



Common properties of bounded linear operators AC and BA : Local spectral theory [☆]



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ABSTRACT

Let X, Y be Banach spaces, $A : X \rightarrow Y$ and $B, C : Y \rightarrow X$ be bounded linear operators satisfying operator equation $ABA = ACA$. In this paper, we show that AC and BA share the local spectral properties such as Bishop's property (β), subscalarity and Dunford's property (C). Also, the quasi-nilpotent part and the analytic core for AC and BA are studied.

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1. Introduction and notations

For any Banach spaces X and Y , let $\mathcal{B}(X, Y)$ denote the set of all bounded linear operators from X to Y . It is well known that if $A \in \mathcal{B}(X, Y)$ and $C \in \mathcal{B}(Y, X)$ then

$$AC - I \text{ is invertible} \iff CA - I \text{ is invertible}, \quad (1.1)$$

see [2–4, 7, 9, 12]. This result is now known as Jacobson's lemma. In [13], we extended (1.1) to various regularities, in the sense of Kordula and Müller, and proved that $AC - I$ and $CA - I$ share common complementability of kernels. Recently, Corach, Duggal and Harte [5] generalized (1.1) and many of its relatives from $CA - I$ to certain $BA - I$ under the assumption

$$ABA = ACA, \quad (1.2)$$

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where $A \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$. They studied common properties of AC and BA in algebraic viewpoint and also obtained some topological analogues. In [14], we investigated common properties of AC and BA from the viewpoint of spectral theory. In this paper, we continue to study this situation from the viewpoint of local spectral theory. We show that AC and BA share many common local spectral properties such as Bishop's property (β), subscalarity and Dunford's property (C). Also, the quasi-nilpotent part and the analytic core for AC and BA are studied.

We first fix some notations in local spectral theory. Throughout this paper, $\mathcal{B}(X) = \mathcal{B}(X, X)$. An operator $T \in \mathcal{B}(X)$ is said to have the *single valued extension property* at $\lambda \in \mathbb{C}$ (SVEP at λ for brevity), if for every open neighborhood $\mathcal{U}_\lambda$ of λ the only analytic function $f : \mathcal{U}_\lambda \rightarrow X$ which satisfies $(T - \mu)f(\mu) = 0$ for all $\mu \in \mathcal{U}_\lambda$ is the function $f \equiv 0$. An operator $T \in \mathcal{B}(X)$ is said to have the SVEP if T has SVEP at every $\lambda \in \mathbb{C}$. For $T \in \mathcal{B}(X)$, the *local resolvent set* $\rho_T(x)$ of T at the point $x \in X$ is defined as the set of all $\lambda \in \mathbb{C}$ for which there exist an open neighborhood $\mathcal{U}_\lambda$ of λ and an analytic function $f : \mathcal{U}_\lambda \rightarrow X$ such that

$$(T - \mu)f(\mu) = x \quad \text{for all } \mu \in \mathcal{U}_\lambda.$$

The *local spectrum* $\sigma_T(x)$ of T at x is then defined as $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$. The local analytic solutions occurring in the definition of the local resolvent set will be unique for all $x \in X$ if and only if T has SVEP.

For every subset F of $\mathbb{C}$, we define the *local spectral subspace* of T associated with F by

$$X_T(F) := \{x \in X : \sigma_T(x) \subseteq F\}.$$

Evidently, $X_T(F)$ is a hyperinvariant subspace of T , but not always closed. An operator $T \in \mathcal{B}(X)$ is said to have *Dunford's property* (C), if the local spectral subspace $X_T(F)$ is closed for every closed set $F \subseteq \mathbb{C}$.

Let $D(\lambda, r)$ be the open disc centred at $\lambda \in \mathbb{C}$ with radius $r > 0$, and let $\mathcal{O}(U, X)$ denote the Fréchet algebra of all X -valued analytic functions on the open subset $U \subseteq \mathbb{C}$ endowed with uniform convergence on compact subsets of U . An operator $T \in \mathcal{B}(X)$ is said to satisfy *Bishop's property* (β) at $\lambda \in \mathbb{C}$ if there exists $r > 0$ such that for every open subset $U \subseteq D(\lambda, r)$ and for any sequence $\{f_n\}_{n=1}^\infty \subseteq \mathcal{O}(U, X)$, $\lim_{n \rightarrow \infty} (T - \mu)f_n(\mu) = 0$ in $\mathcal{O}(U, X)$ implies $\lim_{n \rightarrow \infty} f_n(\mu) = 0$ in $\mathcal{O}(U, X)$. We denote by $\sigma_\beta(T)$ the set where T fails to satisfy (β) and we say that T satisfies *Bishop's property* (β) if $\sigma_\beta(T) = \emptyset$. Dually, $T \in \mathcal{B}(X)$ is said to have the *decomposition property* (δ) if T^* satisfies (β). It is well known that T is *decomposable*, in the sense of Foiaş, if and only if T satisfies both (β) and (δ). We refer the reader to [8] for more details and further definitions.

In a similar way, property (β_ϵ) is defined. To be precise, let $\mathcal{E}(U, X)$ denote the Fréchet algebra of all X -valued infinitely continuously differentiable functions on the open subset $U \subseteq \mathbb{C}$ endowed with the topology of uniform convergence on compact subsets of U of such functions and their derivatives. An operator $T \in \mathcal{B}(X)$ is said to have *property* (β_ϵ) at $\lambda \in \mathbb{C}$ if there exists a neighborhood $\mathcal{U}_\lambda$ of λ such that for each open subset $U \subseteq \mathcal{U}_\lambda$ and for any sequence $\{f_n\}_{n=1}^\infty \subseteq \mathcal{E}(U, X)$, $\lim_{n \rightarrow \infty} (T - \mu)f_n(\mu) = 0$ in $\mathcal{E}(U, X)$ implies $\lim_{n \rightarrow \infty} f_n(\mu) = 0$ in $\mathcal{E}(U, X)$. We denote by $\sigma_{\beta_\epsilon}(T)$ the set where T fails to satisfy (β_ϵ) and we say that T satisfies *property* (β_ϵ) if $\sigma_{\beta_\epsilon}(T) = \emptyset$. It is well known that $T \in \mathcal{B}(X)$ has property (β_ϵ) precisely when T is *subscalar*. For further definitions and more details, we refer the reader to [6]. It is well known that [8]

$$\text{subscalar} \implies \text{Bishop's property } (\beta) \implies \text{Dunford's property } (C) \implies \text{SVEP}.$$

For every closed subset F of $\mathbb{C}$, the *global spectral subspace* $\mathcal{X}_T(F)$ is defined as the set of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus F \rightarrow X$ that satisfies $(T - \lambda)f(\lambda) = x$ for all $\lambda \in \mathbb{C} \setminus F$. Clearly, $\mathcal{X}_T(F)$ is a hyperinvariant subspace of T and $\mathcal{X}_T(F) \subseteq X_T(F)$. Moreover, $\mathcal{X}_T(F) = X_T(F)$ holds for all closed subsets F of $\mathbb{C}$ precisely when T has SVEP, see [8, Proposition 3.3.2].

Associated with $T \in \mathcal{B}(X)$, two important subspaces of X are the *analytic core* of T defined by

$$K(T) := \{x \in X: \text{there exist a sequence } \{x_n\}_{n \geq 1} \text{ in } X \text{ and a constant } \delta > 0 \\ \text{such that } Tx_1 = x, Tx_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n \geq 1\},$$

and the *quasi-nilpotent part* of T defined by

$$H_0(T) := \left\{x \in X: \lim_{n \rightarrow \infty} \|T^n x\|^{1/n} = 0\right\}.$$

These subspaces have been introduced and studied by Mbekhta in [11]. They are closely related to the local spectral subspace and the global spectral subspace. Indeed, $K(T) = X_T(\mathbb{C} \setminus \{0\})$ and $H_0(T) = \mathcal{X}_T(\{0\})$, see [1, Theorems 2.18 and 2.20].

This paper is organized as follows. Suppose that $A \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ satisfy $ABA = ACA$. In Section 2, we show that AC and BA share many common local spectral properties such as Bishop's property (β) , decomposition property (δ) , subscalarity, decomposability and Dunford's property (C) . It extends previous ones in [2,10] from the special case " $B = C$ " to the general case. In Section 3, we show that $K(AC - \lambda)$ is closed if and only if $K(BA - \lambda)$ is closed for all $\lambda \neq 0$ and that $H_0(AC)$ is closed if and only if $H_0(BA)$ is closed. For the special case " $B = C$ ", some further results are given for injective operators.

2. Local spectral properties for AC and BA

In [5, Theorem 9], the authors firstly studied the local spectral properties for AC and BA . They showed that if $A \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ satisfy $ABA = ACA$, then AC has SVEP if and only if BA has SVEP. In this section, we continue to study other local spectral properties for AC and BA . We begin with the following result, which extends [10, Theorems 2.3 and 2.4] from the special case " $B = C$ " to the general case.

Theorem 2.1. *If $A \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ satisfy $ABA = ACA$, then*

- (1) $\sigma_\beta(AC) = \sigma_\beta(BA)$;
- (2) $\sigma_{\beta_e}(AC) = \sigma_{\beta_e}(BA)$.

In particular, AC satisfies Bishop's property (β) (resp. is subscalar) if and only if BA satisfies Bishop's property (β) (resp. is subscalar).

Proof. (1) Suppose that $\lambda \notin \sigma_\beta(AC)$. Then there exists $r > 0$ such that for every open subset $U \subseteq D(\lambda, r)$ and for any sequence $\{g_n\}_{n=1}^\infty \subseteq \mathcal{O}(U, Y)$, $\lim_{n \rightarrow \infty} (AC - \mu)g_n(\mu) = 0$ in $\mathcal{O}(U, Y)$ implies $\lim_{n \rightarrow \infty} g_n(\mu) = 0$ in $\mathcal{O}(U, Y)$. Let $U \subseteq D(\lambda, r)$ be open and $\{f_n\}_{n=1}^\infty \subseteq \mathcal{O}(U, X)$ such that $\lim_{n \rightarrow \infty} (BA - \mu)f_n(\mu) = 0$ in $\mathcal{O}(U, X)$. Thus $\lim_{n \rightarrow \infty} (AC - \mu)Af_n(\mu) = \lim_{n \rightarrow \infty} A(BA - \mu)f_n(\mu) = 0$ in $\mathcal{O}(U, Y)$. It follows then that $\lim_{n \rightarrow \infty} Af_n(\mu) = 0$ in $\mathcal{O}(U, Y)$. Thus $\lim_{n \rightarrow \infty} BAf_n(\mu) = 0$ in $\mathcal{O}(U, X)$, and hence

$$\lim_{n \rightarrow \infty} \mu f_n(\mu) = 0 \quad \text{in } \mathcal{O}(U, X). \quad (2.1)$$

Since $T = 0 \in \mathcal{B}(X)$ is decomposable and hence satisfies Bishop's property (β) , (2.1) may be rewritten as $\lim_{n \rightarrow \infty} (T - \mu)f_n(\mu) = 0$ in $\mathcal{O}(U, X)$, it follows from the definition of Bishop's property (β) that $\lim_{n \rightarrow \infty} f_n(\mu) = 0$ in $\mathcal{O}(U_\lambda, X)$. This shows that $\lambda \notin \sigma_\beta(BA)$.

For the special case " $B = C$ ", the previous paragraph shows that $\sigma_\beta(BA) \subseteq \sigma_\beta(AB)$. Then interchanging A and B , we have $\sigma_\beta(AB) = \sigma_\beta(BA)$.

Suppose that $\lambda \notin \sigma_\beta(BA)$. Then $\lambda \notin \sigma_\beta(AB)$. Interchanging B and C in the first paragraph of this proof, we infer that $\lambda \notin \sigma_\beta(CA)$. Therefore, $\lambda \notin \sigma_\beta(AC)$.

(2) The proof is similar to that of (1), we omit it here. $\square$

Corollary 2.2. *If $A \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ satisfy $ABA = ACA$, then AC satisfies decomposition property (δ) (resp. is decomposable) if and only if BA satisfies decomposition property (δ) (resp. is decomposable).*

Proof. Noting that (β) and (δ) are dual to each other and that $T \in \mathcal{B}(X)$ is decomposable if and only if T satisfies both (β) and (δ) , the conclusion follows immediately from [Theorem 2.1](#). $\square$

Next, we turn to the discussion of Dunford's property (C) for AC and BA . Before this, the relations between the local spectra of AC and BA are needed.

Theorem 2.3. *If $A \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ satisfy $ABA = ACA$, then*

- (1) $\sigma_{AC}(Ax) \subseteq \sigma_{BA}(x) \subseteq \sigma_{AC}(Ax) \cup \{0\}$ for all $x \in X$;
- (2) $\sigma_{BA}(BACy) \subseteq \sigma_{AC}(y) \subseteq \sigma_{BA}(BACy) \cup \{0\}$ for all $y \in Y$.

Proof. (1) Suppose that $\lambda \notin \sigma_{BA}(x)$. Then there exists an analytic function $f : \mathcal{U}_\lambda \rightarrow X$ defined on some neighborhood $\mathcal{U}_\lambda$ of λ such that

$$(BA - \mu)f(\mu) = x \quad \text{for all } \mu \in \mathcal{U}_\lambda.$$

Thus, $A(BA - \mu)f(\mu) = A(CA - \mu)f(\mu) = (AC - \mu)Af(\mu) = Ax$ and hence $\lambda \notin \sigma_{AC}(Ax)$. The first inclusion is established.

To establish the second inclusion, let $\lambda \notin \sigma_{AC}(Ax) \cup \{0\}$ and $g : \mathcal{V}_\lambda \rightarrow Y$ be an analytic function defined on some neighborhood $\mathcal{V}_\lambda$ of λ such that

$$(AC - \mu)g(\mu) = Ax \quad \text{for all } \mu \in \mathcal{V}_\lambda.$$

Therefore, $g(\mu) = \frac{ACg(\mu) - Ax}{\mu}$ and we have

$$\begin{aligned} x &= \frac{BAx - (BA - \mu)x}{\mu} \\ &= \frac{B(AC - \mu)g(\mu) - (BA - \mu)x}{\mu} \\ &= \frac{[(BAC - \mu B)g(\mu) - (BA - \mu)x]}{\mu} \\ &= \frac{[(BAC - \mu B)\frac{ACg(\mu) - Ax}{\mu} - (BA - \mu)x]}{\mu} \\ &= \frac{[BACAC\frac{g(\mu)}{\mu} - BACA\frac{x}{\mu} - \mu BAC\frac{g(\mu)}{\mu} + \mu BA\frac{x}{\mu} - (BA - \mu)x]}{\mu} \\ &= \frac{[BABAC\frac{g(\mu)}{\mu} - BABA\frac{x}{\mu} - \mu BAC\frac{g(\mu)}{\mu} + \mu BA\frac{x}{\mu} - (BA - \mu)x]}{\mu} \end{aligned}$$

$$\begin{aligned}
&= \frac{(BA - \mu)(BAC \frac{g(\mu)}{\mu} - BA \frac{x}{\mu} - x)}{\mu} \\
&= \frac{(BA - \mu)(Bg(\mu) - x)}{\mu}.
\end{aligned}$$

Define

$$h(\mu) = \frac{Bg(\mu) - x}{\mu} \quad \text{for all } \mu \in \mathcal{V}_\lambda.$$

Evidently, $h : \mathcal{V}_\lambda \rightarrow X$ is analytic and $(BA - \mu)h(\mu) = x$, so $\lambda \notin \sigma_{BA}(x)$.

(2) By [4, Proposition 3.1], we have $\sigma_{CA}(Cy) \subseteq \sigma_{AC}(y) \subseteq \sigma_{CA}(Cy) \cup \{0\}$. According to (1) and interchanging B and C , we obtain $\sigma_{AB}(ACy) \subseteq \sigma_{CA}(Cy) \subseteq \sigma_{AB}(ACy) \cup \{0\}$. By [4, Proposition 3.1] again, we have $\sigma_{BA}(BACy) \subseteq \sigma_{AB}(ACy) \subseteq \sigma_{BA}(BACy) \cup \{0\}$. From these inclusions, we conclude that $\sigma_{BA}(BACy) \subseteq \sigma_{AC}(y) \subseteq \sigma_{BA}(BACy) \cup \{0\}$. $\square$

The following Lemmas 2.4–2.6 are essentially inspired by the proof of [2, Theorem 2.5].

Lemma 2.4. *Let F be a closed subset of $\mathbb{C}$ such that $0 \in F$. If $A \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ satisfy $ABA = ACA$, then $Y_{AC}(F)$ is closed if and only if $X_{BA}(F)$ is closed.*

Proof. Suppose that $Y_{AC}(F)$ is closed. Let $\{x_n\}_{n=1}^\infty$ be a sequence in $X_{BA}(F)$ which converges to some $x \in X$. To show that $X_{BA}(F)$ is closed, it suffices to prove that $x \in X_{BA}(F)$. Since $x_n \in X_{BA}(F)$, $\sigma_{BA}(x_n) \subseteq F$ for all $n \in \mathbb{N}$. Hence by Theorem 2.3(1), $\sigma_{AC}(Ax_n) \subseteq F$, that is $Ax_n \in Y_{AC}(F)$. Since $Ax_n \rightarrow Ax$ and $Y_{AC}(F)$ is closed, $Ax \in Y_{AC}(F)$, that is $\sigma_{AC}(Ax) \subseteq F$. By Theorem 2.3(1) again, $\sigma_{BA}(x) \subseteq F$ because $0 \in F$, that is $x \in X_{BA}(F)$.

The converse implication follows by using Theorem 2.3(2) instead of Theorem 2.3(1). $\square$

Lemma 2.5. *Suppose that $T \in \mathcal{B}(X)$ has SVEP, F is a closed subset of $\mathbb{C}$ and $\lambda \notin F$. If $X_T(F \cup \{\lambda\})$ is closed, then $X_T(F)$ is closed.*

Proof. Let $Z := X_T(F \cup \{\lambda\})$ and $S := T|_{X_T(F \cup \{\lambda\})}$. It follows from [8, Proposition 1.2.20] that $\sigma(S) \subseteq F \cup \{\lambda\}$. In order to show that $X_T(F)$ is closed, we consider two cases:

(I) Suppose that $\lambda \notin \sigma(S)$. Then $\sigma(S) \subseteq F$, and we immediately have $Z = Z_S(F)$. By [2, Lemma 2.4], we have $Z_S(F) = X_T(F)$, so $X_T(F)$ is closed.

(II) Suppose that $\lambda \in \sigma(S)$. Let $F_0 = \sigma(S) \cap F$. Then $\sigma(S) = F_0 \cup \{\lambda\}$. Since $\lambda \in \sigma(S)$, it follows from [8, Proposition 3.3.3] that the topological direct sum decomposition $Z = Z_S(F_0) \oplus Z_S(\{\lambda\})$ holds. By [2, Lemma 2.4], we have

$$Z_S(F_0) = Z_S(\sigma(S) \cap F) = Z_S(F) = X_T(F),$$

and hence $X_T(F)$ is closed. $\square$

Lemma 2.6. *Let F be a closed subset of $\mathbb{C}$ such that $0 \notin F$. If $A \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ satisfy $ABA = ACA$ and AC has SVEP, then*

- (1) if $Y_{AC}(F \cup \{0\})$ is closed, then $X_{BA}(F)$ is closed;
- (2) if $X_{BA}(F \cup \{0\})$ is closed, then $Y_{AC}(F)$ is closed.

Proof. (1) Since $0 \in F \cup \{0\}$, by Lemma 2.4, $Z := X_{BA}(F \cup \{0\})$ is closed. Since AC has SVEP, we also know that BA has SVEP by [5, Theorem 9]. Thus by Lemma 2.5, $X_{BA}(F)$ is closed.

(2) It can be proved similarly to (1). $\square$

When “ $B = C$ ” and $0 \notin F$, the readers should compare the above lemma with [2, Theorem 2.5]. And, the careful readers will have observed that there is a difference between them. In [2, Theorem 2.5], the authors claimed that if $X_{BA}(F)$ is closed, then $Y_{AB}(F)$ is closed. The proof was based on a fact that if $X_T(F)$ is closed, then $X_T(F \cup \{0\})$ is closed. As [1, Example 2.32] shows, this fact is not always true because there exists $T \in \mathcal{B}(X)$ such that $X_T(\emptyset)$ is closed but $X_T(\{0\})$ is not closed. So there was a gap in the proof of it. We bypass this gap and state in Lemma 2.6 that if $X_{BA}(F \cup \{0\})$ is closed, then $Y_{AB}(F)$ is closed. Fortunately, if $T \in \mathcal{B}(X)$ has Dunford’s property (C) and F is a closed subset of $\mathbb{C}$ such that $0 \notin F$, then $X_T(F \cup \{0\})$ is closed, since $F \cup \{0\}$ is closed. Thus, we have

Theorem 2.7. *If $A \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ satisfy $ABA = ACA$, then AC has Dunford’s property (C) if and only if BA has Dunford’s property (C).*

Proof. It follows immediately from Lemmas 2.4 and 2.6. $\square$

3. Quasi-nilpotent part and analytic core for AC and BA

Recall that $K(T - \lambda) = X_T(\mathbb{C} \setminus \{\lambda\}) = \{x \in X : \lambda \in \rho_T(x)\}$, see [1, Theorem 2.18]. As an immediate application of Theorem 2.3, we obtain

Theorem 3.1. *If $A \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ satisfy $ABA = ACA$, then for all $\lambda \neq 0$, $K(AC - \lambda)$ is closed if and only if $K(BA - \lambda)$ is closed.*

Proof. Suppose that $K(AC - \lambda)$ is closed. Let $\{x_n\}_{n=1}^\infty$ be a sequence in $K(BA - \lambda)$ which converges to some $x \in X$. To show that $K(BA - \lambda)$ is closed, it suffices to prove that $x \in K(BA - \lambda)$. Since $x_n \in K(BA - \lambda)$, $\lambda \in \rho_{BA}(x_n)$ for all $n \in \mathbb{N}$. Hence by Theorem 2.3(1), $\lambda \in \rho_{AC}(Ax_n)$, that is, $Ax_n \in K(AC - \lambda)$. Since $Ax_n \rightarrow Ax$ and $K(AC - \lambda)$ is closed, $Ax \in K(AC - \lambda)$, that is, $\lambda \in \rho_{AC}(Ax)$. By Theorem 2.3(1) again, $\lambda \in \rho_{BA}(x)$, that is, $x \in K(BA - \lambda)$.

The converse implication follows by using Theorem 2.3(2) instead of Theorem 2.3(1). $\square$

Lemma 3.2. (See [4].) *Let $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, X)$.*

- (1) *If A is injective, then $\sigma_{AB}(Ax) = \sigma_{BA}(x)$ for all $x \in X$.*
- (2) *If B is injective, then $\sigma_{BA}(By) = \sigma_{AB}(y)$ for all $y \in Y$.*

Corollary 3.3. *Let $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, X)$.*

- (1) *$K(AB - \lambda)$ is closed if and only if $K(BA - \lambda)$ is closed, for all $\lambda \neq 0$.*
- (2) *If moreover A and B are injective, then $K(AB)$ is closed if and only if $K(BA)$ is closed.*

Proof. (1) It is a special case of Theorem 3.1.

(2) It can be proved similarly to Theorem 3.1, by using Lemma 3.2. $\square$

Lemma 3.4. *Let F be a closed subset of $\mathbb{C}$ such that $0 \in F$. If $A \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ satisfy $ABA = ACA$, then $\mathcal{Y}_{AC}(F)$ is closed if and only if $\mathcal{X}_{BA}(F)$ is closed.*

Proof. Suppose that $\mathcal{Y}_{AC}(F)$ is closed. Let $\{x_n\}_{n=1}^\infty$ be a sequence in $\mathcal{X}_{BA}(F)$ which converges to some $x \in X$. To show that $\mathcal{X}_{BA}(F)$ is closed, it suffices to prove that $x \in \mathcal{X}_{BA}(F)$. Since $x_n \in \mathcal{X}_{BA}(F)$, for every $n \in \mathbb{N}$ there exists an analytic function $f_n : \mathbb{C} \setminus F \rightarrow X$ such that

$$(BA - \lambda)f_n(\lambda) = x_n \quad \text{for all } \lambda \in \mathbb{C} \setminus F.$$

Thus, $A(BA - \lambda)f_n(\lambda) = A(CA - \lambda)f_n(\lambda) = (AC - \lambda)Af_n(\lambda) = Ax_n$ and hence $Ax_n \in \mathcal{Y}_{AC}(F)$. Since $Ax_n \rightarrow Ax$ and $\mathcal{Y}_{AC}(F)$ is closed, $Ax \in \mathcal{Y}_{AC}(F)$. Thus, there exists an analytic function $g : \mathbb{C} \setminus F \rightarrow Y$ such that

$$(AC - \lambda)g(\lambda) = Ax \quad \text{for all } \lambda \in \mathbb{C} \setminus F.$$

As in the proof of Theorem 2.3(1), we obtain $x = \frac{(BA - \lambda)(Bg(\lambda) - x)}{\lambda}$ for all $\lambda \in \mathbb{C} \setminus F$. Define

$$h(\lambda) = \frac{Bg(\lambda) - x}{\lambda} \quad \text{for all } \lambda \in \mathbb{C} \setminus F.$$

Evidently, $h : \mathbb{C} \setminus F \rightarrow X$ is analytic and $(BA - \lambda)h(\lambda) = x$, so $x \in \mathcal{X}_{BA}(F)$.

For the special case “ $B = C$ ”, the previous paragraph shows that if $\mathcal{Y}_{AB}(F)$ is closed, then $\mathcal{X}_{BA}(F)$ is closed. Then interchanging A and B , we find an equivalence statement: $\mathcal{Y}_{AB}(F)$ is closed if and only if $\mathcal{X}_{BA}(F)$ is closed.

Now suppose that $\mathcal{X}_{BA}(F)$ is closed. Then $\mathcal{Y}_{AB}(F)$ is closed. Interchanging B and C in the first paragraph of this proof, we infer that $\mathcal{X}_{CA}(F)$ is closed. Therefore, $\mathcal{Y}_{AC}(F)$ is closed. $\square$

Recall that $H_0(T - \lambda) = \mathcal{X}_T(\{\lambda\})$, see [1, Theorem 2.20]. As an immediate consequence of Lemma 3.4, we obtain

Theorem 3.5. *If $A \in \mathcal{B}(X, Y)$ and $B, C \in \mathcal{B}(Y, X)$ satisfy $ABA = ACA$, then $H_0(AC)$ is closed if and only if $H_0(BA)$ is closed.*

Lemma 3.6. *Let $A \in \mathcal{B}(X, Y)$, $B \in \mathcal{B}(Y, X)$ and F be a closed subset of $\mathbb{C}$ such that $0 \notin F$. If A and B are injective, then $\mathcal{Y}_{AB}(F)$ is closed if and only if $\mathcal{X}_{BA}(F)$ is closed.*

Proof. Suppose that $\mathcal{Y}_{AB}(F)$ is closed. Let $\{x_n\}_{n=1}^\infty$ be a sequence in $\mathcal{X}_{BA}(F)$ which converges to some $x \in X$. To show that $\mathcal{X}_{BA}(F)$ is closed, it suffices to prove that $x \in \mathcal{X}_{BA}(F)$. As in the proof of Lemma 3.4, we obtain $x = \frac{(BA - \lambda)(Bg(\lambda) - x)}{\lambda}$ for all $\lambda \in \mathbb{C} \setminus (F \cup \{0\})$ and $x = Bg(0)$. Define

$$h(\lambda) = \begin{cases} \frac{Bg(\lambda) - x}{\lambda}, & \lambda \in \mathbb{C} \setminus (F \cup \{0\}); \\ Bg'(0), & \lambda = 0. \end{cases}$$

Evidently, $h : \mathbb{C} \setminus F \rightarrow X$ is analytic and $(BA - \lambda)h(\lambda) = x$, so $x \in \mathcal{X}_{BA}(F)$.

The converse implication follows by interchanging A and B . $\square$

Corollary 3.7. *Let $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, X)$.*

- (1) $H_0(AB)$ is closed if and only if $H_0(BA)$ is closed.
- (2) If moreover A and B are injective, then for all $\lambda \neq 0$, $H_0(AB - \lambda)$ is closed if and only if $H_0(BA - \lambda)$ is closed.

Proof. (1) It is a special case of [Theorem 3.5](#).

(2) It follows from [Lemma 3.6](#). $\square$

Recall that an operator $T \in \mathcal{B}(X)$ is said to have *property (Q)* if $H_0(T - \lambda)$ is closed for all $\lambda \in \mathbb{C}$.

Corollary 3.8. *Let $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, X)$ be injective. Then AB has property (Q) if and only if BA has property (Q).*

As a consequence of [\[2, Theorem 2.5\]](#), the authors claimed that the above result is true without the assumption of the injectivity of A and B . But, as we explain before [Theorem 2.7](#), there was a gap in the proof of [\[2, Theorem 2.5\]](#). So, the following question is still open.

Question 3.9. *Let $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(Y, X)$. Is it true that AB has property (Q) if and only if BA has property (Q)?*

References

- [1] P. Aiena, Fredholm and Local Spectral Theory, with Application to Multipliers, Kluwer Academic Publishers, Dordrecht, 2004.
- [2] P. Aiena, M. González, On the Dunford property (C) for bounded linear operators RS and SR , *Integral Equations Operator Theory* 70 (2011) 561–568.
- [3] B.A. Barnes, Common operator properties of the linear operators RS and SR , *Proc. Amer. Math. Soc.* 126 (1998) 1055–1061.
- [4] C. Benhida, E.H. Zerouali, Local spectral theory of linear operators RS and SR , *Integral Equations Operator Theory* 54 (2006) 1–8.
- [5] G. Corach, B. Duggal, R. Harte, Extensions of Jacobson’s lemma, *Comm. Algebra* 41 (2013) 520–531.
- [6] J. Eschmeier, M. Putinar, Bishop’s condition (β) and rich extensions of linear operators, *Indiana Univ. Math. J.* 37 (1988) 325–348.
- [7] R.E. Harte, *Invertibility and Singularity for Bounded Linear Operator*, Marcel Dekker, 1988.
- [8] K.B. Laursen, M.M. Neumann, *An Introduction to Local Spectral Theory*, Oxford University Press, 2000.
- [9] C. Lin, Z. Yan, Y. Ruan, Common properties of operators RS and SR and p -hyponormal operators, *Integral Equations Operator Theory* 43 (2002) 313–325.
- [10] C. Lin, Y. Ruan, Z. Yan, w -Hyponormal operators are subscalar, *Integral Equations Operator Theory* 50 (2004) 165–168.
- [11] M. Mbekhta, Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux, *Glasg. Math. J.* 29 (1987) 159–175.
- [12] V. Müller, *Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras*, second edition, Birkhäuser, Basel, Boston, Berlin, 2007.
- [13] Q.P. Zeng, H.J. Zhong, New results on common properties of the bounded linear operators RS and SR , *Acta Math. Sin. (Engl. Ser.)* 29 (2013) 1871–1884.
- [14] Q.P. Zeng, H.J. Zhong, Common properties of bounded linear operators AC and BA : Spectral theory, *Math. Nachr.* (2014), <http://dx.doi.org/10.1002/mana201300123>.