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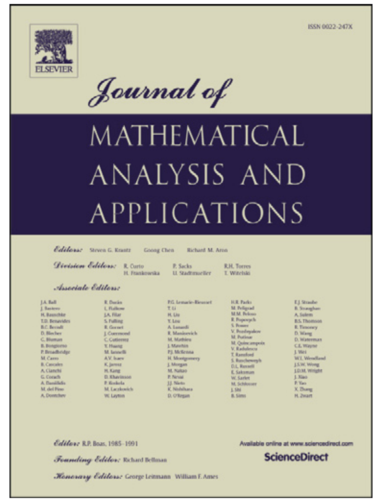

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Fractal Perturbation Preserving Fundamental Shapes: Bounds on the Scale Factors

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Abstract

Fractal interpolation function defined through suitable iterated function system provides a method to perturb a function $f \in \mathcal{C}(I)$ so as to yield a class of functions $f^\alpha \in \mathcal{C}(I)$, where α is a free parameter, called scale vector. For suitable values of scale vector α, the fractal functions f^α simultaneously interpolate and approximate f. Further, the iterated function system can be selected suitably so that the corresponding fractal function f^α shares the quality of smoothness or non-smoothness of f. The objective of the present paper is to choose elements of the iterated function system appropriately in order that f^α preserves fundamental shape properties, namely positivity, monotonicity, and convexity in addition to the regularity of f in the given interval. In particular, the scale factors (elements of the scale vector) must be restricted to satisfy two inequalities that provide numerical lower and upper bounds for the multipliers. As a consequence of this process, fractal versions of some elementary theorems in shape preserving interpolation/approximation are obtained. For instance, positive approximation (that is to say, using a positive function) is extended to the fractal case if the factors verify certain inequalities.

Keywords: α-Fractal Function, Fractals, Shape Preserving Approximation, Müntz Polynomial.
2000 MSC: 41A29, 41A30, 28A80, 65D05, 26A48, 26A51.

1. Introduction

One of the central themes in numerical analysis/approximation theory is to represent an arbitrary function or a data set in terms of functions which are easier to describe and convenient to use. When we have to deal with irregular forms, for instance, real world signals such as financial

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series, time series, climate data, and bioelectric recordings, traditional methods may not provide an approximant with a desired precision. Fractal functions form the basis of a constructive approximation theory for non-differentiable functions.

The idea of interpolation and approximation using fractal methodology first appeared in the work of Barnsley [1, 2]. Barnsley introduced fractal functions as continuous functions interpolating a given set of data points. Since its inception, fractal interpolation function has been developed both in theory and applications by many authors, see for example [3, 4, 6, 9, 11, 12] and references therein.

The methods in fractal approximation theory are based on iterated function system (IFS), which are chosen suitably for different target functions. Given a continuous function f defined on a real compact interval, Barnsley and Navascués have considered suitable IFS to construct continuous functions f^α that simultaneously interpolate and approximate f. The graph of f^α is a union of transformed copies of itself, and, in general, f^α may have noninteger Hausdorff and Minkowski dimensions. Due to these fractal characteristics, f^α may be treated as the fractal perturbation of f. In this way, every continuous function is generalized with a family of fractal functions. The degrees of freedom offered by this procedure may be useful when some problems combined with approximation and optimization have to be approached.

Navascués and group has studied various properties of the fractal perturbation f^α of f and proposed the fractal operator $\mathcal{F}^\alpha : \mathcal{C}(I) \to \mathcal{C}(I)$, where $\mathcal{C}(I)$ denotes the space of continuous functions on a real compact interval I, that maps $f \mapsto f^\alpha$ (see [13, 14, 15, 16, 17]). In this fractal perturbation process, these studies concern mainly on two properties, namely smoothness and approximation order among the various desirable properties of a good approximant. To be precise, given a function $f \in \mathcal{C}^p(I)$, it is known how to select the elements of the IFS so that the corresponding fractal function $f^\alpha \in \mathcal{C}^p(I)$. Similarly, for a given original function Φ with its traditional approximant f, we can perturb f using fractal method so as to yield f^α that has same approximation order as that of f. However, in many problems arising in engineering and science, one requires approximation methods to reproduce physical reality as close as possible. Schematically, given a function or data set with a shape S one desires to represent it by a function that approximates it well, and, in addition, has the same shape S. This kind of approximation is called *shape preserving approximation* and arises quite naturally in fields such as computer aided geometric design, robotics, data visualization, chemical and physical sciences, and reverse engineering. Recently, our group has developed the shape preserving aspects of the cubic Hermite fractal interpolation function (FIF), the rational quadratic FIF and the rational cubic FIF in constructive manner (see [7, 8, 10]).

The aforementioned considerations naturally leads to the question: can we find fractal perturbation f^α of f that retains properties of f? The current article seeks to develop suitable methods to choose elements of the IFS appropriately so that the corresponding fractal functions f^α retain the order of continuity and the fundamental shape properties, namely positivity, monotonicity, and convexity of f. The selection of the parameters involves the boundedness of the scale factors (components of the scale vector α) of the transformation, by means of two appropriate inequalities. Interpolating a given data set within a prescribed frame is a basic requirement in image compression. The parameter identification problem given in this article can be adapted to interpolate a given function f at specified knot points with the help of a fractal function f^α whose graph is contained in a prescribed axis-aligned rectangle. The method used here is more general than that in the parameter identification problems discussed by Dalla et al. in [5, 11], wherein the analysis depends heavily on the affinity of the maps in the IFS.

For the sake of simplicity, we have presented shape preserving aspects of the fractal perturbation with the assumption that the function $f \in \mathcal{C}(I)$ being perturbed possesses uniform shape property on the entire interval (see Theorems 3.1, 4.2, 5.1). However, $f \in \mathcal{C}(I)$ may not have a uniform shape on the entire interval I and we may require the fractal function f^α to preserve the shape of f. For instance, suppose that a function $f \in \mathcal{C}(I)$ switches back and forth between nonnegativity and nonpositivity (say a finite number of times). To obtain a fractal function $f^\alpha \in \mathcal{C}(I)$ that is copositive with f (i.e., $f(x)f^\alpha(x) \geq 0$ for all $x \in I$), we shall proceed as follows. Subdivide I into subintervals I_i, $i = 1, 2, \ldots, r$ such that in a typical subinterval I_s the function f is either nonnegative or nonpositive throughout. In each subinterval I_i, we choose elements of the IFS so as to meet the specifications in Theorem 3.1. Consequently, in each interval I_i we can produce a fractal function f^{α^i} which honour the nonnegativity/nonpositivity of f. Letting α to be the r-rowed matrix whose rows are the scale vector α^i, define fractal function f^α in a piecewise manner as $f^\alpha|_{I_i} = f^{\alpha^i}$, $i = 1, 2, \ldots, r$. The continuity of f^α follows from the fact that each f^{α^i} interpolates f at the end points of the interval I_i. The fractal function f^α is copositive with f on I. Similarly we can construct fractal function f^α which is comonotonic/coconvex with f.

In practice, there are many instances where we desire shape preserving approximants with suitable derivatives of these approximants receiving varying irregularity that can be quantified in terms of fractal dimension, and the introduction of shape preservation to the process of fractal perturbation accomplishes this. In addition to be of independent interest, this shape preserving fractal perturbation paves the way towards a fractal generalization of some of the fundamental results in traditional shape preserving approximation theory. For instance, the uniform approximation of a positive (nonnegative) function by means of a positive polynomial is extended to the fractal analogue if the modulus of the scale vector satisfies a given inequality. The fractal dimensions of these solutions provide an additional quantifier (or index) of the processes under consideration.

Consequently, this article can also be viewed as an attempt for the exposition of fractal functions to the field of shape preserving approximation and as a humble contribution to the claim "fractals are everywhere" [2].

2. Fractal Functions Revisited

In this section we shall reintroduce the fractal interpolation problem which concerns us, recall the associated fractal operator and some of their basic properties.

2.1. Iterated function system

Definition 2.1. *Let $\mathbb{X}$ be a complete metric space. If $w_n : \mathbb{X} \to \mathbb{X}, n = 1, 2, \dots, M$ are continuous mappings, then $\mathcal{I} = \{\mathbb{X}; w_1, w_2, \dots, w_M\}$ is called an* ***iterated function system*** *(IFS). If each of the maps w_n is a contraction, then the IFS $\mathcal{I}$ is termed a* ***contractive*** *or* ***hyperbolic*** *IFS.*

Given a metric $d(.,.)$ on $\mathbb{X}$, there is a corresponding metric $d_{\mathbb{H}}$, called the Hausdorff metric, on the collection $\mathbb{H}(\mathbb{X})$ of all nonempty compact subsets of $\mathbb{X}$:

$$d_{\mathbb{H}}(B, C) := \max\big\{\max_{b\in B} \min_{c\in C} d(b, c), \max_{c\in C} \min_{b\in B} d(c, b)\big\}.$$

The IFS $\mathcal{I}$ induces a set-valued Hutchinson map $W : \mathbb{H}(\mathbb{X}) \to \mathbb{H}(\mathbb{X})$ defined by $W(B) = \bigcup_{n=1}^{M} w_n(B)$. For $B \subseteq \mathbb{X}$, $k \in \mathbb{N}$, let $W^k(B)$ denotes the k-fold composition of W with itself and define $W^0 = I$.

Definition 2.2. *A set $A \in \mathbb{H}(\mathbb{X})$ is said to be an* ***attractor*** *of the IFS $\mathcal{I}$ if $\lim_{k\to\infty} W^k(B) = A$ for all compact set $B \subseteq \mathbb{X}$, where the limit is with respect to the Hausdorff metric. Set A is said to be* ***invariant*** *if $W(A) = A$.*

A basic result in the theory of IFS is the following:

Theorem 2.1. *(Barnsley [2]). If the IFS $\mathcal{I}$ is contractive, then $\mathcal{I}$ has a unique attractor A which is invariant under W.*

2.2. *Fractal interpolation function*

This subsection provides a certain type of IFS whose attractor is the graph of a continuous function.

Let $x_1 < x_2 < \cdots < x_N$ be real numbers, and $I = [x_1, x_N]$ be the closed interval that contains them. Let a set of data points $\{(x_n, y_n) \in I \times \mathbb{R}\}$ be given. Set $I_n = [x_n, x_{n+1}]$, and let $L_n : I \to I_n$, $n \in J = \{1, 2, \ldots, N-1\}$ be contraction homeomorphisms such that:

$$L_n(x_1) = x_n, \ \ L_n(x_N) = x_{n+1}. \tag{2.1}$$

Let $-1 < \alpha_n < 1$, $n \in J$, $K = I \times D$, where D is a large enough compact subset of $\mathbb{R}$. Suppose $N-1$ continuous mappings $F_n : K \to D$ be given satisfying:

$$|F_n(x, y) - F_n(x, y^*)| \leq |\alpha_n||y - y^*|; \ F_n(x_1, y_1) = y_n, \ F_n(x_N, y_N) = y_{n+1}. \tag{2.2}$$

For $n \in J$, define the functions $w_n(x, y) = \big(L_n(x), F_n(x, y)\big)$. The IFS $\{K; w_n, n \in J\}$ is contractive with respect to a metric inducing the same topology as the Euclidean metric on $\mathbb{R}^2$. Let $\mathcal{G} := \{h \in \mathcal{C}(I) : h(x_1) = y_1, h(x_N) = y_N\}$ be endowed with the uniform metric defined as $d_\infty(h_1, h_2) = \max\{|h_1(x) - h_2(x)| : x \in I\}$.

Theorem 2.2. *(Barnsley [1]) We have the following:*

(i) The IFS $\{K; w_n, n \in J\}$ has a unique attractor G such that G is the graph of a continuous function $g : I \to \mathbb{R}$.

(ii) The function g interpolates the data set $\{(x_n, y_n) \in I \times \mathbb{R} : n = 1, 2, \ldots, N\}$, i.e., $g(x_n) = y_n$ for $n = 1, 2, \ldots, N$.

(iii) If $T : \mathcal{G} \to \mathcal{G}$ is defined by $Th(x) = F_n\big(L_n^{-1}(x), h \circ L_n^{-1}(x)\big)$, $x \in I_n$, $n \in J$, and for all $h \in \mathcal{G}$, then T has a unique fixed point g, and $g = \lim_{n\to\infty} T^n(h)$ for any $h \in \mathcal{G}$. Further, the fixed point g is the function satisfying conditions in (i)-(ii).

Definition 2.3. *The function g given in the previous theorem whose graph is the attractor of an IFS is called a* ***fractal interpolation function*** *(FIF) or a* ***self-referential function***. *Further, g satisfies the functional equation:*

$$g\big(L_n(x)\big) = F_n\big(x, g(x)\big), \ \ x \in I, \ \ n \in J.$$

FIFs generated by the following special types of IFS are extensively studied in the literature:

$$L_n(x) = a_n x + b_n, \quad F_n(x, y) = \alpha_n y + q_n(x), \ n \in J. \tag{2.3}$$

Following the prescription in (2.1), we obtain: $a_n = \dfrac{x_{n+1} - x_n}{x_N - x_1}$, $b_n = \dfrac{x_N x_n - x_1 x_{n+1}}{x_N - x_1}$. The multiplier α_n such that $-1 < \alpha_n < 1$ is called a scale factor of the transformation w_n and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_{N-1})$ is called the scale vector of the IFS.

2.3. α-fractal function

Let $f \in \mathcal{C}(I)$. We consider here, the special case

$$q_n(x) = f \circ L_n(x) - \alpha_n b(x), \ x \in I, \tag{2.4}$$

where b is a continuous function satisfying $b(x_1) = f(x_1)$, $b(x_N) = f(x_N)$, and $b \neq f$. This case is proposed by Barnsley [1] and Navascués [12] as generalization of any continuous function.

Definition 2.4. *The continuous function $f^{\alpha}_{\Delta,b} = f^{\alpha}$ whose graph is the attractor of the IFS defined by (2.3)-(2.4) is referred to as α-**fractal function** associated with f with respect to b and the partition Δ.*

The operator T is defined in this case as:

$$Th(x) = F_n\big(L_n^{-1}(x), h \circ L_n^{-1}(x)\big) = f(x) + \alpha_n (h - b) \circ L_n^{-1}(x) \ \ \forall \ x \in I_n, \ \ n \in J,$$

and thus f^{α} satisfies the functional equation:

$$f^{\alpha}(x) = f(x) + \alpha_n (f^{\alpha} - b) \circ L_n^{-1}(x) \quad \forall \, x \in I_n, \ n \in J. \tag{2.5}$$

Since f^{α} may have nonintegral Hausdorff and Minkowski dimensions, the act of obtaining f^{α} from f may be referred to as *fractal perturbation* of f. Further, as $\alpha \in (-1, 1)^{N-1}$ is arbitrary, the above process associates an entire family of continuous functions $F = \{f^{\alpha} : \alpha \in (-1, 1)^{N-1}\}$ with each fixed function $f \in \mathcal{C}(I)$.
Note that for any partition $\Delta : x_1 < x_2 < \cdots < x_N$ of $I = [x_1, x_N]$, f^{α} interpolates f at x_n for $n = 1, 2, \ldots, N$. Let $|\alpha|_{\infty} := \max\{|\alpha_n| : n \in J\}$, and for $g \in \mathcal{C}(I)$, let $\|g\|_{\infty} := \max\{|g(x)| : x \in I\}$. From (2.5), the following bound for the uniform error committed in the process of fractal perturbation can be easily deduced:

$$\|f^{\alpha} - f\|_{\infty} \leq \frac{|\alpha|_{\infty}}{1 - |\alpha|_{\infty}} \|f - b\|_{\infty}. \tag{2.6}$$

Let $\epsilon > 0$. Given $f \in \mathcal{C}(I)$, from (2.6) it follows that there exists $\alpha \in (-1,1)^{N-1}$ satisfying $\|f - f^\alpha\|_\infty < \epsilon$. Consequently, for suitable values of the scale factors, the map f^α simultaneously interpolates and approximates f. Further, depending on the values of the scale vector α, the number of data points, and the choice of the base function b, f^α is smooth or non-smooth, providing more flexibility and diversity in the process of approximation.

Assume that the continuous function b occurring in (2.4) depends linearly on f, that is to say $b_{\lambda g_1 + g_2} = \lambda b_{g_1} + b_{g_2}$, then the map $\mathcal{F}^\alpha : \mathcal{C}(I) \to \mathcal{C}(I)$, $f \mapsto f^\alpha$ defines a linear operator. The following choices of b and properties of the corresponding bounded linear fractal operator $\mathcal{F}^\alpha$ are studied in the literature.

(i) b is a line passing through $(x_1, f(x_1))$ and $(x_N, f(x_N))$ (see [16]).

(ii) $b = f \circ c$, where $c : I \to \mathbb{R}$ is a continuous function satisfying $c(x_1) = x_1$ and $c(x_N) = x_N$ (see reference [12]).

(iii) $b = vf$, where $v : I \to \mathbb{R}$ is a continuous function satisfying $v(x_1) = v(x_N) = 1$ (see [13]).

(iv) slightly more generally, $b = Lf$ where $L : \mathcal{C}(I) \to \mathcal{C}(I)$ is a linear and bounded operator with respect to the uniform norm or $\mathcal{L}^p$-norm on $\mathcal{C}(I)$ ([13, 14, 15]).

It is worth mentioning here that the assumption of linear dependence of b on the map f allows us to preserve the algebraic properties, like for instance, the constitution of Schauder bases for spaces of functions, and helps to contribute to the fields of functional analysis and basic operator theory.

3. Fractal Perturbation Preserving Positivity

Let $f \in \mathcal{C}(I)$. In the previous section, it was noted that for a fixed scale vector $\alpha \in (-1,1)^{N-1}$, the function $b \in \mathcal{C}(I)$, $b \neq f$ with $b(x_1) = f(x_1)$, $b(x_N) = f(x_N)$, and arbitrary partition Δ of I, the fractal perturbation f^α obtained via the IFS defined by

$$L_n(x) = a_n x + b_n, \quad F_n(x,y) = \alpha_n y + f \circ L_n(x) - \alpha_n b(x), \ n \in J, \tag{3.7}$$

preserves continuity of f and interpolates f. Further, with a suitable choice of α, f^α approximates f sufficiently well.

In this section, we begin to associate the transformation $f \mapsto f^\alpha$ with shape preservation requirement and ask the following question. Given $f \in \mathcal{C}(I)$, $f(x) \geq 0$ for all $x \in I$, how do we choose the IFS (3.7) so that the corresponding fractal function fulfills $f^\alpha(x) \geq 0$ for all $x \in I$. Recall that the elements of the IFS that we may have to choose appropriately are: (i) the scale vector α, (ii) the continuous function b interpolating f at the end points of the interval, and perhaps (iii) the partition Δ.

For the duration of this section, let us introduce the following notation:

$$m_* = \min_{x \in I} b(x), \quad M^* = \max_{x \in I} b(x); \quad m_n = \min_{x \in I} f\big(L_n(x)\big), \quad M_n = \max_{x \in I} f\big(L_n(x)\big) \text{ for } n \in J.$$

Note that the existence of these parameters are ensured by the continuity of the corresponding functions and the compactness of the domain. The next theorem establishes the conditions on the elements of the IFS (3.7) so that the corresponding fractal function f^α satisfies $0 \le f^\alpha(x) \le M$, where M is a large enough positive constant. Though the desired positivity (nonnegativity) of f^α does not demand an upper bound for it, the proof of the following theorem should convince the reader of the role of M in admitting negative values for the scale factors whilst maintaining the nonnegativity of f^α. If we do not like being forced to use an upper bound for f^α in exchange for the slightly increased generality of negative scale factors, we can certainly work with $f^\alpha(x) \ge 0$ instead.

Theorem 3.1. *Let $f \in \mathcal{C}(I)$ be such that $f(x) \ge 0$ for all $x \in I$. Let $\Delta = \{x_1, x_2, \dots, x_N\}$ be a partition of I such that $x_1 < x_2 < \cdots < x_N$ and $b \in \mathcal{C}(I)$ satisfy the conditions $b(x_1) = f(x_1)$ and $b(x_N) = f(x_N)$. Then, the range of the fractal function f^α corresponding to the IFS (3.7) is contained in the interval $[0, M]$, provided the scale factors obey $|\alpha_n| < 1$ and*

$$\max\Big\{ -\frac{m_n}{M - m_*}, -\frac{M - M_n}{M^*} \Big\} \le \alpha_n \le \min\Big\{ \frac{m_n}{M^*}, \frac{M - M_n}{M - m_*} \Big\} \;\forall\, n \in J,$$

where the constant M is a positive real number strictly greater than m_* and $\|f\|_\infty$. In case of zero denominator ($M^* = 0$), we avoid the corresponding term in the above inequality. In particular, these conditions ensure positivity of f^α.

Proof. The condition that the function b agrees with the given positive function f at the extremes of the interval $I = [x_1, x_N]$ ensures nonnegativity of the constant M^*. Since the fractal function $f^\alpha \in \mathcal{C}(I)$ is constructed by iterating the functional equation:

$$f^\alpha\big(L_n(x)\big) = F_n\big(x, f^\alpha(x)\big) = \alpha_n f^\alpha(x) + q_n(x) = f\big(L_n(x)\big) + \alpha_n (f^\alpha - b)(x), \; x \in I,$$

for proving $0 \le f^\alpha(x) \le M$ for all $x \in I$, one merely needs to verify that $0 \le F_n(x, y) \le M$ for all $n \in J$ whenever $(x, y) \in I \times [0, M]$.

Assume $(x, y) \in I \times [0, M]$. Consider the scale factors α_n such that $|\alpha_n| < 1$. Firstly, consider $0 \le \alpha_n < 1$.

With this assumption, $0 \leq y \leq M$ implies $q_n(x) \leq \alpha_n y + q_n(x) \leq \alpha_n M + q_n(x)$, where $q_n(x) = f \circ L_n(x) - \alpha_n b(x)$. Therefore, $0 \leq F_n(x, y) = \alpha_n y + q_n(x) \leq M$ holds if:

$$f \circ L_n(x) - \alpha_n b(x) \geq 0, \quad f \circ L_n(x) - \alpha_n b(x) \leq M(1 - \alpha_n) \ \forall\, x \in I. \tag{3.8}$$

Keeping $f \circ L_n(x) \geq m_n$ and $b(x) \leq M^*$ for all $x \in I$ in mind, it can be readily verified that the selection $\alpha_n \leq \frac{m_n}{M^*}$ fulfills the condition $f \circ L_n(x) - \alpha_n b(x) \geq 0$. Note also that if M^* is zero, then no additional conditions on the scale factors are needed to ensure $f \circ L_n(x) - \alpha_n b(x) \geq 0$. Similarly, from $f \circ L_n(x) \leq M_n$ and $b(x) \geq m_*$ for all $x \in I$, we can assert that $\alpha_n \leq \frac{M - M_n}{M - m_*}$ satisfies the second inequality in (3.8). Hence for (3.8) to be valid, we take the scale factors such that $\alpha_n \leq \min\left\{\frac{m_n}{M^*}, \frac{M - M_n}{M - m^*}\right\}$.

Next, let $-1 < \alpha_n \leq 0$. In this case, $0 \leq y \leq M$ implies $\alpha_n M + q_n(x) \leq \alpha_n y + q_n(x) \leq q_n(x)$. Consequently, for $0 \leq F_n(x, y) \leq M$ it suffices to verify:

$$\alpha_n M + f \circ L_n(x) - \alpha_n b(x) \geq 0, \quad f \circ L_n(x) - \alpha_n b(x) \leq M \ \forall\, x \in I. \tag{3.9}$$

Note that $m_n \leq f \circ L_n(x)$ and $m_* \leq b(x)$ for all $x \in I$. From routine calculations, we infer that $\alpha_n \geq \frac{-m_n}{M - m_*}$ satisfies $\alpha_n M + f \circ L_n(x) - \alpha_n b(x) \geq 0$. On similar lines, from $f \circ L_n(x) \leq M_n$ and $b(x) \leq M^*$, it follows that the condition $f \circ L_n(x) - \alpha_n b(x) \leq M$ is satisfied as long as $\alpha_n \geq -\frac{M - M_n}{M^*}$. Combining these we deduce that (3.9) is valid for the scale factors defined by $\alpha_n \geq \max\left\{-\frac{m_n}{M - m_*}, -\frac{M - M_n}{M^*}\right\}$, accomplishing the proof. Note that in case of zero denominator ($M^* = 0$), we avoid the corresponding term. $\square$

Some remarks that supplement and extend the previous theorem are in order.

Remark 3.1. *If it is enough to consider the nonnegative scale factors, then the first part of the proof of the foregoing theorem gives the following condition on the scale factors for the nonnegativity of* $f^\alpha : 0 \leq \alpha_n \leq \frac{m_n}{M^*}$ *for all* $n \in J$*, where of course* $|\alpha_n| < 1$ *is assumed.*

Remark 3.2. *If* $f \in \mathcal{C}(I)$ *is nonpositive (i.e.,* $f(x) \leq 0$ *for all* $x \in I$*), then we may construct fractal perturbation* f^α *satisfying* $f^\alpha(x) \leq 0$ *for all* $x \in I$ *by employing Theorem 3.1 for the positive function* $\tilde{f} = -f$ *and the associated function* $\tilde{b} = -b$*. The conditions for* $m \leq f^\alpha(x) \leq 0$ *for all* $x \in I$ *can be obtained as:* $|\alpha_n| < 1$ *and*

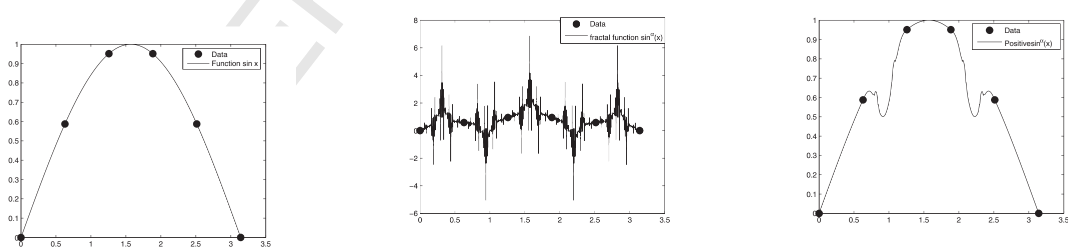
$$\max\left\{-\frac{m - m_n}{m_*}, -\frac{M_n}{m - M^*}\right\} \leq \alpha_n \leq \min\left\{\frac{m - m_n}{m - M^*}, \frac{M_n}{m_*}\right\} \forall\, n \in J.$$

Here the constant m *that bounds* f^α *from below is chosen to be a negative real strictly less than* M^* *and* $\min_{x \in I} f(x)$*, and the term with a zero denominator is avoided.*

Remark 3.3. *Let $f \in \mathcal{C}(I)$ stands for an approximant such as polynomial and spline used in traditional approximation theory. Then, f is, in fact, infinitely differentiable except possibly at a finite number of points. Consequently, f stands less satisfactory for (positive) approximation of an original (positive) function Φ having non-differentiability on a dense subset of the interval I. Assume that derivative f' does not agree with $f(x_N) - f(x_1)$ in a non-empty open subinterval of I. It is known [16] that for the fractal perturbation f^α of f, where α satisfies $|\alpha|_\infty > h_\Delta$, h_Δ being the step of the partition Δ, the set of points of non-differentiability is dense on I. Hence, by appropriate choice of a partition Δ, function b, and scale vector α, we may obtain a positive approximant f^α of Φ having irregularity in a dense set of points on I.*

Examples: Let us consider $I = [0, \pi]$ with a uniform partition of I having step size $h = \frac{\pi}{5}$. Let the original function be $f(x) = \sin x$ whose plot is given in Fig. 1(a). To obtain a fractal function corresponding to f, we take $b(x) = v(x)f(x)$ where $v(x) = \cos 2x$. With scale vector $\alpha = (0.5, -0.5, 0.5, -0.5, 0.5)$, the corresponding α-fractal function $\sin^\alpha x$ is plotted in Fig. 1(b). It can be seen from the figure that this fractal perturbation does not preserve positivity of the original function. Next, we calculate the scale factors as per the prescription in Theorem 3.1, where M is taken to be 1. The corresponding positive α-fractal function with the choice $\alpha = (0, -0.15, 0, -0.15, 0)$ is plotted in Fig. 1(c). Note that the fractal function $\sin^\alpha x$ agrees with original function $\sin x$ in those subintervals wherein the corresponding scale factors are equal to zero. In this way, perturbation can be confined in a small portion of the domain, if in this part the underlying signal displays some complex disturbance. It is also worth noting that the fractal trigonometric function satisfies the condition $0 \leq \sin^\alpha x \leq 1$ for $x \in I$.

Even though the main intent of this section is to produce positivity preserving fractal pertur-



(a) Original function $\sin x$

(b) Non-positive fractal function $\sin^\alpha(x)$

(c) Positive fractal function $\sin^\alpha(x)$

Figure 1: Function $\sin x$ and its fractal perturbations, where the data points are given by the circles and the relevant functions by the solid lines.

bation corresponding to a continuous function, in what follows, we provide two theorems to illustrate the application of this procedure in the field of shape preserving (fractal) approximation.

It is well-known that every continuous function on a real compact interval is uniformly approximable by algebraic polynomials. The fractal version of this Weierstrass theorem is established recently by Navascués [12]. This next theorem incorporates the positivity of the original function in the fractal polynomial approximant.

Theorem 3.2. *(Positive fractal polynomial approximation). Let f be a continuous function defined on I satisfying $f(x) \geq 0$ for all $x \in I$. To each $\epsilon > 0$, there corresponds a fractal polynomial p^α such that $p^\alpha(x) \geq 0$ for all $x \in I$ and $\|f - p^\alpha\|_\infty < \epsilon$.*

Proof. Let $\epsilon > 0$ and $f \in \mathcal{C}(I)$ be such that $f(x) \geq 0$. By classical Weierstrass theorem there exists a polynomial q such that $\|f - q\|_\infty < \frac{\epsilon}{4}$. For $x \in I$, define $p(x) = q(x) + \frac{\epsilon}{4}$. Then,

$$p(x) = q(x) - f(x) + f(x) + \frac{\epsilon}{4} \geq -\|f - q\|_\infty + f(x) + \frac{\epsilon}{4} > f(x) \geq 0.$$

Also, $\|f - p\|_\infty \leq \|f - q\|_\infty + \|q - p\|_\infty \leq \frac{\epsilon}{2}$. Therefore it follows that there exists an algebraic polynomial p satisfying $p(x) \geq 0$ and $\|f - p\|_\infty \leq \frac{\epsilon}{2}$. Let p^α be a positive fractal perturbation of p, where the scale vector is so chosen that $|\alpha|_\infty < \frac{\epsilon}{\epsilon + 2\|p-b\|_\infty}$. We have

$$\begin{aligned} \|f - p^\alpha\|_\infty &\leq \|f - p\|_\infty + \|p - p^\alpha\|_\infty, \\ &\leq \|f - p\|_\infty + \frac{|\alpha|_\infty}{1 - |\alpha|_\infty}\|p - b\|_\infty, \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

The first of the above steps is justified by the triangle inequality, second is consequent upon (2.6), and last one is a matter of direct verification from the prescribed condition on α. □

Remark 3.4. *The above theorem can also be approached as follows. If $f \in \mathcal{C}(I)$, $f(x) \geq 0$, then the Bernstein polynomial of f, $B_n(f)$, is positive for any n. Choose $B_n(f)$ sufficiently close to f. Now find positivity preserving fractal perturbation $B_n^\alpha(f)$ of this $B_n(f)$ so that the uniform error in perturbation is small enough. Then, $B_n^\alpha(f)$ will provide the desired positive fractal polynomial approximant of f.*

Here we consider $I = [0, 1]$. Let $\Lambda = \{\lambda_i\}_{i=0}^\infty$ with $0 = \lambda_0 < \lambda_1 < \dots$ be a sequence of distinct nonnegative real numbers. The collection $\{x^{\lambda_0}, x^{\lambda_1}, x^{\lambda_2}, \dots, x^{\lambda_m}\}$ is called a (finite) Müntz system. The linear space $M_m(\Lambda) = \text{span}\{x^{\lambda_0}, x^{\lambda_1}, x^{\lambda_2}, \dots, x^{\lambda_m}\}$ is called a Müntz space. That is, Müntz space is the collection of Müntz polynomials $p(x) = \sum_{i=0}^{m} c_i x^{\lambda_i}$, $c_i \in \mathbb{R}$. For a scale vector $\alpha = (-1,1)^{N-1}$, the set $M_m^\alpha(\Lambda) = \text{span}\{(x^{\lambda_0})^\alpha, (x^{\lambda_1})^\alpha, (x^{\lambda_2})^\alpha, \dots, (x^{\lambda_m})^\alpha\}$ is the set of fractal Müntz polynomials. This concept of fractal Müntz polynomials and their properties are introduced by Navascués and Chand [13]. Our next theorem deals with the approximation of a continuous function by a copositive fractal Müntz polynomial.

Theorem 3.3. *(Copositive fractal Müntz polynomial approximation). Let $[0,1]$ be partitioned into k subintervals by the points $\Delta : 0 = x_0 < x_1 < \cdots < x_k = 1$. Suppose that f is a continuous function that is alternately nonnegative and nonpositive on the intervals $(0, x_1)$, $(x_1, x_2), \dots, (x_{k-1}, 1)$. Let $\Lambda = \{\lambda_i\}_{i=0}^\infty$ be a sequence of nonnegative real numbers with the following properties: (i) $0, 1, \dots, k-1 \in \Lambda$, (ii) $\lim_{i\to\infty} \lambda_i = \infty$, (iii)$\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty$. Then, for given $\epsilon > 0$, there exists a corresponding piecewise defined fractal Müntz polynomial p^α such that p^α is copositive with f and $\|f - p^\alpha\|_\infty < \epsilon$.*

Proof. With the hypotheses of the theorem, there exists a (classical) Müntz polynomial $p(x) = \sum_{i=0}^{N} a_i x^{\lambda_i}$ that is copositive with f enjoying $\|f - p\|_\infty < \frac{\epsilon}{2}$ (see [19]).
Consider the function p on $[0,1]$ which is alternately nonnegative and nonpositive on the intervals $(0, x_1), (x_1, x_2), \dots, (x_{k-1}, 1)$. Let $I_i = [x_{i-1}, x_i]$, $i \in J^* = \{1, 2, \dots, k\}$. In each I_i, choose a partition Δ_i, continuous function b_i, and scale vector α^i so as to meet the conditions prescribed in Theorem 3.1. In addition, the scale factors are so chosen that $|\alpha^i|_\infty < \frac{\epsilon}{2(\|p\|_\infty + |B|) + \epsilon}$. Here $|B| = \max_{i\in J^*} \|b_i\|_{I_i}$, where $\|.\|_{I_i}$ denotes the uniform norm on I_i. Let p^α be defined on I in a piecewise manner by $p^\alpha|_{I_i} = p^{\alpha^i}$, where α is a matrix whose rows are the scale vectors α^i. Then, p^α is copositive with p (consequently, with f). Using the definition of uniform norm, inequality (2.6), and the condition on the scale vector α^i, we obtain:

$$\begin{aligned}
\|p^\alpha - p\|_\infty &= \max\{|p^\alpha(x) - p(x)| \ : \ x \in I\} = \max_{i\in J} \max\{|p^{\alpha^i}(x) - p(x)| \ : \ x \in I_i\}, \\
&\le \max_{i\in J} \frac{|\alpha^i|_\infty}{1 - |\alpha^i|_\infty} \|p - b_i\|_{I_i} \le \max_{i\in J} \frac{|\alpha^i|_\infty}{1 - |\alpha^i|_\infty} \big(\|p\|_\infty + |B|\big), \\
&\le \big(\|p\|_\infty + |B|\big) \frac{\epsilon}{2\big(\|p\|_\infty + |B|\big)} = \frac{\epsilon}{2}.
\end{aligned}$$

The proof can be completed by invoking the triangle inequality $\|f - p^\alpha\|_\infty \le \|f - p\|_\infty + \|p^\alpha - p\|_\infty$. $\square$

4. Fractal Perturbation Preserving $\mathcal{C}^1$-Continuity and Monotonicity

Let $f \in \mathcal{C}^1(I)$ be monotone, say nondecreasing. In this section, we identify the elements of the IFS (3.7) in order that the corresponding fractal function f^α shares the same regularity and monotonicity as that of f.

We begin our analysis by recalling the following theorem which proves the existence of differentiable FIFs and gives the conditions for their existence.

Theorem 4.1. *(Barnsley and Harrington [4]). Let $\{(x_n, y_n) : n = 1, 2, \dots, N\}$ be a given data set with $x_1 < x_2 < \cdots < x_N$. Let $L_n(x) = a_n x + b_n$, $n \in J$, satisfy (2.1) and $F_n(x, y) = \alpha_n y + q_n(x)$, $n \in J$, satisfy (2.2). Suppose that for some integer $p \geq 0$, $|\alpha_n| < a_n^p$ and $q_n \in \mathcal{C}^p(I)$, $n \in J$. Let*

$$F_{n,k}(x,y) = \tfrac{\alpha_n y + q_n^{(k)}(x)}{a_n^k}, \quad y_{1,k} = \tfrac{q_1^{(k)}(x_1)}{a_1^k - \alpha_1}, \; y_{N,k} = \tfrac{q_{N-1}^{(k)}(x_N)}{a_{N-1}^k - \alpha_{N-1}}, \quad k = 1, 2, \dots, p.$$

If $F_{n-1,k}(x_N, y_{N,k}) = F_{n,k}(x_1, y_{1,k})$ for $n = 2, 3, \dots, N-1$ and $k = 1, 2, \dots, p$, then the IFS $\big\{I \times \mathbb{R}; \big(L_n(x), F_n(x,y)\big), n \in J\big\}$ determines a FIF $f \in \mathcal{C}^p[x_1, x_N]$, and $f^{(k)}$ is the FIF determined by $\big\{I \times \mathbb{R}; \big(L_n(x), F_{n,k}(x,y)\big), n \in J\big\}$ for $k = 1, 2, \dots, p$.

Assume $|\alpha_n| < a_n^p$, $n \in J$. Then, to obtain a fractal perturbation $f^\alpha \in \mathcal{C}^p(I)$ corresponding to a given $f \in \mathcal{C}^p(I)$, it is enough to find the conditions on function b so that the IFS defined by (3.7) fulfills the conditions of the above theorem. In the reference [17], Navascués and Sebastián have undertaken this project, assuming a uniform partition. For the sake of completeness and record, we include a fairly self-contained and expanded rendition of this argument here. The advantage of the present analysis is that we can now allow non-uniform partition and unequal scale factors in different subintervals that cater to situations one encounters in practice. The conditions prescribed in the above theorem are

$$F_{n-1,k}(x_N, y_{N,k}) = F_{n,k}(x_1, y_{1,k}), \text{ for } n = 2, 3, \dots, N-1, \text{and } k = 1, 2, \dots, p, \tag{4.10}$$

where $F_{n,k}(x,y) = \frac{\alpha_n y + q_n^{(k)}(x)}{a_n^k}$. As $L_n(x) = a_n x + b_n$, we have $q_n^{(k)}(x) = a_n^k f^{(k)}\big(L_n(x)\big) - \alpha_n b^{(k)}(x)$ for all $k = 0, 1, \dots, p$. Therefore, (4.10) may be recast as follows:

$$\begin{aligned} &\frac{\frac{\alpha_{n-1}}{a_{N-1}^k - \alpha_{N-1}}[a_{N-1}^k f^{(k)}(x_N) - \alpha_{N-1} b^{(k)}(x_N)] + a_{n-1}^k f^{(k)}(x_n) - \alpha_{n-1} b^{(k)}(x_N)}{a_{n-1}^k} \\ &= \frac{\frac{\alpha_n}{a_1^k - \alpha_1}[a_1^k f^{(k)}(x_1) - \alpha_1 b^{(k)}(x_1)] + a_n^k f^{(k)}(x_n) - \alpha_n b^{(k)}(x_1)}{a_n^k}. \end{aligned} \tag{4.11}$$

If we consider the conditions (4.12), then both terms of (4.11) agree with $f^{(k)}(x_n)$, and the conditions (4.10) are satisfied.

$$b^{(k)}(x_1) = f^{(k)}(x_1),\ b^{(k)}(x_N) = f^{(k)}(x_N),\quad k = 0, 1, \ldots, p. \tag{4.12}$$

In particular, for $f \in \mathcal{C}^1(I)$ the following conditions on $b \in \mathcal{C}^1(I)$ ensure that the corresponding fractal function f^α constructed via the IFS (2.3)-(2.4) belongs to $\mathcal{C}^1(I)$.

$$b(x_1) = f(x_1),\ b(x_N) = f(x_N),\ b'(x_1) = f'(x_1),\ b'(x_N) = f'(x_N). \tag{4.13}$$

For instance, b may be taken as the cubic Hermite interpolant corresponding to f.
As indicated at the start of the section, we now turn to the task of preserving monotonicity of f in the fractal perturbation. The *modus operandi* is already inherent in the proof of Theorem 3.1. In order to best describe it, we need the following notation.

$$d_* = \min_{x\in I} b'(x),\quad D^* = \max_{x\in I} b'(x);\ d_n = \min_{x\in I} f'\big(L_n(x)\big),\ D_n = \max_{x\in I} f'\big(L_n(x)\big) \text{ for } n \in J.$$

Our next theorem may now be stated.

Theorem 4.2. *Let $f \in \mathcal{C}^1(I)$ be a monotone increasing function. Let $\Delta = \{x_1, x_2, \ldots, x_N\}$ be a partition of I satisfying $x_1 < x_2 < \cdots < x_N$ and $b \in \mathcal{C}^1(I)$ satisfy the conditions $b(x_1) = f(x_1)$, $b(x_N) = f(x_N)$, $b'(x_1) = f'(x_1)$, $b'(x_N) = f'(x_N)$. Then, for a large enough positive real M, the fractal function f^α corresponding to the IFS (3.7) is $\mathcal{C}^1$-smooth and satisfies $0 \le (f^\alpha)'(x) \le M$ (and hence, in particular, f^α is monotone), provided the scale factors obey $|\alpha_n| < a_n$ and*

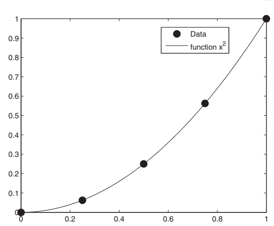
$$\max\Big\{-\frac{a_n d_n}{M - d_*}, -\frac{a_n(M - D_n)}{D^*}\Big\} \le \alpha_n \le \min\Big\{\frac{a_n d_n}{D^*}, \frac{a_n(M - D_n)}{M - d_*}\Big\}\ \forall\, n \in J.$$

Here M may be chosen as a positive real number strictly greater than d_* and $\|f'\|_\infty$. Also, in case of zero denominator ($D^* = 0$), we avoid the corresponding term in the above inequality.

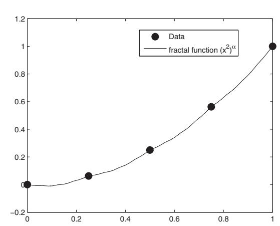
Proof. In light of the discussion we had until now in this section, it follows that the stated conditions on the scale factors and the function b ensure $\mathcal{C}^1$-continuity of the fractal function f^α. Since $(f^\alpha)'$ is a fractal function corresponding to the IFS $\{I \times \mathbb{R}; (L_n(x), F_{n,1}(x,y))\}$ (see Theorem 4.1), by the property of the attractor of the IFS, we have the following: $0 \le (f^\alpha)'(x) \le M$ for all $x \in I$ is true if $0 \le F_{n,1}(x,y) = \frac{\alpha_n y + q_n'(x)}{a_n} \le M$ for all $(x,y) \in I \times [0, M]$. We begin with the assumption $(x,y) \in I \times [0, M]$.
Firstly, let $0 \le \alpha_n < a_n$. With this nonnegativity assumption on the scale factor, we note

that: $0 \leq y \leq M$ implies $q'_n(x) \leq \alpha_n y + q'_n(x) \leq \alpha_n M + q'_n(x)$. Therefore, for the desired $0 \leq F_{n,1}(x,y) = \frac{\alpha_n y + q'_n(x)}{a_n} \leq M$ constraint, it is enough to verify the inequality constraints $q'_n(x) \geq 0$ and $\alpha_n M + q'_n(x) \leq a_n M$, where $q'_n(x) = a_n f'(L_n(x)) - \alpha_n b'(x)$.
The conditions $f'\big(L_n(x)\big) \geq d_n$ and $b'(x) \leq D^*$ being true for all $x \in I$, we deduce that $q'_n(x) \geq 0$ for $\alpha_n \leq \frac{a_n d_n}{D^*}$. It is worth to note here that if $D^* = 0$, then no additional constraint on the scale factor is required for $q'_n(x) \geq 0$. Similarly, since $f'\big(L_n(x)\big) \leq D_n$ and $b'(x) \geq d_*$ for all $x \in I$, we infer that $\alpha_n M + q'_n(x) \leq a_n M$, whenever $\alpha_n \leq \frac{a_n(M-D_n)}{M-d_*}$.
Now assume $-a_n < \alpha_n \leq 0$. In this case, $0 \leq y \leq M$ implies that $\alpha_n M + q'_n(x) \leq \alpha_n y + q'_n(x) \leq q'_n(x)$. Consequently, for $0 \leq F_{n,1}(x,y) = \frac{\alpha_n y + q'_n(x)}{a_n} \leq M$, it suffices to verify the two inequalities $\alpha_n M + q'_n(x) \geq 0$ and $q'_n(x) \leq a_n M$. Once again utilizing the definitions of d_n, d_*, D_n, and D^*, the analysis which run on lines similar to those followed in the first part provides the conditions $\alpha_n \geq \frac{-a_n d_n}{M-d_*}$ and $\alpha_n \geq \frac{a_n(D_n - M)}{D^*}$, which ensure the desired inequalities. Combination of the obtained conditions on the scale factors completes the proof. $\square$

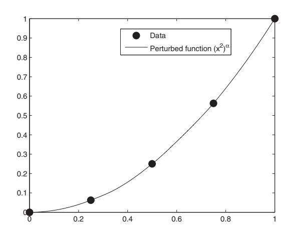
Examples: Fig. 2(a) represents the function $f(x) = x^2$ in the interval $I = [0,1]$. With respect to a partition with step $h = \frac{1}{4}$, function $b(x) = \frac{3x^2 - 2x^3}{1+2x(1-x)}$, and scale vector $\alpha = (0.2, 0.2, 0.2, 0.2)$, its corresponding fractal function $(x^2)^\alpha$ is constructed and displayed in Fig. 2(b). In spite of the fact that the original function is monotone, the corresponding α-fractal function does not satisfy monotonicity condition. We elect $M = 2$ and obtain the scale factors as per the specifications in Theorem 4.2. With $\alpha = (0, 0.06, -0.06, 0)$, the perturbation process yields the monotonicity preserving fractal polynomial $(x^2)^\alpha$ given in Fig. 2(c). Here, the original function and the fractal function almost coincide due to small magnitudes of the scale factors.



(a) Original function x^2



(b) Non-monotone fractal function $(x^2)^\alpha$



(c) Monotone fractal function $(x^2)^\alpha$

Figure 2: Function x^2 and its fractal perturbations, where the data points are given by the circles and the relevant functions by the solid lines.

As a consequence of this monotonicity preserving fractal perturbation process, we have the following fractal analogue of Wolibner's theorem [21].

Theorem 4.3. *(Monotone fractal polynomial interpolation) Let* (x_i, y_i), $i = 1, 2, \dots, n$ *be a set of data such that* $x_1 < x_2 < \cdots < x_n$ *and* $y_i \neq y_{i+1}$, $i = 1, \dots, n-1$. *Then, there exists a fractal function* p^α *defined in a piecewise manner by* $p^\alpha\big|_{I_i=[x_i,x_{i+1}]} = p^{\alpha^i}$, *where each* p^{α^i} *is a fractal polynomial with the following properties:* $p^\alpha(x_i) = y_i$, $i = 1, 2, \dots, n$, *sgn* $[(p^\alpha)'(x)] =$ *sgn* $[\Delta y_i]$, $x \in [x_i, x_{i+1}]$, *where* $\Delta y_i = y_{i+1} - y_i$.

Proof. With the stated assumptions, Wolibner's theorem ensures that there exists an algebraic polynomial p with the properties $p(x_i) = y_i$, $i = 1, 2, \dots, n$, sgn $[p'(x)] =$ sgn $[\Delta y_i]$, $x \in I_i = [x_i, x_{i+1}]$, where $\Delta y_i = y_{i+1} - y_i$. In each subinterval $I_i = [x_i, x_{i+1}]$, we select a partition Δ_i, scale vector α^i, and function b_i so that the fractal perturbation p^{α^i} has same monotonicity property as that of p. Let α be a matrix whose rows are the scale vectors α^i and define the fractal function p^α by $p^\alpha\big|_{I_i} = p^{\alpha^i}$. Then, p^α provides the fractal function sought for. □

Our next theorem points to the approximation of a continuous function by a comonotone (piecewise defined) fractal Müntz polynomial whose classical counterpart can be consulted in [19]. It can be argued in the same way as the corresponding copositive result (see Theorem 3.3), and hence the proof is omitted.

Theorem 4.4. *(Comonotone fractal Müntz polynomial approximation) Let* $[0, 1]$ *be partitioned into* k *subintervals by the points* $\Delta : 0 = x_0 < x_1 < \cdots < x_k = 1$. *Suppose that* f *is a continuous function that is alternately nondecreasing and nonincreasing on the intervals* $(0, x_1), (x_1, x_2), \dots, (x_{k-1}, 1)$. *Let* $\Lambda = \{\lambda_i\}_{i=0}^\infty$ *be a sequence of nonnegative real numbers with the following properties: (i)* $0, 1, \dots, k \in \Lambda$, *(ii)* $\lim_{i\to\infty} \lambda_i = \infty$, *(iii)* $\sum_{i=1}^\infty \frac{1}{\lambda_i} = \infty$. *Then, for given* $\epsilon > 0$, *there is a corresponding fractal function* p^α *defined in a piecewise manner* $p^\alpha\big|_{I_i=[x_{i-1},x_i]} = p^{\alpha^i}$, $i = 1, 2, \dots, k$, *where each* p^{α^i} *is a Müntz polynomial such that* p^α *is comonotone with* f *and* $\|f - p^\alpha\|_\infty < \epsilon$.

5. Fractal Perturbation Preserving $\mathcal{C}^2$-Continuity and Convexity

Given a function $f \in \mathcal{C}^2(I)$ which is convex, the goal of this section is to identify suitable IFS so that the perturbation produced in f retains the $\mathcal{C}^2$-continuity and convexity. That is, we wish $f^\alpha \in \mathcal{C}^2(I)$ and $(f^\alpha)''(x) \geq 0$ for all $x \in I$. We merely state the required conditions in the following theorem, for the proof is now a familiar terrain (the reader is urged to look back at the proof of Theorem 4.2 if needed).

Theorem 5.1. *Let $f \in \mathcal{C}^2(I)$ be a convex function. Let $\Delta : x_1 < x_2 < \cdots < x_N$ be a partition of I and $b \in \mathcal{C}^2(I)$ satisfy the conditions $b^{(k)}(x_1) = f^{(k)}(x_1)$, $b^{(k)}(x_N) = f^{(k)}(x_N)$, for $k = 0, 1, 2$. Then, for a large enough positive real M, the fractal function f^α corresponding to the IFS (3.7) is $\mathcal{C}^2$-continuous and satisfies $0 \le (f^\alpha)''(x) \le M$ (and hence in particular f^α is convex), provided the scale factors obey $|\alpha_n| < a_n^2$ and*

$$\max\Big\{-\frac{a_n^2 s_n}{M - s_*}, -\frac{a_n^2(M - S_n)}{S^*}\Big\} \le \alpha_n \le \min\Big\{\frac{a_n^2 s_n}{S^*}, \frac{a_n^2(M - S_n)}{M - s_*}\Big\}\ \forall\, n \in J,$$

where $s_n = \min_{x \in I} f''\big(L_n(x)\big)$, $S_n = \max_{x \in I} f''\big(L_n(x)\big)$, $s_* = \min_{x \in I} b''(x)$, $S^* = \max_{x \in I} b''(x)$. Here M may be chosen as a positive real number strictly greater than s_* and $\|f''\|_\infty$, and the terms with zero denominator ($S^* = 0$) are avoided.

As a consequence of the above theorem, we have the following fractal version of Pál's theorem [18]. Proof follows on lines similar to Theorem 3.2. It is worthwhile to mention here that Pál's theorem is probably one of the first results on the topic of shape preserving approximation.

Theorem 5.2. *(Convex fractal polynomial approximation) Let f be a convex function on an interval $[a, b]$. Then, for any $\epsilon > 0$, there exists a convex fractal polynomial p^α such that $\|f - p^\alpha\|_\infty < \epsilon$.*

Let us recall a slightly general concept of shape in the following. For a function $f : [0, 1] \to \mathbb{R}$, the j-th forward differences $\Delta_h^j f(x)$, $0 \le h \le \frac{1}{j}$, $x \in [0, 1 - jh]$ is defined as $\Delta_h^j f(x) := \sum_{k=0}^{j} (-1)^{j-k} \binom{j}{k} f(x + kh)$, for all $j = 0, 1, \ldots$. A function f is called j-convex on $[0, 1]$ if all the j-th forward differences are nonnegative. If $f^{(j)}$ exists, a simple application of the mean value theorem shows that the condition $f^{(j)}(x) \ge 0$ for all $x \in [0, 1]$ implies f is j-convex on $[0, 1]$. Given a function $f \in \mathcal{C}^p(I)$ satisfying $f^{(p)}(x) \ge 0$ for all $x \in I$, the methods given here can be extended to obtain fractal perturbation $f^\alpha \in \mathcal{C}^p(I)$ for which $f^{(p)}(x) \ge 0$ for all $x \in I$. As a consequence, we have the following result whose traditional counterpart follows from the properties of the Bernstein polynomials [19].

Theorem 5.3. *(j-convex fractal polynomial approximation) Let f be a continuous function on $[0, 1]$ with the property that the j-th forward difference $\Delta^j f \ge 0$, where j is some nonnegative integer (i.e., f is j-convex on $[0, 1]$). Then, for given $\epsilon > 0$, there exists a fractal polynomial p^α with $\big(p^\alpha\big)^{(j)}(x) \ge 0$ on $[0, 1]$ such that $\|f - p^\alpha\|_\infty < \epsilon$.*

6. Concluding Remarks

The method of generalizing a continuous function f defined on a real compact interval by means of fractal methods so as to obtain a function or rather a family of functions f^{α} is well-known in fractal literature. For suitable values of the scale vector α, the fractal function f^{α} simultaneously interpolates and approximates f. In this paper, we have developed methods to identify the elements of the IFS so that the corresponding FIF f^{α} preserves fundamental shape properties such as positivity, monotonicity, and convexity in addition to the order of continuity inherent in the original function f. The method of finding fractal perturbation retaining basic shape property of an original function turns out to be a cornerstone to obtain fractal versions of some fundamental results in shape preserving approximation. On the other hand, the practical advantage gained by this process is the following. Treating $f \in \mathcal{C}^p(I)$ itself as a shape preserving traditional approximant for an original function or a data set, its shape preserving fractal perturbation $f^{\alpha} \in \mathcal{C}^p(I)$ has the additional advantage that $(f^{\alpha})^{(p)}$ may be of varying irregularity (smooth to nowhere differentiable). In this case, larger the value of $|\alpha|_{\infty}$ with respect to the interpolation step, more pronounced is the irregularity in the fractal function $(f^{\alpha})^{(p)}$ (measured in terms of fractal dimension), and the fractal dimension of $(f^{\alpha})^{(p)}$ may be used as a quantitative parameter for the analysis of the underlying experimental process.
For the reasons of convenience, and because of the fact that most of the monotone functions appearing in the traditional approximation theory are differentiable, we have taken $f'(x) \geq 0$ as the definition of a monotone function. However, it is to be noted that the definition of monotonicity of a function does not presume its differentiability. Therefore, it is of interest to see if monotonicity preserving fractal perturbation can be constructed for a continuous monotone function, which is not known to be smooth. Similar question applies to convexity, and we leave these questions open. In this connection, we would like to point out that Vasilyev [20] has constructed monotone and convex fractal interpolation function $g \in \mathcal{C}(I)$ for a prescribed data set. He considers anyway a particular case, namely, the affine IFS.

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