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Korteweg–de Vries–Burgers system in $\mathbb{R}^N$

Tomasz Dlotko [a], Maria B. Kania [a,*], Shan Ma [b]

[a] Institute of Mathematics, Silesian University, 40-007 Katowice, Bankowa 14, Poland
[b] School of Mathematics and Statistics, Lanzhou University, 730000 Lanzhou, PR China

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ABSTRACT

We study Cauchy problem in $\mathbb{R}^N$ for the Korteweg–de Vries–Burgers system. Parabolic regularization technique is used to prove its global in time solvability. The regularization effect of the Laplacian term is observed for the viscous solutions constructed in the paper. Asymptotic behavior of *weak solutions* is discussed next in the language of the global attractors.

1. Introduction

Cauchy's problem in $\mathbb{R}^N$ for a generalization of the KdV–Burgers system is considered in Sobolev space $(H^{p+1}(\mathbb{R}^N))^m$ (here $u=(u_1,\dots,u_m)$ is a vector valued function of $x\in\mathbb{R}^N$). We want to extend the results obtained recently in [9,10,4] for the 1-D case to the higher space dimension N, concentrating here mainly on the existence and properties of the viscous solution to the KdV–Burgers system (1.2). Such higher dimensional problems were studied earlier in [23,30,29]; the Cauchy problem we study following that references has the form

$$u_t+\sum_{i=1}^N\frac{\partial}{\partial x_i}\nabla\Phi(u)+\sum_{j=1}^N\frac{\partial}{\partial x_j}\sum_{i=1}^N\frac{\partial^{2p}}{\partial x_i^{2p}}u=\alpha\Delta u+\beta\sum_{i=1}^N\frac{\partial}{\partial x_i}u,\quad t>0,\ x\in\mathbb{R}^N,$$
$$u(0,x)=u_0(x),\quad x\in\mathbb{R}^N,\tag{1.1}$$

where $\alpha>0$, $2p>N\geqslant 1$ and Φ is a scalar function of the vector $u(t,x)=(u_1(t,x),\dots,u_m(t,x))$; ∇ denotes the gradient with respect to u. Decay of solutions of (1.1) as $t\to\infty$ together with the sharp decay rates in $L^2(\mathbb{R}^N)$, $L^\infty(\mathbb{R}^N)$ and $H^k(\mathbb{R}^N)$ were studied in [29,23]. Another possible form of asymptotic behavior of solutions is described in [28].

1.1. Setting of the problem and its parabolic regularization

By certain formal manipulations (1.1) will be rewritten in a slightly simpler form. To avoid having all the trajectories convergent to 0 as $t\to\infty$, we will enrich that simplified form of (1.1) adding the term $g(x,u)$ to its right hand side; consult [13, p. 370] and the references there for the discussion of physically relevant damping terms, see also [14,16,18] for

* Corresponding author.
E-mail addresses: tdlotko@math.us.edu.pl (T. Dlotko), mkania@math.us.edu.pl (M.B. Kania), mashan@lzu.edu.cn (S. Ma).

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considerations concerning 1-D KdV equation on $\mathbb{R}$. Assumption I below on $g(x,u)$ covers the choice in [13] (which is like $-\gamma u$ in this case). Consequently we will study the equation:

$$\begin{aligned} & u_t + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \nabla \Phi(u) + \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \sum_{i=1}^{N} \frac{\partial^{2p}}{\partial x_i^{2p}} u = \alpha \Delta u + g(x,u), \quad t>0,\ x \in \mathbb{R}^N, \\ & u(0,x) = u_0(x), \quad x \in \mathbb{R}^N. \end{aligned} \tag{1.2}$$

The parabolic regularization technique called also the *method of vanishing viscosity* is used in this paper to study Eq. (1.2), which allows us to extend the strong regularity properties and estimates of solutions of the $(2p+2)$-order parabolic approximations onto their $(2p+1)$-order limit—the Korteweg–de Vries–Burgers (KdVB) system (1.2).

So, instead of the problem (1.2) we will study first its parabolic regularization having the form:

$$\begin{aligned} & u_t^\epsilon + \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \nabla \Phi\big(u^\epsilon\big) + \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \sum_{i=1}^{N} \frac{\partial^{2p}}{\partial x_i^{2p}} u^\epsilon = \alpha \Delta u^\epsilon + \epsilon(-1)^p (\Delta)^{p+1} u^\epsilon + g\big(x,u^\epsilon\big), \quad t>0,\ x \in \mathbb{R}^N, \\ & u^\epsilon(0,x) = u_0^\epsilon(x), \quad x \in \mathbb{R}^N, \end{aligned} \tag{1.3}$$

where $\epsilon > 0$ is the *viscosity coefficient*, that will later tend to 0^+. We obtain certain estimates for solutions u^ϵ to (1.3), which we then extend on the limit problem (1.2) by letting $\epsilon \to 0^+$. We will study existence for the Cauchy problem (1.3) using the theory of semigroups. Such approach, in case of systems of parabolic equations, was extended about 20 years ago; see in particular [2,3,15].

Our main task will be to study the *viscous solution* u of (1.2), the unique limit of the regular solutions u^ϵ of (1.3) (see Section 4). The term $\alpha\Delta$ appearing in (1.2) caused that the system (1.2) possess some regularization effect. We will focus on obtaining estimates of u expressing that effect; the corresponding estimates of the regularizations u^ϵ are ϵ independent.

In Section 2 we study local and global in time solvability of the auxiliary problem (1.3). Section 3 is devoted to the higher order uniform in ϵ estimates of the solutions u^ϵ of (1.3). Such estimates are next extended to the viscous solutions of (1.2) in Section 4. Asymptotic behavior of solutions to (1.3) and (1.2) is studied in Section 5.

Standard notation of the Sobolev spaces in used throughout the text. Letters c, C are used to denote various positive constants. Sometimes we will also specify the quantities on which the constants depend. For $r \in \mathbb{R}$ the symbol r^- denotes a number strictly less than r (but close to it).

2. Solvability of an auxiliary problem (1.3)

2.1. Local solvability of (1.3)

We will use Dan Henry's approach, extending to higher dimensions the considerations in [9]. To study (1.3) we need to impose the set of conditions on the nonlinear terms Φ and g. The basic requirements are formulated below.

Note first that since Φ appears *under the gradient*, without loosing generality we will assume that $\Phi(0) = 0$ (alternatively replace $\Phi(s)$ by $\Phi(s) - \Phi(0)$). Further we formulate

Assumption I.

- $\Phi : \mathbb{R}^m \to \mathbb{R}$ is $C^3(\mathbb{R}^m)$ and satisfies $\Phi(0) = 0$.
- $g : \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^m$, with $g(x,0) = 0$, is Lipschitz continuous with respect to u uniformly with respect to $x \in \mathbb{R}^N$ (with the Lipschitz constant uniform in the sets $\mathbb{R}^N \times [-M,M]^m$), and fulfills the condition

$$\exists_{\gamma>0} \exists_{0 \leqslant n \in L^2(\mathbb{R}^N)} \quad g(x,s)\cdot s \leqslant -\gamma |s|^2 + n(x)|s|, \quad \text{when } x \in \mathbb{R}^N,\ s \in \mathbb{R}^m, \tag{2.1}$$

  where $g(x,s)\cdot s$ is the scalar product in $\mathbb{R}^m$.

Remark 2.1. The simplest possible choice is $g(x,s) = -\gamma s$. Condition (2.1) is a *one-sided* bound at infinity for the growth of g (when it is a *source term*). We can take *for example*:

$$\begin{aligned} & g_i(x,s) \leqslant -\gamma s_i + n(x)\big(s_i - s_i^3\big) \quad \text{for } s_i \geqslant 0,\ i = 1,\dots,m, \\ & g_i(x,s) \geqslant -\gamma s_i + n(x)\big(s_i - s_i^3\big) \quad \text{for } s_i \leqslant 0, \end{aligned} \tag{2.2}$$

where $0 \leqslant n \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is sufficiently smooth (for the further needs). This makes the global attractor not trivial (equal to zero function); see [10, Appendix], where this problem was discussed. Letting $g_i(x,s) = -\gamma s_i + n(x)(s_i - s_i^3)$ for $|s_i| \geqslant \epsilon$, $i = 1,\dots,m$, $0 < \epsilon < 1$, and 'gluing' these functions near $s_i = 0$ to fulfill further smallness restrictions (2.14), (3.5), we get an example of nonlinearity g suitable throughout the paper. Extended discussion of such structural type assumption on nonlinear term can be found in [6, p. 529].

We need first to consider the main linear (product) operator of (1.3) on the space $(H^{p+1}(\mathbb{R}^N))^m$ with component, on each space $H^{p+1}(\mathbb{R}^N)$, that equals:

$$A_\epsilon = -\epsilon(-1)^p(\Delta)^{p+1} + \gamma I, \quad \gamma > 0. \tag{2.3}$$

It is well known (e.g. [5]), that the operator A_ϵ with the domain $H^{2p+2}(\mathbb{R}^N)$, is sectorial positive on $L^2(\mathbb{R}^N)$. Next, thanks to the properties of the fractional power spaces, operator A_ϵ is also sectorial on the space $H^{-p}(\mathbb{R}^N)$; see [1, Chapter V.1] for details.

It is also a familiar fact, that if A_i, $i = 1,\dots,m$, with domains $D(A_i)$ respectively, are sectorial positive operators on the Banach spaces X_i, then the product operator $\mathbf{A} = (A_1,\dots,A_m)$, considered with the domain $D(A_1)\times\cdots\times D(A_m)$, will be sectorial positive (product) operator on the space $X_1\times\cdots\times X_m$ (see e.g. [7, Example 1.3.2, p. 37]).

Consequently, the product operator $\mathbf{A}_\epsilon = (A_\epsilon,\dots,A_\epsilon)$ (m-times), will be sectorial positive on the product space $(H^{-p}(\mathbb{R}^N))^m$.

The next step is to check that the operator $\mathbf{B} = \sum_{j=1}^N \frac{\partial}{\partial x_j}\sum_{i=1}^N \frac{\partial^{2p}}{\partial x_i^{2p}}$ is *a perturbation* of $\mathbf{A}_\epsilon$ in $(H^{-p}(\mathbb{R}^N))^m$ in a sense of [12, p. 177] or [7, p. 37]. We have:

Lemma 2.2. *The operator* $\mathbf{B}$ *is a perturbation of* $\mathbf{A}_\epsilon$ *in* $(H^{-p}(\mathbb{R}^N))^m$*, that means*

$$\forall_{0<\mu\leqslant\mu_0}\exists_{C(\mu)>0}\forall_{\phi\in(H^{p+2}(\mathbb{R}^N))^m} \quad \|\mathbf{B}\phi\|_{(H^{-p}(\mathbb{R}^N))^m} \leqslant \mu\|\mathbf{A}_\epsilon\phi\|_{(H^{-p}(\mathbb{R}^N))^m} + C(\mu)\|\phi\|_{(H^{-p}(\mathbb{R}^N))^m}, \tag{2.4}$$

$\mu_0 > 0$ *is a constant. Consequently, the operator* $\mathbf{A}_\epsilon + \mathbf{B}$ *is sectorial in* $(H^{-p}(\mathbb{R}^N))^m$*, and the operator* $\mathbf{A}_\epsilon + \mathbf{B} + \omega\mathbf{I}$ *is sectorial positive, provided that* $\omega > 0$ *is chosen sufficiently large.*

Proof. Note first that the operator $\mathbf{A}_\epsilon$ defines a linear isomorphism from $(H^{p+2}(\mathbb{R}^N))^m$ onto $(H^{-p}(\mathbb{R}^N))^m$ (e.g. [5, Proposition 5.1]), and the expressions

$$\|\mathbf{A}_\epsilon\psi\|_{(H^{-p}(\mathbb{R}^N))^m} \quad \text{and} \quad \|\psi\|_{(H^{p+2}(\mathbb{R}^N))^m} \tag{2.5}$$

are equivalent norms of the space $(H^{p+2}(\mathbb{R}^N))^m$. Note also an estimate

$$\|\mathbf{B}\psi\|_{(H^{-p}(\mathbb{R}^N))^m} = \left\| \sum_{j=1}^N \frac{\partial}{\partial x_j}\sum_{i=1}^N \frac{\partial^{2p}\psi}{\partial x_i^{2p}} \right\|_{(H^{-p}(\mathbb{R}^N))^m} \leqslant C\|\psi\|_{(H^{p+1}(\mathbb{R}^N))^m} \tag{2.6}$$

($C > 0$ is a constant), being a consequence of the fact that any partial derivative $\frac{\partial}{\partial x_i}$ is a bounded linear operator from $H^s(\mathbb{R}^N)$ to $H^{s-1}(\mathbb{R}^N)$ (for any $s \in \mathbb{R}$). Using interpolation inequality, (2.5) and (2.6), we find that

$$\begin{aligned} \|\mathbf{B}\psi\|_{(H^{-p}(\mathbb{R}^N))^m} &\leqslant C\|\psi\|_{(H^{p+1}(\mathbb{R}^N))^m} \leqslant c\|\psi\|_{(H^{p+2}(\mathbb{R}^N))^m}^{\frac{2p+1}{2p+2}}\|\psi\|_{(H^{-p}(\mathbb{R}^N))^m}^{\frac{1}{2p+2}} \\ &\leqslant \mu\|\mathbf{A}_\epsilon\psi\|_{(H^{-p}(\mathbb{R}^N))^m} + C(\mu)\|\psi\|_{(H^{-p}(\mathbb{R}^N))^m}, \end{aligned} \tag{2.7}$$

with arbitrary $\mu > 0$. This finishes the proof. $\square$

Remark 2.3. Since evidently the term $\alpha\Delta$ is also a perturbation of A_ϵ in $H^{-p}(\mathbb{R}^N)$, increasing eventually the value of ω, we claim that the operator

$$\mathbf{A}^\epsilon = \mathbf{A}_\epsilon + \mathbf{B} - \alpha\mathbf{\Delta} + \omega\mathbf{I} \tag{2.8}$$

is sectorial positive on the space $(H^{-p}(\mathbb{R}^N))^m$.

Now, the problem (1.3) will be rewritten as an abstract parabolic equation in $X = (H^{-p}(\mathbb{R}^N))^m$ with sectorial positive operator $\mathbf{A}^\epsilon$:

$$\begin{aligned} &u_t^\epsilon + \mathbf{A}^\epsilon u^\epsilon = \mathbf{F}\big(u^\epsilon\big) + (\gamma+\omega)u^\epsilon, \\ &u^\epsilon(0) = u_0, \end{aligned} \tag{2.9}$$

where $\omega > 0$ sufficiently large ($\gamma > 0$ arbitrary; see (2.3)). Here $\mathbf{F}(u^\epsilon) = g(\cdot,u^\epsilon) - \sum_{i=1}^N \frac{\partial}{\partial x_i}\nabla\Phi(u^\epsilon)$ is the nonlinear term.

Our next task is to show that with the conditions of Assumption I the nonlinear term $\mathbf{F}$ is Lipschitz continuous on bounded subsets as a map from $(H^{p+1}(\mathbb{R}^N))^m$ to X. More precisely, we need to prove:

Lemma 2.4. *For arbitrary bounded set* $B \subset (H^{p+1}(\mathbb{R}^N))^m$*, we have an estimate*

$$\exists_{C_B>0}\forall_{\phi,\psi\in B} \quad \big\|\mathbf{F}(\phi) - \mathbf{F}(\psi)\big\|_{(H^{-p}(\mathbb{R}^N))^m} \leqslant C_B\|\phi-\psi\|_{(H^{p+1}(\mathbb{R}^N))^m}. \tag{2.10}$$

Proof. Note first that with our assumption $2p > N \geqslant 1$ the inclusion $H^{p+1}(\mathbb{R}^N) \subset C_b^1(\mathbb{R}^N)$ holds ($C_b^1(\mathbb{R}^N)$ is the space of all $C^1(\mathbb{R}^N)$ functions having first order derivatives bounded on $\mathbb{R}^N$), since $p+1-\frac{N}{2} > 1$. Let $B \subset (H^{p+1}(\mathbb{R}^N))^m$ be bounded. Due to the form of $\mathbf{F}$ we need to estimate:

$$\begin{aligned}
&\left\| g(\cdot,\phi) - g(\cdot,\psi) - \sum_{i=1}^N \frac{\partial}{\partial x_i}\nabla\Phi(\phi) + \sum_{i=1}^N \frac{\partial}{\partial x_i}\nabla\Phi(\psi)\right\|_{(H^{-p}(\mathbb{R}^N))^m} \\
&\quad\leqslant L_B\|\phi-\psi\|_{(C_b^1(\mathbb{R}^N))^m} + \left\|\sum_{i=1}^N \frac{\partial}{\partial x_i}\nabla\Phi(\phi) - \sum_{i=1}^N \frac{\partial}{\partial x_i}\nabla\Phi(\psi)\right\|_{(H^{-p}(\mathbb{R}^N))^m},
\end{aligned}$$

where L_B is the Lipschitz constant of g restricted to arguments from B. In the estimate of the second term we use the assumption that $\Phi \in C^3(\mathbb{R}^m)$,

$$\begin{aligned}
\left\|\sum_{i=1}^N \frac{\partial}{\partial x_i}\nabla\Phi(\phi) - \sum_{i=1}^N \frac{\partial}{\partial x_i}\nabla\Phi(\psi)\right\|_{(H^{-p}(\mathbb{R}^N))^m} &\leqslant C\left\|\sum_{i=1}^N \frac{\partial}{\partial x_i}\nabla\Phi(\phi) - \sum_{i=1}^N \frac{\partial}{\partial x_i}\nabla\Phi(\psi)\right\|_{(L^2(\mathbb{R}^N))^m} \\
&\leqslant C\sum_{i=1}^N\sum_{j=1}^m \left\|\frac{\partial}{\partial x_i}\big(D_j\Phi(\phi) - D_j\Phi(\psi)\big)\right\|_{L^2(\mathbb{R}^N)} \\
&\leqslant c_B\|\phi-\psi\|_{(H^1(\mathbb{R}^N))^m},
\end{aligned}$$

$D_j\Phi$ is the j-th component of $\nabla\Phi$, and we use the global Lipschitz continuity of $D_j\Phi$ and its first derivatives on elements of B. The proof is completed. $\square$

Consequently, we have proved the following local existence result:

Theorem 2.5. *When $\alpha > 0$, $2p > N \geqslant 1$, and Assumption I is satisfied, then the problem (1.3) possess a unique local in time* mild solution:

$$u^\epsilon \in C\big([0,\tau_{u_0}); \big(H^{p+1}\big(\mathbb{R}^N\big)\big)^m\big) \cap C\big((0,\tau_{u_0}); \big(H^{p+2}\big(\mathbb{R}^N\big)\big)^m\big), \qquad u_t^\epsilon \in C\big((0,\tau_{u_0}); \big(H^{(p+2)^-}\big(\mathbb{R}^N\big)\big)^m\big), \tag{2.11}$$

where τ_{u_0} denotes the 'lifetime' of the local solution corresponding to u_0 and $(p+2)^-$ denotes any number strictly less than $(p+2)$. Moreover, such local solution fulfills the Cauchy integral formula:

$$u^\epsilon(t,u_0) = e^{-\mathbf{A}^\epsilon t}u_0 + \int_0^t e^{-\mathbf{A}^\epsilon(t-s)}\big(\mathbf{F}\big(u^\epsilon(s,u_0)\big) + (\gamma+\omega)u^\epsilon(s,u_0)\big)\,ds \quad \textit{for } 0 \leqslant t < \tau_{u_0}. \tag{2.12}$$

Once having stated the local existence result we are ready to study global solvability of (1.3) and existence of a semigroup on $(H^{p+1}(\mathbb{R}^N))^m$ corresponding to it.

2.2. Global solvability of an auxiliary problem (1.3)

As usual, if we have sufficiently good a priori estimates of the above local in time solutions valid over their interval of existence, the solutions will exist *globally in time*. In the sequel we will need the strengthened hypothesis concerning the nonlinear terms.

Assumption II. In addition to Assumption I, we require that:

- $\Phi : \mathbb{R}^m \to \mathbb{R}$ fulfills the growth conditions (D_k denotes the k-th partial derivative)

$$\exists_{c>0}\exists_{q\in[1,1+\frac{2p}{N})}\forall_{s\in\mathbb{R}^m}\forall_{1\leqslant j,k\leqslant m} \quad \big|D_jD_k\Phi(s)\big| \leqslant c\big(|s|+|s|^q\big). \tag{2.13}$$

- $g : \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^m$, with $g(x,0)=0$, fulfills the growth condition

$$\exists_{c>0}\exists_{r\in[2,2+\frac{2(p+1)}{N})}\forall_{x\in\mathbb{R}^N}\forall_{s\in\mathbb{R}^m} \quad \big|g(x,s)\big| \leqslant c\big(|s|^2+|s|^r\big). \tag{2.14}$$

Remark 2.6. For the future use observe, that if we accept a slower growth of nonlinear terms, then the growth restrictions above will be replaced with:

$$\exists_{c>0}\exists_{q<\frac{2p}{N}}\forall_{s\in\mathbb{R}^m}\forall_{1\leqslant j,k\leqslant m} \quad \big|D_jD_k\Phi(s)\big| \leqslant c\big(1+|s|^q\big), \tag{2.15}$$

and

$$\exists_{c>0}\exists_{r\in[1,1+\frac{2(p+1)}{N})}\forall_{x\in\mathbb{R}^N}\forall_{s\in\mathbb{R}^m}\quad \left|g(x,s)\right|\leqslant c\left(|s|+|s|^r\right). \tag{2.16}$$

The estimates given in (2.21)–(2.23) under the assumptions (2.15) and (2.16) are similar as presented there, but instead of the embedding $L^1(\mathbb{R}^N)\subset H^{-p}(\mathbb{R}^N)$ we need to use $L^2(\mathbb{R}^N)\subset H^{-p}(\mathbb{R}^N)$ with small modifications of exponents in these calculations.

Lemma 2.7. *The following estimates of the solution u^ϵ of (1.3) hold*:

$$\begin{aligned}&\left\|u^\epsilon\right\|^2_{(L^2(\mathbb{R}^N))^m}\leqslant \max\left\{\|u_0\|^2_{(L^2(\mathbb{R}^N))^m},\frac{\|n\|^2_{L^2(\mathbb{R}^N)}}{\gamma}\right\},\\ &\limsup_{t\to\infty}\left\|u^\epsilon(t,\cdot)\right\|^2_{(L^2(\mathbb{R}^N))^m}\leqslant\frac{\|n\|^2_{L^2(\mathbb{R}^N)}}{\gamma}.\end{aligned} \tag{2.17}$$

Proof. Taking the scalar product of (1.3) with u^ϵ, we obtain:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|u^\epsilon\right\|^2_{(L^2(\mathbb{R}^N))^m}+\int_{\mathbb{R}^N}u^\epsilon\cdot\sum_{i=1}^N\frac{\partial}{\partial x_i}\nabla\Phi\left(u^\epsilon\right)dx+\int_{\mathbb{R}^N}u^\epsilon\cdot\sum_{j=1}^N\frac{\partial}{\partial x_j}\sum_{i=1}^N\frac{\partial^{2p}u^\epsilon}{\partial x_i^{2p}}\,dx\\ &\quad=-\alpha\int_{\mathbb{R}^N}\left|\nabla u^\epsilon\right|^2dx+\int_{\mathbb{R}^N}\epsilon(-1)^p(\Delta)^{p+1}u^\epsilon\cdot u^\epsilon\,dx+\int_{\mathbb{R}^N}g\left(x,u^\epsilon\right)\cdot u^\epsilon\,dx.\end{aligned}$$

Since the two left hand side components vanish (see the explanation below), and

$$\int_{\mathbb{R}^N}\epsilon(-1)^p(\Delta)^{p+1}u^\epsilon\cdot u^\epsilon\,dx=-\epsilon\int_{\mathbb{R}^N}(-\Delta)^{p+1}u^\epsilon\cdot u^\epsilon\,dx=-\epsilon\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{p+1}{2}}u^\epsilon\right|^2dx\leqslant 0,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|u^\epsilon\right\|^2_{(L^2(\mathbb{R}^N))^m}+\alpha\int_{\mathbb{R}^N}\left|\nabla u^\epsilon\right|^2dx\leqslant\int_{\mathbb{R}^N}g\left(x,u^\epsilon\right)\cdot u^\epsilon\,dx.$$

Here we have used the equality:

$$\int_{\mathbb{R}^N}u^\epsilon\cdot\sum_{j=1}^N\frac{\partial}{\partial x_j}\sum_{i=1}^N\frac{\partial^{2p}u^\epsilon}{\partial x_i^{2p}}\,dx=-\int_{\mathbb{R}^N}\sum_{j=1}^N\frac{\partial}{\partial x_j}\sum_{i=1}^N\frac{\partial^{2p}u^\epsilon}{\partial x_i^{2p}}\cdot u^\epsilon\,dx,$$

which is obtained by the $(2p+1)$ integration by parts (for dense set of smooth solutions), and

$$\int_{\mathbb{R}^N}u^\epsilon\cdot\sum_{i=1}^N\frac{\partial}{\partial x_i}\nabla\Phi\left(u^\epsilon\right)dx=-\int_{\mathbb{R}^N}\sum_{i=1}^N\sum_{k=1}^m\frac{\partial u_k^\epsilon}{\partial x_i}D_k\Phi\left(u^\epsilon\right)dx=-\int_{\mathbb{R}^N}\sum_{i=1}^N\frac{\partial}{\partial x_i}\Phi\left(u^\epsilon\right)dx=0$$

(where $D_k\Phi$ denotes the k-th component of the vector $\nabla\Phi$). Now, by the assumption (2.1),

$$\frac{1}{2}\frac{d}{dt}\left\|u^\epsilon\right\|^2_{(L^2(\mathbb{R}^N))^m}+\alpha\int_{\mathbb{R}^N}\left|\nabla u^\epsilon\right|^2dx+\epsilon\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{p+1}{2}}u^\epsilon\right|^2dx\leqslant-\gamma\left\|u^\epsilon\right\|^2_{(L^2(\mathbb{R}^N))^m}+\int_{\mathbb{R}^N}n(x)\left|u^\epsilon\right|dx, \tag{2.18}$$

which, with the use of the Cauchy inequality, assures uniform in time boundedness of the $(L^2(\mathbb{R}^N))^m$ norm of u^ϵ as reported in (2.17).

Returning to Eq. (2.18) we can also check that

$$\epsilon\int_0^T\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{p+1}{2}}u^\epsilon\right|^2dx\,dt\leqslant\text{const}\quad\text{for arbitrary } T>0.\qquad\square \tag{2.19}$$

With the above introductory $(L^2(\mathbb{R}^N))^m$ a priori estimate (2.17) and the growth restriction imposed on Φ we are able to show, that the nonlinear term $\mathbf{F}(u^\epsilon(t,\cdot))$ is *subordinated* to the main part operator $\mathbf{A}^\epsilon$ (compare [7, p. 72] for more details).

Generally speaking, under such condition the term $\mathbf{F}(u^\epsilon(t,\cdot))$ can be treated as having *sublinear growth* on the solution. Consequently the local solution obtained in Theorem 2.5 will be extended globally in time.

The subordination condition reads:

$$\big\|\mathbf{F}\big(u^\epsilon(t,\cdot)\big)\big\|_{(H^{-p}(\mathbb{R}^N))^m} \leqslant c\big(\big\|u^\epsilon(t,\cdot)\big\|_{(L^2(\mathbb{R}^N))^m}\big)\big(1+\big\|u^\epsilon(t,\cdot)\big\|^\theta_{(H^{p+1}(\mathbb{R}^N))^m}\big), \tag{2.20}$$

where $\theta\in(0,1)$ is a constant.

Here, $X^\alpha=(H^{p+1}(\mathbb{R}^N))^m$, the auxiliary space $Y=(L^2(\mathbb{R}^N))^m$ and, to estimate the left hand side in $(H^{-p}(\mathbb{R}^N))^m$ we will use the embeddings of the *conjugate spaces*. More precisely, since $2p>N\geqslant 1$ then $(H^p(\mathbb{R}^N))^m\subset(L^\infty(\mathbb{R}^N))^m$, consequently $(L^1(\mathbb{R}^N))^m\subset(H^{-p}(\mathbb{R}^N))^m$. We will thus start the estimates:

$$\begin{aligned}\big\|\mathbf{F}\big(u^\epsilon(t,\cdot)\big)\big\|_{(H^{-p}(\mathbb{R}^N))^m} &\leqslant \big\|g\big(\cdot,u^\epsilon\big)\big\|_{(H^{-p}(\mathbb{R}^N))^m}+\left\|\sum_{i=1}^N\frac{\partial}{\partial x_i}\nabla\Phi\big(u^\epsilon\big)\right\|_{(H^{-p}(\mathbb{R}^N))^m}\\ &\leqslant c\big\|g\big(\cdot,u^\epsilon\big)\big\|_{(L^1(\mathbb{R}^N))^m}+c\left\|\sum_{i=1}^N\frac{\partial}{\partial x_i}\nabla\Phi\big(u^\epsilon\big)\right\|_{(L^1(\mathbb{R}^N))^m}.\end{aligned} \tag{2.21}$$

The first component is estimated with the use of Assumption II and the Nirenberg–Gagliardo inequality:

$$\begin{aligned}\big\|g\big(\cdot,u^\epsilon\big)\big\|_{(L^1(\mathbb{R}^N))^m} &\leqslant c\big(\big\|u^\epsilon\big\|^2_{(L^2(\mathbb{R}^N))^m}+\big\|u^\epsilon\big\|^r_{(L^r(\mathbb{R}^N))^m}\big)\\ &\leqslant c\big(\big\||u^\epsilon|\big\|^2_{L^2(\mathbb{R}^N)}+\big\||u^\epsilon|\big\|^{r(1-\theta)}_{L^2(\mathbb{R}^N)}\big\||u^\epsilon|\big\|^{r\theta}_{H^{p+1}(\mathbb{R}^N)}\big),\end{aligned} \tag{2.22}$$

where $\frac{N(r-2)}{2r(p+1)}\leqslant\theta<\frac{1}{r}$, or $r<\frac{2(p+1)}{N}+2$.

To estimate the second component we need to use the assumption (2.13):

$$\begin{aligned}\left\|\sum_{i=1}^N\frac{\partial}{\partial x_i}\nabla\Phi\big(u^\epsilon\big)\right\|_{(L^1(\mathbb{R}^N))^m} &\leqslant c\big\|\big(|u^\epsilon|+|u^\epsilon|^q\big)|\nabla u^\epsilon|\big\|_{L^1(\mathbb{R}^N)}\\ &\leqslant c\big\||u^\epsilon|\big\|_{L^2(\mathbb{R}^N)}\big\||\nabla u^\epsilon|\big\|_{L^2(\mathbb{R}^N)}+c\big\||u^\epsilon|\big\|^q_{L^{2q}(\mathbb{R}^N)}\big\||\nabla u^\epsilon|\big\|_{L^2(\mathbb{R}^N)}.\end{aligned} \tag{2.23}$$

Now, due to the Nirenberg–Gagliardo inequality

$$\big\||\nabla u^\epsilon|\big\|_{L^2(\mathbb{R}^N)}\leqslant c\big\||u^\epsilon|\big\|^{\frac{p}{p+1}}_{L^2(\mathbb{R}^N)}\big\||u^\epsilon|\big\|^{\frac{1}{p+1}}_{H^{p+1}(\mathbb{R}^N)}$$

(since $2p>N$ then $\frac{1}{p+1}<\frac{2}{N+2}<1$). Also, again by the Nirenberg–Gagliardo inequality,

$$\big\||u^\epsilon|\big\|^q_{L^{2q}(\mathbb{R}^N)}\leqslant c'\big\||u^\epsilon|\big\|^{\theta q}_{L^2(\mathbb{R}^N)}\big\||u^\epsilon|\big\|^{(1-\theta)q}_{H^{p+1}(\mathbb{R}^N)},$$

where $(1-\theta)q=\frac{N(q-1)}{2(p+1)}<1-\frac{1}{p+1}$, since $q\in[1,1+\frac{2p}{N})$ by (2.13). Thus, the subordination condition (2.20) holds. Consequently, the local solution to (1.3) constructed in Theorem 2.5 will be extended globally in time defining on the phase space $(H^{p+1}(\mathbb{R}^N))^m$ a semigroup:

$$S^\epsilon(t)u_0:=u^\epsilon(t,u_0),\quad t\geqslant 0,\ \epsilon>0, \tag{2.24}$$

where we temporarily express the dependence of the solution on initial data u_0. We recall here the regularity of that semigroup, as reported in (2.12):

$$\begin{aligned}&S^\epsilon(\cdot)u_0\in C\big([0,\infty);\big(H^{p+1}\big(\mathbb{R}^N\big)\big)^m\big)\cap C\big((0,\infty);\big(H^{p+2}\big(\mathbb{R}^N\big)\big)^m\big),\\ &S^\epsilon(\cdot)u_0\in C^1\big((0,\infty);\big(H^{(p+2)^-}\big(\mathbb{R}^N\big)\big)^m\big).\end{aligned} \tag{2.25}$$

3. Smooth global solutions to (1.3)

We will concentrate on the studies of regular viscous solutions to (1.2). We need first to investigate the smooth solutions of (1.3) and then a limit passage $\epsilon\to 0^+$, where we focus on the regularization effect of the viscosity term $\alpha\Delta u^\epsilon$. From now on, to get the estimates uniform in ϵ, the following sharper assumptions are required.

Assumption III. In addition to Assumption I we require that:

- $N = 1, 2, 3$; the space dimensions that are most interesting in applications.
- $\Phi : \mathbb{R}^m \to \mathbb{R}$ is $C^{p+2}(\mathbb{R}^m)$, with $\Phi(0) = 0$, fulfills the growth conditions

$$\exists_{c>0} \exists_{q' < \frac{4-N}{4}} \forall_{s\in\mathbb{R}^m} \forall_{1\leqslant j_1, j_2, k\leqslant m} \quad \left|D_{j_1} D_{j_2} D_k \Phi(s)\right| \leqslant c\left(1 + |s|^{q'}\right) \quad \text{when } N = 2, 3. \tag{3.1}$$

Moreover we assume

$$\exists_{c>0} \exists_{q < \frac{2}{N}} \forall_{s\in\mathbb{R}^m} \forall_{1\leqslant j, k\leqslant m} \quad \left|D_j D_k \Phi(s)\right| \leqslant c\left(1 + |s|^q\right), \tag{3.2}$$

and the, following from it (see Observation 3.1), consequences

$$\exists_{c>0} \forall_{s\in\mathbb{R}^m} \forall_{1\leqslant k\leqslant m} \quad \left|D_k \Phi(s)\right| \leqslant c\left(|s| + |s|^{q+1}\right), \tag{3.3}$$

$$\exists_{c>0} \forall_{s\in\mathbb{R}^m} \quad \left|\Phi(s)\right| \leqslant c\left(|s|^2 + |s|^{q+2}\right). \tag{3.4}$$

- $g : \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^m$ is $C^p(\mathbb{R}^N \times \mathbb{R}^m)$, with $g(x, 0) = 0$, fulfills the growth condition

$$\exists_{c>0} \exists_{r < 1 + \frac{4}{N}} \forall_{x\in\mathbb{R}^N} \forall_{s\in\mathbb{R}^m} \quad \left|g(x, s)\right| \leqslant c\left(|s|^2 + |s|^r\right). \tag{3.5}$$

- $$\exists_{c>0} \exists_{r' < 4} \forall_{x\in\mathbb{R}^N} \forall_{s\in\mathbb{R}^m} \forall_{i=1,\ldots,m} \quad \left|\frac{\partial g(x, s)}{\partial s_i}\right| \leqslant c\left(|s| + |s|^{r'}\right) \quad \text{when } N = 2, 3. \tag{3.6}$$
- $$\exists_{c>0} \exists_{w\geqslant 1} \forall_{s\in\mathbb{R}^m} \forall_{x\in\mathbb{R}^N} \forall_{i=1,\ldots,N} \quad \left|D_i g(x, s)\right| \leqslant c\left(|s|^2 + |s|^w\right), \tag{3.7}$$

  where $w < 7$ when $N = 3$ (recall that D_i denotes the i-th partial derivative).

The following elementary observation will be used to bound the lower order derivatives of Φ from (3.2):

Observation 3.1. Let $\Phi : \mathbb{R}^m \to \mathbb{R}$ be $C^3(\mathbb{R}^m)$, $\Phi(0) = 0$, and satisfy the growth restriction

$$\exists_{c>0} \exists_{q>0} \forall_{s\in\mathbb{R}^m} \forall_{1\leqslant j, k\leqslant m} \quad \left|D_j D_k \Phi(s)\right| \leqslant c\left(1 + |s|^q\right).$$

Then we have an estimate:

$$\left|D_k \Phi(s)\right| \leqslant cm\left(|s| + |s|^{q+1}\right).$$

Indeed,

$$\begin{aligned} D_k \Phi(s) &= D_k\left(\Phi(s) - \Phi(0)\right) \\ &= D_k\left(\Phi(s_1, s_2, \ldots, s_m) - \Phi(0, s_2, \ldots, s_m) + \Phi(0, s_2, \ldots, s_m) - \Phi(0, 0, s_3, \ldots, s_m) + \cdots - \Phi(0, 0, \ldots, 0)\right) \\ &= D_k\left(\sum_{j=1}^m D_j \Phi(\tilde{s}_j) s_j\right), \end{aligned}$$

and consequently,

$$\left|D_k \Phi(s)\right| \leqslant \sum_{j=1}^m c\left(1 + |\tilde{s}_j|^q\right)|s| \leqslant cm\left(1 + |s|^q\right)|s| = cm\left(|s| + |s|^{q+1}\right).$$

Remark 3.2. There are several restrictions imposed on the constants appearing in the paper: the space dimension N, the exponent p, the exponents of admissible growth of nonlinear terms Φ and g. The limitation imposed on p and N is the, valid throughout the text, assumption $2p > N \geqslant 1$. It was essential in the proof of Lemma 2.4 and Theorem 2.5. Further limitation on the space dimension N and the growth of nonlinear terms is needed for smooth solutions. Assumption III is essential if we want, in the proof of $(H^1(\mathbb{R}^N))^m$ a priori estimate, the term $\alpha \Delta u^\epsilon$ together with the a priori estimate in $(L^2(\mathbb{R}^N))^m$ to control the nonlinear components in Eq. (1.3). In the light of Assumption III the space dimension N will be restricted to $N = 1, 2, 3$ and the function $\Phi(s)$ will grow like $|s|^{(2+\frac{2}{N})^-}$ for $N = 1, 2, 3$ (see also Remark 3.5), while the nonlinear term $g(x, s)$ will grow like $|s|^{(1+\frac{4}{N})^-}$ in these dimensions.

We start with obtaining a priori estimates of u^ϵ, taking care that they are uniform in $\epsilon > 0$. Such estimates will be next inherited by the viscous solutions of (1.2).

Remark 3.3. In this paper, in the process of estimating the $H^q(\mathbb{R}^N)$ type norms of u^ϵ in (3.8) or, even more, in (3.21), we return to an old idea of 'multiplying' Eq. (1.3) by another elliptic operator; see [17, Chapter 3, Remark 8.3], or the source paper by P.E. Sobolevskij [24]. While in that references the order of the two elliptic operators should be the same (and equal to 2), here we need to consider a generalization of that idea when the two elliptic operators (one in the equation and another one we are multiplying by) have eventually different orders.

Lemma 3.4. *Under Assumption III the global solutions u^ϵ of (1.3) are bounded in $(H^1(\mathbb{R}^N))^m$ uniformly in $t \geqslant 0$ and in $\epsilon > 0$.*

Proof. Note first, that the expression

$$\left(\|\phi\|^2_{L^2(\mathbb{R}^N)} + \sum_{i=1}^N \left\|\frac{\partial^l \phi}{\partial x_i^l}\right\|^2_{L^2(\mathbb{R}^N)}\right)^{\frac{1}{2}},$$

defines an equivalent norm of $H^l(\mathbb{R}^N)$ (see e.g. [26, Remark 2.3.3]). To get an exact form of the $(H^1(\mathbb{R}^N))^m$ *a priori estimate* we multiply (1.3) by $-\Delta u^\epsilon$:

$$\begin{aligned} &-\int_{\mathbb{R}^N} u_t^\epsilon \cdot \Delta u^\epsilon\,dx - \int_{\mathbb{R}^N}\left(\sum_{i=1}^N \frac{\partial}{\partial x_i}\nabla\Phi\big(u^\epsilon\big) + \sum_{j=1}^N \frac{\partial}{\partial x_j}\sum_{i=1}^N \frac{\partial^{2p}}{\partial x_i^{2p}} u^\epsilon\right)\cdot \Delta u^\epsilon\,dx \\ &\quad = -\int_{\mathbb{R}^N}\big(\alpha\Delta u^\epsilon + \epsilon(-1)^p(\Delta)^{p+1}u^\epsilon + g\big(x,u^\epsilon\big)\big)\cdot\Delta u^\epsilon\,dx. \end{aligned} \tag{3.8}$$

We will transform the components one by one. We have:

$$-\int_{\mathbb{R}^N} u_t^\epsilon\cdot\Delta u^\epsilon\,dx = \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}\sum_{i=1}^N\left|\frac{\partial}{\partial x_i}u^\epsilon\right|^2 dx,$$

$$\int_{\mathbb{R}^N}\sum_{j=1}^N \frac{\partial}{\partial x_j}\sum_{i=1}^N\frac{\partial^{2p}}{\partial x_i^{2p}}u^\epsilon\cdot\sum_{i=1}^N\frac{\partial^2}{\partial x_i^2}u^\epsilon\,dx = 0,$$

and

$$\epsilon\int_{\mathbb{R}^N}(-1)^{p+1}(\Delta)^{p+1}u^\epsilon\cdot\Delta u^\epsilon\,dx \leqslant 0, \tag{3.9}$$

through $(p+2)$ integrations by parts. To estimate the nonlinear terms we will use Assumption III. We have

$$\begin{aligned} &\int_{\mathbb{R}^N}\left(-\sum_{i=1}^N\frac{\partial}{\partial x_i}\nabla\Phi\big(u^\epsilon\big) + g\big(x,u^\epsilon\big)\right)\cdot\Delta u^\epsilon\,dx \\ &\quad\leqslant c\left(\sum_{i=1}^N\sum_{j,k=1}^m\left\|D_kD_j\Phi\big(u^\epsilon\big)\frac{\partial u_j^\epsilon}{\partial x_i}\right\|_{L^2(\mathbb{R}^N)} + \big\|g\big(\cdot,u^\epsilon\big)\big\|_{(L^2(\mathbb{R}^N))^m}\right)\big\|u^\epsilon\big\|_{(H^2(\mathbb{R}^N))^m}. \end{aligned} \tag{3.10}$$

The term $\|g(\cdot,u^\epsilon)\|_{(L^2(\mathbb{R}^N))^m}$ is estimated as follows

$$\begin{aligned} \big\|g\big(\cdot,u^\epsilon\big)\big\|_{(L^2(\mathbb{R}^N))^m} &\leqslant c\big(\big\||u^\epsilon|\big\|^2_{L^4(\mathbb{R}^N)} + \big\||u^\epsilon|\big\|^r_{L^{2r}(\mathbb{R}^N)}\big) \\ &\leqslant c\big(\big\||u^\epsilon|\big\|_{L^2(\mathbb{R}^N)}\big)\big(\big\||u^\epsilon|\big\|^{\frac{N}{4}}_{H^2(\mathbb{R}^N)} + \big\||u^\epsilon|\big\|^\theta_{H^2(\mathbb{R}^N)}\big), \end{aligned} \tag{3.11}$$

where $\theta = \frac{N(r-1)}{4} < 1$.

Using Agmon's inequality [25, p. 52]

$$\big\||u^\epsilon|\big\|_{L^\infty(\mathbb{R}^N)} \leqslant \begin{cases} c'\||u^\epsilon|\|^{\frac{1}{2}}_{H^{\frac{N}{2}-1}(\mathbb{R}^N)}\||u^\epsilon|\|^{\frac{1}{2}}_{H^{\frac{N}{2}+1}(\mathbb{R}^N)} & N \text{ even}, \\ c'\||u^\epsilon|\|^{\frac{1}{2}}_{H^{\frac{N-1}{2}}(\mathbb{R}^N)}\||u^\epsilon|\|^{\frac{1}{2}}_{H^{\frac{N+1}{2}}(\mathbb{R}^N)} & N \text{ odd}, \end{cases} \tag{3.12}$$

we obtain

$$\left\| D_k D_j \Phi(u^\epsilon) \frac{\partial u_j^\epsilon}{\partial x_i} \right\|_{L^2(\mathbb{R}^N)} \leqslant \| D_k D_j \Phi(u^\epsilon) \|_{L^\infty(\mathbb{R}^N)} \left\| \frac{\partial u_j^\epsilon}{\partial x_i} \right\|_{L^2(\mathbb{R}^N)}$$
$$\leqslant c\big(\| |u^\epsilon| \|^q_{L^\infty(\mathbb{R}^N)} + 1 \big) \| |u^\epsilon| \|^{\frac{1}{2}}_{L^2(\mathbb{R}^N)} \| |u^\epsilon| \|^{\frac{1}{2}}_{H^2(\mathbb{R}^N)} \leqslant c\big(\| |u^\epsilon| \|_{L^2(\mathbb{R}^N)} \big) \| |u^\epsilon| \|^{\frac{Nq+2}{4}}_{H^2(\mathbb{R}^N)}. \tag{3.13}$$

Note that in (3.12) we need to have $\frac{N}{2} + 1 \leqslant 2$ (N even); equivalently $N = 2$. For N odd, we require $\frac{N+1}{2} \leqslant 2$; equivalently $N = 1, 3$. Finally, to bound the $H^2(\mathbb{R}^N)$ norm of the solution u^ϵ, we need to use a version of the *Calderon–Zygmund estimate*

$$-\alpha \int_{\mathbb{R}^N} \Delta u^\epsilon \cdot \Delta u^\epsilon \, dx \leqslant -c(\alpha) \| u^\epsilon \|^2_{(H^2(\mathbb{R}^N))^m} + \alpha \| u^\epsilon \|^2_{(L^2(\mathbb{R}^N))^m}, \tag{3.14}$$

which has the coefficients $\alpha, c(\alpha) > 0$ independent of ϵ.

Collecting the above estimates we obtain:

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} \sum_{i=1}^N \left| \frac{\partial}{\partial x_i} u^\epsilon \right|^2 dx + c(\alpha) \| u^\epsilon \|^2_{(H^2(\mathbb{R}^N))^m}$$
$$\leqslant \alpha \| u^\epsilon \|^2_{(L^2(\mathbb{R}^N))^m} + c\big(\| u^\epsilon(t,\cdot) \|_{(L^2(\mathbb{R}^N))^m} \big) \big(1 + \| u^\epsilon(t,\cdot) \|^\theta_{(H^2(\mathbb{R}^N))^m} \big) \| u^\epsilon \|_{(H^2(\mathbb{R}^N))^m}, \tag{3.15}$$

where $\theta < 1$ as above. Consequently, by the Young inequality,

$$\frac{d}{dt} \int_{\mathbb{R}^N} \sum_{i=1}^N \left| \frac{\partial}{\partial x_i} u^\epsilon \right|^2 dx + c(\alpha) \| u^\epsilon \|^2_{(H^2(\mathbb{R}^N))^m} \leqslant C\big(\| u^\epsilon(t,\cdot) \|_{(L^2(\mathbb{R}^N))^m} \big),$$

the estimate being uniform in $\epsilon > 0$. Together with (2.17) this brings us the, uniform in ϵ and in $t \in [0, \infty)$, estimate of the $(H^1(\mathbb{R}^N))^m$ norm of $u^\epsilon(t,\cdot)$

$$\| u^\epsilon(t,\cdot) \|_{(H^1(\mathbb{R}^N))^m} \leqslant c_0\big(\| u^\epsilon(t,\cdot) \|_{(L^2(\mathbb{R}^N))^m} \big), \quad t \in [0,\infty). \qquad \square \tag{3.16}$$

Remark 3.5. It is easy to see, that the same reasoning (especially (3.13)) allows us to estimate the $(H^1(\mathbb{R}^N))^m$ norm of u^ϵ uniformly in ϵ if we let $\theta = 1$ in (3.15) but require the coefficient $c(\| |u^\epsilon| \|_{L^2(\mathbb{R}^N)})$ there to be sufficiently small compared to $c(\alpha)$ introduced in (3.14). Extending this observation we will describe next the form of Φ admitted by the above estimates.

When $N = 1$ we will assume that

$$\big| \Phi(s) \big| \leqslant C(\alpha) |s|^4 + \text{a polynomial of } |s| \text{ of order } 3, \tag{3.17}$$

in space dimension $N = 2$ we need to have

$$\big| \Phi(s) \big| \leqslant C(\alpha) |s|^3 + \text{a polynomial of } |s| \text{ of order } 2, \tag{3.18}$$

where $C(\alpha) > 0$ are *sufficiently small* constants relatively to the coefficient α in (1.2). So, when $N = 2$, Φ will be in particular a third order polynomial with small positive main coefficient. It is difficult to write explicitly the smallness restriction on the main coefficient. Instead one can think of the nonlinearity Φ satisfying

$$\big| \Phi(s) \big| \leqslant c |s|^{3^-} + \text{a polynomial of } |s| \text{ of order } 2, \tag{3.19}$$

since $|s|^{3^-} \leqslant \epsilon |s|^3 + C(\epsilon)|s|$ holds *for arbitrary small* $\epsilon > 0$. Similar observation is true for the estimate (3.23) below and the resulting estimate of the solutions, when $N = 2$.

For $N = 3$, the situation is a bit more delicate since now the growth of nonlinear term Φ allowed in (3.13) is like

$$\big| \Phi(s) \big| \leqslant C(\alpha) |s|^{\frac{8}{3}} + \text{a polynomial of } |s| \text{ of order } 2, \tag{3.20}$$

where $C(\alpha) > 0$ is *sufficiently small* relatively to the coefficient α. Thus, Φ is not a *pure polynomial* in that case.

3.1. Third a priori estimate of the global solutions to (1.3)

We get the $H^{p+1}(\mathbb{R}^N)$ *a priori estimate* by induction. In such estimates we need to use the formula for high derivative of composite function, as stated e.g. in [11], known under the name of the *generalized Bruno formula*. Since

$$\int_{\mathbb{R}^N} \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \sum_{i=1}^{N} \frac{\partial^{2p}}{\partial x_i^{2p}} u^\epsilon \cdot \sum_{i=1}^{N} \frac{\partial^{2l+2}}{\partial x_i^{2l+2}} u^\epsilon \, dx = 0,$$

multiplying Eq. (1.3) by $(-1)^{l+1} \sum_{i=1}^N \frac{\partial^{2l+2}}{\partial x_i^{2l+2}} u^\epsilon$, integrating by parts, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^N} \sum_{i=1}^{N} \left| \frac{\partial^{l+1}}{\partial x_i^{l+1}} u^\epsilon \right|^2 dx + c\alpha \left\| u^\epsilon \right\|^2_{(H^{l+2}(\mathbb{R}^N))^m} - \alpha \left\| u^\epsilon \right\|^2_{(L^2(\mathbb{R}^N))^m} + \epsilon \int_{\mathbb{R}^N} (-\Delta)^{p+1} u^\epsilon \cdot (-1)^{l+1} \sum_{i=1}^{N} \frac{\partial^{2l+2}}{\partial x_i^{2l+2}} u^\epsilon \, dx \\ &\quad \leqslant c \left(\left\| \nabla \Phi(u^\epsilon) \right\|_{(L^2(\mathbb{R}^N))^m} + \sum_{i=1}^{N} \left\| \frac{\partial^{l+1} \nabla \Phi(u^\epsilon)}{\partial x_i^{l+1}} \right\|_{(L^2(\mathbb{R}^N))^m} \right) \left\| u^\epsilon \right\|_{(H^{l+2}(\mathbb{R}^N))^m} \\ &\qquad + c \left(\left\| g(\cdot, u^\epsilon) \right\|_{(L^2(\mathbb{R}^N))^m} + \sum_{i=1}^{N} \left\| \frac{\partial^{l} g(\cdot, u^\epsilon)}{\partial x_i^{l}} \right\|_{(L^2(\mathbb{R}^N))^m} \right) \left\| u^\epsilon \right\|_{(H^{l+2}(\mathbb{R}^N))^m}. \end{aligned} \tag{3.21}$$

The term $\|g(\cdot, u^\epsilon)\|_{(L^2(\mathbb{R}^N))^m}$ is estimated as in (3.11). To bound the component $\|\nabla \Phi(u^\epsilon)\|_{(L^2(\mathbb{R}^N))^m}$ we need to use the assumption (3.3). We have

$$\left\| \nabla \Phi(u^\epsilon) \right\|_{(L^2(\mathbb{R}^N))^m} \leqslant c\left(\left\| u^\epsilon \right\|_{(L^2(\mathbb{R}^N))^m} + \left\| u^\epsilon \right\|^{q+1}_{(L^{2q+2}(\mathbb{R}^N))^m} \right) \leqslant c\left(\left\| |u^\epsilon| \right\|_{H^l(\mathbb{R}^N)} \right).$$

Note that to bound the term $\|\frac{\partial^l g(\cdot, u^\epsilon)}{\partial x_i^l}\|_{(L^2(\mathbb{R}^N))^m}$ we need to estimate the following expressions:

$$\begin{aligned} &\left\| D_i^l g(\cdot, u^\epsilon) \right\|_{(L^2(\mathbb{R}^N))^m}, \ \sum_{j_1}^{m} \left\| D_i^{l-1} \frac{\partial g(\cdot, u^\epsilon)}{\partial u^\epsilon_{j_1}} \frac{\partial u^\epsilon_{j_1}}{\partial x_i} \right\|_{(L^2(\mathbb{R}^N))^m}, \ \ldots, \ \sum_{j_1 \ldots j_l}^{m} \left\| \frac{\partial^l g(\cdot, u^\epsilon)}{\partial u^\epsilon_{j_1} \ldots \partial u^\epsilon_{j_l}} \frac{\partial u^\epsilon_{j_1}}{\partial x_i} \cdots \frac{\partial u^\epsilon_{j_l}}{\partial x_i} \right\|_{(L^2(\mathbb{R}^N))^m}, \\ &\sum_{j_1}^{m} \left\| D_i^{l-2} \frac{\partial g(\cdot, u^\epsilon)}{\partial u^\epsilon_{j_1}} \frac{\partial^2 u^\epsilon_{j_1}}{\partial x_i^2} \right\|_{(L^2(\mathbb{R}^N))^m}, \ \ldots, \ \sum_{j_1 \ldots j_{l-1}}^{m} \left\| \frac{\partial^{l-1} g(\cdot, u^\epsilon)}{\partial u^\epsilon_{j_1} \ldots \partial u^\epsilon_{j_{l-1}}} \frac{\partial^2 u^\epsilon_{j_1}}{\partial x_i^2} \frac{\partial u^\epsilon_{j_2}}{\partial x_i} \cdots \frac{\partial u^\epsilon_{j_{l-1}}}{\partial x_i} \right\|_{(L^2(\mathbb{R}^N))^m}, \ \ldots, \\ &\sum_{j_1}^{m} \left\| D_i \frac{\partial g(\cdot, u^\epsilon)}{\partial u^\epsilon_{j_1}} \frac{\partial^{l-1} u^\epsilon_{j_1}}{\partial x_i^{l-1}} \right\|_{(L^2(\mathbb{R}^N))^m}, \ \ldots, \ \sum_{j_1, j_2}^{m} \left\| \frac{\partial^2 g(\cdot, u^\epsilon)}{\partial u^\epsilon_{j_1} \partial u^\epsilon_{j_2}} \frac{\partial^{l-1} u^\epsilon_{j_1}}{\partial x_i^{l-1}} \frac{\partial u^\epsilon_{j_2}}{\partial x_i} \right\|_{(L^2(\mathbb{R}^N))^m}, \ \sum_{j_1}^{m} \left\| \frac{\partial g(\cdot, u^\epsilon)}{\partial u^\epsilon_{j_1}} \frac{\partial^l u^\epsilon_{j_1}}{\partial x_i^l} \right\|_{(L^2(\mathbb{R}^N))^m}. \end{aligned} \tag{3.22}$$

Now we present the calculations for $l = 1$. This case is the most complicated for the estimates. The case $l \geqslant 2$ is similar, but easier. Using Assumption III we have

$$\begin{aligned} \left\| \frac{\partial g(\cdot, u^\epsilon)}{\partial x_i} \right\|_{(L^2(\mathbb{R}^N))^m} &\leqslant \left\| D_i g(\cdot, u^\epsilon) \right\|_{(L^2(\mathbb{R}^N))^m} + \sum_{j}^{m} \left\| \frac{\partial g(\cdot, u^\epsilon)}{\partial u^\epsilon_j} \frac{\partial u^\epsilon_j}{\partial x_i} \right\|_{(L^2(\mathbb{R}^N))^m} \\ &\leqslant c\left(\left\| u^\epsilon \right\|^2_{(L^4(\mathbb{R}^N))^m} + \left\| u^\epsilon \right\|^w_{(L^{2w}(\mathbb{R}^N))^m} + \left(\left\| u^\epsilon \right\|_{(L^\infty(\mathbb{R}^N))^m} + \left\| u^\epsilon \right\|^{r'}_{(L^\infty(\mathbb{R}^N))^m} \right) \left\| u^\epsilon \right\|_{(H^1(\mathbb{R}^N))^m} \right) \end{aligned}$$

when $N = 2, 3$ (here $w < 7$ when $N = 3$; $r' < 4$). Next, thanks to the estimate (3.12) and Nirenberg–Gagliardo inequality, we obtain

$$\begin{aligned} \left\| \frac{\partial g(\cdot, u^\epsilon)}{\partial x_i} \right\|_{(L^2(\mathbb{R}^N))^m} &\leqslant c\left(\left\| u^\epsilon \right\|_{(H^1(\mathbb{R}^N))^m} \right) \\ &\quad \times \begin{cases} 1 & N = 1, \\ 1 + \|u^\epsilon\|^{\frac{1}{4}}_{(H^3(\mathbb{R}^N))^m} + \|u^\epsilon\|^{\frac{r'}{4}}_{(H^3(\mathbb{R}^N))^m} & N = 2, \\ 1 + \|u^\epsilon\|^{w\theta}_{(H^3(\mathbb{R}^N))^m} + \|u^\epsilon\|^{\frac{1}{4}}_{(H^3(\mathbb{R}^N))^m} + \|u^\epsilon\|^{\frac{r'}{4}}_{(H^3(\mathbb{R}^N))^m} & N = 3, \ \theta = \max\{0, \frac{w-3}{4w}\}. \end{cases} \end{aligned}$$

The term $\|\frac{\partial^2 \nabla\Phi(u^\epsilon)}{\partial x_i^2}\|_{(L^2(\mathbb{R}^N))^m}$ is estimated similarly to the term $\|\frac{\partial g(\cdot,u^\epsilon)}{\partial x_i}\|_{(L^2(\mathbb{R}^N))^m}$. We have

$$\left\| D_k D_{j_1}\Phi\big(u^\epsilon\big)\frac{\partial^2 u_{j_1}^\epsilon}{\partial x_i^2}\right\|_{L^2(\mathbb{R}^N)} \leqslant c\big\|u^\epsilon\big\|_{(H^1(\mathbb{R}^N))^m}^{\frac{1}{2}}\big\|u^\epsilon\big\|_{(H^3(\mathbb{R}^N))^m}^{\frac{1}{2}}\big(1+\big\|u^\epsilon\big\|_{(L^\infty(\mathbb{R}^N))^m}^q\big)$$
$$\leqslant c\big(\big\|u^\epsilon\big\|_{(H^1(\mathbb{R}^N))^m}\big)\times\begin{cases}\|u^\epsilon\|_{(H^3(\mathbb{R}^N))^m}^{\frac{1}{2}} & N=1,\\ \|u^\epsilon\|_{(H^3(\mathbb{R}^N))^m}^{\frac{1}{2}}+\|u^\epsilon\|_{(H^3(\mathbb{R}^N))^m}^{\frac{q+2}{4}} & N=2,3,\end{cases}$$

and

$$\left\| D_k D_{j_1} D_{j_2}\Phi\big(u^\epsilon\big)\frac{\partial u_{j_1}^\epsilon}{\partial x_i}\frac{\partial u_{j_2}^\epsilon}{\partial x_i}\right\|_{(L^2(\mathbb{R}^N))^m} \leqslant c\big\|D_k D_{j_1} D_{j_2}\Phi\big(u^\epsilon\big)\big\|_{L^\infty(\mathbb{R}^N)}\left(\left\|\frac{\partial u_{j_1}^\epsilon}{\partial x_i}\right\|_{L^4(\mathbb{R}^N)}^2+\left\|\frac{\partial u_{j_2}^\epsilon}{\partial x_i}\right\|_{L^4(\mathbb{R}^N)}^2\right)$$
$$\leqslant c\big(\big\|u^\epsilon\big\|_{(H^1(\mathbb{R}^N))^m}\big)\times\begin{cases}\|u^\epsilon\|_{(H^3(\mathbb{R}^N))^m}^{\frac{N}{4}} & N=1,\\ \|u^\epsilon\|_{(H^3(\mathbb{R}^N))^m}^{\frac{N}{4}}+\|u^\epsilon\|_{(H^3(\mathbb{R}^N))^m}^{\frac{q'+N}{4}} & N=2,3.\end{cases}$$

By (3.16) we already have an estimate of u^ϵ in $(H^1(\mathbb{R}^N))^m$. Consequently, when $N=1$ the solution is bounded in $(L^\infty(\mathbb{R}^N))^m$ and since $\Phi\in C^3(\mathbb{R}^m)$ the term $\|D_k D_{j_1} D_{j_2}\Phi(u^\epsilon)\|_{L^\infty(\mathbb{R}^N)}$ above is bounded. When $N=2,3$, we need to use Assumption III, (3.1). The second step of induction ($l=1$) shows that the $(H^2(\mathbb{R}^N))^m$ norm of u^ϵ is bounded uniformly in $\epsilon>0$. Since, in Assumption III, we limit the space dimension to $N\leqslant 3$ such estimate gives also the $(L^\infty(\mathbb{R}^N))^m$ estimate of u^ϵ. Consequently, when $N=2,3$, in further considerations the functional arguments of Φ and g as well as of their derivatives are bounded.

When $l\geqslant 2$ we present next the calculations for a few components; the way of handling another components is very similar. Since the term containing first order derivative of g will always be present in the calculations, we start with that term.

Using Assumption III, the estimate (3.12) and Nirenberg–Gagliardo inequality, we obtain

$$\left\|\frac{\partial g(\cdot,u^\epsilon)}{\partial u_{j_1}^\epsilon}\frac{\partial^l u_{j_1}^\epsilon}{\partial x_i^l}\right\|_{(L^2(\mathbb{R}^N))^m} \leqslant c\big(\big\||u^\epsilon|\big\|_{L^\infty(\mathbb{R}^N)}+\big\||u^\epsilon|\big\|_{L^\infty(\mathbb{R}^N)}^{r'}\big)\big\||u^\epsilon|\big\|_{H^l(\mathbb{R}^N)}\leqslant c''\big(\big\||u^\epsilon|\big\|_{H^l(\mathbb{R}^N)}\big).$$

We estimate another components of (3.22). We have,

$$\left\|\frac{\partial^2 g(\cdot,u^\epsilon)}{\partial u_{j_1}^\epsilon\partial u_{j_2}^\epsilon}\frac{\partial^{l-k} u_{j_1}^\epsilon}{\partial x_i^{l-k}}\frac{\partial^k u_{j_2}^\epsilon}{\partial x_i^k}\right\|_{(L^2(\mathbb{R}^N))^m}\leqslant\left\|\frac{\partial^2 g(\cdot,u^\epsilon)}{\partial u_{j_1}^\epsilon\partial u_{j_2}^\epsilon}\right\|_{(L^\infty(\mathbb{R}^N))^m}\left\|\frac{\partial^{l-k}u_{j_1}^\epsilon}{\partial x_i^{l-k}}\right\|_{L^4(\mathbb{R}^N)}\left\|\frac{\partial^k u_{j_2}^\epsilon}{\partial x_i^k}\right\|_{L^4(\mathbb{R}^N)}\leqslant c\big(\big\||u^\epsilon|\big\|_{H^l(\mathbb{R}^N)}\big),$$

and when $l\geqslant 3$ and $k_1+k_2+k_3=l$,

$$\left\|\frac{\partial^3 g(\cdot,u^\epsilon)}{\partial u_{j_1}^\epsilon\partial u_{j_2}^\epsilon\partial u_{j_3}^\epsilon}\frac{\partial^{k_1}u_{j_1}^\epsilon}{\partial x_i^{k_1}}\frac{\partial^{k_2}u_{j_2}^\epsilon}{\partial x_i^{k_2}}\frac{\partial^{k_3}u_{j_2}^\epsilon}{\partial x_i^{k_3}}\right\|_{(L^2(\mathbb{R}^N))^m}$$
$$\leqslant\left\|\frac{\partial^3 g(\cdot,u^\epsilon)}{\partial u_{j_1}^\epsilon\partial u_{j_2}^\epsilon\partial u_{j_3}^\epsilon}\right\|_{(L^\infty(\mathbb{R}^N))^m}\left\|\frac{\partial^{k_1}u_{j_1}^\epsilon}{\partial x_i^{k_1}}\right\|_{L^4(\mathbb{R}^N)}\left\|\frac{\partial^{k_2}u_{j_2}^\epsilon}{\partial x_i^{k_2}}\right\|_{L^8(\mathbb{R}^N)}\left\|\frac{\partial^{k_3}u_{j_2}^\epsilon}{\partial x_i^{k_3}}\right\|_{L^8(\mathbb{R}^N)}\leqslant c\big(\big\||u^\epsilon|\big\|_{H^l(\mathbb{R}^N)}\big).$$

The term $\|\frac{\partial^{l+1}\nabla\Phi(u^\epsilon)}{\partial x_i^{l+1}}\|_{(L^2(\mathbb{R}^N))^m}$ is estimated similarly to the term $\|\frac{\partial^l g(\cdot,u^\epsilon)}{\partial x_i^l}\|_{(L^2(\mathbb{R}^N))^m}$. We have (compare (3.12))

$$\left\|D_k D_{j_1}\Phi\big(u^\epsilon\big)\frac{\partial^{l+1}u_{j_1}^\epsilon}{\partial x_i^{l+1}}\right\|_{L^2(\mathbb{R}^N)}\leqslant c\big\||u^\epsilon|\big\|_{H^l(\mathbb{R}^N)}^{\frac{1}{2}}\big\||u^\epsilon|\big\|_{H^{l+2}(\mathbb{R}^N)}^{\frac{1}{2}}\big(1+\big\||u^\epsilon|\big\|_{L^\infty(\mathbb{R}^N)}^q\big)$$
$$\leqslant c\big(\big\||u^\epsilon|\big\|_{H^l(\mathbb{R}^N)}\big)\big\||u^\epsilon|\big\|_{H^{l+2}(\mathbb{R}^N)}^{\frac{1}{2}}.$$

Using the Nirenberg–Gagliardo inequality we get, for $1\leqslant\nu\leqslant l-1$,

$$\left\|D_k D_{j_1} D_{j_2}\Phi\big(u^\epsilon\big)\frac{\partial^{l-\nu}u_{j_1}^\epsilon}{\partial x_i^{l-\nu}}\frac{\partial^\nu u_{j_2}^\epsilon}{\partial x_i^\nu}\right\|_{L^2(\mathbb{R}^N)}\leqslant\frac{1}{2}\left\|D_k D_{j_1} D_{j_2}\Phi\big(u^\epsilon\big)\right\|_{L^\infty(\mathbb{R}^N)}\left(\left\|\frac{\partial^{l-\nu}u_{j_1}^\epsilon}{\partial x_i^{l-\nu}}\right\|_{L^4(\mathbb{R}^N)}^2+\left\|\frac{\partial^\nu u_{j_2}^\epsilon}{\partial x_i^\nu}\right\|_{L^4(\mathbb{R}^N)}^2\right)$$
$$\leqslant c\big(\big\||u^\epsilon|\big\|_{H^l(\mathbb{R}^N)}\big)\big\||u^\epsilon|\big\|_{W^{l-1,4}(\mathbb{R}^N)}^2\leqslant c\big(\big\||u^\epsilon|\big\|_{H^l(\mathbb{R}^N)}\big). \tag{3.23}$$

Collecting the above estimates we obtain:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}\sum_{i=1}^N\left|\frac{\partial^{l+1}}{\partial x_i^{l+1}}u^\epsilon\right|^2 dx + c\alpha\|u^\epsilon\|^2_{(H^{l+2}(\mathbb{R}^N))^m} + \epsilon\int_{\mathbb{R}^N}(-\Delta)^{p+1}u^\epsilon\cdot(-1)^{l+1}\sum_{i=1}^N\frac{\partial^{2l+2}}{\partial x_i^{2l+2}}u^\epsilon\,dx$$
$$\leqslant \alpha\|u^\epsilon\|^2_{(L^2(\mathbb{R}^N))^m} + c\big(\|u^\epsilon(t,\cdot)\|_{(H^l(\mathbb{R}^N))^m}\big)\big(1+\|u^\epsilon(t,\cdot)\|^\theta_{(H^{l+2}(\mathbb{R}^N))^m}\big)\|u^\epsilon\|_{(H^{l+2}(\mathbb{R}^N))^m}, \tag{3.24}$$

where $\theta < 1$. Consequently,

$$\frac{d}{dt}\int_{\mathbb{R}^N}\sum_{i=1}^N\left|\frac{\partial^{l+1}}{\partial x_i^{l+1}}u^\epsilon\right|^2 dx + c'(\alpha)\|u^\epsilon\|^2_{(H^{l+2}(\mathbb{R}^N))^m} \leqslant C\big(\|u^\epsilon(t,\cdot)\|_{(H^l(\mathbb{R}^N))^m}\big),$$

with constant independent of ϵ. Together with earlier $(H^l(\mathbb{R}^N))^m$ estimate this gives us the, uniform in ϵ and in $t\in[0,\infty)$, estimate of the $(H^{l+1}(\mathbb{R}^N))^m$ norm of $u^\epsilon(t,\cdot)$ for all $1<l\leqslant p$:

$$\|u^\epsilon(t,\cdot)\|_{(H^{l+1}(\mathbb{R}^N))^m)} \leqslant C_0\big(\|u^\epsilon(t,\cdot)\|_{(H^l(\mathbb{R}^N))^m}\big), \quad t\in[0,\infty). \tag{3.25}$$

In particular we have thus estimated the $(H^{p+1}(\mathbb{R}^N))^m$ norm of u^ϵ, uniformly in $\epsilon>0$ and in $t\geqslant 0$. Since, due to the *Calderon–Zygmund estimate*

$$-\epsilon\int_{\mathbb{R}^N}(-\Delta)^{p+1}u^\epsilon\cdot(-1)^{p+1}\sum_{i=1}^N\frac{\partial^{2p+2}}{\partial x_i^{2p+2}}u^\epsilon\,dx \leqslant -c\epsilon\|u^\epsilon\|^2_{(H^{2p+2}(\mathbb{R}^N))^m} + \epsilon\|u^\epsilon\|^2_{(L^2(\mathbb{R}^N))^m},$$

returning to (3.24), integrating it over $(0,T)$ and using (3.25), we obtain also the estimates

$$\alpha\int_0^T\|u^\epsilon(t,\cdot)\|^2_{(H^{p+2}(\mathbb{R}^N))^m}\,dt \leqslant C_1\big(\|u^\epsilon(t,\cdot)\|_{(H^1(\mathbb{R}^N))^m},T\big), \tag{3.26}$$

$$\epsilon\int_0^T\|u^\epsilon(t,\cdot)\|^2_{(H^{2p+2}(\mathbb{R}^N))^m}\,dt \leqslant C_1\big(\|u^\epsilon(t,\cdot)\|_{(H^1(\mathbb{R}^N))^m},T\big). \tag{3.27}$$

3.2. Higher order, uniform in ϵ, estimates of u^ϵ

We will finally describe the, uniform in $\epsilon>0$, estimate of the $(H^{p+2}(\mathbb{R}^N))^m$ norms of u^ϵ. Such estimates are possible thanks to the presence of the viscosity term $\alpha\Delta$ in Eq. (1.3). Since, for the global solutions constructed in Theorem 2.5, the initial data $u_0\in(H^{p+1}(\mathbb{R}^N))^m$ only, we need first to justify the uniform in $\epsilon>0$ smoothing action of Eq. (1.3). More precisely, we need to show that $u^\epsilon(t,\cdot)$ enters $(H^{p+2}(\mathbb{R}^N))^m$ for $t>0$, uniformly with respect to ϵ. Let $\Phi\in C^{p+4}(\mathbb{R}^N)$ and $g\in C^{p+2}(\mathbb{R}^N\times\mathbb{R}^m)$.

The way of getting such estimate is known. We need to multiply Eq. (1.3) by $t(-1)^{p+2}\sum_{i=1}^N\frac{\partial^{2p+4}u^\epsilon}{\partial x_i^{2p+4}}$ and proceed as in the third a priori estimate. The only difference is connected with the term containing time derivative, which is transformed as follows:

$$\int_{\mathbb{R}^N}u_t^\epsilon\cdot t(-1)^{p+2}\sum_{i=1}^N\frac{\partial^{2p+4}u^\epsilon}{\partial x_i^{2p+4}}\,dx = \frac{1}{2}\int_{\mathbb{R}^N}t\frac{\partial}{\partial t}\sum_{i=1}^N\left|\frac{\partial^{p+2}u^\epsilon}{\partial x_i^{p+2}}\right|^2 dx$$
$$= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}t\sum_{i=1}^N\left|\frac{\partial^{p+2}u^\epsilon}{\partial x_i^{p+2}}\right|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N}\sum_{i=1}^N\left|\frac{\partial^{p+2}u^\epsilon}{\partial x_i^{p+2}}\right|^2 dx. \tag{3.28}$$

Calculations very similar to the third a priori estimate lead us to the conclusion valid, however, on bounded time intervals:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}t\sum_{i=1}^N\left|\frac{\partial^{p+2}u^\epsilon}{\partial x_i^{p+2}}\right|^2 dx \leqslant -c(\alpha)t\||u^\epsilon|\|^2_{H^{p+3}(\mathbb{R}^N))} + \alpha t\||u^\epsilon|\|^2_{L^2(\mathbb{R}^N)} + c\big(\sqrt{t}\||u^\epsilon|\|^\theta_{H^{p+3}(\mathbb{R}^N)}\big)\sqrt{t}\||u^\epsilon|\|_{H^{p+3}(\mathbb{R}^N)}$$
$$+\frac{1}{2}\int_{\mathbb{R}^N}\sum_{i=1}^N\left|\frac{\partial^{p+2}u^\epsilon}{\partial x_i^{p+2}}\right|^2 dx, \quad t\in[0,1], \tag{3.29}$$

where we take $t \in [0,1]$ for example and the last term was estimated already in (3.26), uniformly in ϵ. This gives us the, uniform in $\epsilon > 0$ and $t \in [0,1]$, estimate

$$\sqrt{t}\big\| |u^\epsilon| \big\|_{H^{p+2}(\mathbb{R}^N)} \leqslant \text{const}, \tag{3.30}$$

expressing the regularization effect of Eq. (1.3).

Thanks to (3.30) we can assume that the values $\| |u^\epsilon(1,\cdot)| \|_{H^{p+2}(\mathbb{R}^N)}$ are bounded uniformly in $\epsilon > 0$, and consider our global solutions of (1.3) for $t \geqslant 1$ with 'initial data' $u^\epsilon(1,\cdot) \in (H^{p+2}(\mathbb{R}^N))^m$. An estimate, very similar to the third a priori estimate, valid for $t \geqslant 1$ will give us next the uniform in time $t \geqslant 1$ bound

$$\big\|u^\epsilon\big\|_{(H^{p+2}(\mathbb{R}^N))^m} \leqslant \text{const}, \quad t \geqslant 1, \tag{3.31}$$

with the constant uniform in $\epsilon > 0$ and for $t \geqslant 1$.

4. Viscous solutions

In this section we get the solution to (1.2) as a limit, when $\epsilon \to 0^+$, of the global smooth solutions to parabolic regularization (1.3). In the process of passing to the limit, $\epsilon \to 0^+$, an essential role is played by the estimate of the difference $(u^{\epsilon_1} - u^{\epsilon_2})$ in $(L^2(\mathbb{R}^N))^m$. Such estimate will prove uniqueness of such limit, and also uniqueness of the viscous solution obtained in such a way.

Lemma 4.1. *The following estimate of the difference $(u^{\epsilon_1} - u^{\epsilon_2})$ of the two solutions to (1.3) corresponding to different ϵ's holds*:

$$\big\|u^{\epsilon_1} - u^{\epsilon_2}\big\|_{(L^2(\mathbb{R}^N))^m} \leqslant c|\epsilon_1 - \epsilon_2|^{\frac{1}{2}}, \quad t \in [0,T], \tag{4.1}$$

where the constant $c = c(T)$ is independent of ϵ.

Proof. Subtracting (1.3) written for u^{ϵ_2} from (1.3) written for u^{ϵ_1}, defining $U := u^{\epsilon_1} - u^{\epsilon_2}$, and choosing $\epsilon_1 \geqslant \epsilon_2 > 0$, we obtain:

$$\begin{aligned}
&U_t + \sum_{i=1}^N \frac{\partial}{\partial x_i}\big(\nabla\Phi\big(u^{\epsilon_1}\big) - \nabla\Phi\big(u^{\epsilon_2}\big)\big) + \sum_{j=1}^N \frac{\partial}{\partial x_j} \sum_{i=1}^N \frac{\partial^{2p}}{\partial x_i^{2p}} U \\
&\quad = \alpha\Delta U + \epsilon_1(-1)^p(\Delta)^{p+1}U + (\epsilon_1 - \epsilon_2)(-1)^p(\Delta)^{p+1}u^{\epsilon_2} + \big(g\big(x,u^{\epsilon_1}\big) - g\big(x,u^{\epsilon_2}\big)\big), \quad t > 0,\ x \in \mathbb{R}^N, \\
&U(0,x) = 0, \quad x \in \mathbb{R}^N.
\end{aligned} \tag{4.2}$$

Next we multiply (4.2) by U in $(L^2(\mathbb{R}^N))^m$, to get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} |U|^2\,dx + \int_{\mathbb{R}^N} \sum_{i=1}^N \frac{\partial}{\partial x_i}\big(\nabla\Phi\big(u^{\epsilon_1}\big) - \nabla\Phi\big(u^{\epsilon_2}\big)\big)\cdot U\,dx + \alpha\int_{\mathbb{R}^N} |\nabla U|^2\,dx + \epsilon_1 \int_{\mathbb{R}^N} \big|(-\Delta)^{\frac{p+1}{2}}U\big|^2\,dx \\
&\quad \leqslant (\epsilon_2 - \epsilon_1)\int_{\mathbb{R}^N} (-\Delta)^{\frac{p+1}{2}}u^{\epsilon_2}\cdot(-\Delta)^{\frac{p+1}{2}}U\,dx + \int_{\mathbb{R}^N} \big(g\big(x,u^{\epsilon_1}\big) - g\big(x,u^{\epsilon_2}\big)\big)\cdot U\,dx,
\end{aligned} \tag{4.3}$$

where we neglect the vanishing components, and use the equality (valid for smooth solutions)

$$(-1)^p\int_{\mathbb{R}^N} (\Delta)^{p+1}u^{\epsilon_2}\cdot U\,dx = (-1)^{2p+1}\int_{\mathbb{R}^N} (-\Delta)^{p+1}u^{\epsilon_2}\cdot U\,dx = -\int_{\mathbb{R}^N} (-\Delta)^{\frac{p+1}{2}}u^{\epsilon_2}\cdot(-\Delta)^{\frac{p+1}{2}}U\,dx. \tag{4.4}$$

Now, thanks to the uniform in time and in ϵ estimates of u^ϵ in $(C_b^1(\mathbb{R}^N))^m$ and in $(H^{p+1}(\mathbb{R}^N))^m$, and the assumed regularity of Φ and g, using Cauchy's inequality, we get:

$$\begin{aligned}
-\int_{\mathbb{R}^N} \sum_{i=1}^N \frac{\partial}{\partial x_i}\big(\nabla\Phi\big(u^{\epsilon_1}\big) - \nabla\Phi\big(u^{\epsilon_2}\big)\big)\cdot U\,dx &= \int_{\mathbb{R}^N} \sum_{i=1}^N \big(\nabla\Phi\big(u^{\epsilon_1}\big) - \nabla\Phi\big(u^{\epsilon_2}\big)\big)\cdot\frac{\partial}{\partial x_i}U\,dx \\
&\leqslant L\int_{\mathbb{R}^N} |U||\nabla U|\,dx \leqslant \alpha\int_{\mathbb{R}^N} |\nabla U|^2\,dx + c_\alpha\int_{\mathbb{R}^N} |U|^2\,dx,
\end{aligned} \tag{4.5}$$

with similar estimate for the term containing g. Note that $\epsilon_1 - \epsilon_2 \leqslant \epsilon_1$, so that even when $\epsilon_1 \to 0$, we have a subordination:

$$(\epsilon_1-\epsilon_2)\left|\int_{\mathbb{R}^N}(-\Delta)^{\frac{p+1}{2}}u^{\epsilon_2}\cdot(-\Delta)^{\frac{p+1}{2}}U\,dx\right|\leqslant\epsilon_1\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{p+1}{2}}U\right|^2dx+\frac{\epsilon_1-\epsilon_2}{4}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{p+1}{2}}u^{\epsilon_2}\right|^2dx. \tag{4.6}$$

Consequently, we obtain an estimate

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}|U|^2\,dx\leqslant c\int_{\mathbb{R}^N}|U|^2\,dx+\frac{\epsilon_1-\epsilon_2}{4}\text{const}, \tag{4.7}$$

which, through explicit integration, using (3.27), gives the required bound (4.1), since $U(0)=0$. Note that the const depends on the bound for $\|(-\Delta)^{\frac{p+1}{2}}u^{\epsilon_2}\|_{L^2(\mathbb{R}^N)}$ and the Lipschitz constants for $\nabla\Phi$ and g. $\square$

Remark 4.2. There are several ways of passing with ϵ to 0^+ to get the *viscous solution* of (1.2). Further we will follow the classical approach of J.L. Lions [19], which relates such a limit to a solution of the limiting equation. But, if we wish to stay inside the frame of the dynamical systems only, we will consider a *convergence of the trajectories*. More precisely, consider the limits of the global solutions of (1.3) in the space $C([0,T];(H^p(\mathbb{R}^N))^m)$. Once we stated the estimates (4.1) and (3.25) saying that the solutions to (1.3) are bounded in $(H^{p+1}(\mathbb{R}^N))^m$ uniformly in $\epsilon>0$, by interpolation argument we find that:

$$\begin{aligned}&\sup_{t\in[0,T]}\left\|u^{\epsilon_1}(t,\cdot)-u^{\epsilon_2}(t,\cdot)\right\|_{(H^p(\mathbb{R}^N))^m}\\&\quad\leqslant\sup_{t\in[0,T]}\left\|u^{\epsilon_1}(t,\cdot)-u^{\epsilon_2}(t,\cdot)\right\|^{\frac{p}{p+1}}_{(H^{p+1}(\mathbb{R}^N))^m}\left\|u^{\epsilon_1}(t,\cdot)-u^{\epsilon_2}(t,\cdot)\right\|^{\frac{1}{p+1}}_{(L^2(\mathbb{R}^N))^m}\\&\quad\leqslant c|\epsilon_1-\epsilon_2|^{\frac{1}{p+1}}\sup_{t\in[0,T]}\left(\left\|u^{\epsilon_1}(t,\cdot)\right\|^{\frac{p}{p+1}}_{(H^{p+1}(\mathbb{R}^N))^m}+\left\|u^{\epsilon_2}(t,\cdot)\right\|^{\frac{p}{p+1}}_{(H^{p+1}(\mathbb{R}^N))^m}\right)\leqslant\text{const}\,|\epsilon_1-\epsilon_2|^{\frac{1}{p+1}},\end{aligned} \tag{4.8}$$

with the const uniform in ϵ's. The last estimate shows that $\{u^{\epsilon_i}\}$ is a *Cauchy sequence* in the space $C([0,T];(H^p(\mathbb{R}^N))^m)$. Letting $\epsilon_2\to 0$, we can get the limiting trajectory $u(t,\cdot)$ and some of its properties.

Similar estimate, based on (3.31), shows the convergence in $(H^{(p+2)^-}(\mathbb{R}^N))^m$:

$$\sup_{t\in[1,T]}\left\|u^{\epsilon_1}(t,\cdot)-u^{\epsilon_2}(t,\cdot)\right\|_{(H^{(p+2)^-}(\mathbb{R}^N))^m}\leqslant\text{const}\,|\epsilon_1-\epsilon_2|^{1-\frac{(p+2)^-}{p+2}}, \tag{4.9}$$

valid, however, on compact time intervals $t\in[\tau,T]$, separated from $t=0$ (we set $t\geqslant 1$ above, for simplicity, recall also that $T>1$ is an arbitrary fixed number).

4.1. Viscous solutions of (1.2)

We will describe now the process of passing with ϵ to 0^+ to get the *viscous solutions of* (1.2). That notion is similar to the *viscosity solutions* introduced by P.L. Lions and G. Barles for the Hamilton–Jacobi type equations. The, classical nowadays, procedure of passing to the limit was used first in the studies of the Burgers equation (see e.g. [22]) and extended later significantly in [19]. Thus, we will sketch it here only without discussing the details.

We start with formulating the existence result.

Theorem 4.3. *There exists a unique* viscous solution u *of* (1.2) *having the following regularity properties*:

$$\begin{aligned}&\|u\|_{L^\infty(0,\infty;(H^p(\mathbb{R}^N))^m)}\leqslant const,\quad t\geqslant 0,\qquad \|u_t\|_{L^\infty(0,T;(H^{-p}(\mathbb{R}^N))^m)}\leqslant const,\quad T>0,\\&\|u\|_{L^\infty(0,\infty;(H^{(p+2)^-}(\mathbb{R}^N))^m)}\leqslant const\quad for\ t\geqslant 1,\end{aligned}$$

where const are the same as for u^ϵ's and T is arbitrary.

Proof. In (2.17), (3.25), (4.9), we have shown the ϵ independent estimates of the approximating solutions u^ϵ:

$$\left\|u^\epsilon(t,\cdot)\right\|_{(L^2(\mathbb{R}^N))^m}\leqslant\text{const},\qquad\left\|u^\epsilon(t,\cdot)\right\|_{(H^{p+1}(\mathbb{R}^N))^m}\leqslant\text{const}, \tag{4.10}$$

and

$$\left\|u^\epsilon(t,\cdot)\right\|_{(H^{p+2}(\mathbb{R}^N))^m}\leqslant\text{const},\quad t\geqslant 1, \tag{4.11}$$

with the constants independent of ϵ. It follows further from (3.27) that

$$\sqrt{\epsilon}\left\|u^\epsilon\right\|_{L^2(0,T;(H^{2p+2}(\mathbb{R}^N))^m)}\leqslant\text{const}(T), \tag{4.12}$$

again with the constant independent of ϵ. The above bounds, uniform in $\epsilon > 0$, allow us to pass to the limit over a subsequence, in the following spaces (compare [19]):

$$\begin{aligned} &u^\epsilon \to u \quad \text{weak star in } L^\infty\big(0,T;\big(H^p\big(\mathbb{R}^N\big)\big)^m\big),\\ &u^\epsilon \to u \quad \text{weak star in } L^\infty\big(1,T;\big(H^{(p+2)^-}\big(\mathbb{R}^N\big)\big)^m\big),\\ &\epsilon u^\epsilon \to 0 \quad \text{in } L^2\big(0,T;\big(H^{2p+2}\big(\mathbb{R}^N\big)\big)^m\big). \end{aligned} \tag{4.13}$$

It follows next from the estimates (4.8), (3.25) and the embedding $C([0,T];(H^{(p+1)^-}(\mathbb{R}^N))^m) \subset C([0,T];(C_b^1(\mathbb{R}^N))^m)$ (where $2p > N$ and $(p+1)^-$ is close to $(p+1)$), that

$$u^\epsilon \to u \quad \text{uniformly in } \big([0,T]\times\mathbb{R}^N\big)^m. \tag{4.14}$$

This information allows us to pass to the limit in nonlinear terms.

Now, we can look at (1.3) as an equation in $(H^{-p}(\mathbb{R}^N))^m$. In Section 2.1 formula (2.9) the Cauchy problem (1.3) was written in a form

$$u_t^\epsilon + \mathbf{A}^\epsilon u^\epsilon = \mathbf{F}\big(u^\epsilon\big) + (\gamma+\omega)u^\epsilon, \tag{4.15}$$

as an equation in $(H^{-p}(\mathbb{R}^N))^m$. According to Lemma 2.4, the nonlinear term $\mathbf{F}: (H^{p+1}(\mathbb{R}^N))^m \to (H^{-p}(\mathbb{R}^N))^m$. Further, by Lemma 2.2, the operator $\mathbf{A}_\epsilon$ defines a linear isomorphism from $(H^{p+2}(\mathbb{R}^N))^m$ onto $(H^{-p}(\mathbb{R}^N))^m$, and $\mathbf{B}$ is a perturbation of $\mathbf{A}_\epsilon$ in $(H^{-p}(\mathbb{R}^N))^m$, consequently $\mathbf{A}^\epsilon$ acts from $(H^{p+2}(\mathbb{R}^N))^m$ onto $(H^{-p}(\mathbb{R}^N))^m$ as well. Therefore (4.15) can be seen as an equality of functionals on an arbitrary '*test function*' $\phi \in (H^p(\mathbb{R}^N))^m$. In particular u_t^ϵ calculated from (4.15):

$$u_t^\epsilon = -\mathbf{A}^\epsilon u^\epsilon + \mathbf{F}\big(u^\epsilon\big) + (\gamma+\omega)u^\epsilon$$

will have the norm in $L^\infty(0,T;(H^{-p}(\mathbb{R}^N))^m)$ bounded uniformly in ϵ. Consequently, since $(H^{-p}(\mathbb{R}^N))^m$ is Hilbert,

$$u_t^\epsilon \to u_t \quad \text{weak star in } L^\infty\big(0,T;\big(H^{-p}\big(\mathbb{R}^N\big)\big)^m\big). \tag{4.16}$$

We thus claim that (which is, however, weaker than (4.14))

$$u \in C\big([0,T];\big(L^2\big(\mathbb{R}^N\big)\big)^m\big), \tag{4.17}$$

by the *Lions lemma* (e.g. [25, p. 71]).

Next, using the uniform in ϵ estimates (2.17), (3.25), (3.27), we can let $\epsilon \to 0^+$ in (4.15) written as an equation in $(H^{-p}(\mathbb{R}^N))^m$. Moreover, due to (4.12), the term

$$\big|\big(\epsilon(-\Delta)^{p+1}u^\epsilon,\phi\big)_{-p}\big| \leqslant \epsilon\big\|(-\Delta)^{p+1}u^\epsilon\big\|_{(H^{-p}(\mathbb{R}^N))^m}\|\phi\|_{(H^p(\mathbb{R}^N))^m} \leqslant \epsilon c\big\|u^\epsilon\big\|_{(H^{p+2}(\mathbb{R}^N))^m}\|\phi\|_{(H^p(\mathbb{R}^N))^m} \tag{4.18}$$

(where $(\cdot,\cdot)_{-p}$ denotes the duality between $(H^{-p}(\mathbb{R}^N))^m$ and $(H^p(\mathbb{R}^N))^m$) will vanish when $\epsilon \to 0^+$; see (4.13). Consequently, the limit function $u = \lim_{\epsilon\to 0^+} u^\epsilon$, announced in Remark 4.2, will be an $(H^{-p}(\mathbb{R}^N))^m$ solution of the limit problem (1.2) having additional properties inherited from the corresponding properties of u^ϵ:

$$\begin{aligned} &\big\|u(t,\cdot)\big\|_{(H^{p+1}(\mathbb{R}^N))^m} \leqslant \text{const}, \qquad \big\|u_t(t,\cdot)\big\|_{(H^{-p}(\mathbb{R}^N))^m} \leqslant \text{const}, \quad t \geqslant 0,\\ &\big\|u(t,\cdot)\big\|_{(H^{(p+2)^-}(\mathbb{R}^N))^m} \leqslant \text{const} \quad \text{for } t \geqslant 1, \end{aligned} \tag{4.19}$$

with the same constants as for the u^ϵ. Our task now is to study its properties in more details. □

4.2. Further properties of the viscous solutions of (1.2)

Recall first that, due to Lemma 4.1, viscosity limit in (1.3) is independent of the choice of a sequence $\epsilon_n \to 0$. We discuss next continuity of the viscous solution with respect to initial data u_0.

Remark 4.4. Viscous solution of (1.2) is continuous in $(H^p(\mathbb{R}^N))^m$ with respect to initial data. Let u_1^ϵ, u_2^ϵ be two solutions of (1.3), with the same $\epsilon > 0$ but different initial data $u_1^\epsilon(0,x)$, $u_2^\epsilon(0,x)$. By a reasoning very similar to the proof of Lemma 4.1 (but with $\epsilon_1 = \epsilon_2 = \epsilon$ and $V(0,x) = u_1^\epsilon(0,x) - u_2^\epsilon(0,x) \neq 0$) one can get the estimate of the difference $V = u_1^\epsilon - u_2^\epsilon$ having the form:

$$\big\|V(t,\cdot)\big\|_{(L^2(\mathbb{R}^N))^m} \leqslant \big\|V(0,\cdot)\big\|_{(L^2(\mathbb{R}^N))^m} e^{ct}, \quad t \geqslant 0. \tag{4.20}$$

Using (3.25), the last estimate will be strengthened to an estimate in $(H^p(\mathbb{R}^N))^m$:

$$\big\|V(t,\cdot)\big\|_{(H^p(\mathbb{R}^N))^m} \leqslant \big\|V(t,\cdot)\big\|_{(L^2(\mathbb{R}^N))^m}^{\frac{1}{p+1}} \big\|V(t,\cdot)\big\|_{(H^{p+1}(\mathbb{R}^N))^m}^{\frac{p}{p+1}} \leqslant \text{const}\big(\big\|V(0,\cdot)\big\|_{(L^2(\mathbb{R}^N))^m} e^{ct}\big)^{\frac{1}{p+1}}, \tag{4.21}$$

justifying continuity in $(H^p(\mathbb{R}^N))^m$ of the viscous solutions with respect to initial data.

Following [10] we introduce now the notion of the *weak solution* of (1.2).

Definition 4.5. If $u_0 \in (H^p(\mathbb{R}^N))^m$, then u is called a *weak solution* of (1.2) corresponding to u_0, if:

- $u \in C([0,T]; (H^p(\mathbb{R}^N))^m) \cap L^2(0,T; (H^{p+1}(\mathbb{R}^N))^m)$ for any $T > 0$, and $u(0) = u_0$,
- there is a sequence of initial data $\{u_{0n}\} \subset (H^{p+1}(\mathbb{R}^N))^m$ convergent to u_0 in $(H^p(\mathbb{R}^N))^m$, such that the corresponding viscous solutions u_n of (1.2) fulfill

$$u_n \to u \quad \text{in } C\big([0,T]; \big(H^p\big(\mathbb{R}^N\big)\big)^m\big).$$

- Eq. (1.2) is fulfilled by u in $(H^{-p}(\mathbb{R}^N))^m$.

Note, that weak solutions are global in time and define a semigroup $S(t)u_0 = u(t,u_0)$, $t \geqslant 0$, on the phase space $(H^p(\mathbb{R}^N))^m$. Existence and uniqueness of such weak solutions is evident thanks to the properties of viscous solutions constructed in previous section. We will use weak solutions constructing the *global attractor* for (1.2).

5. Asymptotic behavior of solutions to (1.2)

We will study existence of the global attractor for the semigroups generated by (1.2) and the approximating problems (1.3) on $(H^p(\mathbb{R}^N))^m$.

5.1. Tail estimates in $L^2(\mathbb{R})$

Thanks to the regularity estimates given in Section 3.2, the necessary asymptotic compactness will be deduced by the *tail estimates* (as in [21,27,10]), and interpolation inequalities.

In the following we will obtain for solutions u^ϵ of (1.3) the, so called, *tail estimates* in $(L^2(\mathbb{R}^N))^m$ as introduced in [27]:

Lemma 5.1. *Let Assumption III be satisfied and let $\epsilon \in (0,1]$. Then, for each $\eta > 0$ and arbitrary $u_0 \in (H^{p+1}(\mathbb{R}^N))^m$, there exist $k = k(\eta, \|u_0\|_{(L^2(\mathbb{R}^N))^m})$ and $T = T(\eta, \|u_0\|_{(L^2(\mathbb{R}^N))^m})$ such that the corresponding to u_0 solution $u^\epsilon(t)$ of (1.3) satisfies*

$$\int_{\mathcal{O}_k} \big|u^\epsilon(t)\big|^2\,dx \leqslant \eta \quad \textit{for all } t \geqslant T, \tag{5.1}$$

where $\mathcal{O}_k = \{x \in \mathbb{R}\colon |x| \geqslant k\}$.

Proof. Choose a smooth function $\theta(\cdot)$ such that $0 \leqslant \theta(s) \leqslant 1$ for any $s \in \mathbb{R}^+$, and

$$\theta(s) = 0 \quad \text{for } 0 \leqslant s \leqslant 1, \quad \text{and} \quad \theta(s) = 1 \quad \text{for } s \geqslant 2.$$

Then there exists a constant C_0 such that $|\theta'(s)| + |\theta''(s)| + \cdots + |\theta^{(p)}(s)| \leqslant C_0$ for any $s \in \mathbb{R}^+$. We recall here also the, uniform in $\epsilon > 0$ and for $t \geqslant 1$, estimates (3.31) of u^ϵ in $(H^{p+2}(\mathbb{R}^N))^m$.

Taking the scalar product of (1.3) with $\theta(\frac{|x|^2}{k^2})u^\epsilon$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} \theta\left(\frac{|x|^2}{k^2}\right)\big|u^\epsilon\big|^2\,dx + \int_{\mathbb{R}^N} \theta\left(\frac{|x|^2}{k^2}\right)u^\epsilon \cdot \sum_{i=1}^N \frac{\partial}{\partial x_i}\nabla\Phi\big(u^\epsilon\big)\,dx + \int_{\mathbb{R}^N} \theta\left(\frac{|x|^2}{k^2}\right)u^\epsilon \cdot \sum_{j=1}^N \frac{\partial}{\partial x_j}\sum_{i=1}^N \frac{\partial^{2p}u^\epsilon}{\partial x_i^{2p}}\,dx \\
&\quad = \alpha\int_{\mathbb{R}^N} \theta\left(\frac{|x|^2}{k^2}\right)u^\epsilon \cdot \Delta u^\epsilon\,dx + \int_{\mathbb{R}^N} \epsilon(-1)^p(\Delta)^{p+1}u^\epsilon \cdot \theta\left(\frac{|x|^2}{k^2}\right)u^\epsilon\,dx + \int_{\mathbb{R}^N} g\big(x,u^\epsilon\big)\cdot\theta\left(\frac{|x|^2}{k^2}\right)u^\epsilon\,dx.
\end{aligned} \tag{5.2}$$

We will transform the components one by one. At first, integrating by parts, we have

$$\int_{\mathbb{R}^N} \theta\left(\frac{|x|^2}{k^2}\right)u^\epsilon \cdot \sum_{i=1}^N \frac{\partial}{\partial x_i}\nabla\Phi\big(u^\epsilon\big)\,dx = -\int_{\mathbb{R}^N}\sum_{i=1}^N \frac{2x_i}{k^2}\theta'\left(\frac{|x|^2}{k^2}\right)\big(u^\epsilon \cdot \nabla\Phi\big(u^\epsilon\big) - \Phi\big(u^\epsilon\big)\big)\,dx. \tag{5.3}$$

Then, combining with (3.3) and (3.4), we deduce that

$$\left|\int_{\mathbb{R}^N} \theta\left(\frac{|x|^2}{k^2}\right) u^\epsilon \cdot \sum_{i=1}^N \frac{\partial}{\partial x_i} \nabla \Phi\left(u^\epsilon\right) dx\right| \leqslant \frac{2\sqrt{2}C_0}{k}\left(\int_{\mathbb{R}^N} \left|u^\epsilon \cdot \nabla \Phi\left(u^\epsilon\right)\right| dx + \int_{\mathbb{R}^N} \left|\Phi\left(u^\epsilon\right)\right| dx\right)$$
$$\leqslant \frac{c}{k}\left(\left\||u^\epsilon|\right\|_{L^2(\mathbb{R}^N)} \left\||\nabla \Phi\left(u^\epsilon\right)|\right\|_{L^2(\mathbb{R}^N)} + \left\||u^\epsilon|\right\|^2_{L^2(\mathbb{R}^N)} + \left\||u^\epsilon|\right\|^{q+2}_{L^{q+2}(\mathbb{R}^N)}\right)$$
$$\leqslant \frac{c(\||u^\epsilon|\|_{H^1(\mathbb{R}^N)})}{k}. \tag{5.4}$$

Secondly, note that

$$\int_{\mathbb{R}^N} \theta\left(\frac{|x|^2}{k^2}\right) u^\epsilon \cdot \sum_{j=1}^N \frac{\partial}{\partial x_j} \sum_{i=1}^N \frac{\partial^{2p} u^\epsilon}{\partial x_i^{2p}}\, dx = -\sum_{j=1}^N \sum_{i=1}^N \left(\theta'\left(\frac{|x|^2}{k^2}\right) \frac{2x_j}{k^2} u^\epsilon + \theta\left(\frac{|x|^2}{k^2}\right) \frac{\partial u^\epsilon}{\partial x_j}\right) \cdot \frac{\partial^{2p} u^\epsilon}{\partial x_i^{2p}}\, dx. \tag{5.5}$$

By $(p-1)$ integrations by parts the last component is transformed below. The next to last component will be estimated in a similar way.

$$-\int_{\mathbb{R}^N} \theta\left(\frac{|x|^2}{k^2}\right) \sum_{j=1}^N \frac{\partial u^\epsilon}{\partial x_j} \cdot \sum_{i=1}^N \frac{\partial^{2p} u^\epsilon}{\partial x_i^{2p}}\, dx$$
$$= \int_{\mathbb{R}^N} \sum_{j=1}^N \sum_{i=1}^N \theta'\left(\frac{|x|^2}{k^2}\right) \frac{2x_i}{k^2} \frac{\partial u^\epsilon}{\partial x_j} \cdot \frac{\partial^{2p-1} u^\epsilon}{\partial x_i^{2p-1}}\, dx + \int_{\mathbb{R}^N} \sum_{j=1}^N \sum_{i=1}^N \theta\left(\frac{|x|^2}{k^2}\right) \frac{\partial^2 u^\epsilon}{\partial x_j \partial x_i} \cdot \frac{\partial^{2p-1} u^\epsilon}{\partial x_i^{2p-1}}\, dx$$
$$= -\int_{\mathbb{R}^N} \sum_{j=1}^N \sum_{i=1}^N \theta''\left(\frac{|x|^2}{k^2}\right) \frac{4x_i^2}{k^4} \frac{\partial u^\epsilon}{\partial x_j} \cdot \frac{\partial^{2p-2} u^\epsilon}{\partial x_i^{2p-2}}\, dx - \int_{\mathbb{R}^N} \sum_{j=1}^N \sum_{i=1}^N \theta'\left(\frac{|x|^2}{k^2}\right) \frac{2}{k^2} \frac{\partial u^\epsilon}{\partial x_j} \cdot \frac{\partial^{2p-2} u^\epsilon}{\partial x_i^{2p-2}}\, dx$$
$$- 2\int_{\mathbb{R}^N} \sum_{j=1}^N \sum_{i=1}^N \theta'\left(\frac{|x|^2}{k^2}\right) \frac{2x_i}{k^2} \frac{\partial^2 u^\epsilon}{\partial x_i \partial x_j} \cdot \frac{\partial^{2p-2} u^\epsilon}{\partial x_i^{2p-2}}\, dx - \int_{\mathbb{R}^N} \sum_{j=1}^N \sum_{i=1}^N \theta\left(\frac{|x|^2}{k^2}\right) \frac{\partial^3 u^\epsilon}{\partial x_j \partial x_i^2} \cdot \frac{\partial^{2p-2} u^\epsilon}{\partial x_i^{2p-2}}\, dx = \cdots$$
$$= (-1)^p \int_{\mathbb{R}^N} \sum_{j=1}^N \sum_{i=1}^N \left(\theta^{(p-1)}\left(\frac{|x|^2}{k^2}\right) \frac{(2x_i)^{p-1}}{k^{2(p-1)}} \frac{\partial u^\epsilon}{\partial x_j} \cdot \frac{\partial^{p+1} u^\epsilon}{\partial x_i^{p+1}} + \cdots + \theta\left(\frac{|x|^2}{k^2}\right) \frac{\partial^p u^\epsilon}{\partial x_j \partial x_i^{p-1}} \cdot \frac{\partial^{p+1} u^\epsilon}{\partial x_i^{p+1}}\right) dx.$$

The last component above is transformed next:

$$\int_{\mathbb{R}^N} \theta\left(\frac{|x|^2}{k^2}\right) \sum_{j=1}^N \sum_{i=1}^N \frac{\partial^p u^\epsilon}{\partial x_j \partial x_i^{p-1}} \cdot \frac{\partial^{p+1} u^\epsilon}{\partial x_i^{p+1}}\, dx$$
$$= -\int_{\mathbb{R}^N} \theta'\left(\frac{|x|^2}{k^2}\right) \sum_{j=1}^N \sum_{i=1}^N \frac{2x_i}{k^2} \frac{\partial^p u^\epsilon}{\partial x_j \partial x_i^{p-1}} \cdot \frac{\partial^p u^\epsilon}{\partial x_i^p}\, dx - \int_{\mathbb{R}^N} \theta\left(\frac{|x|^2}{k^2}\right) \sum_{j=1}^N \sum_{i=1}^N \frac{\partial^{p+1} u^\epsilon}{\partial x_j \partial x_i^p} \cdot \frac{\partial^p u^\epsilon}{\partial x_i^p}\, dx$$
$$= -\int_{\mathbb{R}^N} \theta'\left(\frac{|x|^2}{k^2}\right) \sum_{j=1}^N \sum_{i=1}^N \frac{2x_i}{k^2} \frac{\partial^p u^\epsilon}{\partial x_j \partial x_i^{p-1}} \cdot \frac{\partial^p u^\epsilon}{\partial x_i^p}\, dx + \frac{1}{2}\int_{\mathbb{R}^N} \theta'\left(\frac{|x|^2}{k^2}\right) \sum_{j=1}^N \sum_{i=1}^N \frac{2x_j}{k^2} \left|\frac{\partial^p u^\epsilon}{\partial x_i^p}\right|^2 dx.$$

Consequently, we obtain an estimate extending (5.5)

$$\left|\int_{\mathbb{R}^N} \theta\left(\frac{|x|^2}{k^2}\right) u^\epsilon \cdot \sum_{j=1}^N \frac{\partial}{\partial x_j} \sum_{i=1}^N \frac{\partial^{2p} u^\epsilon}{\partial x_i^{2p}}\, dx\right|$$
$$\leqslant \frac{C(p)C_0}{k} \int_{\mathbb{R}^N} \sum_{i=1}^N \left(\sum_{j=1}^N \left|\frac{\partial^{p+1} u^\epsilon}{\partial x_j \partial x_i^p}\right| \left|\frac{\partial^p u^\epsilon}{\partial x_i^p}\right| + \cdots + \left|u^\epsilon\right| \left|\frac{\partial^p u^\epsilon}{\partial x_i^p}\right|\right) dx \leqslant \frac{C(p)C_0}{k} \left\||u^\epsilon|\right\|^2_{H^{p+1}(\mathbb{R}^N)}. \tag{5.6}$$

Further components in (5.2) are transformed as follows:

$$\alpha \int_{\mathbb{R}^N} \theta\left(\frac{|x|^2}{k^2}\right) u^\epsilon \cdot \Delta u^\epsilon\, dx = -\alpha \int_{\mathbb{R}^N} \sum_{i=1}^N \theta'\left(\frac{|x|^2}{k^2}\right) \frac{2x_i}{k^2} u^\epsilon \cdot \nabla u^\epsilon\, dx - \alpha \int_{\mathbb{R}^N} \theta\left(\frac{|x|^2}{k^2}\right) \left|\nabla u^\epsilon\right|^2 dx, \tag{5.7}$$

and, due to (2.1),

$$\int_{\mathbb{R}^N} g(x,u^\epsilon)\cdot\theta\left(\frac{|x|^2}{k^2}\right)u^\epsilon\,dx \leqslant \int_{\mathbb{R}^N}\theta\left(\frac{|x|^2}{k^2}\right)\left(-\frac{\gamma}{2}|u^\epsilon|^2+n^2(x)\right)dx. \tag{5.8}$$

Finally, we deal with the viscosity term. We have

$$\begin{aligned}
&-\int_{\mathbb{R}^N}(-\Delta)^{p+1}u^\epsilon\cdot\theta\left(\frac{|x|^2}{k^2}\right)u^\epsilon\,dx\\
&\quad=-\int_{\mathbb{R}^N}\sum_{i=1}^N\left(\theta'\left(\frac{|x|^2}{k^2}\right)\frac{2x_i}{k^2}u^\epsilon+\theta\left(\frac{|x|^2}{k^2}\right)\frac{\partial u^\epsilon}{\partial x_i}\right)\cdot\frac{\partial}{\partial x_i}(-\Delta)^p u^\epsilon\,dx\\
&\quad=\int_{\mathbb{R}^N}\sum_{i=1}^N\theta''\left(\frac{|x|^2}{k^2}\right)\frac{4x_i^2}{k^4}u^\epsilon\cdot(-\Delta)^p u^\epsilon\,dx+2\int_{\mathbb{R}^N}\sum_{i=1}^N\theta'\left(\frac{|x|^2}{k^2}\right)\frac{2x_i}{k^2}\frac{\partial u^\epsilon}{\partial x_i}\cdot(-\Delta)^p u^\epsilon\,dx\\
&\qquad+\int_{\mathbb{R}^N}\theta'\left(\frac{|x|^2}{k^2}\right)\frac{2N}{k^2}u^\epsilon\cdot(-\Delta)^p u^\epsilon\,dx+\int_{\mathbb{R}^N}\theta\left(\frac{|x|^2}{k^2}\right)\Delta u^\epsilon\cdot(-\Delta)^p u^\epsilon\,dx,
\end{aligned} \tag{5.9}$$

and further

$$\begin{aligned}
&\int_{\mathbb{R}^N}\theta\left(\frac{|x|^2}{k^2}\right)\Delta u^\epsilon\cdot(-\Delta)^p u^\epsilon\,dx\\
&\quad=\int_{\mathbb{R}^N}\sum_{i=1}^N\left(\theta'\left(\frac{|x|^2}{k^2}\right)\frac{2x_i}{k^2}\Delta u^\epsilon+\theta\left(\frac{|x|^2}{k^2}\right)\frac{\partial}{\partial x_i}\Delta u^\epsilon\right)\cdot\frac{\partial}{\partial x_i}(-\Delta)^{p-1}u^\epsilon\,dx\\
&\quad=\int_{\mathbb{R}^N}\sum_{i=1}^N\theta''\left(\frac{|x|^2}{k^2}\right)\frac{4x_i^2}{k^4}(-\Delta)u^\epsilon\cdot(-\Delta)^{p-1}u^\epsilon\,dx+\int_{\mathbb{R}^N}\sum_{i=1}^N\theta'\left(\frac{|x|^2}{k^2}\right)\frac{2N}{k^2}(-\Delta)u^\epsilon\cdot(-\Delta)^{p-1}u^\epsilon\,dx\\
&\qquad+2\int_{\mathbb{R}^N}\sum_{i=1}^N\theta'\left(\frac{|x|^2}{k^2}\right)\frac{2x_i}{k^2}\frac{\partial}{\partial x_i}(-\Delta)u^\epsilon\cdot(-\Delta)^{p-1}u^\epsilon\,dx-\int_{\mathbb{R}^N}\theta\left(\frac{|x|^2}{k^2}\right)(-\Delta)^{p-1}u^\epsilon\cdot(-\Delta)^2u^\epsilon\,dx.
\end{aligned} \tag{5.10}$$

Integrating by parts the components in (5.9) and (5.10) many times we obtain

$$-\epsilon\int_{\mathbb{R}^N}(-\Delta)^{p+1}u^\epsilon\cdot\theta\left(\frac{|x|^2}{k^2}\right)u^\epsilon\,dx\leqslant\frac{c\epsilon}{k}\big\||u^\epsilon|\big\|^2_{H^{p+1}(\mathbb{R}^N)}. \tag{5.11}$$

Consequently, we can rewrite (5.2) as

$$\frac{d}{dt}\int_{\mathbb{R}^N}\theta\left(\frac{x^2}{k^2}\right)|u^\epsilon|^2\,dx+\gamma\int_{\mathbb{R}^N}\theta\left(\frac{x^2}{k^2}\right)|u^\epsilon|^2\,dx+2\alpha\int_{\mathbb{R}^N}\theta\left(\frac{|x|^2}{k^2}\right)|\nabla u^\epsilon|^2\,dx\leqslant\frac{c}{k}\big\||u^\epsilon|\big\|^2_{H^{p+1}(\mathbb{R}^N)}+2\int_{|x|\geqslant k}n^2(x)\,dx. \tag{5.12}$$

On the other hand, since $n\in L^2(\mathbb{R})$, then

$$\int_{|x|\geqslant k}n^2(x)\,dx\to 0\quad\text{as } k\to\infty.$$

Hence, combining with the $H^{p+1}(\mathbb{R})$ estimate (3.25), we have that: given $\eta>0$, then for $t\geqslant Q_2(\|u_0\|_{(L^2(\mathbb{R}^N))^m})$ and k large enough,

$$\frac{d}{dt}\int_{\mathbb{R}^N}\theta\left(\frac{x^2}{k^2}\right)|u^\epsilon|^2\,dx+\gamma\int_{\mathbb{R}^N}\theta\left(\frac{x^2}{k^2}\right)|u^\epsilon|^2\,dx+2\alpha\int_{\mathbb{R}^N}\theta\left(\frac{x^2}{k^2}\right)|\nabla u^\epsilon|^2\,dx\leqslant\frac{\eta\gamma}{2}, \tag{5.13}$$

which implies (5.1) through a direct application of the Gronwall inequality. □

Lemma 5.2. *Under Assumption III, the semigroups $\{S^\epsilon(t)\}_{t\geqslant 0}$ are uniformly in $\epsilon > 0$ asymptotically compact in $(H^{p+1}(\mathbb{R}^N))^m$.*

Proof. From Lemmas 3.4 and 5.1 we know that $\{S^\epsilon(t)\}_{t\geqslant 0}$ introduced in (2.24) are uniformly in $\epsilon > 0$ asymptotically compact in $(L^2(\mathbb{R}^N))^m$. Then, combining with the, uniform in $\epsilon > 0$ and for $t \geqslant 1$, estimate (3.31) of u^ϵ in $(H^{p+2}(\mathbb{R}^N))^m$, the asymptotic compactness in $(H^{p+1}(\mathbb{R}^N))^m$ follows directly from interpolation inequality. □

Remark 5.3. As a consequence of the abstract results in [25,20], asymptotic compactness reported in Lemma 5.2 and the uniform in $\epsilon > 0$ estimate (3.31), the semigroups $\{S^\epsilon(t)\}_{t\geqslant 0}$ possess $(H^{p+1}(\mathbb{R}^N))^m$ global attractors $\mathcal{A}_\epsilon$, $\epsilon > 0$.

A corresponding result concerning the limiting semigroup $S(t)$ follows directly from our previous considerations. It states that (see [8] for the definition of the *bi-spaces global attractor*)

Lemma 5.4. *Let Assumption III be satisfied. Then the semigroup $S(t)$ of global weak solutions of (1.2) has an $((H^p(\mathbb{R}^N))^m, (H^{p+1}(\mathbb{R}^N))^m)$ global attractor $\mathcal{A}$. The attractor $\mathcal{A}$ is invariant, compact in $(H^{p+1}(\mathbb{R}^N))^m$ and attracts every $(H^p(\mathbb{R}^N))^m$ bounded set in $(H^{p+1}(\mathbb{R}^N))^m$ topology. It is also bounded in $(H^{(p+2)^-}(\mathbb{R}^N))^m$.*

Remark 5.5. We will discuss further relations between the attractors $\mathcal{A}_\epsilon$, $\epsilon > 0$, and $\mathcal{A}$. Using an abstract criterion in [20, p. 916, (H.1a)], it is easy to check that the family of the global attractors $\{\mathcal{A}_\epsilon\}_{\epsilon\in[0,1]}$, where we denote $\mathcal{A} = \mathcal{A}_0$, is *upper semicontinuous* at $\epsilon = 0$. Indeed, thanks to the estimate (3.31), the approximating solutions u^ϵ, $\epsilon > 0$, enter for $t \geqslant 1$ a bounded set $B \subset (H^{p+2}(\mathbb{R}^N))^m$, uniformly in $\epsilon > 0$. It is next evident from Definition 4.5 and (4.19), that the global attractor $\mathcal{A}_0$ corresponding to (1.2) is bounded in $(H^{(p+2)^-}(\mathbb{R}^N))^m$. Consequently,

$$A = \bigcup_{\lambda\in[0,1]} \mathcal{A}_\epsilon \tag{5.14}$$

is bounded in $(H^{(p+2)^-}(\mathbb{R}^N))^m$. Together with the, following from the tail estimates, $(L^2(\mathbb{R}^N))^m$ asymptotic compactness, this shows that the set A is precompact in $(H^{p+1}(\mathbb{R}^N))^m$. The second part of the condition (H.1a) in [20] follows directly from the Definition 4.5. Thus, the family of global attractors $\{\mathcal{A}\}_{\epsilon\in[0,1]}$ is *upper semicontinuous* at $\epsilon = 0$ in $(H^{p+1}(\mathbb{R}^N))^m$.

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