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Solid extensions of the Cesàro operator on the Hardy space $H^2(\mathbb{D})$

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We introduce and study the largest Banach space of analytic functions on the unit disc which is *solid* for the coefficient-wise order and to which the classical Cesàro operator $\mathcal{C}: H^2 \to H^2$ can be continuously extended, while still maintaining its values in H^2. Properties of this Banach space $\mathcal{H}(ces_2)$ as well as a characterization of individual analytic functions which belong to $\mathcal{H}(ces_2)$ are presented. In addition, both the multiplier space of $\mathcal{H}(ces_2)$ and the spectrum of $\mathcal{C} : \mathcal{H}(ces_2) \to \mathcal{H}(ces_2)$ are determined.

1. Introduction

The study of optimal domains for certain operators is a tool for dealing with refinement of inequalities and extensions of such operators. For instance, the Hausdorff–Young inequality asserts $\|\hat{f}\|_{p'} \le \|f\|_p$ for $f \in L^p(\mathbb{T})$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $1 \le p \le 2$. Via a study of the optimal domain for the underlying kernel operator it is shown in [15] that there exists a *largest* Banach function space $\mathbf{F}^p(\mathbb{T})$ having order continuous norm and satisfying $L^p(\mathbb{T}) \subsetneq \mathbf{F}^p(\mathbb{T}) \subsetneq L^1(\mathbb{T})$, with continuous inclusions, such that $\|\hat{f}\|_{p'} \le \|f\|_{\mathbf{F}^p(\mathbb{T})}$ for all $f \in \mathbf{F}^p(\mathbb{T})$. A similar approach leads to a sharpening of Sobolev's inequality in rearrangement invariant spaces, with applications to compactness properties of Sobolev embeddings [4,5]. In [8] the Hardy integral operator $S: f \mapsto \frac{1}{x}\int_0^x f(y)\,dy$, $x \in (0,\infty)$, for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$, considered with values in a rearrangement invariant space, is also treated from this viewpoint.

Consider now the Cesàro operator, given by

$$\mathcal{C}(f)(z) := \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z^n \tag{1}$$

with $f(z) = \sum_0^\infty a_k z^k \in H(\mathbb{D})$ (the space of all analytic functions on the open unit disc $\mathbb{D}$), which is bounded on the Hardy space $H^p := H^p(\mathbb{D})$ for every $1 \le p < \infty$; see [18] and the references therein. In [6] it is shown that $\mathcal{C}$ has a continuous *optimal extension* $\mathcal{C}: [\mathcal{C}, H^p] \to H^p$ where, relative to $\|f\|_{[\mathcal{C},H^p]} := \|\mathcal{C}(f)\|_{H^p}$ as its norm, $[\mathcal{C}, H^p]$ is a Banach space of analytic functions on $\mathbb{D}$ determined by the property of being the largest amongst all Banach spaces of analytic functions X such that $\mathcal{C}$ maps X continuously into H^p. For the particular case $p = 2$, this *optimal domain* $[\mathcal{C}, H^2]$ is a Hilbert space

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characterized by

$$\sum_{n=0}^{\infty} a_n z^n \in [\mathcal{C}, H^2] \iff \left(\frac{1}{n+1}\sum_{k=0}^{n} a_k\right) \in \ell^2, \tag{2}$$

with norm $\|\sum_0^\infty a_n z^n\|_{[\mathcal{C},H^2]} := \|(\frac{1}{n+1}\sum_0^n a_k)_0^\infty\|_{\ell^2}$ [6, Theorem 3.8(iii)].

Unlike for H^2, it is a priori unclear whether $[\mathcal{C}, H^2]$, which is genuinely larger than H^2, is solid for the pointwise order, i.e., whether $g \in [\mathcal{C}, H^2]$ whenever $g \in H(\mathbb{D})$ satisfies $|g(z)| \le |f(z)|$, for all $z \in \mathbb{D}$, with $f \in [\mathcal{C}, H^2]$. That this is so follows from a growth characterization for elements of $[\mathcal{C}, H^2]$, [6, Corollary 3.3], namely, $f \in H(\mathbb{D})$ belongs to $[\mathcal{C}, H^2]$ if and only if

$$\int_0^{2\pi}\int_0^1 \frac{|f(re^{i\theta})|^2}{|1-re^{i\theta}|^2}(1-r)\,drd\theta < \infty. \tag{3}$$

There is also a significant interest in subspaces $X \subset H(\mathbb{D})$ which are *solid for the coefficient-wise order*, i.e., the function $g(z) = \sum_0^\infty b_n z^n \in X$ whenever $|b_n| \le |a_n|$, for $n \ge 0$, with $f(z) = \sum_0^\infty a_n z^n \in X$; see for example [14] and the references therein. Whereas H^2 is solid for this order, this property does *not* transfer to its optimal domain space $[\mathcal{C}, H^2]$. This follows from (2) by considering $g(z) = (1-z)^{-1}$ and $f(z) = (1+z)^{-1}$.

So, it is meaningful to consider the *solid core* of the optimal domain space $[\mathcal{C}, H^2]$, namely the largest of all subspaces within $[\mathcal{C}, H^2]$ which are solid for the coefficient-wise order. Direct inspection of (2) shows that this solid core of $[\mathcal{C}, H^2]$ is precisely the space

$$\mathcal{H}(ces_2) := \left\{\sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}) : \left(\frac{1}{n+1}\sum_{k=0}^{n} |a_k|\right)_{n=0}^{\infty} \in \ell^2\right\},$$

which contains H^2 as a (solid) subspace.

The aim of this paper is to study the space $\mathcal{H}(ces_2)$ and the operator $\mathcal{C}$ acting on it. If $\mathcal{H}(ces_2)$ is equipped with its natural norm (cf. (5) below), then $\mathcal{C}$ actually maps $\mathcal{H}(ces_2)$ continuously into H^2. In Section 2 we undertake a detailed analysis of the Banach space of analytic functions $\mathcal{H}(ces_2)$. For this purpose we need to consider the Cesàro operator acting on various sequence spaces in $\mathbb{C}^{\mathbb{N}}$. In particular, we characterize those analytic functions belonging to $\mathcal{H}(ces_2)$ via a monotonicity property of their Taylor coefficients (Theorem 2.8). Section 3 is devoted to identifying the continuous multiplication operators on $\mathcal{H}(ces_2)$, i.e., the *multiplier space* of $\mathcal{H}(ces_2)$; these turn out to be precisely those given by multiplication via analytic functions with absolutely summable Taylor coefficients (Theorem 3.1). There is a significant interest in identifying the spectrum of the Cesàro operator acting in various Banach spaces of analytic functions; see [1,16], and the references therein. In Section 4 we show that the spectrum of $\mathcal{C}: \mathcal{H}(ces_2) \to \mathcal{H}(ces_2)$ is $\sigma(\mathcal{C}) = \{z \in \mathbb{C} : |1-z| \le 1\}$; see Theorem 4.1. In view of these results we are in the interesting situation where the spectrum of $\mathcal{C}: \mathcal{H}(ces_2) \to \mathcal{H}(ces_2)$ coincides with that of the initial operator $\mathcal{C}: H^2 \to H^2$, but the multiplier space of $\mathcal{H}(ces_2)$ is significantly smaller (being isomorphic to ℓ^1) than that of H^2 (namely, H^∞). It is also noteworthy that the solid space $\mathcal{H}(ces_2)$ is in a certain sense "maximal". Namely, if one considers $\mathcal{C}: \mathcal{H}(ces_2) \to \mathcal{H}(ces_2)$, rather than $\mathcal{C}: H^2 \to H^2$, then its optimal domain space $[\mathcal{C}, \mathcal{H}(ces_2)]$ contains $[\mathcal{C}, H^2]$ as a *proper* subspace. Remarkably, however, the solid core of the larger space $[\mathcal{C}, \mathcal{H}(ces_2)]$ is again $\mathcal{H}(ces_2)$, that is, *no* further solid extension occurs; see Proposition 2.10.

2. The Banach space of analytic functions $\mathcal{H}(ces_2)$

A precise description of the analytic functions belonging to $\mathcal{H}(ces_2)$ is possible. To establish this we need to study in some detail the Cesàro operator acting on sequence spaces. We use the same notation for the Cesàro operator acting on functions (via (1)) as for the Cesàro operator acting on sequences. Thus, writing elements of $\mathbb{C}^{\mathbb{N}}$ as $a = (a_n)_{n=0}^\infty$, the Cesàro operator $\mathcal{C}: \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ is given by

$$a = (a_n)_0^\infty \longmapsto \mathcal{C}(a) := \left(\frac{1}{n+1}\sum_{k=0}^{n} a_k\right)_{n=0}^{\infty}.$$

It is a bijection on $\mathbb{C}^{\mathbb{N}}$ with inverse $\mathcal{C}^{-1}((b_n)_0^\infty) = \big((n+1)b_n - nb_{n-1}\big)_0^\infty$, where $b_{-1} := 0$. Let $\mathbb{C}_+^{\mathbb{N}}$ denote the cone of all non-negative sequences, in which case we have $\mathcal{C}(a) \in \mathbb{C}_+^{\mathbb{N}}$ whenever $a \in \mathbb{C}_+^{\mathbb{N}}$. Moreover, $|\mathcal{C}(a)| \le \mathcal{C}(|a|)$ for $a \in \mathbb{C}^{\mathbb{N}}$, where $|a| := (|a_n|)_0^\infty \in \mathbb{C}^{\mathbb{N}}$ is the modulus of a in the complex vector lattice $\mathbb{C}^{\mathbb{N}}$ and $\le$ is the coordinate-wise order in $\mathbb{R}^{\mathbb{N}}$.

Recall that $\mathcal{C}: \ell^2 \to \ell^2$ continuously with operator norm $\|\mathcal{C}\|_2 = 2$ [11, Theorem 326]. Thus, we may also consider its *optimal domain*, namely

$$[\mathcal{C}, \ell^2] := \left\{a = (a_n)_0^\infty \in \mathbb{C}^{\mathbb{N}} : \mathcal{C}(a) = \left(\frac{1}{n+1}\sum_{k=0}^{n} a_k\right)_{n=0}^{\infty} \in \ell^2\right\},$$

which can be shown to be a Banach space for the norm

$$\|a\|_{[\mathcal{C},\ell^2]} := \|\mathcal{C}(a)\|_{\ell^2} = \left(\sum_{n=0}^{\infty}\left|\frac{1}{n+1}\sum_{k=0}^{n} a_k\right|^2\right)^{1/2}.$$

Via the Cesàro operator $\mathcal{C}\colon [\mathcal{C},\ell^2] \to \ell^2$ the Banach sequence space $[\mathcal{C},\ell^2]$ is linearly isomorphic and isometric to ℓ^2. However, unlike ℓ^2, $[\mathcal{C},\ell^2]$ is not solid for the coordinate-wise order (consider $a := ((-1)^n)_0^\infty \in [\mathcal{C},\ell^2]$ whereas $|a| \notin [\mathcal{C},\ell^2]$). The *solid core* of $[\mathcal{C},\ell^2]$, with respect to the coordinate-wise order, is clearly the space

$$\left\{a = (a_n)_0^\infty \in \mathbb{C}^{\mathbb{N}} : \left(\frac{1}{n+1}\sum_{k=0}^{n}|a_k|\right)_{n=0}^{\infty} \in \ell^2\right\}.$$

It is the known Banach sequence space ces_2, equipped with the norm

$$\|a\|_{ces_2} := \left(\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}|a_k|\right)^2\right)^{1/2} = \|\mathcal{C}(|a|)\|_{\ell^2}, \tag{4}$$

which is thoroughly treated in [2]. Note that the positive cone of ces_2 and that of $[\mathcal{C},\ell^2]$ coincide. From the continuity of $\mathcal{C}$ on ℓ^2 it follows that $\ell^2 \subseteq ces_2 \subseteq [\mathcal{C},\ell^2]$, with each inclusion continuous. Moreover, both embeddings are strict. It follows from $|\mathcal{C}(a)| \le \mathcal{C}(|a|)$ and (4) that $\mathcal{C}\colon ces_2 \to \ell^2$ continuously.

Remark 2.1. (i) The largest amongst the spaces ℓ^p, for $1 \le p \le \infty$, which satisfy $\ell^p \subseteq ces_2$ is ℓ^2. The space $ces_2 \not\subset \ell^\infty$; actually, it contains sequences with arbitrarily large terms. Indeed, given any increasing sequence of positive integers $(k_n)_0^\infty$, the element $a = \sum_{n=0}^{\infty} k_n e_{i_n}$, where $i_n = k_n^2(n+1)^4$, belongs to ces_2. Here $e_n := (\delta_{in})_{i=0}^\infty$ for $n \ge 0$.

(ii) Despite (i) there is still some control on the growth of the partial sums of elements from ces_2. Indeed,

$$\lim_{n\to\infty}\frac{\sum_{k=0}^{n}|a_k|}{\sqrt{n+1}} = 0, \quad a \in ces_2.$$

To see this, let $n \in \mathbb{N}$ and observe that

$$\begin{aligned}\|a\|_{ces_2}^2 &\ge \sum_{m=n}^{\infty}\left(\frac{1}{m+1}\sum_{k=0}^{m}|a_k|\right)^2 \ge \sum_{m=n}^{\infty}\left(\frac{1}{m+1}\sum_{k=0}^{n}|a_k|\right)^2\\ &\ge \left(\sum_{k=0}^{n}|a_k|\right)^2\sum_{m=n}^{\infty}\frac{1}{(m+1)^2} \ge \left(\sum_{k=0}^{n}|a_k|\right)^2\frac{1}{n+1}.\end{aligned}$$

The claim now follows because $\sum_{m=n}^{\infty}(\frac{1}{m+1}\sum_{k=0}^{m}|a_k|)^2 \to 0$ as $n \to \infty$.

Given any subset $A \subseteq \{a \in \mathbb{C}^{\mathbb{N}} : \limsup \sqrt[n]{|a_n|} \le 1\}$ we denote by $\mathcal{H}(A)$ the subset of $H(\mathbb{D})$ consisting of those analytic functions whose sequence of Taylor coefficients belongs to A. In this manner $\mathcal{H}(ces_2)$ arises from the sequence space ces_2. Moreover, $\mathcal{H}(ces_2)$ becomes a Banach space of analytic functions on $\mathbb{D}$ relative to the norm $\|\cdot\|_{\mathcal{H}(ces_2)}$, where for $f(z) = \sum_0^\infty a_n z^n \in \mathcal{H}(ces_2)$,

$$\|f\|_{\mathcal{H}(ces_2)} := \left(\sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}|a_k|\right)^2\right)^{1/2} = \|(a_n)_0^\infty\|_{ces_2}. \tag{5}$$

In particular, $\mathcal{H}(ces_2)$ and ces_2 are isometrically isomorphic. From $\ell^2 \subsetneq ces_2 \subsetneq [\mathcal{C},\ell^2]$ we have $H^2 \subsetneq \mathcal{H}(ces_2) \subsetneq [\mathcal{C},H^2]$, with continuous inclusions.

Remark 2.2. For $1 \le p < 2$, we have $H^p \not\subseteq \mathcal{H}(ces_2)$, since it was shown in [6, p. 280] that $H^p \not\subseteq [\mathcal{C},H^2]$, for $1 \le p < 2$. Unlike for H^2, there exist functions in $\mathcal{H}(ces_2)$ which fail to have a.e. boundary values. This follows from $H^2 \subsetneq \mathcal{H}(ces_2)$ and a classical result of Littlewood stating that if $(a_n) \notin \ell^2$ then, for almost all choices of signs (ε_n), with $\varepsilon_n = \pm 1$, the function $\sum_0^\infty \varepsilon_n a_n z^n$ fails to have a.e. boundary values [9, Theorem A.5].

We now collect various Banach space properties of $\mathcal{H}(ces_2)$.

Proposition 2.3. *For $\mathcal{H}(ces_2)$ the following assertions hold.*

(i) *The monomial functions $\{z^n : n \ge 0\}$ are an unconditional, boundedly complete and shrinking basis for $\mathcal{H}(ces_2)$.*
(ii) *$\mathcal{H}(ces_2)$ is reflexive.*

(iii) *Every* $f \in \mathcal{H}(ces_2)$ *is the sum, in* $\mathcal{H}(ces_2)$, *of its Taylor series.*
(iv) *Point evaluations on* $\mathcal{H}(ces_2)$ *are continuous.*

Proof. (i)–(iii). These assertions follow from the fact that there is an isometric isomorphism between $\mathcal{H}(ces_2)$ and ces_2.

(iv) This follows from the continuous inclusion $\mathcal{H}(ces_2) \subseteq [\mathcal{C}, H^2]$ and continuity of point evaluations in $[\mathcal{C}, H^2]$ [6, Section 3]. □

Remark 2.4. In $[\mathcal{C}, H^2]$ the set $\{z^{n+1} - z^n : n \geq 0\}$ constitutes a basis [6, Proposition 3.8(iv)]. On the other hand, $f \in [\mathcal{C}, H^2]$ is the sum (in $[\mathcal{C}, H^2]$) of its Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ if and only if

$$\lim_{n\to\infty} \frac{\left|\sum_{k=0}^{n} a_k\right|}{\sqrt{n+1}} = 0. \tag{6}$$

To see this first note, for $f \in [\mathcal{C}, H^2]$, that $\mathcal{C}f$ is the sum of its Taylor series in H^2, i.e., $\mathcal{C}f(z) = \sum_{n=0}^{\infty} b_n z^n$ with $(b_n)_0^\infty := \mathcal{C}((a_n)_0^\infty)$. Since $\mathcal{C}$ is a (topological) isomorphism of the Frechét space $H(\mathbb{D})$ onto itself and $\mathcal{C}^{-1}(z^n) = (n+1)(z^n - z^{n+1})$, we have $f(z) = \sum_{n=0}^{\infty} b_n (n+1)(z^n - z^{n+1})$ in $[\mathcal{C}, H^2]$. Rearranging the partial sums of this last series yields

$$\sum_{k=0}^{n} b_k (k+1)(z^k - z^{k+1}) = \sum_{k=0}^{n} a_k z^k - \left(\sum_{k=0}^{n} a_k\right) z^{n+1}, \quad n \geq 0.$$

Since the norm of z^{n+1} in $[\mathcal{C}, H^2]$ is equivalent to $1/\sqrt{n+1}$ for $n \geq 0$, the claim follows. Condition (6) implies, via Remark 2.1 (ii), that functions in $\mathcal{H}(ces_2)$ are always the sum, in $[\mathcal{C}, H^2]$, of their Taylor series. This also follows from Proposition 2.3 (iii) and $\mathcal{H}(ces_2) \subseteq [\mathcal{C}, H^2]$.

We now describe those functions which belong to $\mathcal{H}(ces_2)$. For this we first need to describe the range $\mathcal{C}(ces_2) \subseteq \ell^2$ of $\mathcal{C}: ces_2 \to \ell^2$.

Denote by $\mathbb{C}_\star^\mathbb{N}$ the set of all $(b_n)_0^\infty \in \mathbb{C}_+^\mathbb{N}$ such that the sequence $\big((n+1)b_n\big)_0^\infty$ is increasing, in which case

$$\mathcal{C}(\mathbb{C}_+^\mathbb{N}) = \mathbb{C}_\star^\mathbb{N}. \tag{7}$$

Note that $\mathbb{C}_\star^\mathbb{N}$ is a cone in $\mathbb{C}^\mathbb{N}$ generating the full space $\mathbb{C}^\mathbb{N}$, that is, $\mathbb{C}^\mathbb{N} = (\mathbb{C}_\star^\mathbb{N} - \mathbb{C}_\star^\mathbb{N}) + i(\mathbb{C}_\star^\mathbb{N} - \mathbb{C}_\star^\mathbb{N})$. The set of all non-negative sequences $(b_n)_0^\infty \in \ell^2$ for which the sequence $\big((n+1)b_n\big)_0^\infty$ is increasing will be denoted by $\ell_\star^2$, i.e., $\ell_\star^2 = \ell^2 \cap \mathbb{C}_\star^\mathbb{N}$.

The following striking property of the Cesàro operator and the space ces_2 is due to G. Bennett [2, Theorem 20.31].

Theorem 2.5. *Let* $a \in \mathbb{C}^\mathbb{N}$. *Then* $a \in ces_2$ *if and only if* $\mathcal{C}(|a|) \in ces_2$.

With this property we can now prove the following result.

Proposition 2.6. *The range of* $\mathcal{C}: ces_2 \to \ell^2$ *is given by*

$$\mathcal{C}(ces_2) = \left\{ b \in \mathbb{C}^\mathbb{N} : b = (b^1 - b^2) + i(b^3 - b^4), \text{ with } b^j \in \ell_\star^2 \right\}.$$

Proof. Let us first establish that

$$\mathcal{C}(ces_2) \cap \mathbb{C}_\star^\mathbb{N} = \ell^2 \cap \mathbb{C}_\star^\mathbb{N} = ces_2 \cap \mathbb{C}_\star^\mathbb{N}. \tag{8}$$

A chain of embeddings follows from $\mathcal{C}(ces_2) \subseteq \ell^2 \subseteq ces_2$. Let $b \in ces_2 \cap \mathbb{C}_\star^\mathbb{N}$. By (7) there exists $a \in \mathbb{C}_+^\mathbb{N}$ such that $b = \mathcal{C}(a)$. That is, $\mathcal{C}(|a|) = \mathcal{C}(a) = b \in ces_2$ which implies that $a \in ces_2$; see Theorem 2.5. Consequently, $b = \mathcal{C}(a) \in \mathcal{C}(ces_2)$. Recall that $\ell_\star^2 = \ell^2 \cap \mathbb{C}_\star^\mathbb{N}$. This, together with (8), shows that $(b^1 - b^2) + i(b^3 - b^4) \in \mathcal{C}(ces_2)$ whenever $b^j \in \ell_\star^2$.

Let now $b \in \mathcal{C}(ces_2)$ and set $a = \mathcal{C}^{-1}(b) \in ces_2$. Observe that $(\Re a)^+$ and $(\Re a)^-$ are disjointly supported sequences. Thus, $|a| \geq |\Re a| \geq \max\{(\Re a)^+, (\Re a)^-\}$. As ces_2 is solid, $a \in ces_2$ implies that $(\Re a)^+, (\Re a)^- \in ces_2$. A similar argument applies to $\Im a$. Since $(\Re a)^+ \in ces_2 \cap \mathbb{C}_+^\mathbb{N}$, by (8) we have

$$\mathcal{C}((\Re a)^+) \in \mathcal{C}(ces_2) \cap \mathcal{C}(\mathbb{C}_+^\mathbb{N}) = \mathcal{C}(ces_2) \cap \mathbb{C}_\star^\mathbb{N} = \ell^2 \cap \mathbb{C}_\star^\mathbb{N} = \ell_\star^2.$$

A similar argument applies to $(\Re a)^-$, $(\Im a)^+$, and $(\Im a)^-$. Since

$$b = \mathcal{C}(a) = \mathcal{C}((\Re a)^+) - \mathcal{C}((\Re a)^-) + i\mathcal{C}((\Im a)^+) - i\mathcal{C}((\Im a)^-),$$

the claim is established. □

Corollary 2.7. *For* $g \in H(\mathbb{D})$ *we have* $g \in \mathcal{H}(\mathcal{C}(ces_2))$ *precisely when*

$$g = (g_1 - g_2) + i(g_3 - g_4), \quad g_j \in \mathcal{H}(\ell_\star^2).$$

Let $\ell^2_\bullet$ denote the set of all non-negative increasing sequences $(a_n)_0^\infty \in \mathbb{C}^\mathbb{N}$ satisfying $(a_n/(n+1))_0^\infty \in \ell^2$.

Theorem 2.8. *A function $f \in H(\mathbb{D})$ belongs to $\mathcal{H}(ces_2)$ if and only if*

$$f(z) = (1-z)\big((h_1(z) - h_2(z)) + i(h_3(z) - h_4(z))\big), \quad h_j \in \mathcal{H}(\ell^2_\bullet).$$

Proof. Since $\mathcal{C}$ is injective, $a \in ces_2$ if and only if $\mathcal{C}(a) \in \mathcal{C}(ces_2)$. Thus, $f \in \mathcal{H}(ces_2)$ if and only if $\mathcal{C}(f) \in \mathcal{H}(\mathcal{C}(ces_2))$. From Corollary 2.7 this occurs precisely when

$$\mathcal{C}(f) = (g_1 - g_2) + i(g_3 - g_4), \quad g_j \in \mathcal{H}(\ell^2_\star).$$

The known integral expression $\mathcal{C}(f)(z) = \frac{1}{z}\int_0^z \frac{f(\xi)}{1-\xi}\,d\xi$, for $z \in \mathbb{D}$, yields $f(z) = (1-z)(z\mathcal{C}(f)(z))'$. Thus, $f \in \mathcal{H}(ces_2)$ if and only if

$$f(z) = (1-z)\Big(z \cdot \big((g_1 - g_2) + i(g_3 - g_4)\big)\Big)', \quad g_j \in \mathcal{H}(\ell^2_\star).$$

Given $(b_n)_0^\infty \in \mathbb{C}^\mathbb{N}$ set $a_n := (n+1)b_n$, for $n \geq 0$. Then $(b_n)_0^\infty \in \ell^2_\star$ if and only if $(a_n)_0^\infty \in \ell^2_\bullet$. For $g(z) := \sum_0^\infty b_n z^n$ we have

$$\big(zg(z)\big)' = \sum_{n=0}^\infty (n+1)b_n z^n = \sum_{n=0}^\infty a_n z^n.$$

Then $g \in \mathcal{H}(\ell^2_\star)$ if and only if $(b_n)_0^\infty \in \ell^2_\star$, which occurs if and only if $(a_n)_0^\infty \in \ell^2_\bullet$, equivalently if and only if $h(z) := (zg(z))' \in \mathcal{H}(\ell^2_\bullet)$. $\square$

Remark 2.9. The previous result has an analogue for the optimal domain $[\mathcal{C}, H^2]$. To see this, let $f \in H(\mathbb{D})$. Then $h(z) := f(z)/(1-z) \in H(\mathbb{D})$ and so $h(z) = \sum_0^\infty a_n z^n$. In view of condition (3) it follows that $f \in [\mathcal{C}, H^2]$ if and only if

$$\int_0^{2\pi}\int_0^1 \frac{|f(re^{i\theta})|^2}{|1-re^{i\theta}|^2}(1-r)\,dr d\theta = 2\pi \sum_{n=0}^\infty \frac{|a_n|^2}{(2n+2)(2n+1)} < \infty, \tag{9}$$

which holds if and only if $(a_n/(n+1))_0^\infty \in \ell^2$; see also [6, Proposition 3.2]. Thus, the difference between the solid core $\mathcal{H}(ces_2)$ of $[\mathcal{C}, H^2]$ and the optimal domain $[\mathcal{C}, H^2]$ itself arises from the fact that

$$\Big\{(a^1 - a^2) + i(a^3 - a^4) : a^j \in \ell^2_\bullet\Big\} \subsetneq \Big\{(x_n) : (x_n/(n+1))_0^\infty \in \ell^2\Big\},$$

where $a^j \in \mathbb{C}_+^\mathbb{N}$ are *increasing* sequences with $(a_n^j/(n+1))_0^\infty \in \ell^2$. Note, via (9) with $M_2(r, h) := \big(\int_0^{2\pi} |h(re^{i\theta})|^2\,d\theta\big)^{1/2}$, that

$$[\mathcal{C}, H^2] = (1-z) \cdot \left\{h \in H(\mathbb{D}) : \int_0^1 (1-r)(M_2(r,h))^2\,dr < \infty\right\}.$$

The space $\mathcal{H}(ces_2)$ also arises via a different procedure. The operator $\mathcal{C}: H^2 \to H^2$ is a *positive* operator when considered as acting between complex Banach lattices (for the coefficient-wise order in H^2). So, we may look for continuous H^2-valued extensions of $\mathcal{C}$ to larger Banach spaces of analytic functions (i.e., containing H^2 continuously) which are *solid* for the coefficient-wise order. The (optimal) continuous extension $\mathcal{C}: [\mathcal{C}, H^2] \to H^2$ is *not* such an extension because $[\mathcal{C}, H^2]$ is not solid. Of course, the *largest* of these solid spaces for which such a continuous extension is possible is the solid core $\mathcal{H}(ces_2)$ of $[\mathcal{C}, H^2]$.

We end this section with a remarkable stability property of $\mathcal{H}(ces_2)$. The optimal domain $[\mathcal{C}, \mathcal{H}(ces_2)]$ of the continuous Cesàro operator $\mathcal{C}: \mathcal{H}(ces_2) \to \mathcal{H}(ces_2)$ contains $[\mathcal{C}, H^2]$ as a proper subspace (as $[\mathcal{C}, \ell^2] \subsetneq [\mathcal{C}, ces_2]$). Surprisingly, even though the target space $\mathcal{H}(ces_2)$ is now substantially larger than H^2, no further solid extension occurs. This is a consequence of the particular property of the Cesàro operator stated in Theorem 2.5.

Proposition 2.10. *The largest solid space of analytic functions on $\mathbb{D}$ which $\mathcal{C}$ maps continuously into $\mathcal{H}(ces_2)$ is $\mathcal{H}(ces_2)$ itself.*

Proof. Let X be any solid subspace of $[\mathcal{C}, \mathcal{H}(ces_2)]$, i.e., $\mathcal{C}(X) \subset \mathcal{H}(ces_2)$ with X solid for the coefficient-wise order. If $f(z) = \sum_0^\infty a_n z^n \in X$, then also $h(z) := \sum_0^\infty |a_n| z^n \in X$ and hence, $\mathcal{C}(h) = \sum_0^\infty (\mathcal{C}(|a|))_n z^n \in \mathcal{H}(ces_2)$, i.e., $\mathcal{C}(|a|) \in ces_2$. Then Theorem 2.5 implies that $a \in ces_2$ and so $f \in \mathcal{H}(ces_2)$. Accordingly, $X \subseteq \mathcal{H}(ces_2)$. $\square$

3. The multipliers of $\mathcal{H}(ces_2)$

Given any Banach space of analytic functions it is always desirable to identify its multipliers. For $\mathcal{H}(ces_2)$ this means to determine all continuous operators M_φ given by multiplication via a function $\varphi \in H(\mathbb{D})$:

$$\mathcal{H}(ces_2) \ni f \mapsto M_\varphi(f) := f\varphi \in \mathcal{H}(ces_2). \tag{10}$$

Denote by $\mathcal{M}(\mathcal{H}(ces_2))$ the space of all such continuous multiplication operators on $\mathcal{H}(ces_2)$. It is a subspace of the Banach space $\mathcal{L}(\mathcal{H}(ces_2))$ of all bounded linear operators on $\mathcal{H}(ces_2)$ and is closed for the operator norm $\|\cdot\|_{op}$. Since point evaluations on $\mathcal{H}(ces_2)$ are continuous, it follows from the Closed Graph Theorem that if $\varphi \in H(\mathbb{D})$ satisfies $f\varphi \in \mathcal{H}(ces_2)$ whenever $f \in \mathcal{H}(ces_2)$, then necessarily $M_\varphi \in \mathcal{M}(\mathcal{H}(ces_2))$. By an abuse of language, on occasions we will identify the multiplication operator M_φ with the function (symbol) φ and refer to φ as a *multiplier* on $\mathcal{H}(ces_2)$. Thus, in [6, Theorem 3.7] it was shown that $\mathcal{M}([\mathcal{C}, H^2]) = H^\infty$. Consider now ℓ^1 as a commutative, unital Banach algebra relative to convolution and equipped with its usual norm $\|(b_n)_0^\infty\|_1 = \sum_0^\infty |b_n|$ for $(b_n)_0^\infty \in \ell^1$.

Theorem 3.1. *As Banach algebras and with equality of norms,*

$$\mathcal{M}(\mathcal{H}(ces_2)) = \left\{ \varphi(z) = \sum_{n=0}^{\infty} b_n z^n : (b_n)_0^\infty \in \ell^1 \right\} \subsetneq H^\infty.$$

Moreover, the spectrum

$$\sigma(M_\varphi) = \varphi(\overline{\mathbb{D}}), \quad \varphi \in \mathcal{M}(\mathcal{H}(ces_2)).$$

Proof. Multiplication of functions in $H(\mathbb{D})$ corresponds to convolution of their sequences of Taylor coefficients, i.e.,

$$\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{m=0}^{\infty} b_m z^m\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_j b_{k-j}\right) z^k.$$

Consequently, an analytic function $\varphi(z) = \sum_0^\infty b_n z^n$ defines an element of $\mathcal{M}(\mathcal{H}(ces_2))$ via (10) precisely when the sequence $b = (b_n)_0^\infty$ of its Taylor coefficients defines a bounded convolution operator T_b on ces_2:

$$ces_2 \ni a = (a_n)_0^\infty \longmapsto T_b(a) := a * b = \left(\sum_{j=0}^{k} a_j b_{k-j}\right)_{k=0}^{\infty} \in ces_2.$$

Due to the isometric isomorphism between $\mathcal{H}(ces_2)$ and ces_2, we have $\|M_\varphi\|_{op} = \|T_b\|_{op}$.

Suppose first that $b = (b_n)_0^\infty \in \ell^1$. Let $a = (a_n)_0^\infty \in ces_2$. To show that $a * b \in ces_2$ we need to verify that $\mathcal{C}(|a * b|) \in \ell^2$, where

$$\mathcal{C}(|a * b|) = \left(\frac{1}{n+1}\sum_{k=0}^{n}\left|\sum_{j=0}^{k} a_j b_{k-j}\right|\right)_{n=0}^{\infty}. \tag{11}$$

Now, the n-th coordinate of $\mathcal{C}(|a * b|)$ satisfies

$$\begin{aligned}
\big(\mathcal{C}(|a * b|)\big)_n &= \frac{1}{n+1}\sum_{k=0}^{n}\left|\sum_{j=0}^{k} a_j b_{k-j}\right| \le \frac{1}{n+1}\sum_{k=0}^{n}\sum_{j=0}^{k} |a_j|\,|b_{k-j}| \\
&= \frac{1}{n+1}\sum_{j=0}^{n} |a_j| \sum_{i=0}^{n-j} |b_i| \le \|b\|_{\ell^1} \cdot \frac{1}{n+1}\sum_{j=0}^{n} |a_j| \\
&= \|b\|_{\ell^1} \cdot \big(\mathcal{C}(|a|)\big)_n.
\end{aligned}$$

Since $a \in ces_2$, we have $\mathcal{C}(|a|) \in \ell^2$. Consequently, $\mathcal{C}(|a*b|) \in \ell^2$, so that $a*b \in ces_2$. Moreover, $\|a*b\|_{ces_2} \le \|b\|_{\ell^1} \cdot \|a\|_{ces_2}$, i.e., $\|T_b\|_{op} \le \|b\|_{\ell^1}$.

Now let T_b, with $b = (b_n)_0^\infty \in \mathbb{C}^{\mathbb{N}}$, be a bounded convolution operator on ces_2. To show that $b \in \ell^1$ we prove, for every $N \in \mathbb{N}$, that $\sum_0^N |b_i| \le \|T_b\|_{op}$. So, let $N \in \mathbb{N}$ be fixed.

In order to estimate $\|T_b\|_{op}$ from below, recall that

$$\|T_b\|_{op} = \sup_{a \in ces_2} \frac{\|a * b\|_{ces_2}}{\|a\|_{ces_2}} = \sup_{a \in ces_2} \frac{\|\mathcal{C}(|a * b|)\|_{\ell^2}}{\|\mathcal{C}(|a|)\|_{\ell^2}}. \tag{12}$$

Fix $M \in \mathbb{N}$. For each $(a_M, \dots, a_{M+N}) \in \mathbb{C}^{N+1}$ set

$$a := \sum_{i=M}^{M+N} a_i e_i \in ces_2. \tag{13}$$

First we estimate the norm of a in ces_2 from above. Since $a_n = 0$, for $n < M$ and also for $n > M + N$, we conclude that

$$(\mathcal{C}(|a|))_n = \begin{cases} 0, & \text{if } n < M, \\ \dfrac{1}{n+1} \displaystyle\sum_{j=M}^{n} |a_j|, & \text{if } M \le n < M+N, \\ \dfrac{1}{n+1} \displaystyle\sum_{j=M}^{M+N} |a_j|, & \text{if } n \ge M+N. \end{cases}$$

Consequently, we have that

$$\mathcal{C}(|a|) \le \left(\sum_{j=M}^{M+N} |a_j| \right) \sum_{i=M}^{\infty} \frac{e_i}{i+1} = \|a\|_{\ell^1} \sum_{i=M}^{\infty} \frac{e_i}{i+1}.$$

Hence, for any a of the form (13),

$$\|\mathcal{C}(|a|)\|_{\ell^2} \le \|a\|_{\ell^1} \left\| \sum_{i=M}^{\infty} \frac{e_i}{i+1} \right\|_{\ell^2} \le \frac{1}{\sqrt{M}} \|a\|_{\ell^1}. \tag{14}$$

Next we estimate the norm of $a * b$ in ces_2 from below. Since $a_n = 0$, for $n < M$, it is clear that $(a * b)_n = 0$ whenever $n < M$. Thus, for the Cesàro means (11) we have

$$\big(\mathcal{C}(|a * b|)\big)_n = \begin{cases} 0, & \text{if } n < M, \\ \dfrac{1}{n+1} \displaystyle\sum_{k=M}^{n} |a * b|_k, & \text{if } M \le n < M+N, \\ \dfrac{1}{n+1} \displaystyle\sum_{k=M}^{M+N} |a * b|_k, & \text{if } n = M+N, \\ \dfrac{1}{n+1} \left(\displaystyle\sum_{k=M}^{M+N} |a * b|_k + \sum_{k=M+N+1}^{n} |a * b|_k \right), & \text{if } n > M+N. \end{cases}$$

Consequently,

$$\big(\mathcal{C}(|a * b|)\big)_n \ge \begin{cases} 0, & \text{if } n < M+N, \\ \dfrac{1}{n+1} \displaystyle\sum_{k=M}^{M+N} |a * b|_k, & \text{if } n \ge M+N. \end{cases}$$

Setting

$$S := \sum_{k=M}^{M+N} |a * b|_k,$$

it follows that

$$\mathcal{C}(|a * b|) \ge S \sum_{i=M+N}^{\infty} \frac{e_i}{i+1}.$$

Hence,

$$\|\mathcal{C}(|a * b|)\|_{\ell^2} \ge S \left\| \sum_{i=M+N}^{\infty} \frac{e_i}{i+1} \right\|_{\ell^2} \ge \frac{S}{\sqrt{M+N+1}}. \tag{15}$$

We now require an alternative expression for S. Since $a_n = 0$ for $n < M$, observe that

$$S = \sum_{k=M}^{M+N} |a * b|_k = \sum_{k=M}^{M+N} \left| \sum_{j=0}^{k} a_j b_{k-j} \right| = \sum_{k=M}^{M+N} \left| \sum_{j=M}^{k} a_j b_{k-j} \right|,$$

that is,

$$S = |a_M b_0| + |a_M b_1 + a_{M+1} b_0| + \cdots + |a_M b_N + \cdots + a_{M+N} b_0|.$$

A close inspection shows, by setting

$$B_N a := \begin{pmatrix} b_0 & 0 & \cdots & 0 \\ b_1 & b_0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ b_N & b_{N-1} & \cdots & b_0 \end{pmatrix} \begin{pmatrix} a_M \\ a_{M+1} \\ \vdots \\ a_{M+N} \end{pmatrix},$$

that $S = \|B_N a\|_{\ell^1}$. Then (15) becomes

$$\|\mathcal{C}(|a * b|)\|_{\ell^2} \geq \frac{\|B_N a\|_{\ell^1}}{\sqrt{M+N+1}}.$$

This inequality and (14), together with (12), yield for all a of the form (13) with $M \in \mathbb{N}$ arbitrary that

$$\|T_b\|_{op} \geq \sup_{a,M} \frac{\sqrt{M}}{\sqrt{M+N+1}} \frac{\|B_N a\|_{\ell^1}}{\|a\|_{\ell^1}} = \|B_N\|_{\ell^1_{N+1} \to \ell^1_{N+1}}.$$

The matrix B_N has operator norm $\|B_N\|_{\ell^1_{N+1} \to \ell^1_{N+1}} = \sum_0^N |b_i|$. Consequently, $b = (b_n)_0^\infty \in \ell^1$ and $\|T_b\|_{op} \geq \|b\|_{\ell^1}$.

Concerning the spectrum, it is routine to check that $\mathcal{M}(\mathcal{H}(ces_2))$ is a closed subalgebra of $\mathcal{L}(\mathcal{H}(ces_2))$ which is also inverse closed. Hence, $\sigma(M_\varphi)$ coincides with the spectrum of φ as an element of the Banach algebra ℓ^1. Since ℓ^1 is generated (as a Banach algebra) by $e_1 = (0, 1, 0, \ldots)$ and the maximal ideal space of ℓ^1 can be identified with $\overline{\mathbb{D}}$ [7, Theorem 4.6.9] it follows from Theorem 2.3.30 (see also pp. 158–160 of [7]) that the spectrum of $\varphi(z) = \sum_0^\infty b_n z^n$, considered as the element $(b_n)_0^\infty$ of ℓ^1, is precisely $\varphi(\overline{\mathbb{D}})$. □

Remark 3.2. (i) Every function $h(z) = \sum_0^\infty b_n z^n$ from $H(\mathbb{D})$ induces a linear map $T_b \colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ via the *Toeplitz matrix*

$$T_b := \begin{pmatrix} b_0 & 0 & 0 & \cdots \\ b_1 & b_0 & 0 & \cdots \\ b_2 & b_1 & b_0 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix}.$$

Theorem 3.1 implies that T_b maps ces_2 into itself if and only if $b \in \ell^1$.

(ii) Theorem 3.1 shows that the multipliers for $\mathcal{H}(ces_2)$ form a lattice for the coefficient-wise order. This is not so for the multipliers of H^2 and $[\mathcal{C}, H^2]$, both of which have H^∞ as their multiplier space.

4. The operator $\mathcal{C}$ on $\mathcal{H}(ces_2)$

In a classical paper the authors study $\mathcal{C} \colon \ell^2 \to \ell^2$ and show that its spectrum is $\sigma(\mathcal{C}) = \{z \in \mathbb{C} : |1-z| \leq 1\}$ [3, Theorem 2(6)]. Recall that its operator norm is $\|\mathcal{C}\|_2 = 2$. In order to treat $\mathcal{C} \colon \mathcal{H}(ces_2) \to \mathcal{H}(ces_2)$ it is useful to identify the dual space $\mathcal{H}(ces_2)'$ of $\mathcal{H}(ces_2)$. The isometric isomorphism between $\mathcal{H}(ces_2)$ and ces_2 yields the identification $\mathcal{H}(ces_2)' = \mathcal{H}(ces_2')$ (where ces_2' is the dual space of ces_2) with the duality given by

$$\langle f, g \rangle := \sum_{n=0}^\infty a_n b_n,$$

for elements $f(z) = \sum_0^\infty a_n z^n \in \mathcal{H}(ces_2)$ and $g(z) = \sum_0^\infty b_n z^n \in \mathcal{H}(ces_2')$.

The dual space of ces_2 was identified by Jagers [13]. G. Bennett has given a more tractable *isomorphic* identification [2, Corollary 12.17]. Namely, consider the Banach space

$$d(2) := \left\{ b = (b_n)_0^\infty : \left(\sup_{k \geq n} |b_k| \right)_{n=0}^\infty \in \ell^2 \right\},$$

equipped with the norm

$$\|b\|_{d(2)} := \left(\sum_{n=0}^\infty \sup_{k \geq n} |b_k|^2 \right)^{1/2}.$$

Then ces_2' is isomorphic to $d(2)$ and its duality with ces_2 is given by

$$\langle a, b \rangle := \sum_{n=0}^\infty a_n b_n, \quad a \in ces_2, \ b \in d(2).$$

However, the dual norm of ces_2' is only equivalent to that of $d(2)$. This fact will not interfere with any of our results. The sequence $\tilde{b} = (\tilde{b}_n)_0^\infty$ given by $\tilde{b}_n := \sup_{k\geq n} |b_k|$, $n \geq 0$, is called the least decreasing majorant of $(b_n)_0^\infty \in \mathbb{C}^{\mathbb{N}}$. Note that, even though $ces_2' \simeq d(2) \subsetneq \ell^2$ (e.g. $x = \sum_0^\infty 2^{-n} e_{2^{2n}} \in \ell^2 \setminus ces_2'$), all non-negative and eventually decreasing sequences from ℓ^2 do belong to ces_2'.

Theorem 4.1. *For the Cesàro operator $\mathcal{C}\colon \mathcal{H}(ces_2) \to \mathcal{H}(ces_2)$ we have that $\|\mathcal{C}\|_{op} = 2$ and*

$$\sigma(\mathcal{C}) = \{z \in \mathbb{C} : |1 - z| \leq 1\}.$$

Proof. Due to the isometric isomorphism between the spaces $\mathcal{H}(ces_2)$ and ces_2, it suffices to study the problem for $\mathcal{C}\colon ces_2 \to ces_2$.

We first show that $\|\mathcal{C}\|_{op} \leq 2$ for the continuous operator $\mathcal{C}\colon ces_2 \to ces_2$. Since $|\mathcal{C}(a)| \leq \mathcal{C}(|a|)$, for $a \in ces_2$, we have

$$\|\mathcal{C}(a)\|_{ces_2} = \|\mathcal{C}(|\mathcal{C}(a)|)\|_{\ell^2} \leq \|\mathcal{C}(\mathcal{C}(|a|))\|_{\ell^2} \leq \|\mathcal{C}\|_2 \cdot \|\mathcal{C}(|a|)\|_{\ell^2} = 2\|a\|_{ces_2},$$

that is, $\|\mathcal{C}\|_{op} \leq 2$.

To show that the point spectrum $\sigma_p(\mathcal{C})$ of $\mathcal{C}$ is empty, suppose that $\lambda \in \sigma_p(\mathcal{C})$. Choose $0 \neq a \in ces_2$ such that $\mathcal{C}(a) = \lambda a$. A direct algebraic calculation (as in the proof of part (3) of Theorem 2 of [3]) yields $|a_n| \geq |a_{n-1}|$ for $n \geq 1$. Thus, $|a_n| \geq |a_N| > 0$ for $n \geq N$ with N the smallest natural number such that $a_N \neq 0$. This implies that $\mathcal{C}(|a|) \geq u$ with $u_n = 0$ for $n < N$ and $u_n = |a_N| > 0$ for $n \geq N$. But, $u \notin \ell^2$ and so $\mathcal{C}(|a|) \notin \ell^2$ which is a contradiction. Hence, $\sigma_p(\mathcal{C}) = \emptyset$.

Let $\mathcal{C}'\colon ces_2' \to ces_2'$ be the dual operator of $\mathcal{C}$, i.e.,

$$\mathcal{C}'((x_n)_0^\infty) = \left(\sum_{k=n}^\infty \frac{x_k}{k+1}\right)_{n=0}^\infty, \quad (x_n)_0^\infty \in ces_2'.$$

We now show that $\sigma_p(\mathcal{C}') = \mathcal{U}$ where

$$\mathcal{U} = \{z \in \mathbb{C} : |1 - z| < 1\}.$$

Let $\lambda \in \sigma_p(\mathcal{C}')$. Then $\mathcal{C}'(x) = \lambda x$ for some $0 \neq x = (x_n)_0^\infty$ in ces_2'. But, $ces_2' \subseteq \ell^2$ and so $x \in \ell^2$, i.e., λ is an eigenvalue of the Hilbert space adjoint operator $\mathcal{C}^* = \mathcal{C}'$ of $\mathcal{C}\colon \ell^2 \to \ell^2$. By Theorem 2(5) of [3] we have that λ belongs to $\mathcal{U}$, i.e., $\sigma_p(\mathcal{C}') \subseteq \mathcal{U}$. In order to establish the reverse inclusion, observe that direct algebraic calculations (as in the proof of part (4) of Theorem 2 of [3]) yield: if $\mathcal{C}'(x) = \lambda x$ with $x \neq 0$, then necessarily $\lambda \neq 0$, $x_0 \neq 0$ and

$$x_n = x_0 \cdot \prod_{j=1}^n \left(1 - \frac{1}{j\lambda}\right), \quad n \geq 1.$$

We need to verify that this x belongs to $ces_2' = d(2)$ whenever $\lambda \in \mathcal{U}$, i.e., the least decreasing majorant $\tilde{x}$ of x belongs to ℓ^2. The identities

$$\frac{|x_n|^2}{|x_{n-1}|^2} = \left|1 - \frac{1}{n\lambda}\right|^2 = 1 - \frac{2\Re(\lambda)}{n|\lambda|^2} + \frac{1}{n^2|\lambda|^2}, \tag{16}$$

imply that $|x_n| \leq |x_{n-1}|$ whenever the right-side of (16) is at most 1, i.e., if $\Re(\lambda) \geq 1/(2n)$ for all $n \geq 1$. For each $\lambda \in \mathcal{U}$ we have $\Re(\lambda) > 0$. Thus there exists n_0 such that, for $n \geq n_0$, we have $|x_n| \leq |x_{n-1}|$. Hence $(\tilde{x})_n = |x_n|$ for $n \geq n_0$, and so $\tilde{x} \in \ell^2$ if and only if $x \in \ell^2$. But, (16) and the Raabe–Duhamel criterion [12, Chapter I, Sections 20–21] imply that $x \in \ell^2$ whenever $2\frac{\Re(\lambda)}{|\lambda|^2} > 1$. This last condition is precisely $|1 - \lambda| < 1$, that is, $\lambda \in \mathcal{U}$. Accordingly, $\sigma_p(\mathcal{C}') = \mathcal{U}$.

It follows from $\sigma_p(\mathcal{C}) = \emptyset$ and Corollary II.5.3(iii) and (vi) of [10] that $\sigma_r(\mathcal{C}) = \sigma_p(\mathcal{C}') = \mathcal{U}$. Accordingly, we have that $\mathcal{U} = \sigma_r(\mathcal{C}) \cup \sigma_p(\mathcal{C}) \subseteq \sigma(\mathcal{C})$ and so $\overline{\mathcal{U}} \subseteq \sigma(\mathcal{C})$. In particular, $2 \in \sigma(\mathcal{C})$ and so also $\|\mathcal{C}\|_{op} \geq 2$. Hence, $\|\mathcal{C}\|_{op} = 2$.

For the reverse inclusion $\sigma(\mathcal{C}) \subseteq \overline{\mathcal{U}}$ consider first $(\mathcal{C} - \lambda I)^{-1}\colon \mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$. For $\lambda \in \mathbb{C} \setminus \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ the matrix for $(\mathcal{C} - \lambda I)^{-1} = (c_{nm})_{n,m=0}^\infty$ is given by, [17, Theorem 3],

$$c_{nm} = \begin{cases} \dfrac{-1}{(n+1)\lambda^2 \prod\limits_{k=m}^n \left(1 - \frac{1}{(k+1)\lambda}\right)}, & \text{if } 0 \leq m < n, \\[2ex] \dfrac{n+1}{1 - (n+1)\lambda}, & \text{if } m = n, \\[2ex] 0, & \text{if } n < m. \end{cases}$$

We write

$$(\mathcal{C}-\lambda I)^{-1}=D_\lambda-\frac{1}{\lambda^2}E_\lambda,$$

where the diagonal operator $D_\lambda=(d_{nm})_{n,m=0}^\infty$ is given by $d_{nn}=\frac{n+1}{1-(n+1)\lambda}$ with $d_{nm}=0$ in all other cases, and the lower triangular operator $E_\lambda=(e_{nm})_{n,m=0}^\infty$ is specified as follows: for each $n\geq 0$ define

$$e_{nm}=\begin{cases}\dfrac{1}{(n+1)\prod_{k=m}^{n}\left(1-\frac{1}{(k+1)\lambda}\right)}, & \text{if } 0\leq m<n,\\[2ex] 0, & \text{if } n\leq m,\end{cases}\tag{17}$$

for all $m\geq 0$.

Let $d_\lambda:=\inf\{|\lambda-\frac{1}{k}|:k\geq 1\}>0$. Then D_λ is bounded on ces_2 since

$$\left|\big(D_\lambda(a)\big)_n\right|=\left|\frac{(n+1)a_n}{1-(n+1)\lambda}\right|=\left|\frac{a_n}{\frac{1}{n+1}-\lambda}\right|\leq\frac{|a_n|}{d_\lambda},\quad n\geq 0.$$

Thus, $|D_\lambda(a)|\leq\frac{1}{d_\lambda}|a|$. Since ces_2 is solid, if $a\in ces_2$ it follows that $D_\lambda(a)\in ces_2$ and $\|D_\lambda(a)\|_{ces_2}\leq\frac{1}{d_\lambda}\|a\|_{ces_2}$. Accordingly, $D_\lambda\colon ces_2\to ces_2$ is bounded with $\|D_\lambda\|_{op}\leq\frac{1}{d_\lambda}$. Consequently, $(\mathcal{C}-\lambda I)^{-1}$ is bounded on ces_2 precisely when E_λ is.

For the case when $\Re\lambda\leq 0$, with $\lambda\neq 0$, we claim that E_λ is bounded on ces_2 and so λ is in the resolvent set of $\mathcal{C}$. To see this observe that

$$\left|\frac{1}{\left(1-\frac{1}{(k+1)\lambda}\right)}\right|\leq 1\iff|\lambda|\leq\left|\lambda-\frac{1}{k+1}\right|\iff\Re\lambda\leq\frac{1}{2(k+1)}.$$

It follows from (17) that $|e_{nm}|\leq\frac{1}{n+1}$, for $0\leq m<n$. Consequently, $|E_\lambda(a)|\leq\mathcal{C}(|a|)$. Again, due to the fact that ces_2 is solid, if $a\in ces_2$ then $\mathcal{C}(|a|)\in\ell^2\subseteq ces_2$ and so $E_\lambda(a)\in ces_2$ with $\|E_\lambda(a)\|_{ces_2}\leq\|\mathcal{C}(|a|)\|_{ces_2}\leq\|\mathcal{C}\|_{op}\|a\|_{ces_2}$. Accordingly, $E_\lambda\colon ces_2\to ces_2$ is bounded and $\|E_\lambda\|_{op}\leq\|\mathcal{C}\|_{op}=2$.

Since $\|\mathcal{C}\|_{op}=2$ ensures that $\{z\in\mathbb{C}:|z|>2\}$ is a subset of the resolvent set $\rho(\mathcal{C})$ of $\mathcal{C}$, it remains to show that the set

$$\mathcal{D}:=\left\{z\in\mathbb{C}:|z|\leq 2,\ \Re(z)>0,\ |z-1|>1\right\}\subseteq\rho(\mathcal{C}).$$

For this purpose $\mathcal{D}$ can be described via the family of circles $\Gamma_\alpha=\{z\in\mathbb{C}:\Re(\frac{1}{z})=\alpha\}$ for $0<\alpha<\frac{1}{2}$ with Γ_α having centre $\frac{1}{2\alpha}>1$ and radius $\frac{1}{2\alpha}$ (note, for any $z\in\mathbb{C}$, that $|z-1|>1$ if and only if $\Re(\frac{1}{z})<\frac{1}{2}$). Indeed,

$$\mathcal{D}=\left\{z\in\mathbb{C}:|z|\leq 2\right\}\cap\left(\bigcup_{0<\alpha<\frac{1}{2}}\Gamma_\alpha\right).$$

Let $\lambda\in\mathcal{D}$. Then $\lambda\in\Gamma_\alpha$ for some $\alpha=\Re(\frac{1}{\lambda})\in(0,\frac{1}{2})$. Consequently, $\Re(1-\frac{1}{k\lambda})=1-\frac{\alpha}{k}$, for $k\geq 1$. Given $a\in ces_2$ it follows from (17) that $\big(E_\lambda(a)\big)_0=0$ and, for $n\geq 1$, that

$$\begin{aligned}\left|\big(E_\lambda(a)\big)_n\right|&=\left|\frac{1}{n+1}\sum_{m=0}^{n-1}\frac{a_m}{\prod_{k=m}^{n}\left(1-\frac{1}{(k+1)\lambda}\right)}\right|\\&\leq\frac{1}{n+1}\sum_{m=0}^{n-1}\frac{|a_m|}{\prod_{k=m}^{n}\left|1-\frac{1}{(k+1)\lambda}\right|}\\&\leq\frac{1}{n+1}\sum_{m=0}^{n-1}\frac{|a_m|}{\prod_{k=m}^{n}\left|\Re\left(1-\frac{1}{(k+1)\lambda}\right)\right|}\end{aligned}$$

$$= \frac{1}{n+1} \sum_{m=0}^{n-1} \frac{|a_m|}{\prod_{k=m}^{n} \left(1 - \frac{\alpha}{k+1}\right)}$$

$$= \left(E_{\frac{1}{\alpha}}(|a|)\right)_n.$$

Thus, $|E_\lambda(a)| \leq E_{\frac{1}{\alpha}}(|a|)$. Since $0 < \alpha < \frac{1}{2}$ we have that $\frac{1}{\alpha} > 2$ and so $\frac{1}{\alpha}$ belongs to the resolvent set of $\mathcal{C}$. Thus, $E_{\frac{1}{\alpha}}$ is bounded on ces_2. Since $a \in ces_2$ implies $|a| \in ces_2$, we have $E_{\frac{1}{\alpha}}(|a|) \in ces_2$. Thus, $E_\lambda(a) \in ces_2$ with $\|E_\lambda(a)\|_{ces_2} \leq \|E_{\frac{1}{\alpha}}\|_{op}\|a\|_{ces_2}$. Accordingly, $E_\lambda : ces_2 \to ces_2$ is bounded (with $\|E_\lambda\|_{op} \leq \|E_{\frac{1}{\alpha}}\|_{op}$) and so also $(\mathcal{C} - \lambda I)^{-1}$ is bounded. □

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