2),\end{cases}$$

and (ρ, u, H) satisfies (1.1) a.e. in $\mathbb{R}^3 \times (0, T)$.

The local well-posedness theorem of strong solutions to the compressible MHD with vacuum was proved by Fan–Yu [6].

Theorem 1.1. *Assume that for some $q \in (3, 6]$ the initial data (ρ_0, u_0, H_0) satisfies*

$$\rho_0 \ge 0, \quad \rho_0 \in L^1 \cap H^1 \cap W^{1,q}, \qquad u_0 \in D^1 \cap D^2, \qquad H_0 \in H^2, \quad \mathrm{div} H_0 = 0, \tag{1.3}$$

and the compatibility condition

$$-\mu \Delta u_0 - (\mu + \lambda)\nabla \mathrm{div} u_0 + \nabla \left(P(\rho_0) + \frac{1}{2}|H_0|^2\right) - H_0 \cdot \nabla H_0 = \rho_0^{1/2} g \tag{1.4}$$

for some $g \in L^2$. Then there exist a positive time $\overline{T} \in (0, \infty)$ and a unique strong solution (ρ, u, H) to the problem (1.1) and (1.2) in $\mathbb{R}^3 \times (0, \overline{T}]$.

We wonder whether the strong solution blows up in finite time. The first attempt toward such problem is to investigate the possible blow-up mechanism of the solution. Recently, many works are devoted to this subject for compressible MHD or Navier–Stokes equations; cf. [9,10,18,23]. In particular, Sun–Wang–Zhang in [18] established the blow-up criterion of strong solutions to 3D Navier–Stokes equations as follows: if $T^* < \infty$ is the maximal time of the existence of the strong solutions, then

$$\lim_{T \to T^*} \|\rho\|_{L^\infty(0,T;L^\infty)} = \infty.$$

Just because of the similarity of compressible MHD with the Navier–Stokes equations, some ideas used to get the blow-up criterion of the strong solutions for the Navier–Stokes equations will be applied to deal with the MHD system. In our present paper, we want to obtain a similar result for the compressible MHD system. This work is motivated by Xu–Zhang [23].

Our main result in this paper reads as follows.

Theorem 1.2. *Assume that for some $q \in (3, 6]$ the initial data (ρ_0, u_0, H_0) satisfies (1.3) and (1.4). Let (ρ, u, H) be a strong solution of the problem (1.1)–(1.2) satisfying Definition 1.1. If $0 < T^* < \infty$ is the maximal time of existence, then*

$$\lim_{T \to T^*} (\|\rho\|_{L^\infty(0,T;L^\infty)} + \|H\|_{L^\infty(0,T;L^r)}) = \infty, \tag{1.5}$$

for any $\frac{24}{5} \le r \le \infty$, provided $3\mu > \lambda$.

We should mention that the decomposition of the velocity $u = v + \omega$ plays a key role in our paper. More precisely, let v solve the elliptic system

$$\begin{cases} \mu\Delta v + (\mu+\lambda)\nabla \mathrm{div} v = \nabla\left(P + \frac{1}{2}|H|^2\right) \quad \text{in } \mathbb{R}^3, \\ v(x,t) \to 0 \qquad |x| \to \infty, \end{cases} \tag{1.6}$$

then, from the momentum equation $(1.1)_2$ and (1.6), it is easy to verify that ω satisfies

$$\begin{cases} \mu\Delta \omega + (\mu+\lambda)\nabla \mathrm{div} \omega = \rho\dot{u} - H\cdot\nabla H \quad \text{in } \mathbb{R}^3, \\ \omega(x,t) \to 0 \qquad |x| \to \infty. \end{cases} \tag{1.7}$$

2. Preliminaries

In this section, we state some known auxiliary lemmas which will be frequently used in the proof of Theorem 1.2. We begin with the following well-known Gagliardo–Nirenberg inequality which can be found in [16].

Lemma 2.1. *For $p \in [2, 6]$, $q \in (1, \infty)$, and $r \in (3, \infty)$, there exists some generic constant $C > 0$ which may depend on q, r such that for $f \in H^1$ and $g \in L^q \cap D^{1,r}$, we have*

$$\|f\|_{L^p} \le C\|f\|_{L^2}^{(6-p)/2p}\|\nabla f\|_{L^2}^{(3p-6)/2p}, \tag{2.1}$$

$$\|g\|_{L^\infty} \le C\|g\|_{L^q}^{q(r-3)/(3r+q(r-3))}\|\nabla g\|_{L^r}^{3r/(3r+q(r-3))}. \tag{2.2}$$

Let U solve the following boundary value problem

$$\begin{cases} \mu\Delta U + (\mu+\lambda)\nabla \mathrm{div} U = F \quad \text{in } \mathbb{R}^3, \\ U(x,t) \to 0 \qquad |x| \to \infty, \end{cases} \tag{2.3}$$

we state some classical estimates for the above strongly elliptic system.

Lemma 2.2. *Let $p \in (1, \infty)$ and U be a solution of (2.3), then there exists a constant C depending only on μ, λ and p such that the following estimates hold:*

(1) *if $F \in L^p$, then*

$$\|\nabla^2 U\|_{L^p} \le C\|F\|_{L^p}; \tag{2.4}$$

(2) *if $F = \mathrm{div} f$ with $f = (f_{i,j})_{2\times 2}, f_{i,j} \in L^p$, then*

$$\|\nabla U\|_{L^p} \le C\|f\|_{L^p}; \tag{2.5}$$

(3) *if $F = \mathrm{div} f$ with $f = (f_{i,j})_{2\times 2}, f_{i,j} \in L^\infty \cap L^2$, then $\nabla U \in BMO(\Omega)$ and*

$$\|\nabla U\|_{BMO(\Omega)} \le C(\|f\|_{L^2} + \|f\|_{L^\infty}). \tag{2.6}$$

Here $BMO(\Omega)$ stands for the John–Nirenberg space of bounded mean oscillation whose norm is defined by

$$\|f\|_{BMO(\Omega)} \triangleq \|f\|_{L^2} + [f]_{BMO(\Omega)},$$

with

$$[f]_{BMO(\Omega)} \triangleq \sup_{x\in\Omega, r\in(0,d)} \frac{1}{|\Omega_r(x)|}\int_{\Omega_r(x)} |f(y) - f_{\Omega_r(x)}|dy,$$

and

$$f_{\Omega_r(x)} = \frac{1}{|\Omega_r(x)|}\int_{\Omega_r(x)} f(y)dy,$$

where $\Omega_r(x) = B_r(x) \cap \Omega$, $B_r(x)$ is a ball with center x and radius r, d is the diameter of Ω and $|\Omega_r(x)|$ denotes the Lebesgue measure of $\Omega_r(x)$.

The next lemma is a variant of the Brezis–Wainger inequality [3], which together with Lemma 2.2 will be used to give the gradient estimate of ρ.

Lemma 2.3. *Let $f \in W^{1,q}$ with $q \in (3, \infty)$, then there exists a constant C depending on q such that*

$$\|f\|_{L^\infty} \le C(1 + \|f\|_{BMO}\ln(e + \|f\|_{W^{1,q}})). \tag{2.7}$$

The proof of Lemmas 2.2 and 2.3 can be found in [18].

3. Regularity of ρ, u, H

Let (ρ, u, H) be a strong solution to the initial boundary problem (1.1)–(1.2) as described in Theorem 1.1. Here we take a contradiction argument to complete the proof of Theorem 1.2. Thus, we now assume otherwise that

$$\lim_{T\to T^*}(\|\rho\|_{L^\infty(0,T;L^\infty)} + \|H\|_{L^\infty(0,T;L^r)}) \le M, \tag{3.1}$$

where M is independent of T, $\frac{24}{5} \le r \le \infty$. Hereafter, we denote by C a general positive constant which may depend on the initial data, M and the maximal existence time T^*.

Then, it follows from the standard energy estimate that

Lemma 3.1. *Under the conditions of Theorem 1.1, for any $0 \le T < T^*$, it holds that*

$$\sup_{0\le t\le T}\left(\left\|\sqrt{\rho}u\right\|_{L^2}^2 + \|H\|_{L^2}^2 + \|\rho\|_{L^\gamma\cap L^1}\right) + \int_0^T \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 dt \le C. \tag{3.2}$$

Lemma 3.2. *Under the assumption of (3.1), if $3\mu > \lambda$, then one has for any $0 \le T < T^*$ that*

$$\int \rho|u|^4dx + \int_0^T \|u\nabla u\|_{L^2}^2 dt \le C. \tag{3.3}$$

Proof. Multiplying $(1.1)_2$ by $4|u|^2u$, and integrating by parts over $\mathbb{R}^3$, one has from Lemma 4.2 in [21] that

$$\frac{d}{dt}\int \rho|u|^4dx + c_1\|u\nabla u\|_{L^2}^2 \le C\int \rho|u|^4dx + \int 4|u|^2u\cdot\left(-\frac{1}{2}\nabla|H|^2 + H\cdot\nabla H\right)dx + C. \tag{3.4}$$

For the second term on the right-hand side of (3.4), by integrating by parts and Young's inequality, one has

$$\begin{aligned}
\int 4|u|^2u\cdot\left(-\frac{1}{2}\nabla|H|^2 + H\cdot\nabla H\right)dx &\le C\int |u|^2|\nabla u||H|^2dx \\
&\le C\left(\int |u|^2|\nabla u|^2dx\right)^{\frac{1}{2}}\left(\int |u|^2|H|^4dx\right)^{\frac{1}{2}} \\
&\le \frac{\epsilon}{2}\int |u|^2|\nabla u|^2dx + C(\epsilon)\left(\int (|u|^2)^6dx\right)^{\frac{1}{6}}\left(\int (|H|^4)^{\frac{6}{5}}dx\right)^{\frac{5}{6}} \\
&\le \frac{\epsilon}{2}\int |u|^2|\nabla u|^2dx + C(\epsilon)\left(\int |u|^2|\nabla u|^2dx\right)^{\frac{1}{2}} \\
&\le \epsilon\int |u|^2|\nabla u|^2dx + C(\epsilon).
\end{aligned} \tag{3.5}$$

Choosing ϵ small enough, adding (3.5) to (3.4), and applying Gronwall's inequality, one obtains the lemma immediately. □

Let $\dot{f}$ and G be the material derivative, and effective viscous flux, which are defined, respectively, as follows:

$$\dot{f} = f_t + u\cdot\nabla f, \qquad G = (2\mu+\lambda)\mathrm{div}u - P - \frac{1}{2}|H|^2. \tag{3.6}$$

We have from $(1.1)_2$ that

$$\Delta G = \mathrm{div}(\rho\dot{u} - H\cdot\nabla H). \tag{3.7}$$

Now we can prove the following lemma which gives the L^2-estimate of ∇u and ∇H.

Lemma 3.3. *Under the assumption of (3.1) and $3\mu > \lambda$, for any $0 \le T < T^*$, it holds that*

$$\sup_{0\le t\le T}(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) + \int_0^T \left\|\sqrt{\rho}u_t\right\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\nabla H\|_{H^1}^2 dt \le C. \tag{3.8}$$

Proof. Multiplying (1.1)$_2$, (1.1)$_3$ by u_t and H_t in L^2, respectively, integrating the resulting equation by parts, we obtain after summing up that

$$\begin{aligned}
&\frac{d}{dt}\int \frac{\mu}{2}|\nabla u|^2+\frac{\mu+\lambda}{2}(\mathrm{div}u)^2+\frac{\nu}{2}|\nabla H|^2dx+\int \rho|u_t|^2+|H_t|^2dx\\
&=\int P\mathrm{div}u_t dx+\int\left(H\cdot\nabla H-\frac{1}{2}\nabla|H|^2\right)\cdot u_t dx-\int \rho u\cdot\nabla u\cdot u_t dx\\
&\quad+\int H\cdot\nabla u\cdot H_t dx-\int u\cdot\nabla H\cdot H_t dx-\int H\cdot H_t\mathrm{div}u dx\\
&\triangleq\sum_{i=1}^{6}K_i.
\end{aligned}\tag{3.9}$$

To deal with the first term on the right-hand side of (3.9), we notice that

$$P_t+\mathrm{div}(Pu)+(\gamma-1)P(\mathrm{div}u)=0.\tag{3.10}$$

Hence, we infer from integration by parts and the definition of G that

$$\begin{aligned}
\int P\mathrm{div}u_t dx&=\frac{d}{dt}\int P\mathrm{div}u dx-\int Pu\cdot\nabla\mathrm{div}u dx+(\gamma-1)\int P(\mathrm{div}u)^2dx\\
&=\frac{d}{dt}\int P\mathrm{div}u dx-\frac{1}{2\mu+\lambda}\int Pu\cdot\nabla G dx+\frac{1}{2(2\mu+\lambda)}\int P^2\mathrm{div}u dx\\
&\quad-\frac{1}{2(2\mu+\lambda)}\int Pu\cdot\nabla|H|^2dx+(\gamma-1)\int P(\mathrm{div}u)^2dx\\
&\le\frac{d}{dt}\int P\mathrm{div}u dx+C\left\|\sqrt{\rho}u\right\|_{L^2}\|\nabla G\|_{L^2}+C\|H\|_{L^2}\|u\nabla H\|_{L^2}+C\|\nabla u\|_{L^2}^2+C\\
&\le\frac{d}{dt}\int P\mathrm{div}u dx+\epsilon_1\|\nabla G\|_{L^2}^2+\|u\nabla H\|_{L^2}^2+C\|\nabla u\|_{L^2}^2+C,
\end{aligned}\tag{3.11}$$

where we have used (3.1), (3.6) and Young's inequality. Combining this with (3.7) and choosing ϵ_1 small enough, one has

$$K_1\le\frac{d}{dt}\int P\mathrm{div}u dx+\frac{1}{4}\left\|\sqrt{\rho}u_t\right\|_{L^2}^2+C(1+\|u\nabla u\|_{L^2}^2+\|H\nabla H\|_{L^2}^2+\|u\nabla H\|_{L^2}^2+\|\nabla u\|_{L^2}^2).\tag{3.12}$$

Noticing $\mathrm{div}H=0$, one has from integrating by parts that

$$\begin{aligned}
K_2&=-\int H\cdot\nabla u_t\cdot H-\frac{1}{2}|H|^2\mathrm{div}u_t dx\\
&=-\frac{d}{dt}\int H\cdot\nabla u\cdot H-\frac{1}{2}|H|^2\mathrm{div}u dx+\int H_t\cdot\nabla u\cdot H+H\cdot\nabla u\cdot H_t-H\cdot H_t\mathrm{div}u dx\\
&\le-\frac{d}{dt}\int H\cdot\nabla u\cdot H-\frac{1}{2}|H|^2\mathrm{div}u dx+\frac{1}{4}\|H_t\|_{L^2}^2+C\|H\nabla u\|_{L^2}^2.
\end{aligned}\tag{3.13}$$

For the last four terms on the right-hand side of (3.9), one has

$$\left|\sum_{i=3}^{6}K_i\right|\le\frac{1}{4}\left(\left\|\sqrt{\rho}u_t\right\|_{L^2}^2+\|H_t\|_{L^2}^2\right)+C(\|u\nabla H\|_{L^2}^2+\|u\nabla u\|_{L^2}^2+\|H\nabla u\|_{L^2}^2).\tag{3.14}$$

Thus, combining (3.9)–(3.14), one obtains

$$\begin{aligned}
&\frac{d}{dt}\int \frac{\mu}{2}|\nabla u|^2+\frac{\mu+\lambda}{2}(\mathrm{div}u)^2+\frac{\nu}{2}|\nabla H|^2dx+\frac{1}{2}\int \rho|u_t|^2+|H_t|^2dx\\
&\le\frac{d}{dt}\int P\mathrm{div}u-H\cdot\nabla u\cdot H+\frac{1}{2}|H|^2\mathrm{div}u dx+C\|\nabla u\|_{L^2}^2+C\\
&\quad+C(\|u\nabla H\|_{L^2}^2+\|u\nabla u\|_{L^2}^2+\|H\nabla u\|_{L^2}^2+\|H\nabla H\|_{L^2}^2).
\end{aligned}\tag{3.15}$$

For the last term on the right-hand side of (3.15), one has from (3.1) and the interpolation inequality that

$$\begin{aligned}
&\|u\nabla H\|_{L^2}+\|u\nabla u\|_{L^2}+\|H\nabla u\|_{L^2}+\|H\nabla H\|_{L^2}\\
&\le \||u|^2\|_{L^6}^{\frac12}\left(\|\nabla u\|_{L^{\frac{12}{5}}}+\|\nabla H\|_{L^{\frac{12}{5}}}\right)+\|H\|_{L^4}(\|\nabla u\|_{L^4}+\|\nabla H\|_{L^4})\\
&\le C\|u\nabla u\|_{L^2}^{\frac12}\left(\|\nabla u\|_{L^2}^{\frac34}\|\nabla u\|_{L^6}^{\frac14}+\|\nabla H\|_{L^2}^{\frac34}\|\nabla H\|_{L^6}^{\frac14}\right)+C(\|\nabla u\|_{L^4}+\|\nabla H\|_{L^4})\\
&\le \delta(\|\nabla u\|_{L^6}+\|\nabla H\|_{L^6})+C\left(\|u\nabla u\|_{L^2}^{\frac23}+1\right)(\|\nabla u\|_{L^2}+\|\nabla H\|_{L^2}).
\end{aligned}\tag{3.16}$$

By the standard L^p-estimate, one deduces from (1.6) and (1.7) that

$$\begin{aligned}
\|\nabla u\|_{L^6}&\le \|\nabla v\|_{L^6}+\|\nabla\omega\|_{L^6}\\
&\le C(\|P\|_{L^6}+\||H|^2\|_{L^6}+\|\nabla^2\omega\|_{L^2})\\
&\le C\left(1+\|H\nabla H\|_{L^2}+\left\|\sqrt{\rho}u_t\right\|_{L^2}+\|u\nabla u\|_{L^2}\right).
\end{aligned}\tag{3.17}$$

Similarly, one has

$$\|\nabla H\|_{L^6}\le\|\nabla^2 H\|_{L^2}\le C(\|H_t\|_{L^2}+\|u\nabla H\|_{L^2}+\|H\nabla u\|_{L^2}).\tag{3.18}$$

Thus, putting (3.17) and (3.18) into (3.16), choosing δ small enough, one gets

$$\begin{aligned}
&\|u\nabla H\|_{L^2}+\|u\nabla u\|_{L^2}+\|H\nabla u\|_{L^2}+\|H\nabla H\|_{L^2}\\
&\le\frac14\left(\left\|\sqrt{\rho}u_t\right\|_{L^2}+\|H_t\|_{L^2}\right)+C\left(\|u\nabla u\|_{L^2}^{\frac23}+1\right)(\|\nabla u\|_{L^2}+\|\nabla H\|_{L^2})+C.
\end{aligned}\tag{3.19}$$

Then, putting (3.16) and (3.19) into (3.15), by Gronwall's inequality, and noticing that

$$\int P\mathrm{div}u-H\cdot\nabla u\cdot H+\frac12|H|^2\mathrm{div}u dx\le\frac{\mu}{4}\|\nabla u\|_{L^2}^2+C,$$

one gets from (3.3) that

$$\sup_{0\le t\le T}(\|\nabla u\|_{L^2}^2+\|\nabla H\|_{L^2}^2)+\int_0^T\left\|\sqrt{\rho}u_t\right\|_{L^2}^2+\|H_t\|_{L^2}^2dt\le C.\tag{3.20}$$

On the other hand, (3.18) combined with (3.19) and (3.20), completes the proof of Lemma 3.3. □

Next we prove the boundedness of $\left\|\sqrt{\rho}\dot u\right\|_{L^2}$, $\|H_t\|_{L^2}$ and $\|\nabla H\|_{H^1}$ by using the compatibility condition (1.4).

Lemma 3.4. *Under the assumption of (3.1) and $3\mu>\lambda$, for any $0\le T<T^*$, one has*

$$\sup_{0\le t\le T}\left(\left\|\sqrt{\rho}\dot u\right\|^2+\|H_t\|_{L^2}^2+\|\nabla H\|_{H^1}^2\right)+\int_0^T\|\nabla\dot u\|_{L^2}^2+\|\nabla H_t\|_{L^2}^2dt\le C.\tag{3.21}$$

Proof. Operating $\dot u^j[\partial/\partial t+\mathrm{div}(u\cdot)]$ to both sides of $(1.1)_2$, summing with respect to j, and integrating the resulting equation over $\mathbb{R}^3$, one obtains

$$\begin{aligned}
\frac12\frac{d}{dt}\int\rho|\dot u|^2dx&=-\int\dot u^j[\partial_jP_t+\mathrm{div}(\partial_jPu)]dx-\int\dot u^j[\partial_j(H^iH^i_t)+\mathrm{div}(H^i\partial_jH^iu)]dx\\
&\quad+\int\dot u^j[\partial_t(H^i\partial_iH^j)+\mathrm{div}(H^i\partial_iH^ju)]dx+\mu\int\dot u^j[\Delta u^j_t+\mathrm{div}(u\Delta u^j)]dx\\
&\quad+(\mu+\lambda)\int\dot u^j[\partial_t\partial_j\mathrm{div}u+\mathrm{div}(u\partial_j\mathrm{div}u)]dx\triangleq\sum_{i=1}^5 I_i.
\end{aligned}\tag{3.22}$$

Using (3.10), (3.1) and integrating by parts, one has

$$\begin{aligned}
I_1&=\int[-\rho P'(\rho)\partial_iu^i\partial_j\dot u^j+P(\rho)\partial_i(u^i\partial_j\dot u^j)-P(\rho)\partial_j(u^i\partial_i\dot u^j)]dx\\
&=\int[-\rho P'(\rho)\partial_iu^i\partial_j\dot u^j+P(\rho)\partial_iu^i\partial_j\dot u^j-\partial_ju^i\partial_i\dot u^jP(\rho)]dx\\
&\le\delta\|\nabla\dot u\|_{L^2}^2+C(\delta).
\end{aligned}$$

Integrating by parts, one obtains from (3.8), Hölder and interpolation inequalities that

$$\begin{aligned} I_2 &= \int (\partial_j \dot{u}^j H^i H_t^i + \partial_k \dot{u}^j u^k \partial_j H^i H^i) dx \\ &\leq \|\nabla \dot{u}\|_{L^2} (\|H\|_{L^6} \|H_t\|_{L^3} + \|u\|_{L^6} \|H\|_{L^6} \|\nabla H\|_{L^6}) \\ &\leq \|\nabla \dot{u}\|_{L^2} \left(\|H_t\|_{L^2}^{\frac{1}{2}} \|\nabla H_t\|_{L^2}^{\frac{1}{2}} + \|\nabla H\|_{H^1} \right) \\ &\leq \delta \|\nabla \dot{u}\|_{L^2}^2 + C(\delta)(\|H_t\|_{L^2} \|\nabla H_t\|_{L^2} + \|\nabla H\|_{H^1}^2). \end{aligned}$$

Similarly,

$$\begin{aligned} I_3 &= \int -\partial_i \dot{u}^j (H^j H_t^i + H^i H_t^j) - \partial_k \dot{u}^j H^i \partial_i H^j u^k dx \\ &\leq \delta \|\nabla \dot{u}\|_{L^2}^2 + C(\delta)(\|H_t\|_{L^2} \|\nabla H_t\|_{L^2} + \|\nabla H\|_{H^1}^2) \end{aligned}$$

where we have used the fact that $\text{div}H = 0$. Integrating by parts, we obtain

$$\begin{aligned} I_4 &= -\mu \int (\partial_k \dot{u}^j \partial_k u_t^j + \partial_i \dot{u}^j u^i \Delta u^j) dx \\ &= -\mu \int (\partial_k \dot{u}^j \partial_k u_t^j - \partial_{ik}^2 \dot{u}^j u^i \partial_k u^j - \partial_i \dot{u}^j \partial_k u^i \partial_k u^j) dx \\ &= -\mu \int (|\nabla \dot{u}|^2 + \partial_k \dot{u}^j \partial_i u^i \partial_k u^j - \partial_k \dot{u}^j \partial_k u^i \partial_i u^j - \partial_i \dot{u}^j \partial_k u^i \partial_k u^j) dx \\ &\leq -\frac{3\mu}{4} \|\nabla \dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4, \end{aligned}$$

and similarly,

$$I_5 \leq -\frac{\mu + \lambda}{2} \|\text{div}\dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4.$$

Then, putting the estimates of I_i $(i = 1, \ldots, 5)$ into (3.22) and taking δ small enough, we obtain

$$\frac{d}{dt} \int \rho |\dot{u}|^2 dx + \int |\nabla \dot{u}|^2 dx \leq C(\|H_t\|_{L^2} \|\nabla H_t\|_{L^2} + \|\nabla H\|_{H^1}^2 + \|\nabla u\|_{L^4}^4 + 1). \tag{3.23}$$

To estimate $\|H_t\|_{L^2}$, we differentiate $(1.1)_3$ with respect to t, multiply the resulting equations by H_t in L^2, and integrate by parts over $\mathbb{R}^3$ to get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int |H_t| dx + \nu \int |\nabla H_t|^2 dx \\ &= \int (H \cdot \nabla u_t - u_t \cdot \nabla H - H \text{div} u_t) \cdot H_t dx + \int (H_t \cdot \nabla u - u \cdot \nabla H_t - H_t \text{div} u) \cdot H_t \\ &\triangleq J_1 + J_2. \end{aligned} \tag{3.24}$$

Since $u_t = \dot{u} - u \cdot \nabla u$, integrating by parts and using (3.8), we deduce

$$\begin{aligned} J_1 &= \int (H \cdot \nabla \dot{u} - \dot{u} \cdot \nabla H - H \text{div} \dot{u}) \cdot H_t dx + \int (H_i \partial_i H_t^j - H^k \partial_j H_t^k)(u \cdot \nabla u^j) dx \\ &\leq C(\|H\|_{L^6} \|H_t\|_{L^3} \|\nabla \dot{u}\|_{L^2} + \|\dot{u}\|_{L^6} \|\nabla H\|_{L^2} \|H_t\|_{L^3}) + \|H\|_{L^{12}} \|\nabla H_t\|_{L^2} \|\nabla u\|_{L^4} \|u\|_{L^6} \\ &\leq C\left(\|H_t\|_{L^2}^{\frac{1}{2}} \|\nabla H_t\|_{L^2}^{\frac{1}{2}} \|\nabla \dot{u}\|_{L^2} + \|\nabla \dot{u}\|_{L^2} \|H_t\|_{L^2}^{\frac{1}{2}} \|\nabla H_t\|_{L^2}^{\frac{1}{2}} \right) + C\|\nabla H_t\|_{L^2} \|\nabla u\|_{L^4} \\ &\leq \epsilon_2 \|\nabla H_t\|_{L^2}^2 + \epsilon_3 \|\nabla \dot{u}\|_{L^2}^2 + C(\epsilon_2, \epsilon_3)(\|H_t\|_{L^2}^2 + \|\nabla u\|_{L^4}^2). \end{aligned}$$

Next, integrating by parts and using the interpolation inequality, we have

$$J_2 = \int \left(H_t \cdot \nabla u - \frac{1}{2} H_t \text{div} u \right) \cdot H_t \leq C\|H_t\|_{L^4}^2 \leq \epsilon_2 \|\nabla H_t\|_{L^2}^2 + C(\epsilon_2) \|H_t\|_{L^2}^2.$$

Thus, plugging the estimates of J_1 and J_2 into (3.24) and choosing $\epsilon_2 > 0$ small enough, we have

$$\frac{1}{2} \frac{d}{dt} \int |H_t| dx + \frac{\nu}{2} \int |\nabla H_t|^2 dx \leq \epsilon_3 \|\nabla \dot{u}\|_{L^2}^2 + C(\epsilon_3)(\|H_t\|_{L^2}^2 + \|\nabla u\|_{L^4}^2).$$

This, together with (3.23), taking $\epsilon_3 > 0$ small enough, yields

$$\frac{d}{dt}\int \rho|\dot{u}|^2 + |H_t|^2 dx + \int |\nabla\dot{u}|^2 + |\nabla H_t|^2 dx \le C(\|H_t\|_{L^2}^2 + \|\nabla H\|_{H^1}^2 + \|\nabla u\|_{L^4}^4 + 1). \tag{3.25}$$

To deal with the right-hand side of (3.25), we first make use of (1.6)–(1.7) to infer from the standard L^p-estimate and (2.4)–(2.5) that

$$\begin{aligned}
\|\nabla u\|_{L^4}^4 &\le C(\|\nabla v\|_{L^4}^4 + \|\nabla\omega\|_{L^4}^4)\\
&\le C(\|P\|_{L^4}^4 + \|H^2\|_{L^4}^4 + \|\nabla\omega\|_{L^2}^2\|\nabla\omega\|_{L^\infty}^2)\\
&\le C\left(1 + \|H\|_{L^\infty}^4\|H\|_{L^4}^4 + (\|\nabla u\|_{L^2} + \|\nabla v\|_{L^2})^{\frac{5}{2}}\|\nabla^2\omega\|_{L^6}^{\frac{3}{2}}\right)\\
&\le C\left(1 + \|H\|_{L^6}^2\|\nabla H\|_{L^6}^2 + (\|\rho\dot{u}\|_{L^6} + \|H\nabla H\|_{L^6})^{\frac{3}{2}}\right)\\
&\le \delta\|\nabla\dot{u}\|_{L^2}^2 + C(1 + \|\nabla H\|_{H^1}^3),
\end{aligned} \tag{3.26}$$

where one has used (2.2). Note that from (3.18) and (2.2),

$$\begin{aligned}
\|\nabla H\|_{H^1} &\le C(\|H_t\|_{L^2} + \|u\|_{L^6}\|\nabla H\|_{L^3} + \|H\|_{L^\infty}\|\nabla u\|_{L^2} + \|\nabla H\|_{L^2})\\
&\le C\left(1 + \|H_t\|_{L^2} + \|\nabla H\|_{H^1}^{\frac{1}{2}}\right),
\end{aligned}$$

and hence,

$$\|\nabla H\|_{H^1} \le C(1 + \|H_t\|_{L^2}). \tag{3.27}$$

This, together with (3.26), leads to

$$\|\nabla u\|_{L^4}^4 \le \delta\|\nabla\dot{u}\|_{L^2}^2 + C(1 + \|H_t\|_{L^2}^3). \tag{3.28}$$

Now, choosing δ small enough, we see from (3.23), (3.25) and (3.28) that

$$\frac{d}{dt}\int \rho|\dot{u}|^2 + |H_t|^2 dx + \int |\nabla\dot{u}|^2 + |\nabla H_t|^2 dx \le C(1 + \|H_t\|_{L^2}^3),$$

from which and the fact that $\|H_t\|_{L^2}^2 \in L^1(0,T)$ due to (3.8), we immediately obtain (3.21) by applying Gronwall's inequality and the compatibility condition. We also deduce the boundedness of $\|\nabla H\|_{H^1}$ from (3.27) and (3.21). □

4. High order regularity estimates

With the estimates obtained in Section 3 in hand, we start to improve the regularity of (ρ, u, H) in the following lemmas. First, we give the higher regularity of ω.

Lemma 4.1. *Under the assumption of (3.1) and $3\mu > \lambda$, for any $0 \le T < T^*$ and $3 < q \le 6$, it holds that*

$$\int_0^T (\|\nabla^2 w\|_{L^q}^2 + \|\nabla\omega\|_{L^\infty}^2)dt \le C. \tag{4.1}$$

Proof. We will use the following interpolation inequality

$$\|f\|_{L^q} \le C\|f\|_{L^2} + C\|\nabla f\|_{L^2}, \quad 2 \le q \le 6. \tag{4.2}$$

From (4.2) and the Sobolev embedding $W^{1,q} \hookrightarrow L^\infty$ for $3 < q \le 6$ and the interpolation inequality, one has

$$\begin{aligned}
\|\nabla\omega\|_{L^\infty} &\le C(\|\nabla\omega\|_{L^q} + \|\nabla^2\omega\|_{L^q})\\
&\le C(\|\nabla\omega\|_{L^2} + \|\nabla^2\omega\|_{L^2} + \|\nabla^2\omega\|_{L^q})\\
&\le C\|\rho\dot{u}\|_{L^q} + C\|\nabla H\|_{L^q} + C\\
&\le C(\|\rho\dot{u}\|_{L^2} + \|\rho\dot{u}\|_{L^6}) + C\\
&\le C\|\nabla\dot{u}\|_{L^2} + C.
\end{aligned} \tag{4.3}$$

By (3.21) and (4.3), one completes the proof of Lemma 4.1. □

With Lemma 4.1 in hand, we will give the gradient estimates of the density.

Lemma 4.2. *Under the assumption of (3.1) and $3\mu > \lambda$, one has for any $0 \le T < T^*$ and $3 < q \le 6$ that*

$$\sup_{0\le t\le T}(\|\rho\|_{H^1\cap W^{1,q}} + \|\nabla u\|_{H^1}) + \int_0^T \|\nabla^2 u\|_{L^q}^2 + \|\nabla^2 H\|_{L^q}^2 + \|\nabla u\|_{L^\infty}^2 dt \le C. \tag{4.4}$$

Proof. The proof follows the idea in [18]. First, differentiating $(1.1)_1$ with respect to x_i and multiplying the resulting equation by $|\partial_i\rho|^{q-2}\partial_i\rho$ in L^2, we have after integrating by parts and summing them up

$$\frac{d}{dt}\|\nabla\rho\|_{L^q}^q \le C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^q}^q + C\|\nabla^2 u\|_{L^q}\|\nabla\rho\|_{L^q}^{q-1},$$

that is

$$\begin{aligned}\frac{d}{dt}\|\nabla\rho\|_{L^q} &\le C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^q} + C\|\nabla^2 u\|_{L^q}\\ &\le C(\|\nabla v\|_{L^\infty} + \|\nabla\omega\|_{L^\infty})\|\nabla\rho\|_{L^q} + C(\|\nabla^2 v\|_{L^q} + \|\nabla^2\omega\|_{L^q})\\ &\le C(1 + \|\nabla v\|_{L^\infty} + \|\nabla\omega\|_{L^\infty})\|\nabla\rho\|_{L^q} + C(1 + \|\nabla^2 w\|_{L^q}),\end{aligned} \tag{4.5}$$

where one has used (2.4) and (4.2). To close (4.5), one has to bound $\|\nabla v\|_{L^\infty}$. In fact, (2.6) and (2.7) show that if $3 < q \le 6$ then

$$\begin{aligned}\|\nabla v\|_{L^\infty} &\le C(1 + \|\nabla v\|_{BMO}\ln(e + \|\nabla^2 v\|_{L^q}))\\ &\le C\left(1 + \left\|P + \frac{1}{2}|H|^2\right\|_{L^\infty} + \left\|P + \frac{1}{2}|H|^2\right\|_{L^2}\right)\ln(e + \|\nabla^2 v\|_{L^q})\\ &\le C(1 + \ln(e + \|\nabla\rho\|_{L^q}) + \ln(e + \|\nabla H\|_{L^q}))\\ &\le C(1 + \ln(e + \|\nabla\rho\|_{L^q})).\end{aligned} \tag{4.6}$$

Substituting (4.6) into (4.5), we obtain

$$\frac{d}{dt}(e + \|\nabla\rho\|_{L^q}) \le C(1 + \|\nabla\omega\|_{L^\infty})\|\nabla\rho\|_{L^q} + C\ln(e + \|\nabla\rho\|_{L^q})\|\nabla\rho\|_{L^q} + C(1 + \|\nabla^2\omega\|_{L^q}). \tag{4.7}$$

Set

$$f(t) = e + \|\nabla\rho\|_{L^q}, \qquad g(t) = 1 + \|\nabla\omega\|_{L^\infty} + \|\nabla^2\omega\|_{L^q}.$$

By (4.7), we get

$$f'(t) \le Cg(t)f(t) + Cf(t)\ln f(t) + Cg(t),$$

which yields

$$(\ln f(t))' \le Cg(t) + C\ln f(t), \tag{4.8}$$

due to $f(t) > 1$. Note that (4.1) implies

$$\int_0^T g(t)dt \le C,$$

which together with (4.8) and Gronwall's inequality yields

$$\sup_{0\le t\le T}\ln f(t) \le C.$$

Consequently,

$$\sup_{0\le t\le T}\|\nabla\rho\|_{L^q} \le C \quad \text{for any } q \in (3, 6]. \tag{4.9}$$

From (4.9) and the Sobolev embedding theorem, for $3 < q \le 6$, we get

$$\int_0^T \|\nabla v\|_{L^\infty}^2 dt \le \int_0^T \|\nabla v\|_{L^q}^2 + \|\nabla^2 v\|_{L^q}^2 dt \le C. \tag{4.10}$$

This, combining with (4.1), deduces

$$\int_0^T \|\nabla^2 u\|_{L^q}^2 + \|\nabla u\|_{L^\infty}^2 dt \le C. \tag{4.11}$$

Moreover, the standard L^2-estimate of the elliptic system and (1.1)$_2$, together with (3.21), imply

$$\|\nabla^2 u\|_{L^2} \le C\left(\left\|\sqrt{\rho}\dot{u}\right\|_{L^2} + \|\nabla\rho\|_{L^2} + \|H\nabla H\|_{L^2}\right) \le C(1 + \|\nabla\rho\|_{L^2}),$$

which, combined with (4.5) (with $q = 2$), (4.11) and Gronwall's inequality, yields

$$\sup_{0\le t\le T} \|\nabla\rho\|_{L^2} \le C.$$

From this, we have

$$\|\nabla^2 u\|_{L^2} \le C.$$

We now estimate $\|\nabla^2 H\|_{L^q}$. Indeed, using (3.4), (3.21) and the interpolation inequality, we have from (1.1)$_3$ and the L^p-estimate of the elliptic system that

$$\int_0^T \|\nabla^2 H\|_{L^q}^2 dt \le C\int_0^T \|H_t\|_{L^q}^2 + \|\nabla u\|_{L^q}^2 + \|\nabla H\|_{L^q}^2 dt \le C\int_0^T \|\nabla H_t\|_{L^2}^2 dt + C \le C.$$

This completes the proof of Lemma 4.2. □

As in [23], we have the following lemma.

Lemma 4.3. *Under the assumption of (3.1) and $3\mu > \lambda$, for any $0 \le T < T^*$, it holds that*

$$\sup_{0\le t\le T} \left\|\sqrt{\rho}u_t\right\|_{L^2}^2 + \int_0^T \|\nabla u_t\|_{L^2}^2 dt \le C. \tag{4.12}$$

5. Proof of Theorem 1.2

In view of (4.4) and (3.21), it is clear that the functions $(\rho, u, H)(x, t = T^*) = \lim_{t\to T^*}(\rho, u, H)$ have the same regularities imposed on the initial data (1.3) at the time $t = T^*$. Therefore, we can take $(\rho, u, H)|_{t=T^*}$ as initial data and apply the local existence theorem (cf. Theorem 1.1) to extend the local strong solutions beyond T^*. This contradicts the assumption on T^*. The proof of Theorem 1.2 is therefore completed.

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