



On discrete symplectic systems: Associated maximal and minimal linear relations and nonhomogeneous problems



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ABSTRACT

In this paper we characterize the definiteness of the discrete symplectic system, study a nonhomogeneous discrete symplectic system, and introduce the minimal and maximal linear relations associated with these systems. Fundamental properties of the corresponding deficiency indices, including a relationship between the number of square summable solutions and the dimension of the defect subspace, are also derived. Moreover, a sufficient condition for the existence of a densely defined operator associated with the symplectic system is provided.

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1. Introduction

The spectral theory for difference equations and systems, which include discrete analogs of Sturm–Liouville and Hamiltonian systems of differential equations, has a long history and a considerable literature which we will not attempt to delineate here other than to cite the following works, and the references therein, to give the reader a sense of the scope of the subject in time and content, cf. [2,7,10–13,34,35,39]. In connection with this, the development of a Weyl–Titchmarsh theory for discrete Hamiltonian and symplectic systems parallel to that which exists for Hamiltonian systems of differential equations is of relatively recent origin, cf. [3–5,9,10,14,26,29,33,36,37]. Our current paper contributes to this ongoing development.

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We investigate the nonhomogeneous problem as well as the basic development of linear relations associated with discrete symplectic systems written in the so-called time-reversed form given by

$$z_k(\lambda) = \mathbb{S}_k(\lambda)z_{k+1}(\lambda) \quad \text{with } \mathbb{S}_k(\lambda) := \mathcal{S}_k + \lambda\mathcal{V}_k \text{ and } k \in \mathbb{N}_0, \quad (\text{S}_\lambda)$$

where $\lambda \in \mathbb{C}$ is the spectral parameter, $\mathbb{N}_0 := [0, \infty) \cap \mathbb{Z}$, and \mathcal{S}_k and \mathcal{V}_k are complex $2n \times 2n$ matrices such that

$$\mathcal{S}_k^* \mathcal{J} \mathcal{S}_k = \mathcal{J}, \quad \mathcal{S}_k^* \mathcal{J} \mathcal{V}_k \text{ is Hermitian}, \quad \mathcal{V}_k^* \mathcal{J} \mathcal{V}_k = 0, \quad \text{and } \Psi_k := \mathcal{J} \mathcal{S}_k \mathcal{J} \mathcal{V}_k^* \mathcal{J} \geq 0, \quad (1.1)$$

where \mathcal{J} represents a $2n \times 2n$ skew-symmetric matrix given by $\mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.

We note (viz. Lemma 2.2) that the first, second, and third identities in (1.1), i.e., (1.1)(i)–(iii), can be combined into the single equality

$$\mathbb{S}_k^*(\bar{\lambda}) \mathcal{J} \mathbb{S}_k(\lambda) = \mathcal{J} \quad \text{for all } \lambda \in \mathbb{C} \text{ and } k \in \mathbb{N}_0, \quad (1.2)$$

where $\mathbb{S}_k^*(\bar{\lambda}) := (\mathbb{S}_k(\bar{\lambda}))^*$. Identity (1.2) justifies the terminology *symplectic system* for (S_λ) , though system (S_λ) corresponds to the well-known time-reversed discrete symplectic system introduced in [8, Remark 4] only when $\lambda \in \mathbb{R}$; particularly, the case when $\lambda = 0$. In addition, system (S_λ) can also be viewed as a perturbation of the original symplectic system $z_k = \mathcal{S}_k z_{k+1}$, i.e., of (S_λ) with $\lambda = 0$, but for which the fundamental properties of symplectic systems remain true with appropriate, natural, modifications.

In [9,14], the Weyl–Titchmarsh theory was first established for discrete symplectic systems given by

$$z_{k+1}(\lambda) = \mathbb{S}_k^{-1} z_k(\lambda), \quad k \in \mathbb{N}_0, \quad (1.3)$$

in which a special form for \mathcal{V}_k is assumed; the proper generalization is later derived in [37], see also [38]. The results given in [37] for system (1.3) remain valid for system (S_λ) with standard changes given for the definition of the semi-inner product, viz. (2.13) (cf. [37, Theorem 2.8 and Section 4]), and for the associated weight function, viz. (1.1)(iv) (cf. [37, Identity (1.1)(iv)]).

Consideration here of the time-reversed form given in system (S_λ) , rather than that given in system (1.3), is motivated, in part, by a desire to produce more natural calculations involving the semi-inner product and in particular a more natural form for a Green function associated with nonhomogeneous discrete symplectic systems, viz. Lemma 4.2. We can also associate with system (S_λ) a densely defined operator, because there is no shift in the associated semi-inner product (cf. Theorem 5.4 and [28]). Moreover, this approach will enable us in subsequent research to generalize these results and unify them with the continuous time case by means of the time scale theory.

Given the inherent semi-definiteness of the function Ψ defined in (1.1) (cf. (2.2)), it is natural to consider the construction of *linear relations* in association with (S_λ) , their extensions and their associated spectral theory. The theory of linear relations provides powerful tools for the study of *multivalued* linear operators in a Hilbert space, especially for non-densely defined linear operators. The study of linear relations in this context traces back to [1]; see also [15–17,20] and the references therein. For linear Hamiltonian differential systems given by

$$-\mathcal{J}z'(t) = [H(t) + \lambda W(t)]z(t), \quad (1.4)$$

where $H(t)$ and $W(t)$ are Hermitian and $W(t)$ is positive semi-definite, this approach was initiated in [27] and further developed, e.g., in [6,21,22,25].

In [30], linear relations are considered in association with the linear Hamiltonian difference system

$$\Delta \begin{pmatrix} x_k \\ u_k \end{pmatrix} = (H_k + \lambda W_k) \begin{pmatrix} x_{k+1} \\ u_k \end{pmatrix}, \quad H_k := \begin{pmatrix} A_k & B_k \\ C_k & -A_k^* \end{pmatrix}, \quad W_k := \begin{pmatrix} 0 & W_k^{[1]} \\ -W_k^{[2]} & 0 \end{pmatrix}, \quad (1.5)$$

where B_k and C_k are Hermitian matrices, $W_k^{[j]} \geq 0$, $j = 1, 2$, and the matrix $\tilde{A}_k := I - A_k$ is invertible. Here, we note that the invertibility assumption and the additional requirement $W_k^{[1]}(I - A_k)^{-1}W_k^{[2]} \equiv 0$ imply that system (1.5) can be written as a discrete system whose form is given by (S_λ) ; cf. [31, Formula (2.3)]. On the other hand, with the supplementary assumption concerning the invertibility of the $n \times n$ matrix in the left upper block of the matrix $S_k(\lambda)$, $k \in \mathbb{N}_0$, system (S_λ) can be written as the linear Hamiltonian difference system but with a *nonlinear* dependence on the spectral parameter, see [38]. Moreover, using the time scale theory, see e.g. [18], it can be seen that the discrete symplectic systems given by (S_λ) provide a proper discrete analogue of linear Hamiltonian differential systems. Finally, let us note that system (S_λ) includes any even order Sturm–Liouville difference equation; cf. [8, Remark 2], [40], and see also Example 3.4.

Hence, we shall introduce minimal and maximal linear relations associated with our discrete symplectic system and establish fundamental properties for them in analogy with [25, Section 2] for system (1.4) and [30, Section 5] for system (1.5). Moreover, the reader can observe an essential difference, one which appears natural in the context of the time scale theory, in the assumptions for systems (1.4) and (S_λ) concerning the invertibility of W and Ψ , respectively. While the matrix W can be invertible, the matrix Ψ_k is singular for every $k \in \mathbb{N}_0$, see (1.1).

The remainder of this paper is organized as follows. In Section 2, we present fundamental properties of system (S_λ) and recall some basic facts from the theory of linear relations. In Section 3 we discuss the definiteness condition for system (S_λ) , which plays a crucial role in the spectral theory, and derive some equivalent characterizations. A nonhomogeneous discrete symplectic system is studied in Section 4. Concluding with Section 5, the maximal and minimal linear relations associated with the discrete symplectic systems are introduced and their fundamental properties, such as a relationship between the deficiency indices of the minimal relation in a suitable Hilbert space and the number of square summable solutions of system (S_λ) , are established. In this final section, we also present a sufficient condition providing the existence of a densely defined operator associated with the discrete symplectic system.

2. Preliminaries

2.1. Notation

Throughout this paper, matrices will be considered over the field of complex numbers \mathbb{C} . For any $\lambda \in \mathbb{C}$ we denote its imaginary part by $\Im(\lambda)$. By $\mathbb{C}^{r \times s}$, $r, s \in \mathbb{N}$, we mean the space of $r \times s$ complex matrices and $\mathbb{C}^{r \times 1}$ will be denoted by \mathbb{C}^r for $r \in \mathbb{N}$. With $M \in \mathbb{C}^{r \times s}$, let M^\top denote the transpose, and let M^* denote the adjoint or conjugate transpose of the matrix M ; for parameter dependent matrices, $M^*(\lambda) := M(\lambda)^*$. Let $M \geq 0$ and $M \leq 0$ indicate that M is positive or negative semi-definite, respectively. Similarly, $M > 0$ (respectively, $M < 0$) denotes a positive definite (respectively, negative definite) matrix. By \mathcal{J} we denote the real $2n \times 2n$ skew-symmetric matrix given as

$$\mathcal{J} := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad (2.1)$$

where I_n is the $n \times n$ identity matrix.

If \mathcal{I} denotes an interval in \mathbb{R} , then the associated discrete interval in the set of integers, \mathbb{Z} , is denoted by $\mathcal{I}_{\mathbb{Z}} := \mathcal{I} \cap \mathbb{Z}$. In particular, $\mathbb{N} = [1, \infty)_{\mathbb{Z}}$, and $\mathbb{N}_0 = [0, \infty)_{\mathbb{Z}}$. However, in practice, $\mathcal{I}_{\mathbb{Z}}$ will be referred as

the discrete interval \mathcal{I} . Hence, for the discrete interval \mathcal{I} , by $\mathbb{C}(\mathcal{I})^{r \times s}$ we denote the space of sequences, defined on \mathcal{I} , of complex $r \times s$ matrices, where typically $r \in \{n, 2n\}$ and $1 \leq s \leq 2n$. If $\mathcal{S} \in \mathbb{C}(\mathcal{I})^{r \times s}$, then $\mathcal{S}(k) := \mathcal{S}_k$ for $k \in \mathcal{I}$; if $\mathcal{S}(\lambda) \in \mathbb{C}(\mathcal{I})^{r \times s}$, then $\mathcal{S}(\lambda, k) := \mathcal{S}_k(\lambda)$ for $k \in \mathcal{I}$. When $\mathcal{S} \in \mathbb{C}(\mathcal{I})^{m \times s}$, and $\mathcal{T} \in \mathbb{C}(\mathcal{I})^{s \times n}$, then $\mathcal{ST} \in \mathbb{C}(\mathcal{I})^{m \times n}$, where $(\mathcal{ST})_k := \mathcal{S}_k \mathcal{T}_k$, $k \in \mathcal{I}$. The set $\mathbb{C}_0(\mathcal{I})^{r \times s}$ represents the subspace of $\mathbb{C}(\mathcal{I})^{r \times s}$ consisting of sequences compactly supported in the discrete interval \mathcal{I} . The symbol Δ means the forward difference operator acting on $\mathbb{C}(\mathcal{I})^{r \times s}$, where $(\Delta z)_k := z_{k+1} - z_k$, for all $k \in \mathcal{I}$ and all $z \in \mathbb{C}(\mathcal{I})^{r \times s}$. We shall also use the customary equivalence given by $(\Delta z)_k := \Delta z_k$. Moreover, we let $z_k|_m^n := z_n - z_m$.

2.2. Time-reversed discrete symplectic systems

The basic relations given in the next lemma are easily shown and used throughout.

Lemma 2.1. *The following is true with \mathcal{J} defined as in (2.1), and $\mathcal{S}, \mathcal{V}, \Psi \in \mathbb{C}^{2n \times 2n}$:*

- (i) $\mathcal{S}^* \mathcal{J} \mathcal{S} = \mathcal{J}$ if and only if $\mathcal{S} \mathcal{J} \mathcal{S}^* = \mathcal{J}$;
- (ii) $\mathcal{S}^* \mathcal{J} \mathcal{S} = \mathcal{J}$ if and only if $\mathcal{S}^{-1} = -\mathcal{J} \mathcal{S}^* \mathcal{J}$;
- (iii) if $\mathcal{V} := \mathcal{J}^* \Psi \mathcal{S} = -\mathcal{J} \Psi \mathcal{S}$, where $\mathcal{S}^* \mathcal{J} \mathcal{S} = \mathcal{J}$, and

$$\Psi^* = \Psi, \quad \Psi \mathcal{J} \Psi = 0, \quad (2.2)$$

then

$$(\mathcal{V}^* \mathcal{J} \mathcal{S})^* = \mathcal{V}^* \mathcal{J} \mathcal{S}, \quad (\mathcal{V} \mathcal{J} \mathcal{S}^*)^* = \mathcal{V} \mathcal{J} \mathcal{S}^*, \quad \mathcal{V}^* \mathcal{J} \mathcal{V} = \mathcal{V} \mathcal{J} \mathcal{V}^* = 0, \quad (2.3)$$

and $\Psi = \mathcal{J} \mathcal{S} \mathcal{J} \mathcal{V}^* \mathcal{J}$. However, if $\mathcal{S}^* \mathcal{J} \mathcal{S} = \mathcal{J}$ and $\mathcal{V} \in \mathbb{C}^{2n \times 2n}$ satisfies (2.3), then $\Psi := \mathcal{J} \mathcal{S} \mathcal{J} \mathcal{V}^* \mathcal{J}$ satisfies (2.2), and $\mathcal{V} = \mathcal{J}^* \Psi \mathcal{S}$.

Lemma 2.1 establishes a correspondence between the matrix pairs $\{\mathcal{S}, \Psi\}$ and $\{\mathcal{S}, \mathcal{V}\}$ in which \mathcal{S} satisfies $\mathcal{S}^* \mathcal{J} \mathcal{S} = \mathcal{J}$, Ψ satisfies (2.2) and \mathcal{V} satisfies (2.3). The first part of the next result is also easily verified, while the second part follows as in the proof of [37, Lemma 2.2]. Let us note a typo in the latter reference, where the absolute value is missing.

Lemma 2.2. *Let $\mathbb{S}(\lambda) = \mathcal{S} + \lambda \mathcal{V}$, where $\mathcal{S}, \mathcal{V} \in \mathbb{C}^{2n \times 2n}$, $\lambda \in \mathbb{C}$. Then $\mathbb{S}^*(\bar{\lambda}) \mathcal{J} \mathbb{S}(\lambda) = \mathcal{J}$ for all $\lambda \in \mathbb{C}$ if and only if $\mathcal{S}^* \mathcal{J} \mathcal{S} = \mathcal{J}$, and \mathcal{V} satisfies (2.3). Moreover, $|\det \mathbb{S}(\lambda)| = 1$.*

Relevant to these two lemmas, we assume that the next conditions hold for the remainder of the paper.

Hypothesis 2.3. For $\mathcal{S}, \Psi \in \mathbb{C}(\mathbb{N}_0)^{2n \times 2n}$, and for each $k \in \mathbb{N}_0$, the following is assumed:

- (i) $\mathcal{S}_k^* \mathcal{J} \mathcal{S}_k = \mathcal{J}$;
- (ii) Ψ_k satisfies (2.2) and $\Psi_k \geq 0$;
- (iii) $\mathbb{S}_k(\lambda) := \mathcal{S}_k + \lambda \mathcal{V}_k$, where $\mathcal{V}_k := -\mathcal{J} \Psi_k \mathcal{S}_k$.

Given this hypothesis, note that \mathcal{V}_k satisfies (2.3) for all $k \in \mathbb{N}_0$, then by Lemma 2.1 we have

$$\Psi_k = \mathcal{J} \mathcal{S}_k \mathcal{J} \mathcal{V}_k^* \mathcal{J}, \quad k \in \mathbb{N}_0,$$

and by Lemma 2.2 also $\mathbb{S}_k^*(\bar{\lambda}) \mathcal{J} \mathbb{S}_k(\lambda) = \mathcal{J}$ for all $k \in \mathbb{N}_0$ and all $\lambda \in \mathbb{C}$.

The discrete symplectic system in time-reversed form is now given by

$$z_k(\lambda) = \mathbb{S}_k(\lambda)z_{k+1}(\lambda), \quad k \in \mathbb{N}_0, \tag{S_\lambda}$$

where $z(\lambda) \in \mathbb{C}(\mathbb{N}_0)^{2n \times m}$, and $1 \leq m \leq 2n$. In the future, (S_ν) will denote a system of the form given in (S_λ) with the parameter λ replaced by ν . Note that identity (1.2), the invertibility of $\mathbb{S}_k(\lambda)$ from Lemma 2.2, and $\mathcal{J}^{-1} = -\mathcal{J}$ imply, for all $k \in \mathbb{N}_0$ and all $\lambda \in \mathbb{C}$, that

$$\mathbb{S}_k^{-1}(\lambda) = -\mathcal{J}\mathbb{S}_k^*(\bar{\lambda})\mathcal{J}, \tag{2.4}$$

with the consequent existence of unique solutions on \mathbb{N}_0 , $z(\lambda) \in \mathbb{C}^{2n \times m}(\mathbb{N}_0)$, for system (S_λ) .

Lemma 2.4 (*Wronskian-type identity*). *Let $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$. For solutions $z(\lambda) \in \mathbb{C}(\mathbb{N}_0)^{2n \times m}$ and $z(\bar{\lambda}) \in \mathbb{C}(\mathbb{N}_0)^{2n \times m}$ of (S_λ) and $(S_{\bar{\lambda}})$, respectively, we have, for all $k \in \mathbb{N}_0$,*

$$z_k^*(\lambda)\mathcal{J}z_k(\bar{\lambda}) = z_0^*(\lambda)\mathcal{J}z_0(\bar{\lambda}).$$

Proof. This follows directly from Lemma 2.2 since

$$z_k^*(\lambda)\mathcal{J}z_k(\bar{\lambda}) = z_{k+1}^*(\lambda)\mathbb{S}_k^*(\lambda)\mathcal{J}\mathbb{S}_k(\bar{\lambda})z_{k+1}(\bar{\lambda}) = z_{k+1}^*(\lambda)\mathcal{J}z_{k+1}(\bar{\lambda}),$$

for all $k \in \mathbb{N}_0$, $\lambda \in \mathbb{C}$. \square

Throughout, we shall let $\Phi(\lambda) \in \mathbb{C}(\mathbb{N}_0)^{2n \times 2n}$ denote a fundamental system of solutions for (S_λ) . If this fundamental system is such that, for some $k_0 \in \mathbb{N}_0$,

$$\Phi_{k_0}^*(\lambda)\mathcal{J}\Phi_{k_0}(\bar{\lambda}) = \mathcal{J}, \tag{2.5}$$

then, as an immediate consequence of the preceding lemmas,

$$\Phi_k^*(\lambda)\mathcal{J}\Phi_k(\bar{\lambda}) = \mathcal{J}, \quad \Phi_k^{-1}(\lambda) = -\mathcal{J}\Phi_k^*(\bar{\lambda})\mathcal{J}, \quad \Phi_k(\lambda)\mathcal{J}\Phi_k^*(\bar{\lambda}) = \mathcal{J}, \tag{2.6}$$

for all $k \in \mathbb{N}_0$. When $\Phi_{k_0}(\lambda) = \Phi_{k_0}(\bar{\lambda}) = C \in \mathbb{C}^{2n \times 2n}$, note that (2.5) simply states that C is a symplectic matrix.

We associate with the homogeneous system (S_λ) the following nonhomogeneous system

$$z_k(\lambda) = \mathbb{S}_k(\lambda)z_{k+1}(\lambda) - \mathcal{J}\Psi_k f_k, \quad k \in \mathbb{N}_0, \tag{S_\lambda^f}$$

where $f \in \mathbb{C}(\mathbb{N}_0)^{2n \times m}$, $1 \leq m \leq 2n$. In analogy to previous notation, (S_ν^g) denotes the nonhomogeneous system of the form given in (S_λ^f) , but with λ replaced by ν and f replaced by g . When convenient, we shall suppress the dependence of z on λ when $\lambda = 0$. Note also that (S_λ) is equivalent to (S_λ^0) .

The next result presents an essential tool used throughout; cf. [37, Theorem 2.6] and [14, Identity (5.3)].

Theorem 2.5 (*Extended Lagrange identity*). *Let $\lambda, \nu \in \mathbb{C}$ and $1 \leq m \leq 2n$. If $z(\lambda) \in \mathbb{C}(\mathbb{N}_0)^{2n \times m}$ and $u(\nu) \in \mathbb{C}(\mathbb{N}_0)^{2n \times m}$ are solutions of systems (S_λ^f) and (S_ν^g) , respectively, with $f, g \in \mathbb{C}(\mathbb{N}_0)^{2n \times m}$. Then,*

$$\Delta[z_k^*(\lambda)\mathcal{J}u_k(\nu)] = (\bar{\lambda} - \nu)z_k^*(\lambda)\Psi_k u_k(\nu) + f_k^*\Psi_k u_k(\nu) - z_k^*(\lambda)\Psi_k g_k, \tag{2.7}$$

$$z_j^*(\lambda)\mathcal{J}u_j(\nu)|_0^{k+1} = (\bar{\lambda} - \nu)\sum_{j=0}^k z_j^*(\lambda)\Psi_j u_j(\nu) + \sum_{j=0}^k f_j^*\Psi_j u_j(\nu) - \sum_{j=0}^k z_j^*(\lambda)\Psi_j g_j. \tag{2.8}$$

Proof. By Lemma 2.1 (cf. (1.1)) and (2.4), we see that

$$\begin{aligned}\Delta[z_k^*(\lambda)\mathcal{J}u_k(\nu)] &= [\mathbb{S}_k^{-1}(\lambda)z_k(\lambda) - \mathbb{S}_k^{-1}(\lambda)\mathcal{J}\Psi_k f_k]^* \mathcal{J}[\mathbb{S}_k^{-1}(\nu)u_k(\nu) - \mathbb{S}_k^{-1}(\nu)\mathcal{J}\Psi_k g_k] - z_k^*(\lambda)\mathcal{J}u_k(\nu) \\ &= z_k^*(\lambda)[\mathbb{S}_k^{*-1}(\lambda)\mathcal{J}\mathbb{S}_k^{-1}(\nu) - \mathcal{J}]u_k(\nu) + f_k^* \Psi_k u_k(\nu) - z_k^*(\lambda)\Psi_k g_k \\ &= (\bar{\lambda} - \nu)z_k^*(\lambda)\Psi_k u_k(\nu) + f_k^* \Psi_k u_k(\nu) - z_k^*(\lambda)\Psi_k g_k,\end{aligned}$$

which yields (2.7); which in turn, by summation, yields (2.8). \square

Finally, the following equivalent expression of system (S_λ^f) will play a key role in the results established in Section 5.

Lemma 2.6. System (S_λ^f) can be written equivalently as

$$\mathcal{J}(z_k(\lambda) - \mathcal{S}_k z_{k+1}(\lambda)) = \lambda \Psi_k z_k(\lambda) + \Psi_k f_k, \quad k \in \mathbb{N}_0. \quad (2.9)$$

Proof. Since one can easily observe from the identities in (2.3) that it holds

$$\mathbb{S}_k(\lambda) = (I - \lambda \mathcal{J}\Psi_k)\mathcal{S}_k \quad \text{for all } k \in \mathbb{N}_0, \quad (2.10)$$

system (S_λ^f) can be also expressed as

$$z_k(\lambda) = (I - \lambda \mathcal{J}\Psi_k)\mathcal{S}_k z_{k+1}(\lambda) - \mathcal{J}\Psi_k f_k, \quad k \in \mathbb{N}_0. \quad (2.11)$$

Since $(I - \lambda \mathcal{J}\Psi_k)^{-1} = (I + \lambda \mathcal{J}\Psi_k)$, which follows immediately from (2.2), we obtain from (2.11) that

$$\mathcal{S}_k z_{k+1}(\lambda) = z_k(\lambda) + \lambda \mathcal{J}\Psi_k z_k(\lambda) + \mathcal{J}\Psi_k f_k, \quad k \in \mathbb{N}_0,$$

which after multiplication by the matrix \mathcal{J} from the left yields (2.9). \square

If we denote the left side of (2.9) by

$$\mathcal{L}(z(\lambda))_k := \mathcal{J}(z_k(\lambda) - \mathcal{S}_k z_{k+1}(\lambda)), \quad (2.12a)$$

then it follows from Lemma 2.6 that system (S_λ^f) is equivalent to

$$\mathcal{L}(z(\lambda))_k = \lambda \Psi_k z_k(\lambda) + \Psi_k f_k, \quad k \in \mathbb{N}_0. \quad (2.12b)$$

Hence \mathcal{L} represents a linear map on $\mathbb{C}(\mathbb{N}_0)^{2n \times m}$ with $1 \leq m \leq 2n$, and (2.12) will be summarized by writing $\mathcal{L}(z(\lambda)) = \lambda \Psi z(\lambda) + \Psi f$.

2.3. Sequence spaces

With respect to the assumption in Hypothesis 2.3 that $\Psi \in \mathbb{C}(\mathbb{N}_0)^{2n \times 2n}$ is a sequence with positive semi-definite terms, we define a semi-inner product for $\mathbb{C}(\mathbb{N}_0)^{2n}$ by

$$\langle z, w \rangle_{\Psi, \mathcal{I}} := \sum_{k \in \mathcal{I}} z_k^* \Psi_k w_k, \quad (2.13)$$

where $z, w \in \mathbb{C}(\mathbb{N}_0)^{2n}$, and $\mathcal{I} \subseteq \mathbb{N}_0$ is a discrete subinterval. The associated linear space of all square summable sequences is denoted by

$$\ell_{\Psi}^2(\mathcal{I}) := \{z \in \mathbb{C}(\mathbb{N}_0)^{2n} \mid \|z\|_{\Psi, \mathcal{I}} < \infty\},$$

where $\|\cdot\|_{\Psi, \mathcal{I}} := \sqrt{\langle \cdot, \cdot \rangle_{\Psi, \mathcal{I}}}$ denotes the associated semi-norm. When $\mathcal{I} = \mathbb{N}_0$ the corresponding semi-inner product and semi-norm will be denoted by $\langle \cdot, \cdot \rangle_{\Psi}$, and $\|\cdot\|_{\Psi}$, respectively. The quotient space obtained by factoring out the kernel of the semi-norm is denoted by

$$\tilde{\ell}_{\Psi}^2(\mathcal{I}) := \ell_{\Psi}^2(\mathcal{I}) / \{z \in \mathbb{C}(\mathbb{N}_0)^{2n} \mid \|z\|_{\Psi, \mathcal{I}} = 0\}.$$

Then, with respect to the equivalence classes $\tilde{z} \in \tilde{\ell}_{\Psi}^2(\mathcal{I})$, we see that $\tilde{\ell}_{\Psi}^2(\mathcal{I})$ is a Banach space with respect to the norm generated by the quotient space map, $\pi(z) = \tilde{z}$, and a Hilbert space with respect to the associated inner product where $\langle \tilde{z}, \tilde{w} \rangle_{\Psi, \mathcal{I}} := \langle z, w \rangle_{\Psi, \mathcal{I}}$ (cf. [31, Lemma 2.5]).

In light of this notation, we define

$$\ell_{\Psi, 0}^2(\mathbb{N}_0) := \{z \in \mathbb{C}_0(\mathbb{N}_0)^{2n} \mid z_0 = 0\},$$

and

$$\ell_{\Psi, 1}^2(\mathbb{N}_0) := \{z \in \mathbb{C}(\mathbb{N}_0)^{2n} \mid \exists N \in \mathbb{N}_0 \text{ such that } \Psi_k z_k = 0 \text{ for } k \geq N\}.$$

2.4. Linear relations

We now recall some basic facts from the theory of linear relations relevant to the case at hand; cf. [15–17,20]. Let \mathcal{H} denote a Hilbert space over the field of complex numbers, \mathbb{C} , with an inner product $\langle \cdot, \cdot \rangle$. A (closed) linear relation T in \mathcal{H} is a (closed) linear subspace of the product space $\mathcal{H}^2 := \mathcal{H} \times \mathcal{H}$, i.e., the Hilbert space of all ordered pairs $\{z, f\}$ such that $z, f \in \mathcal{H}$. The domain, range, kernel, and the multivalued part of T are respectively defined as

$$\begin{aligned} \text{dom } T &:= \{z \in \mathcal{H} \mid \exists f \in \mathcal{H}, \{z, f\} \in T\}, & \text{ran } T &:= \{f \in \mathcal{H} \mid \exists z \in \mathcal{H}, \{z, f\} \in T\}, \\ \text{ker } T &:= \{z \in \mathcal{H} \mid \{z, 0\} \in T\}, & \text{mul } T &:= \{f \in \mathcal{H} \mid \{0, f\} \in T\}. \end{aligned}$$

In general, we let $T(z) := \{f \in \mathcal{H} \mid \{z, f\} \in T\}$, and note that a linear relation, T , is the graph of a linear operator in \mathcal{H} when $T(0) = \{0\}$, i.e., when the subspace $\text{mul } T$ is trivial. The inverse of T , denoted as T^{-1} , is the linear relation

$$T^{-1} := \{\{f, z\} \mid \{z, f\} \in T\}$$

and note that $\text{dom } T^{-1} = \text{ran } T$, $\text{ran } T^{-1} = \text{dom } T$, $\text{ker } T^{-1} = \text{mul } T$, and $\text{mul } T^{-1} = \text{ker } T$. The adjoint T^* of the linear relation T is defined by

$$T^* := \{\{y, g\} \in \mathcal{H}^2 \mid \langle z, g \rangle = \langle f, y \rangle, \forall \{z, f\} \in T\}. \tag{2.14}$$

The definition of T^* reduces to the standard definition for the graph of the adjoint operator when T is a densely defined operator. Then T^* is a closed linear relation and we have

$$T^* = (\bar{T})^*, \quad T^{**} = \bar{T}, \quad \text{ker } T^* = (\text{ran } T)^{\perp} = (\text{ran } \bar{T})^{\perp}, \quad \text{and} \quad (\text{dom } T)^{\perp} = \text{mul } T^*, \tag{2.15}$$

where \bar{T} denotes the closure of T . A linear relation T is said to be symmetric (or Hermitian) if $T \subseteq T^*$, and it is said to be self-adjoint if $T^* = T$. It is easily seen that T is a symmetric linear relation if and only if $\langle z, g \rangle = \langle f, y \rangle$ for all $\{z, f\}, \{y, g\} \in T$.

With $\lambda \in \mathbb{C}$, and with T a linear relation, we define the linear relation, $T - \lambda I$, by

$$T - \lambda I := \{ \{z, f - \lambda z\} \mid \{z, f\} \in T \}, \quad (2.16)$$

and note that $(T - \lambda I)^* = T^* - \bar{\lambda}I$. Then,

$$M_\lambda(T) := \text{ran}(T - \bar{\lambda}I)^\perp \quad (2.17)$$

is said to be the *defect subspace* of T and λ , and its dimension, i.e.,

$$d_\lambda(T) := \dim(\text{ran}(T - \bar{\lambda}I)^\perp), \quad (2.18)$$

is said to be the *deficiency index* of T and λ . Since

$$\text{ran}(T - \bar{\lambda}I)^\perp = \ker(T^* - \lambda I) = \{z \in \mathcal{H} \mid \{z, \lambda z\} \in T^*\},$$

the deficiency indices of T and \bar{T} with the same λ are equal by (2.15); cf. [32, Lemma 2.4]. We let

$$d_+(T) := d_i(T), \quad d_-(T) := d_{-i}(T),$$

denote the so-called *positive* and *negative* deficiency indices of T , respectively. If T is a symmetric linear relation, the values of $d_\lambda(T)$ are constant in the open upper and lower half-planes of \mathbb{C} ; cf. [32, Theorem 2.13]. The linear relation T has self-adjoint extensions if and only if the positive and negative deficiency indices are equal; cf. [17, Corollary 6.4]. Finally, for a closed symmetric linear relation T , it was shown in [25, Lemma 2.25], when $\lambda \in \mathbb{R}$ and $\ker(T - \lambda I) = \{0\}$, that

$$d_\lambda(T) \leq d_\pm(T). \quad (2.19)$$

3. Definiteness

In this section, we characterize the definiteness condition associated with the semi-inner product (2.13) when the sequence, $\Psi \in \mathbb{C}(\mathbb{N}_0)^{2n \times 2n}$, possesses positive semi-definite elements. This condition plays a significant role in the spectral theory, particularly the Weyl–Titchmarsh theory, associated with the linear expression \mathcal{L} given in (2.12); cf. [37]. The condition is frequently called as the *Atkinson condition*; cf. [2, Inequality (3.7.10)]. Similar treatment in connection with the linear Hamiltonian differential and difference systems can be found in [6,25,30], respectively.

Definition 3.1. System (S_λ) is said to be *definite* on a nonempty discrete interval $\mathcal{I} \subseteq \mathbb{N}_0$ if, for each $\lambda \in \mathbb{C}$ and for every nontrivial solution $z(\lambda)$ of system (S_λ) , i.e., $\mathcal{L}(z(\lambda))_k = \lambda \Psi_k z_k(\lambda)$ for $k \in \mathcal{I}$,

$$\sum_{k \in \mathcal{I}} z_k^*(\lambda) \Psi_k z_k(\lambda) > 0. \quad (3.1)$$

Remark 3.2. Alternatively, the definiteness condition for (S_λ) can be stated in the following way: System (S_λ) is definite on the discrete interval $\mathcal{I} \subseteq \mathbb{N}_0$ when, for every $\lambda \in \mathbb{C}$, every solution $z(\lambda)$ of (S_λ) , for which

$$\sum_{k \in \mathcal{I}} z_k^*(\lambda) \Psi_k z_k(\lambda) = 0, \quad (3.2)$$

is trivial on \mathcal{I} , i.e. $z_k(\lambda) = 0$, $k \in \mathcal{I}$, and as a consequence of invertibility for $S_k(\lambda)$ (cf. (2.4)) trivial on \mathbb{N}_0 . Furthermore, from the assumption that $\Psi_k \geq 0$, $k \in \mathbb{N}_0$, it follows from

$$\sum_{k \in \mathcal{I}} z_k^*(\lambda) \Psi_k z_k(\lambda) \leq \sum_{k \in \widehat{\mathcal{I}}} z_k^*(\lambda) \Psi_k z_k(\lambda), \quad \mathcal{I} \subseteq \widehat{\mathcal{I}} \subseteq \mathbb{N}_0,$$

that the definiteness of (S_λ) on \mathcal{I} implies definiteness of (S_λ) on every discrete interval superset $\widehat{\mathcal{I}}$, and in particular on \mathbb{N}_0 . Hence, definiteness of system (S_λ) on some finite discrete subinterval \mathcal{I} implies, for every $\lambda \in \mathbb{C}$, that every nontrivial solution of (S_λ) has a nonzero semi-norm $\|\cdot\|_\Psi$. The converse of this last statement will be shown in Lemma 3.5.

In the next lemma, we see that for (S_λ) to be definite on a discrete interval \mathcal{I} , it suffices to verify (3.1) for nontrivial solutions of the system for only one $\lambda \in \mathbb{C}$.

Lemma 3.3. *System (S_λ) is definite on the discrete interval $\mathcal{I} \subseteq \mathbb{N}_0$ if and only if, for some $\lambda_0 \in \mathbb{C}$, each solution $z(\lambda_0)$ of (S_{λ_0}) , which satisfies*

$$\sum_{k \in \mathcal{I}} z_k^*(\lambda_0) \Psi_k z_k(\lambda_0) = 0, \tag{3.3}$$

is trivial on \mathcal{I} , i.e., $z_k(\lambda_0) = 0$ for $k \in \mathcal{I}$.

Proof. We begin by assuming, for $\lambda_0 \in \mathbb{C}$, that each solution $z(\lambda_0)$ of (S_{λ_0}) satisfying (3.3) is necessarily trivial on \mathcal{I} , i.e., $z_k(\lambda_0) = 0, k \in \mathcal{I}$. Let $\lambda \in \mathbb{C}$ be arbitrary and let $z(\lambda)$ be a solution of system (S_λ) such that (3.2) holds. Given Remark 3.2, it suffices to show that $z(\lambda)$ is trivial on \mathcal{I} . However, since $\Psi_k z_k(\lambda) = 0$ for $k \in \mathcal{I}$, we see, by (2.12), that $z(\lambda)$ is also a solution of system (S_{λ_0}) on \mathcal{I} . Then, by the assumed definiteness of (S_{λ_0}) on \mathcal{I} , (3.2) indeed implies that $z_k(\lambda) = 0$, for $k \in \mathcal{I}$. The converse is trivial. \square

Example 3.4. The particular discrete symplectic systems investigated in [37, viz. Examples 5.1, 5.2, and 5.3] are not definite, because they possess nontrivial solutions with semi-norm equal to zero. On the other hand, we can provide a simple example of system (S_λ) being definite on \mathbb{N}_0 ; cf. Theorem 3.11. Consider the following system:

$$\begin{pmatrix} x_k \\ u_k \end{pmatrix} = \begin{pmatrix} 1 & -1/p_{k+1} \\ q_k - \lambda w_k & 1 + (\lambda w_k - q_k)/p_{k+1} \end{pmatrix} \begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} \quad \text{with} \quad \Psi_k = \begin{pmatrix} w_k & 0 \\ 0 & 0 \end{pmatrix}, \quad k \in \mathbb{N}_0, \tag{3.4}$$

where $\{p_k\}_{k=0}^\infty, \{q_k\}_{k=0}^\infty$, and $\{w_k\}_{k=0}^\infty$ are real-valued sequences such that $w_k \geq 0, p_{k+1} > 0$ for $k \in \mathbb{N}_0$ and $w_k > 0$ for $k \in [a - 1, b]_{\mathbb{Z}}$, where $a, b \in [1, \infty)_{\mathbb{Z}}$ and $a \leq b$. System (3.4) corresponds to the second order Sturm–Liouville difference equation $\Delta(p_k \Delta y_{k-1}) + q_k y_k = \lambda w_k y_k$. With respect to Lemma 3.3 it suffices to focus on the definiteness of this system only for $\lambda = 0$. Denote by $z = (x, u)^T$ a solution of system (3.4) with $\lambda = 0$ such that $\Psi_k z_k = 0$ for $k \in \mathbb{N}_0$, i.e., it holds

$$\Delta x_k = \frac{u_{k+1}}{p_{k+1}}, \quad \Delta u_k = -q_k x_{k+1} + \frac{q_k}{p_{k+1}} u_{k+1}, \quad \text{and} \quad w_k x_k = 0, \quad k \in \mathbb{N}_0. \tag{3.5}$$

From the assumptions and the third condition in (3.5) we obtain $x_k \equiv 0$ for $k \in [a - 1, b]_{\mathbb{Z}}$ and then by the first equality in (3.5) also $u_k \equiv 0$ for $k \in [a, b]_{\mathbb{Z}}$. Hence $z \equiv 0$ on $[a, b]_{\mathbb{Z}}$, which implies $z \equiv 0$ on \mathbb{N}_0 , because the coefficient matrix in (3.4) is invertible. Therefore, system (3.4) is definite on $[a, b]_{\mathbb{Z}}$ and, in fact, on the whole interval \mathbb{N}_0 by Remark 3.2. \square

For the remainder of this section, we assume that $\Phi(\lambda)$ represents a fundamental system of the solutions for (S_λ) such that $\Phi_{k_0}(\lambda) = I_{2n}$, for some $k_0 \in \mathbb{N}_0$. Note, as a consequence, that the terms of $\Phi(\lambda)$ satisfy (2.6) for all $k \in \mathbb{N}_0$.

The next result provides a characterization for definiteness of (S_λ) which is analogous to that for (1.4), as seen in [6, Proposition 2.11].

Lemma 3.5. *System (S_λ) is definite on \mathbb{N}_0 if and only if there exists a finite discrete interval \mathcal{I} over which the system is definite.*

Proof. From Remark 3.2 we see that definiteness of (S_λ) on a finite discrete interval \mathcal{I} implies definiteness on \mathbb{N}_0 . Thus, it remains to show the converse.

Assume that (S_λ) is definite on \mathbb{N}_0 . In light of Lemma 3.3, we need only to show the existence of a finite discrete interval \mathcal{I} over which (S_λ) is definite for one value $\lambda_0 \in \mathbb{C}$. Thus, let $\lambda_0 \in \mathbb{C}$ and for each finite discrete subinterval \mathcal{I} , define the set $s(\mathcal{I})$:

$$s(\mathcal{I}) := \left\{ \xi \in \mathbb{C}^{2n} \mid \|\xi\| = 1, \sum_{k \in \mathcal{I}} \xi^* \Phi_k^*(\lambda_0) \Psi_k \Phi_k(\lambda_0) \xi = 0 \right\},$$

where $\|\cdot\|$ denotes a norm for \mathbb{C}^{2n} ; $s(\mathcal{I})$ is compact and $s(\widehat{\mathcal{I}}) \subseteq s(\mathcal{I})$ whenever $\mathcal{I} \subseteq \widehat{\mathcal{I}}$.

Consider a collection $\{\mathcal{I}_m\}_{m \in \mathbb{N}}$ of nested, finite, discrete intervals such that $\bigcup_{m \in \mathbb{N}} \mathcal{I}_m = \mathbb{N}_0$. Next, assume that there exists a vector $\xi \in \mathbb{C}^{2n}$ with $\|\xi\| = 1$ such that for every $m \in \mathbb{N}$,

$$\sum_{k \in \mathcal{I}_m} \xi^* \Phi_k^*(\lambda_0) \Psi_k \Phi_k(\lambda_0) \xi = 0.$$

As a consequence,

$$\sum_{k \in \mathbb{N}_0} \xi^* \Phi_k^*(\lambda_0) \Psi_k \Phi_k(\lambda_0) \xi = 0.$$

Then, definiteness on \mathbb{N}_0 of (S_λ) implies that $\Phi_k(\lambda_0)\xi = 0$ for all $k \in \mathbb{N}_0$, and hence that $\xi = 0$; thus contradicting the assumption that $\|\xi\| = 1$. As consequence, for the nested collection of compact sets given by $\{s(\mathcal{I}_m)\}_{m \in \mathbb{N}}$,

$$\bigcap_{m \in \mathbb{N}} s(\mathcal{I}_m) = \emptyset.$$

Thus, $s(\mathcal{I}_{m_0}) = \emptyset$ for some $m_0 \in \mathbb{N}$, hence demonstrating the definiteness of (S_λ) on the finite discrete interval \mathcal{I}_{m_0} . \square

Remark 3.6. Definiteness of (S_λ) on \mathbb{N}_0 plays a significant role in the development of the Weyl–Titchmarsh theory for the case of a positive semi-definite weight in the definition of semi-inner product. In [14, Hypothesis 2.4] and [37, Hypothesis 4.11], existence is assumed for an $N_0 \in \mathbb{N}_0$ such that

$$\sum_{k=0}^{N_0} z_k^*(\lambda) \Psi_k z_k(\lambda) > 0$$

for every $\lambda \in \mathbb{C}$ and every nontrivial solution, $z(\lambda)$, of (S_λ) . With respect to Lemma 3.5, this assumption is equivalent to the definiteness of (S_λ) on \mathbb{N}_0 for some $\lambda \in \mathbb{C}$. Note also that [9, Assumption 2.2] requires satisfaction of inequality (3.1) on every nonempty finite subinterval of \mathbb{N}_0 : a condition significantly stronger than requiring definiteness of (S_λ) on \mathbb{N}_0 as seen when (S_λ) is definite on $[0, N_0]_{\mathbb{Z}} \subset \mathbb{N}_0$ and $\Psi_k \equiv 0$ for $k \geq N_0 + 1$.

We now give another characterization of the definiteness for (S_λ) , analogous to that given for systems (1.4) and (1.5) in [25, Sections 2.3 and 2.4] and [30, Sections 3 and 4], respectively.

For a discrete finite subinterval $\mathcal{I} \subset \mathbb{N}_0$, with $k_0 \in \mathcal{I}$, we define the $2n \times 2n$ positive semi-definite matrix

$$\varphi(\lambda, \mathcal{I}) := \sum_{k \in \mathcal{I}} \Phi_k^*(\lambda) \Psi_k \Phi_k(\lambda) \tag{3.6}$$

in terms of the fundamental system $\Phi(\lambda)$ for (S_λ) where $\Phi_{k_0}(\lambda) = I_{2n}$, for all $\lambda \in \mathbb{C}$. While $\varphi(\lambda, \mathcal{I})$ depends on λ and \mathcal{I} , we next show that the kernel and the range of $\varphi(\lambda, \mathcal{I})$ do not depend on λ , and hence that the rank of φ is independent of λ .

Lemma 3.7. *For a discrete finite subinterval $\mathcal{I} \subset \mathbb{N}_0$, the subspaces $\ker \varphi(\lambda, \mathcal{I})$ and $\text{ran } \varphi(\lambda, \mathcal{I})$ are independent of $\lambda \in \mathbb{C}$.*

Proof. Fix $\lambda \in \mathbb{C}$ and $\xi \in \ker \varphi(\lambda, \mathcal{I})$, and let $z := \Phi(\lambda)\xi$. Then z solves (S_λ) on \mathcal{I} , i.e., $\mathcal{L}(z)_k = \lambda \Psi_k z_k$ for $k \in \mathcal{I}$, while satisfying the initial condition $z_{k_0} = \xi$. As a consequence, $\Psi_k z_k = \Psi_k \Phi_k(\lambda)\xi = 0$ for $k \in \mathcal{I}$. Note, for an arbitrary $\nu \in \mathbb{C}$, that z also solves (S_ν) on \mathcal{I} , i.e., $\mathcal{L}(z)_k = \nu \Psi_k z_k$, $k \in \mathcal{I}$, with $z_{k_0} = \Phi_{k_0}(\nu)\xi = \xi$. Hence, $z = \Phi(\lambda)\xi = \Phi(\nu)\xi$, which implies

$$0 = \xi^* \varphi(\lambda, \mathcal{I}) \xi = \sum_{k \in \mathcal{I}} z_k^* \Psi_k z_k = \xi^* \varphi(\nu, \mathcal{I}) \xi.$$

Therefore, $\ker \varphi(\lambda, \mathcal{I}) \subseteq \ker \varphi(\nu, \mathcal{I})$. By reversing the roles of λ and ν we obtain $\ker \varphi(\lambda, \mathcal{I}) = \ker \varphi(\nu, \mathcal{I})$ for all $\lambda, \nu \in \mathbb{C}$. The independence of $\text{ran } \varphi(\lambda, \mathcal{I})$ on $\lambda \in \mathbb{C}$ follows from the fact that as defined in (3.6), $\varphi(\lambda, \mathcal{I})$ is Hermitian; thus, $\text{ran } \varphi(\lambda, \mathcal{I}) = \text{ran } \varphi^*(\lambda, \mathcal{I}) = \ker \varphi(\lambda, \mathcal{I})^\perp$. \square

In general, given Lemma 3.7, we shall suppress λ in the following notation: $\ker \varphi(\lambda, \mathcal{I}) \equiv \ker \varphi(\mathcal{I})$, $\text{ran } \varphi(\lambda, \mathcal{I}) \equiv \text{ran } \varphi(\mathcal{I})$, and $\text{rank } \varphi(\lambda, \mathcal{I}) \equiv \text{rank } \varphi(\mathcal{I})$ for $\lambda \in \mathbb{C}$.

Lemma 3.8. *There exists a discrete finite interval \mathcal{I} , with $k_0 \in \mathcal{I} \subset \mathbb{N}_0$, such that for any discrete finite interval $\widehat{\mathcal{I}}$ satisfying $\mathcal{I} \subseteq \widehat{\mathcal{I}} \subset \mathbb{N}_0$,*

$$\text{rank } \varphi(\mathcal{I}) = \text{rank } \varphi(\widehat{\mathcal{I}}), \quad \text{ran } \varphi(\mathcal{I}) = \text{ran } \varphi(\widehat{\mathcal{I}}). \tag{3.7}$$

Proof. For discrete finite intervals \mathcal{I} and $\widehat{\mathcal{I}}$ of \mathbb{N}_0 , where $k_0 \in \mathcal{I} \subseteq \widehat{\mathcal{I}} \subset \mathbb{N}_0$, by the definition of φ in (3.6), we see that $\ker \varphi(\widehat{\mathcal{I}}) \subseteq \ker \varphi(\mathcal{I})$. Then, given that φ is Hermitian, we see that

$$\text{ran } \varphi(\mathcal{I}) = \ker \varphi(\mathcal{I})^\perp \subseteq \ker \varphi(\widehat{\mathcal{I}})^\perp = \text{ran } \varphi(\widehat{\mathcal{I}}),$$

and hence that $\text{rank } \varphi(\mathcal{I}) \leq \text{rank } \varphi(\widehat{\mathcal{I}})$. Since $\text{rank } \varphi(\cdot) \leq 2n$, there must be a finite discrete interval \mathcal{I} such that $\text{rank } \varphi(\widehat{\mathcal{I}}) = \text{rank } \varphi(\mathcal{I})$, and $\text{ran } \varphi(\widehat{\mathcal{I}}) = \text{ran } \varphi(\mathcal{I})$ for all finite discrete intervals $\widehat{\mathcal{I}}$ containing \mathcal{I} . \square

Next, we describe the connection between the definiteness of (S_λ) and the matrix $\varphi(\lambda, \mathcal{I})$ for a finite discrete interval $\mathcal{I} \subset \mathbb{N}_0$.

Theorem 3.9. *For a discrete finite interval $\mathcal{I} \subset \mathbb{N}_0$, and with $\varphi(\lambda, \mathcal{I})$ defined in (3.6), the following statements are equivalent:*

- (i) $\text{rank } \varphi(\mathcal{I}) = 2n$.
- (ii) $\ker \varphi(\mathcal{I}) = \{0\}$.

- (iii) For some $\lambda \in \mathbb{C}$, every nontrivial solution, $z(\lambda)$, of (S_λ) satisfies $\sum_{k \in \mathcal{I}} z_k^*(\lambda) \Psi_k z_k(\lambda) > 0$.
 (iv) For some $\lambda \in \mathbb{C}$, a solution, $z(\lambda)$, of (S_λ) is necessarily trivial, i.e. $z_k(\lambda) = 0$, $k \in \mathcal{I}$, when $\sum_{k \in \mathcal{I}} z_k^*(\lambda) \Psi_k z_k(\lambda) = 0$.

Proof. Equivalence of (i) and (ii) is clear, while equivalence of (iii) and (iv) follows from Lemma 3.3. It suffices then to show that statements (ii) and (iii) are equivalent.

Assume that (ii) is true. Any nontrivial solution $z(\lambda)$ of (S_λ) can be expressed as $z(\lambda) = \Phi(\lambda)\xi$ for some $\xi \in \mathbb{C}^{2n} \setminus \{0\}$. By (ii), $\varphi(\lambda, \mathcal{I})\xi \neq 0$, and by the positive semi-definiteness of $\varphi(\lambda, \mathcal{I})$, we see that

$$\sum_{k \in \mathcal{I}} z_k^*(\lambda) \Psi_k z_k(\lambda) = \sum_{k \in \mathcal{I}} \xi^* \Phi_k^*(\lambda) \Psi_k \Phi_k(\lambda) \xi = \xi^* \varphi(\lambda, \mathcal{I}) \xi > 0.$$

Conversely, assume that (iii) is true, and for $\xi \in \mathbb{C}^{2n} \setminus \{0\}$ let $z(\lambda) := \Phi(\lambda)\xi$. Then $z(\lambda)$ is a nontrivial solution of (S_λ) and, by (iii),

$$\xi^* \varphi(\lambda, \mathcal{I}) \xi = \sum_{k \in \mathcal{I}} z_k^*(\lambda) \Psi_k z_k(\lambda) > 0.$$

Thus $\varphi(\lambda, \mathcal{I})\xi \neq 0$, and hence $\ker \varphi(\mathcal{I}) = \{0\}$; implying that statement (ii) is satisfied. \square

As a consequence of Lemma 3.5 and Theorem 3.9 we get the following corollary, cf. [25, Definition 2.14].

Corollary 3.10. *System (S_λ) is definite on \mathbb{N}_0 if and only if, for some finite discrete interval $\mathcal{I} \subset \mathbb{N}_0$, one of the conditions listed in Theorem 3.9 is satisfied.*

For the special case of linear dependence on λ as studied in [9,14], i.e., where the spectral parameter λ appears only in the second equation of the system, we can show the following sufficient condition for the definiteness of system (S_λ) on \mathbb{N}_0 ; q.v. Example 3.4.

Theorem 3.11. *Let $\lambda \in \mathbb{C}$ and $\mathbb{S}_k(\lambda) = \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k - \lambda \mathcal{W}_k \mathcal{A}_k & \mathcal{D}_k - \lambda \mathcal{W}_k \mathcal{B}_k \end{pmatrix}$ with $\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k, \mathcal{D}_k, \mathcal{W}_k \in \mathbb{C}^{n \times n}$ satisfy identity (1.2) for all $k \in \mathbb{N}_0$. If there exists an index $l \in [1, \infty)_{\mathbb{Z}}$ such that the matrices \mathcal{B}_{l-1} , \mathcal{W}_{l-1} , and \mathcal{W}_l are invertible (in fact, \mathcal{W}_{l-1} and \mathcal{W}_l are positive definite), then system (S_λ) is definite on \mathbb{N}_0 .*

Proof. First we note that the form of the matrix $\mathbb{S}_k(\lambda)$ implies $\Psi_k = \text{diag}\{\mathcal{W}_k, 0\}$ for all $k \in \mathbb{N}_0$ by (1.1)(iv). Let $z(\lambda) = (x(\lambda), u(\lambda))^T$ be any nontrivial solution of (S_λ) such that $\Psi_k z_k(\lambda)$, i.e., $\mathcal{W}_k x_k(\lambda) = 0$, for $k \in \mathbb{N}_0$. By Lemma 3.3 we have to show that $z(\lambda) \equiv 0$ on \mathbb{N}_0 . From the invertibility of \mathcal{W}_{l-1} and \mathcal{W}_l we obtain $x_{l-1}(\lambda) = x_l(\lambda) = 0$. Hence

$$0 = x_{l-1}(\lambda) = \mathcal{A}_{l-1} x_l(\lambda) + \mathcal{B}_{l-1} u_l(\lambda) = \mathcal{B}_{l-1} u_l(\lambda)$$

and the invertibility of \mathcal{B}_{l-1} implies also $u_l(\lambda) = 0$, i.e., $z_l(\lambda) = 0$. Since the matrix $\mathbb{S}_k(\lambda)$ is invertible for every $k \in \mathbb{N}_0$, it follows $z(\lambda) \equiv 0$ on \mathbb{N}_0 . \square

4. Nonhomogeneous problem

The origin of the Weyl–Titchmarsh theory for discrete symplectic systems can be found in [9,14], where the system with the spectral parameter appearing only in the second equation was studied. Recently, these results were generalized and further extended for discrete symplectic systems with general linear dependence on the spectral parameter in [37]. Given that Weyl–Titchmarsh theory has been established for discrete symplectic systems of the form (1.3), one can verify that these results remain valid for the time-reversed

symplectic systems (S_λ) with appropriate changes in the definition of the semi-inner product and its weight matrix as noted in the introduction.

In this section, we take the nonhomogeneous problem into consideration as in [14, Section 5] and begin by recalling some fundamental results from [37] which are related to the study of system (S_λ^f) and refer the reader to [37] for more details.

Throughout this section, we assume that system (S_λ) is definite on \mathbb{N}_0 , and fix the matrix $\alpha \in \Gamma$, where

$$\Gamma := \{ \alpha \in \mathbb{C}^{n \times 2n} : \alpha \alpha^* = I, \alpha \mathcal{J} \alpha^* = 0 \}.$$

By $\Phi(\lambda, \alpha)$ we denote the fundamental matrix of system (S_λ) such that $\Phi_0(\lambda, \alpha) = (\alpha^*, -\mathcal{J} \alpha^*)$, and we emphasize its partition into $2n \times n$ blocks by the notation $\Phi(\lambda, \alpha) := (Z(\lambda, \alpha), \tilde{Z}(\lambda, \alpha))$.

For $M \in \mathbb{C}^{n \times n}$ the function

$$\mathcal{X}_k(\lambda) := \Phi_k(\lambda, \alpha) (I \quad M^*)^* \tag{4.1}$$

represents a *Weyl solution* (cf. [37, Definition 2.11]), and the set

$$D_k(\lambda) := \{ M \in \mathbb{C}^{n \times n} : \mathcal{E}_k(M) \leq 0 \}, \quad \text{where } \mathcal{E}_k(M) := i\delta(\lambda) \mathcal{X}_k^*(\lambda) \mathcal{J} \mathcal{X}_k(\lambda)$$

and $\delta(\lambda) := \operatorname{sgn} \Im(\lambda)$, is said to be the *Weyl disk*; cf. [37, Definition 3.1]. Since system (S_λ) is assumed to be definite on \mathbb{N}_0 , the set $D_+(\lambda) := \lim_{k \rightarrow \infty} D_k(\lambda)$ exists and it is closed, convex and nonempty; cf. [37, Definition 3.10]. It is also well known that the columns of the Weyl solution $\mathcal{X}(\lambda)$ defined as in (4.1) for $k \in \mathbb{N}_0$ through $M \in D_+(\lambda)$ are linearly independent square summable solutions of (S_λ) ; cf. [37, Theorem 4.2]. Thus system (S_λ) has at least n (*the limit point case*) and at most $2n$ (*the limit circle case*) linearly independent square summable solutions.

By $M_+(\lambda)$ we denote the *half-line Weyl–Titchmarsh $M(\lambda)$ -functions*, defined in accordance with [37, Remark 3.17], and note that

$$M_+^*(\lambda) = M_+(\bar{\lambda}), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{4.2}$$

If systems (S_λ) and (S_ν) , where $\lambda, \nu \in \mathbb{C} \setminus \mathbb{R}$, are both in the limit point or in the limit circle case, then

$$\lim_{k \rightarrow \infty} \mathcal{X}_k^{+*}(\lambda) \mathcal{J} \mathcal{X}_k^+(\nu) = 0, \tag{4.3}$$

where $\mathcal{X}^+(\lambda)$ and $\mathcal{X}^+(\nu)$ represent Weyl solutions for systems (S_λ) and (S_ν) defined by (4.1) for $k \in \mathbb{N}_0$, corresponding to the matrices $M_+(\lambda)$ and $M_+(\nu)$, respectively; cf. [37, Theorem 4.12].

For $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $k, l \in \mathbb{N}_0$ we introduce the *Green function*

$$G_{k,l}(\lambda) := \begin{cases} \tilde{Z}_k(\lambda) \mathcal{X}_l^{+*}(\bar{\lambda}), & k \in [0, l]_{\mathbb{Z}}, \\ \mathcal{X}_k^+(\lambda) \tilde{Z}_l^*(\bar{\lambda}), & k \in [l + 1, \infty)_{\mathbb{Z}}. \end{cases} \tag{4.4}$$

In the literature, we also find the terminology *resolvent kernel* for an analogous function in the continuous time case; cf. [23, p. 15]. The function $G_{k,l}(\lambda)$ can be equivalently written as

$$G_{k,l}(\lambda) = \begin{cases} \mathcal{X}_k^+(\lambda) \tilde{Z}_l^*(\bar{\lambda}), & l \in [0, k - 1]_{\mathbb{Z}}, \\ \tilde{Z}_k(\lambda) \mathcal{X}_l^{+*}(\bar{\lambda}), & l \in [k, \infty)_{\mathbb{Z}}. \end{cases} \tag{4.5}$$

Lemma 4.1. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and system (S_λ) be definite on \mathbb{N}_0 . Then*

$$\mathcal{X}_k^+(\lambda) \tilde{Z}_k^*(\bar{\lambda}) - \tilde{Z}_k(\lambda) \mathcal{X}_k^{+*}(\bar{\lambda}) = \mathcal{J} \quad \text{for all } k \in \mathbb{N}_0. \tag{4.6}$$

Proof. Identity (4.6) then follows by a direct calculation from the definition of $\mathcal{X}^+(\cdot)$ and identities (2.6)(iii) and (4.2). \square

In the next lemma, some fundamental properties of the Green function, $G(\lambda)$, are established. We note that the given identities are presented in a more symmetric form, with respect to the variables k, l , than the corresponding identities for the Green function in the case of the discrete symplectic system with the special linear dependence on the spectral parameter given in [14, Lemma 5.1].

Lemma 4.2. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and system (S_λ) be definite on \mathbb{N}_0 . Then the function $G_{\cdot, \cdot}(\lambda)$ possesses the following properties:*

- (i) $G_{k,l}^*(\lambda) = G_{l,k}(\bar{\lambda})$ for all $k, l \in \mathbb{N}_0$ such that $k \neq l$;
- (ii) $G_{k,k}^*(\lambda) = G_{k,k}(\bar{\lambda}) + \mathcal{J}$ for all $k \in \mathbb{N}_0$;
- (iii) for every $k, l \in \mathbb{N}_0$ such that $k \notin \mathcal{T}(l)$, the function $G_{\cdot, l}(\lambda)$ solves the homogeneous system (S_λ) on the set $\mathcal{T}(l)$, where

$$\mathcal{T}(l) := \{\tau \in \mathbb{N}_0 : \tau \neq l\};$$

- (iv) $G_{k,k}(\lambda) = \mathbb{S}_k(\lambda)G_{k+1,k}(\lambda) - \mathcal{J}$ for every $k \in \mathbb{N}_0$;
- (v) the columns of $G_{\cdot, l}(\lambda)$ belong to $\ell_{\Psi}^2(\mathbb{N}_0)$ for every $l \in \mathbb{N}_0$ and the columns of $G_{k, \cdot}(\lambda)$ belong to $\ell_{\Psi}^2(\mathbb{N}_0)$ for every $k \in \mathbb{N}_0$.

Proof. The first property follows directly from the definition of $G_{k,l}(\lambda)$ in (4.4). The second property can be obtained from (4.4) by means of identity (4.6). To prove the third property, it is necessary to distinguish between k and l in the relation: the statement follows from the fact that the functions $\mathcal{X}^+(\lambda)$ and $\tilde{Z}(\lambda)$ solve system (S_λ) with respect to k . Property (iv) can be proven using the definition of $G(\lambda)$ in (4.4) and identities (4.2) and (2.6). Finally, the columns of $G_{\cdot, l}(\lambda)$ belong to $\ell_{\Psi}^2(\mathbb{N}_0)$ for every $l \in \mathbb{N}_0$ by the definition of the Green function, because

$$\begin{aligned} \|G_{\cdot, l}(\lambda)e_j\|_{\Psi}^2 &= e_j^* \mathcal{X}_l^+(\bar{\lambda}) \left(\sum_{k=0}^l \tilde{Z}_k^*(\lambda) \Psi_k \tilde{Z}_k(\lambda) \right) \mathcal{X}_l^{+*}(\bar{\lambda}) e_j \\ &\quad + e_j^* \tilde{Z}_l(\bar{\lambda}) \left(\sum_{k=l+1}^{\infty} \mathcal{X}_k^{**}(\lambda) \Psi_k \mathcal{X}_k^+(\lambda) \right) \tilde{Z}_k^*(\bar{\lambda}) e_j < \infty, \end{aligned}$$

while the columns of $G_{k, \cdot}(\lambda)$ are in $\ell_{\Psi}^2(\mathbb{N}_0)$ for every $k \in \mathbb{N}_0$ by part (i). \square

We associate with the nonhomogeneous system (S_λ^f) the function

$$\hat{z}_k(\lambda) = \sum_{l=0}^{\infty} G_{k,l}(\lambda) \Psi_l f_l, \quad k \in \mathbb{N}_0, \quad (4.7)$$

which is well defined for all $f \in \ell_{\Psi}^2(\mathbb{N}_0)$, because the columns of $G_{k, \cdot}(\lambda)$ are square summable by the previous lemma. In addition, by (4.5), we can write

$$\hat{z}_k(\lambda) = \left\{ \sum_{l=0}^{k-1} + \sum_{l=k}^{\infty} \right\} G_{k,l}(\lambda) \Psi_l f_l = \mathcal{X}_k^+(\lambda) \sum_{l=0}^{k-1} \tilde{Z}_l^*(\bar{\lambda}) \Psi_l f_l + \tilde{Z}_k(\lambda) \sum_{l=k}^{\infty} \mathcal{X}_l^{+*}(\bar{\lambda}) \Psi_l f_l. \quad (4.8)$$

Similarly as in [14, Theorem 5.2] we show that the above defined function $\hat{z}(\lambda)$ represents a square summable solution of system (S_λ^f) .

Theorem 4.3. Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $f \in \ell^2_{\Psi}(\mathbb{N}_0)$, and system (S_{λ}) be definite on \mathbb{N}_0 . The function $\hat{z}(\lambda)$ defined in (4.7) solves system (S_{λ}^f) , satisfies the initial condition $\alpha z_0(\lambda) = 0$, is square summable, i.e., $\hat{z}(\lambda) \in \ell^2_{\Psi}(\mathbb{N}_0)$, and it holds

$$\|\hat{z}(\lambda)\|_{\Psi} \leq \frac{1}{|\Im(\lambda)|} \|f\|_{\Psi}. \tag{4.9}$$

In addition, if system (S_{λ}) is in the limit point or limit circle case for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then

$$\lim_{k \rightarrow \infty} \mathcal{X}_k^{+*}(\nu) \mathcal{J} \hat{z}_k(\lambda) = 0 \quad \text{for every } \nu \in \mathbb{C} \setminus \mathbb{R}. \tag{4.10}$$

Proof. The form of $\hat{z}_k(\lambda)$ given in (4.8) together with a similar expression of $\hat{z}_{k+1}(\lambda)$, the facts that $\mathcal{X}^+(\lambda)$ and $\tilde{Z}(\lambda)$ solve (S_{λ}) , and identity (4.6) yield

$$\begin{aligned} \hat{z}_k(\lambda) - \mathbb{S}_k(\lambda) \hat{z}_{k+1}(\lambda) &= \mathcal{X}_k^+(\lambda) \sum_{l=0}^{k-1} \tilde{Z}_l^*(\bar{\lambda}) \Psi_l f_l + \tilde{Z}_k(\lambda) \sum_{l=k}^{\infty} \mathcal{X}_l^{+*}(\bar{\lambda}) \Psi_l f_l \\ &\quad - \mathbb{S}_k(\lambda) \mathcal{X}_{k+1}^+(\lambda) \sum_{l=0}^k \tilde{Z}_l^*(\bar{\lambda}) \Psi_l f_l - \mathbb{S}_k(\lambda) \tilde{Z}_{k+1}(\lambda) \sum_{l=k+1}^{\infty} \mathcal{X}_l^{+*}(\bar{\lambda}) \Psi_l f_l \\ &= -[\mathcal{X}_k^+(\lambda) \tilde{Z}_k^*(\bar{\lambda}) - \tilde{Z}_k(\lambda) \mathcal{X}_k^{+*}(\bar{\lambda})] \Psi_k f_k = -\mathcal{J} \Psi_k f_k, \end{aligned}$$

i.e., the function $\hat{z}(\lambda)$ solves system (S_{λ}^f) . The fulfillment of the boundary condition follows by the simple calculation

$$\alpha \hat{z}_0(\lambda) = \alpha \tilde{Z}_0(\lambda) \sum_{l=0}^{\infty} \mathcal{X}_l^{+*}(\bar{\lambda}) \Psi_l f_l = -\alpha \mathcal{J} \alpha^* \sum_{l=0}^{\infty} \mathcal{X}_l^{+*}(\bar{\lambda}) \Psi_l f_l = 0,$$

because $\tilde{Z}_0(\lambda) = -\mathcal{J} \alpha^*$ and $\alpha \in \Gamma$.

Next, we prove the estimate in (4.9) which together with the assumption $f \in \ell^2_{\Psi}(\mathbb{N}_0)$ will imply that $\hat{z}(\lambda) \in \ell^2_{\Psi}(\mathbb{N}_0)$. For every $r \in \mathbb{N}_0$ we define the function

$$f_k^{[r]} := \begin{cases} f_k, & k \in [0, r]_{\mathbb{Z}}, \\ 0, & k \in [r + 1, \infty)_{\mathbb{Z}}, \end{cases}$$

and the function

$$\hat{z}_k^{[r]}(\lambda) := \sum_{l=0}^{\infty} G_{k,l}(\lambda) \Psi_l f_l = \sum_{l=0}^r G_{k,l}(\lambda) \Psi_l f_l.$$

The function $\hat{z}_k^{[r]}(\lambda)$ solves system (S_{λ}^f) with f replaced by $f^{[r]}$. Applying the extended Lagrange identity from Theorem 2.5, we obtain

$$\begin{aligned} \lim_{k \rightarrow \infty} \hat{z}_{k+1}^{[r]*}(\lambda) \mathcal{J} \hat{z}_{k+1}^{[r]}(\lambda) &= \hat{z}_0^{[r]*}(\lambda) \mathcal{J} \hat{z}_0^{[r]*}(\lambda) + (\bar{\lambda} - \lambda) \sum_{k=0}^{\infty} \hat{z}_k^{[r]*}(\lambda) \Psi_k \hat{z}_k^{[r]}(\lambda) \\ &\quad + \sum_{k=0}^{\infty} f_k^{[r]*} \Psi_k \hat{z}_k^{[r]}(\lambda) - \sum_{k=0}^{\infty} \hat{z}_k^{[r]*}(\lambda) \Psi_k f_k^{[r]}. \end{aligned} \tag{4.11}$$

Since $\tilde{Z}_0(\lambda) = -\mathcal{J} \alpha^*$ and $\alpha \in \Gamma$, we see that

$$\hat{z}_0^{[r]*}(\lambda)\mathcal{J}\hat{z}_0^{[r]*}(\lambda) = \left(\sum_{l=0}^r \mathcal{X}_l^{+*}(\bar{\lambda})\Psi_l f_l\right)^* \tilde{Z}_0^*(\lambda)\mathcal{J}\tilde{Z}_0(\lambda) \left(\sum_{l=0}^r \mathcal{X}_l^{+*}(\bar{\lambda})\Psi_l f_l\right) = 0,$$

where, for every $k \in [r + 1, \infty)_{\mathbb{Z}}$, we can write

$$\hat{z}_k^{[r]}(\lambda) = \mathcal{X}_k^+(\lambda)g_r(\lambda), \quad \text{where } g_r(\lambda) := \sum_{l=0}^r \tilde{Z}_l^*(\bar{\lambda})\Psi_l f_l. \tag{4.12}$$

This, together with the fact $M_+(\lambda) \in D_+(\lambda)$, yields

$$\begin{aligned} \frac{1}{\bar{\lambda} - \lambda} \lim_{k \rightarrow \infty} \hat{z}_{k+1}^{[r]*}(\lambda)\mathcal{J}\hat{z}_{k+1}^{[r]}(\lambda) &= \frac{i\delta(\lambda)}{2|\Im(\lambda)|} g_r^*(\lambda) \left(\lim_{k \rightarrow \infty} \mathcal{X}_{k+1}^{+*}(\lambda)\mathcal{J}\mathcal{X}_{k+1}^+(\lambda)\right)g_r(\lambda) \\ &= \frac{1}{2|\Im(\lambda)|} g_r^*(\lambda) \left(\lim_{k \rightarrow \infty} \mathcal{E}_{k+1}(M_+(\lambda))\right)g_r(\lambda) \leq 0. \end{aligned}$$

Using the Cauchy–Schwarz inequality, $\Psi \geq 0$, and identity (4.11), we see that

$$\begin{aligned} \|\hat{z}^{[r]}(\lambda)\|_{\Psi}^2 &= \sum_{k=0}^{\infty} \hat{z}_k^{[r]*}(\lambda)\Psi_k\hat{z}_k^{[r]}(\lambda) \leq \frac{1}{2i\Im(\lambda)} \left(\sum_{k=0}^r f_k^{[r]*}\Psi_k\hat{z}_k^{[r]}(\lambda) - \sum_{k=0}^r \hat{z}_k^{[r]*}(\lambda)\Psi_k f_k^{[r]}\right) \\ &\leq \frac{1}{|\Im(\lambda)|} \left|\sum_{k=0}^r \hat{z}_k^{[r]*}(\lambda)\Psi_k f_k^{[r]}\right| \\ &\leq \frac{1}{|\Im(\lambda)|} \left(\sum_{k=0}^r \hat{z}_k^{[r]*}(\lambda)\Psi_k\hat{z}_k^{[r]}(\lambda)\right)^{1/2} \left(\sum_{k=0}^r f_k^{[r]*}\Psi_k f_k^{[r]}\right)^{1/2} \\ &\leq \frac{1}{|\Im(\lambda)|} \|\hat{z}^{[r]}(\lambda)\|_{\Psi} \|f^{[r]}\|_{\Psi}, \end{aligned}$$

thereby yielding the inequality

$$\|\hat{z}^{[r]}(\lambda)\|_{\Psi} \leq \frac{1}{|\Im(\lambda)|} \|f^{[r]}\|_{\Psi} \leq \frac{1}{|\Im(\lambda)|} \|f\|_{\Psi}. \tag{4.13}$$

For any $k, r \in \mathbb{N}_0$, we now see that

$$\hat{z}_k(\lambda) - \hat{z}_k^{[r]}(\lambda) = \sum_{l=r+1}^{\infty} G_{k,l}(\lambda)\Psi_l f_l.$$

Let $m \in [0, r]_{\mathbb{Z}}$ be fixed. By the definition of $G(\lambda)$ in (4.4), we obtain for every $k \in [0, m]_{\mathbb{Z}}$ that

$$\hat{z}_k(\lambda) - \hat{z}_k^{[r]}(\lambda) = \tilde{Z}_k(\lambda) \sum_{l=r+1}^{\infty} \mathcal{X}_l^{+*}(\bar{\lambda})\Psi_l f_l. \tag{4.14}$$

Since the columns of $\mathcal{X}^+(\bar{\lambda})$ and the function f belong to $\ell_{\Psi}^2(\mathbb{N}_0)$, it follows that the right-hand side of (4.14) tends to zero as $r \rightarrow \infty$ for every $k \in [0, m]_{\mathbb{Z}}$. Hence $\hat{z}^{[r]}$ converges uniformly to the function $\hat{z}(\lambda)$ on the interval $[0, m]_{\mathbb{Z}}$. Since $\hat{z}(\lambda) = \hat{z}^{[r]}(\lambda)$ on $[0, m]_{\mathbb{Z}}$ and $\Psi \geq 0$, we see by (4.13) that

$$\sum_{k=0}^m \hat{z}_k^{[r]*}(\lambda)\Psi_k\hat{z}_k^{[r]}(\lambda) \leq \|\hat{z}^{[r]}(\lambda)\|_{\Psi}^2 \leq \frac{1}{|\Im(\lambda)|^2} \|f\|_{\Psi}^2.$$

Then, as a consequence of the uniform convergence for $r \rightarrow \infty$ on $[0, m]_{\mathbb{Z}}$, we see that

$$\sum_{k=0}^m \hat{z}_k^*(\lambda) \Psi_k \hat{z}_k(\lambda) \leq \frac{1}{|\Im(\lambda)|^2} \|f\|_{\Psi}^2. \tag{4.15}$$

Upon taking the limit for $m \rightarrow \infty$ in (4.15) the desired estimate in (4.9) follows.

Finally, to establish the existence of the limit in (4.10), assume that system (S_{λ}) is in the limit point case for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. From the extended Lagrange identity in Theorem 2.5, for any $k, r \in \mathbb{N}_0$, we obtain

$$[\mathcal{X}_j^{+*}(\nu) \mathcal{J} \hat{z}_j^{[r]}(\lambda)]_0^{k+1} = (\bar{\nu} - \lambda) \sum_{j=0}^k \mathcal{X}_j^{+*}(\nu) \Psi_j \hat{z}_j^{[r]}(\lambda) - \sum_{j=0}^k \mathcal{X}_j^{+*}(\nu) \Psi_j f_j^{[r]}(\lambda). \tag{4.16}$$

For $k \in [r + 1, \infty)_{\mathbb{Z}}$ identity (4.12) holds, and, by (4.3), we obtain

$$\lim_{k \rightarrow \infty} \mathcal{X}_{k+1}^{+*}(\nu) \mathcal{J} \hat{z}_{k+1}^{[r]}(\lambda) = \lim_{k \rightarrow \infty} \mathcal{X}_{k+1}^{+*}(\nu) \mathcal{J} \mathcal{X}_{k+1}^+(\lambda) g_r(\lambda) = 0.$$

In the limit, as $k \rightarrow \infty$, (4.16) yields

$$\mathcal{X}_0^{+*}(\nu) \mathcal{J} \hat{z}_0^{[r]}(\lambda) = (\lambda - \bar{\nu}) \sum_{j=0}^{\infty} \mathcal{X}_j^{+*}(\nu) \Psi_j \hat{z}_j^{[r]}(\lambda) + \sum_{j=0}^{\infty} \mathcal{X}_j^{+*}(\nu) \Psi_j f_j^{[r]}(\lambda). \tag{4.17}$$

It was proven earlier that $\hat{z}^{[r]}(\lambda)$ converges uniformly on finite subintervals of \mathbb{N}_0 as $r \rightarrow \infty$. Then, from (4.17), we see that

$$\mathcal{X}_0^{+*}(\nu) \mathcal{J} \hat{z}_0(\lambda) = (\lambda - \bar{\nu}) \sum_{j=0}^{\infty} \mathcal{X}_j^{+*}(\nu) \Psi_j \hat{z}_j(\lambda) + \sum_{j=0}^{\infty} \mathcal{X}_j^{+*}(\nu) \Psi_j f_j(\lambda). \tag{4.18}$$

On the other hand, by the extended Lagrange identity, for every $k \in \mathbb{N}_0$, we obtain

$$[\mathcal{X}_j^{+*}(\nu) \mathcal{J} \hat{z}_j(\lambda)]_0^{k+1} = (\bar{\nu} - \lambda) \sum_{j=0}^k \mathcal{X}_j^{+*}(\nu) \Psi_j \hat{z}_j(\lambda) - \sum_{j=0}^k \mathcal{X}_j^{+*}(\nu) \Psi_j f_j(\lambda). \tag{4.19}$$

Letting $k \rightarrow \infty$ in (4.19), and using inequality (4.18), the limit in (4.10) is established.

An argument, similar to that given above, can be used in the limit circle case to show the existence of the limit in (4.10), because all solutions of (S_{λ}) are square summable. However, an alternative and more direct method of the proof is available.

By (4.8) we have for every $k \in \mathbb{N}_0$ that

$$\mathcal{X}_k^{+*}(\nu) \mathcal{J} \hat{z}_k(\lambda) = \mathcal{X}_k^{+*}(\nu) \mathcal{J} \mathcal{X}_k^+(\lambda) \sum_{l=0}^{k-1} \tilde{Z}_l(\bar{\lambda}) \Psi_l f_l + \mathcal{X}_k^{+*}(\nu) \mathcal{J} \tilde{Z}_k(\lambda) \sum_{l=k}^{\infty} \mathcal{X}_l^{+*}(\bar{\lambda}) \Psi_l f_l. \tag{4.20}$$

The limit of the first term on the right-hand side in (4.20) is equal to zero because $\mathcal{X}_k^{+*}(\nu) \mathcal{J} \mathcal{X}_k^+(\lambda)$ tends to zero by (4.3) and it is multiplied by a sum which converges as $k \rightarrow \infty$. The second term on the right-hand side of (4.20) tends also to zero, because the columns of $\tilde{Z}(\lambda)$ are square summable. This implies that the function $\mathcal{X}^{+*}(\nu) \mathcal{J} \tilde{Z}(\lambda)$ is bounded and it is multiplied by a sum converging to zero as $k \rightarrow \infty$; thereby establishing the limit in (4.10). \square

In the last result of this section, we extend [14, Corollary 5.3] to the case of general linear dependence on the spectral parameter.

Corollary 4.4. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $f \in \ell^2_{\Psi}(\mathbb{N}_0)$, and $v \in \mathbb{C}^n$. Let system (S_{λ}) be definite on \mathbb{N}_0 , and define the function*

$$\hat{y}_k(\lambda) := \mathcal{X}_k^+(\lambda)v + \hat{z}_k(\lambda), \quad k \in \mathbb{N}_0, \tag{4.21}$$

where $\hat{z}(\lambda)$ is given in (4.7). Then, $\hat{y}(\lambda)$ represents a square summable solution of system (S_{λ}^f) satisfying the initial condition $\alpha \hat{y}_0(\lambda) = v$ and

$$\|\hat{y}(\lambda)\|_{\Psi} \leq \frac{1}{|\Im(\lambda)|} \|f\|_{\Psi} + \|\mathcal{X}^+(\lambda)v\|_{\Psi}. \tag{4.22}$$

If system (S_{λ}) is in the limit point or in the limit circle case for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, we have

$$\lim_{k \rightarrow \infty} \mathcal{X}_k^{+*}(\nu) \mathcal{J} \hat{y}_k(\lambda) = 0 \quad \text{for every } \nu \in \mathbb{C} \setminus \mathbb{R}. \tag{4.23}$$

Moreover, in the limit point case, the function $\hat{y}(\lambda)$ is the unique square summable solution of system (S_{λ}^f) satisfying $\alpha \hat{y}_0(\lambda) = v$, while in the limit circle case $\hat{y}(\lambda)$ is the unique solution of (S_{λ}^f) being in $\ell^2_{\Psi}(\mathbb{N}_0)$ such that $\alpha \hat{y}_0(\lambda) = v$ and

$$\lim_{k \rightarrow \infty} \mathcal{X}_k^{+*}(\bar{\lambda}) \mathcal{J} \hat{y}_k(\lambda) = 0. \tag{4.24}$$

Proof. Since $\mathcal{X}^+(\lambda)v$ solves system (S_{λ}) and $\Phi_0(\lambda, \alpha) = (\alpha^*, -\mathcal{J}\alpha^*)$, it follows from Theorem 4.3 that $\hat{y}(\lambda)$ solves the nonhomogeneous system (S_{λ}^f) and satisfies $\alpha \hat{y}_0(\lambda) = \alpha \mathcal{X}_0^+(\lambda)v = v$. The estimate in (4.22) follows directly from (4.21) and (4.9) by the triangle inequality. The limit in (4.23) follows in the limit circle or in the limit point case from (4.3), (4.10), and from the calculation

$$\lim_{k \rightarrow \infty} \mathcal{X}_k^{+*}(\bar{\lambda}) \mathcal{J} \hat{y}_k(\lambda) = \lim_{k \rightarrow \infty} \{ \mathcal{X}_k^{+*}(\bar{\lambda}) \mathcal{J} \mathcal{X}_k^+(\lambda) + \mathcal{X}_k^{+*}(\bar{\lambda}) \mathcal{J} \hat{z}_k(\lambda) \} = 0.$$

Finally, we prove the uniqueness of the solution in the limit point and in the limit circle case. Assume that $y^{[1]}(\lambda)$ and $y^{[2]}(\lambda)$ are two square summable solutions of (S_{λ}^f) satisfying $\alpha y_0^{[1]}(\lambda) = v = \alpha y_0^{[2]}(\lambda)$. Then the function $y_k(\lambda) := y_k^{[1]}(\lambda) - y_k^{[2]}(\lambda)$, $k \in \mathbb{N}_0$, represents a square summable solution of system (S_{λ}) , which satisfies $\alpha y_0(\lambda) = 0$. Since $y(\lambda) = \Phi(\lambda, \alpha)c$ for some $c \in \mathbb{C}^{2n}$, the initial condition $\alpha y_0(\lambda) = 0$ implies that $y_k(\lambda) = \tilde{Z}_k(\lambda)d$ for some $d \in \mathbb{C}^n$. If system (S_{λ}) is in the limit point case, we have $y(\lambda) \notin \ell^2_{\Psi}(\mathbb{N}_0)$ for $d \neq 0$, because the columns of $\tilde{Z}(\lambda)$ do not belong to $\ell^2_{\Psi}(\mathbb{N}_0)$ in this case; cf. [37, Theorem 4.4]. Therefore $d = 0$ and the uniqueness follows. On the other hand, in the limit circle case we obtain from the previous part of the proof, from the limit in (4.24), and from the first identity in (2.6) that

$$0 = \lim_{k \rightarrow \infty} \mathcal{X}_k^{+*}(\bar{\lambda}) \mathcal{J} \Phi_k(\lambda, \alpha) \begin{pmatrix} 0 \\ d \end{pmatrix} = \lim_{k \rightarrow \infty} (I \quad M_+^*(\bar{\lambda})) \Phi_k^*(\bar{\lambda}, \alpha) \mathcal{J} \Phi_k(\lambda, \alpha) \begin{pmatrix} 0 \\ d \end{pmatrix} = d,$$

which implies the uniqueness of the solution $\hat{y}(\lambda)$ also in this case. \square

5. Maximal and minimal linear relations

5.1. Linear relations and definiteness

We now return to the topic of linear relations introduced in Section 2.4 and focus on a pair of linear relations defined in terms of the linear map, \mathcal{L} , introduced in (2.12) in association with the discrete symplectic system (S_{λ}) . We point out that similar results in association with linear Hamiltonian differential

and difference systems can be found in [6,25,30]. We first introduce the maximal linear relation, T_{\max} , as a subspace of $\tilde{\ell}_{\Psi}^2(\mathbb{N}_0)^2 := \tilde{\ell}_{\Psi}^2(\mathbb{N}_0) \times \tilde{\ell}_{\Psi}^2(\mathbb{N}_0)$ defined by

$$T_{\max} := \{ \{ \tilde{z}, \tilde{f} \} \in \tilde{\ell}_{\Psi}^2(\mathbb{N}_0)^2 \mid \exists u \in \tilde{z} \text{ such that } \mathcal{L}(u) = \Psi f \}. \tag{5.1}$$

Note that when $\mathcal{L}(u) = \Psi f$, then $\mathcal{L}(u) = \Psi g$ for all $g \in \tilde{f}$. Similarly, we define a pre-minimal linear relation, T_0 , by

$$T_0 := \{ \{ \tilde{z}, \tilde{f} \} \in \tilde{\ell}_{\Psi}^2(\mathbb{N}_0)^2 \mid \exists u \in \tilde{z} \cap \ell_{\Psi,0}^2(\mathbb{N}_0) \text{ such that } \mathcal{L}(u) = \Psi f \} \subseteq T_{\max}, \tag{5.2}$$

and, by (2.16), we define

$$T_0 - \lambda I := \{ \{ \tilde{z}, \tilde{f} \} \in \tilde{\ell}_{\Psi}^2(\mathbb{N}_0)^2 \mid \exists u \in \tilde{z} \cap \ell_{\Psi,0}^2(\mathbb{N}_0) \text{ such that } \mathcal{L}(u) - \lambda \Psi u = \Psi f \} \subseteq T_{\max} - \lambda I. \tag{5.3}$$

The consideration of linear relations in our current context is natural given that the weight represented by Ψ , present in (S_{λ}^f) and the sequence spaces associated with T_{\max} and T_{\min} , has terms none of which are positive definite, but all of which are positive semi-definite; cf. (2.2). A simple example for (S_{λ}^f) , analogous to that found in [25, Section 2] in association with (1.4), illustrates.

Example 5.1. For the system (S_0^f) with $n = 1$, let

$$S_k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Psi_k = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad f_k = \begin{pmatrix} f_k^{[1]} \\ f_k^{[2]} \end{pmatrix}, \quad z_k = \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \quad k \in \mathbb{N}_0. \tag{5.4}$$

Then $\mathcal{L}(z)$ and (S_0^f) , respectively, can be written as

$$\mathcal{L}(z) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \Delta z, \quad \Delta z = \begin{pmatrix} \Delta x \\ \Delta u \end{pmatrix} = \begin{pmatrix} 0 \\ -f^{[1]} \end{pmatrix}. \tag{5.5}$$

Then, for any $f \in \ell_{\Psi}^2(\mathbb{N}_0)$,

$$z_0 = 0, \quad z_k = \begin{pmatrix} 0 \\ -\sum_{j=0}^{k-1} f_j^{[1]} \end{pmatrix}, \quad k \in \mathbb{N},$$

is a solution of (S_0^f) in (5.5). In particular, $z \in \tilde{0}$ and as a consequence, $\{ \tilde{0}, \tilde{f} \} \in T_{\max}$ for any $\tilde{f} \neq \tilde{0}$. Hence, $\text{mul } T_{\max}$, as defined in Section 2.4, is nontrivial, and T_{\max} , while a linear relation, is not a linear operator. Note also that a solution of (S_0^f) in (5.5) which is an element of $\ell_{\Psi,0}^2(\mathbb{N}_0)$ of necessity is such that $x_k = 0$ for all $k \in \mathbb{N}_0$ with the consequence that $\text{dom } T_0 = \{ \tilde{0} \}$. \square

The system (S_{λ}) given by (5.4) in Example 5.1 is not definite. The next result, characterizes definiteness of (S_{λ}) in terms of the domain of T_{\max} .

Theorem 5.2. *System (S_{λ}) is definite on \mathbb{N}_0 if and only if for any $\{ \tilde{z}, \tilde{f} \} \in T_{\max}$ there exists a unique $u \in \tilde{z}$ such that $\mathcal{L}(u) = \Psi f$.*

Proof. Assume that system (S_{λ}) be definite. Let $\{ \tilde{z}, \tilde{f} \} \in T_{\max}$, let $z^{[1]}, z^{[2]} \in \tilde{z}$, and let $y := z^{[1]} - z^{[2]}$. Then $y \in \tilde{0} \in \tilde{\ell}_{\Psi}^2(\mathbb{N}_0)$, and hence $\Psi_k y_k = 0$, for all $k \in \mathbb{N}_0$. Moreover, $\mathcal{L}(y) = 0$ implies that y is a solution of (S_{λ}) for $\lambda = 0$. Then, by Remark 3.2, the definiteness of (S_{λ}) implies that $y = 0$.

To show the converse, assume that there is only one $u \in \tilde{z}$ for which $\mathcal{L}(u) = \Psi f$, when $\{ \tilde{z}, \tilde{f} \} \in T_{\max}$. Let $\mathcal{I} \subset \mathbb{N}_0$ be a discrete finite interval such that $\text{rank } \varphi(\mathcal{I})$ is maximal; cf. Lemma 3.8. If $\text{rank } \varphi(\mathcal{I}) < 2n$,

then there is $\eta \in \mathbb{C}^{2n} \setminus \{0\}$ such that $\eta^* \varphi(0, \mathcal{I}) \eta = \sum_{k \in \mathcal{I}} \eta^* \Phi_k^* \Psi_k \Phi_k \eta = 0$, where $\Phi_k = \Phi_k(0)$. If $\sum_{k \in \mathbb{N}_0} \eta^* \Phi_k^* \Psi_k \Phi_k \eta = 0$, then $u = \Phi \eta \in \ell_{\Psi}^2(\mathbb{N}_0)$. Moreover, $\mathcal{L}(u) = 0$, and $u \in \tilde{0}$. Given that 0 is the unique sequence in $\tilde{0}$ satisfying $\mathcal{L}(z) = 0$, then $u = \Phi \eta = 0$ and as a consequence, $\eta = 0$, which contradicts the assumption that $\eta \neq 0$. Hence, there is a discrete finite interval superset, $\widehat{\mathcal{I}}$, of \mathcal{I} such that $\varphi(0, \widehat{\mathcal{I}}) \eta \neq 0$. As a result, $\ker \varphi(\widehat{\mathcal{I}}) \subset \ker \varphi(\mathcal{I})$, and hence, $\text{ran } \varphi(\mathcal{I}) \subset \text{ran } \varphi(\widehat{\mathcal{I}})$; thereby contradicting the maximality of $\text{rank } \varphi(\mathcal{I})$. Thus, $\text{rank } \varphi(\mathcal{I}) = 2n$ and (S_{λ}) is definite by [Theorem 3.9](#). \square

Remark 5.3. [Example 5.1](#) shows that the linear relation given by \mathcal{L} does not determine an operator; neither its domain is dense in $\tilde{\ell}_{\Psi}^2(\mathbb{N}_0)$. Moreover, consider the definite system in [\(3.4\)](#) from [Example 3.4](#) with $p_k \equiv 1$, $q_k \equiv 0$, and $w_k \equiv 1$ for all $k \in \mathbb{N}_0$. If we put $f_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $f_k = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $k \in \mathbb{N}$ with the notation $f_k = (f_k^{[1]}, f_k^{[2]})^{\top}$ for $k \in \mathbb{N}_0$, then the corresponding nonhomogeneous system (S_0^f) , i.e.,

$$\begin{pmatrix} x_k \\ u_k \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} + \begin{pmatrix} 0 \\ f_k^{[1]} \end{pmatrix},$$

possesses the solution $z_k = \begin{pmatrix} x_k \\ u_k \end{pmatrix}$ such that $z_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $z_k = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $k \in \mathbb{N}$, i.e., there exists $\tilde{f} \neq \tilde{0}$ such that $\{\tilde{0}, \tilde{f}\} \in T_{\max}$. This shows that definiteness of system (S_{λ}) does not suffice to prove that $\text{mul } T_{\max} = \{\tilde{0}\}$, cf. [\[33\]](#).

Thus, to guarantee that the linear relation defines an operator, we need to assume explicitly that $\text{mul } T_{\max} = \{\tilde{0}\}$, i.e., if there exists $z \in \tilde{0}$ such that $\mathcal{L}(z) = \Psi f$ for $f \in \ell_{\Psi}^2(\mathbb{N}_0)$ on \mathbb{N}_0 , then $z \equiv 0$, cf. [\[24, p. 666\]](#). In other words, we assume the definiteness of system (S_0^f) for every $f \in \ell_{\Psi}^2(\mathbb{N}_0)$. In addition, this assumption implies that system (S_{λ}) is definite as it follows from the choice $f \equiv 0$, by [Theorem 3.9\(iii\)](#) and [Corollary 3.10](#). This assumption establishes the density of $\text{dom } T_0$ in $\tilde{\ell}_{\Psi}^2(\mathbb{N}_0)$, cf. [\[24, Theorem 7.6\]](#). As noted in [\[3\]](#), a similar assumption is also needed for operators associated with system [\(1.5\)](#) in [\[31\]](#).

Theorem 5.4. *If system (S_0^f) is definite on \mathbb{N}_0 , then $\text{dom } T_0$ is dense in $\tilde{\ell}_{\Psi}^2(\mathbb{N}_0)$.*

Proof. Assume that $\text{dom } T_0$ is not dense in $\tilde{\ell}_{\Psi}^2(\mathbb{N}_0)$. Then there exists $\tilde{f} \in \text{dom}(T_0)^{\perp}$ such that $\|\tilde{f}\|_{\Psi} \neq 0$. Let $\tilde{z} \in \text{dom } T_0$ be such that $\mathcal{L}(z) = \Psi f$ and $\tilde{y} \in \text{dom } T_0$ be such that $\mathcal{L}(y) = \Psi g$ for some $g \in \ell_{\Psi}^2(\mathbb{N}_0)$. Then, by the extended Lagrange identity from [Theorem 2.5](#), we obtain

$$\langle g, z \rangle_{\Psi} - \langle y, f \rangle_{\Psi} = \lim_{k \rightarrow \infty} y_{k+1}^* \mathcal{J} z_{k+1} - y_0^* \mathcal{J} z_0 = 0,$$

because $y, z \in \ell_{\Psi,0}^2(\mathbb{N}_0)$, i.e., we have

$$\langle g, z \rangle_{\Psi} = \langle y, f \rangle_{\Psi} = 0. \tag{5.6}$$

Since $g \in \ell_{\Psi}^2(\mathbb{N}_0)$ was chosen arbitrarily and $z \in \ell_{\Psi,0}^2(\mathbb{N}_0)$, we can take $g = z$ and the solution of $\mathcal{L}(y) = \Psi z$ can be obtained as $y_k = \Phi_k \mathcal{J} \sum_{j=0}^{k-1} \Phi_j^* \Psi_j z_j$. Then, [\(5.6\)](#) implies that $\langle z, z \rangle_{\Psi} = 0$, i.e., $\Psi_k z_k = 0$ on \mathbb{N}_0 . Thus we have $\mathcal{L}(z) = \Psi f$ and $z \in \tilde{0}$, which yields $z \equiv 0$ by the definiteness assumption for system (S_0^f) . Then $\Psi_k f_k = 0$ on \mathbb{N}_0 , i.e., $f \in \tilde{0}$, and the density of $\text{dom } T_0$ in $\tilde{\ell}_{\Psi}^2(\mathbb{N}_0)$ is thus established. \square

5.2. The orthogonal decomposition of sequence spaces

Next, we introduce a linear map which will allow an orthogonal decomposition of $\tilde{\ell}_{\Psi}^2(\mathcal{I})$, where $\mathcal{I} \subset \mathbb{N}_0$ denotes a discrete finite interval. First, note that the sum $\sum_{k \in \mathcal{I}} \Phi_k^*(\bar{\lambda}) \Psi_k u_k$ is independent of $u \in \tilde{z} \in \tilde{\ell}_{\Psi}^2(\mathcal{I})$, and let $\mathcal{K}_{\lambda, \mathcal{I}}$ denote the linear map defined by

$$\mathcal{K}_{\lambda, \mathcal{I}} : \tilde{\ell}_{\Psi}^2(\mathcal{I}) \rightarrow \mathbb{C}^{2n}, \quad \mathcal{K}_{\lambda, \mathcal{I}}(\tilde{z}) := \sum_{k \in \mathcal{I}} \Phi_k^*(\bar{\lambda}) \Psi_k z_k,$$

which we will abbreviate as $\mathcal{K}_{\mathcal{I}}$ when $\lambda = 0$.

Lemma 5.5. *Let $\mathcal{I} \subset \mathbb{N}_0$ denote a finite discrete subinterval, and let $\lambda \in \mathbb{C}$. Then, $\text{ran } \mathcal{K}_{\lambda, \mathcal{I}}$ is independent of $\lambda \in \mathbb{C}$; in particular,*

$$\text{ran } \mathcal{K}_{\lambda, \mathcal{I}} = \{ \xi \in \mathbb{C}^{2n} \mid \Psi_k \Phi_k(\bar{\lambda}) \xi = 0, \forall k \in \mathcal{I} \}^{\perp} = \text{ran } \varphi(\mathcal{I}). \tag{5.7}$$

Furthermore, $\tilde{\ell}_{\Psi}^2(\mathcal{I})$ admits the following orthogonal sum decomposition:

$$\tilde{\ell}_{\Psi}^2(\mathcal{I}) = \ker \mathcal{K}_{\lambda, \mathcal{I}} \oplus \{ \tilde{z} \in \tilde{\ell}_{\Psi}^2(\mathcal{I}) \mid z = \Phi(\bar{\lambda}) \xi, \xi \in \text{ran } \varphi(\mathcal{I}) \}. \tag{5.8}$$

Proof. For any $\xi \in \mathbb{C}^{2n}$ and $\tilde{u} \in \tilde{\ell}_{\Psi}^2(\mathcal{I})$,

$$\langle \xi, \mathcal{K}_{\lambda, \mathcal{I}}(\tilde{u}) \rangle_{\mathbb{C}^{2n}} = \sum_{k \in \mathcal{I}} \xi^* \Phi_k^*(\bar{\lambda}) \Psi_k u_k = \langle \Phi(\bar{\lambda}) \xi, u \rangle_{\Psi, \mathcal{I}}.$$

Hence, $\mathcal{K}_{\lambda, \mathcal{I}}^* : \mathbb{C}^{2n} \rightarrow \tilde{\ell}_{\Psi}^2(\mathcal{I})$, where $\mathcal{K}_{\lambda, \mathcal{I}}^*(\xi) = \tilde{z}$, and $z = \Phi(\bar{\lambda}) \xi$, for $\xi \in \mathbb{C}^{2n}$. In particular,

$$\text{ran } \mathcal{K}_{\lambda, \mathcal{I}}^* = \{ \tilde{z} \in \tilde{\ell}_{\Psi}^2(\mathcal{I}) \mid z = \Phi(\bar{\lambda}) \xi, \xi \in \mathbb{C}^{2n} \},$$

and

$$\ker \mathcal{K}_{\lambda, \mathcal{I}}^* = \{ \xi \in \mathbb{C}^{2n} \mid \|\Phi(\bar{\lambda}) \xi\|_{\Psi, \mathcal{I}} = 0 \} = \{ \xi \in \mathbb{C}^{2n} \mid \Psi_k \Phi_k(\bar{\lambda}) \xi = 0, \forall k \in \mathcal{I} \}.$$

Thus, the first of the equalities in (5.7) follows from the fact that $\text{ran } \mathcal{K}_{\lambda, \mathcal{I}} = \ker(\mathcal{K}_{\lambda, \mathcal{I}}^*)^{\perp}$.

Next, note that $\tilde{\ell}_{\Psi}^2(\mathcal{I}) = \ker \mathcal{K}_{\lambda, \mathcal{I}} \oplus \ker(\mathcal{K}_{\lambda, \mathcal{I}})^{\perp} = \ker \mathcal{K}_{\lambda, \mathcal{I}} \oplus \text{ran } \mathcal{K}_{\lambda, \mathcal{I}}^*$. Let $\xi \in \mathbb{C}^{2n}$, and $\xi = \eta + \zeta$, where $\eta \in \text{ran } \mathcal{K}_{\lambda, \mathcal{I}}$, and $\zeta \in \ker(\mathcal{K}_{\lambda, \mathcal{I}})^{\perp} = \ker \mathcal{K}_{\lambda, \mathcal{I}}^*$. Then, let $z = \Phi(\bar{\lambda}) \xi$, $u = \Phi(\bar{\lambda}) \eta$, and $v = \Phi(\bar{\lambda}) \zeta$, and note that $\|v\|_{\Psi, \mathcal{I}} = \|\Phi(\bar{\lambda}) \zeta\|_{\Psi, \mathcal{I}} = 0$. Thus, $\tilde{v} = \tilde{0} \in \tilde{\ell}_{\Psi}^2(\mathcal{I})$, and as a consequence, $\tilde{z} = \tilde{u}$, and hence

$$\text{ran } \mathcal{K}_{\lambda, \mathcal{I}}^* = \{ \tilde{z} \in \tilde{\ell}_{\Psi}^2(\mathcal{I}) \mid z = \Phi(\bar{\lambda}) \eta, \eta \in \text{ran } \mathcal{K}_{\lambda, \mathcal{I}} \}.$$

Thus, to complete the demonstration of (5.7) and (5.8), it remains to show that $\text{ran } \mathcal{K}_{\lambda, \mathcal{I}} = \text{ran } \varphi(\mathcal{I})$.

Let $\tilde{z} \in \tilde{\ell}_{\Psi}^2(\mathcal{I})$. Then $\tilde{z} = \tilde{v} + \tilde{u}$, where $\tilde{v} \in \ker \mathcal{K}_{\lambda, \mathcal{I}}$, and $u = \Phi(\bar{\lambda}) \eta$, where $\eta \in \text{ran } \mathcal{K}_{\lambda, \mathcal{I}}$. Then,

$$\mathcal{K}_{\lambda, \mathcal{I}}(\tilde{z}) = \mathcal{K}_{\lambda, \mathcal{I}}(\tilde{u}) = \sum_{k \in \mathcal{I}} \Phi_k^*(\bar{\lambda}) \Psi_k \Phi_k(\bar{\lambda}) \eta = \varphi(\lambda, \mathcal{I}) \eta,$$

and hence, $\text{ran } \mathcal{K}_{\lambda, \mathcal{I}} \subseteq \text{ran } \varphi(\mathcal{I})$. On the other hand, if $\xi \in \mathbb{C}^{2n}$ and $z = \Phi(\bar{\lambda}) \xi$, then

$$\varphi(\lambda, \mathcal{I}) \xi = \sum_{k \in \mathcal{I}} \Phi_k^*(\bar{\lambda}) \Psi_k \Phi_k(\bar{\lambda}) \xi = \sum_{k \in \mathcal{I}} \Phi_k^*(\bar{\lambda}) \Psi_k z_k = \mathcal{K}_{\lambda, \mathcal{I}}(\tilde{z}),$$

thus showing that $\text{ran } \mathcal{K}_{\lambda, \mathcal{I}} = \text{ran } \varphi(\mathcal{I})$. \square

In analogy with $\mathcal{K}_{\lambda, \mathcal{I}}$, we define the linear map

$$\mathcal{K}_{\lambda} : \tilde{\ell}_{\Psi, 1}^2(\mathbb{N}_0) \rightarrow \mathbb{C}^{2n}, \quad \mathcal{K}_{\lambda}(\tilde{z}) := \sum_{k \in \mathbb{N}_0} \Phi_k^*(\bar{\lambda}) \Psi_k z_k,$$

which we abbreviate as \mathcal{K} when $\lambda = 0$. We note that the sum $\sum_{k \in \mathbb{N}_0} \Phi_k^*(\bar{\lambda}) \Psi_k u_k$ is independent of $u \in \tilde{\ell}_{\Psi,1}^2(\mathbb{N}_0)$.

For the next two lemmas, recall from Lemma 3.8 (viz. (3.7)) that there is a discrete finite interval $\mathcal{I} \subset \mathbb{N}_0$ for which $\text{rank } \varphi(\mathcal{I})$ is maximal; that is, $\text{rank } \varphi(\widehat{\mathcal{I}}) = \text{rank } \varphi(\mathcal{I})$, for any discrete finite interval $\widehat{\mathcal{I}}$ such that $\mathcal{I} \subseteq \widehat{\mathcal{I}}$. By (5.7), it also follows that if \mathcal{I} is a discrete finite interval for which $\text{rank } \varphi(\mathcal{I})$ is maximal, then $\text{ran } \mathcal{K}_{\lambda, \widehat{\mathcal{I}}} = \text{ran } \mathcal{K}_{\lambda, \mathcal{I}}$, when $\widehat{\mathcal{I}}$ is a discrete finite interval for which $\mathcal{I} \subseteq \widehat{\mathcal{I}}$.

Lemma 5.6. *Let $\mathcal{I} \subset \mathbb{N}_0$ be a discrete finite interval for which $\text{rank } \varphi(\mathcal{I})$ is maximal. Then, $\text{ran } \mathcal{K}_\lambda = \text{ran } \varphi(\mathcal{I})$, for all $\lambda \in \mathbb{C}$; in particular, $\text{ran } \mathcal{K}_\lambda$ is independent of λ .*

Proof. Let $\tilde{z} \in \text{dom } \mathcal{K}_\lambda = \tilde{\ell}_{\Psi,1}^2(\mathbb{N}_0)$. Then, there exists an $N \in \mathbb{N}_0$ such that $\Psi_k z_k = 0$ for $k \geq N$; hence, $z \in \ell_{\Psi}^2([0, N]_{\mathbb{Z}})$, and $\tilde{z} \in \tilde{\ell}_{\Psi}^2([0, N]_{\mathbb{Z}}) = \text{dom } \mathcal{K}_{\lambda, [0, N]_{\mathbb{Z}}}$. In particular, $\mathcal{K}_{\lambda, [0, N]_{\mathbb{Z}}}(\tilde{z}) = \mathcal{K}_\lambda(\tilde{z})$, and thus, $\text{ran } \mathcal{K}_\lambda \subseteq \text{ran } \mathcal{K}_{\lambda, [0, N]_{\mathbb{Z}}}$. Without loss of generality, we may assume that $\mathcal{I} \subseteq [0, N]_{\mathbb{Z}}$, and hence that $\text{ran } \mathcal{K}_{\lambda, \mathcal{I}} = \text{ran } \mathcal{K}_{\lambda, [0, N]_{\mathbb{Z}}} \supseteq \text{ran } \mathcal{K}_\lambda$.

Conversely, suppose that $\tilde{z} \in \tilde{\ell}_{\Psi}^2(\mathcal{I}) = \text{dom } \mathcal{K}_{\lambda, \mathcal{I}}$. Let $u \in \ell_{\Psi,1}^2(\mathbb{N}_0)$ be defined by

$$u_k = \begin{cases} z_k, & k \in \mathcal{I}, \\ 0, & k \notin \mathcal{I}. \end{cases}$$

Then, $\mathcal{K}_{\lambda, \mathcal{I}}(\tilde{z}) = \sum_{k \in \mathcal{I}} \Phi_k^*(\bar{\lambda}) \Psi_k z_k = \sum_{k \in \mathbb{N}_0} \Phi_k^*(\bar{\lambda}) \Psi_k u_k = \mathcal{K}_\lambda(\tilde{u})$. Hence, $\text{ran } \mathcal{K}_{\lambda, \mathcal{I}} \subseteq \text{ran } \mathcal{K}_\lambda$, thus $\text{ran } \mathcal{K}_\lambda = \text{ran } \mathcal{K}_{\lambda, \mathcal{I}}$; which by Lemma 5.5 yields $\text{ran } \mathcal{K}_\lambda = \text{ran } \varphi(\mathcal{I})$. \square

Lemma 5.7. *Let $\mathcal{I} \subset \mathbb{N}_0$ be a discrete finite interval for which $\text{rank } \varphi(\mathcal{I})$ is maximal. Then,*

$$\tilde{\ell}_{\Psi,1}^2(\mathbb{N}_0) = \ker \mathcal{K}_\lambda \oplus \{ \tilde{z} \in \tilde{\ell}_{\Psi,1}^2(\mathbb{N}_0) \mid z = \Phi(\bar{\lambda})\xi, \xi \in \text{ran } \varphi(\mathcal{I}) \}$$

and $\text{codim}(\ker \mathcal{K}_\lambda) = \text{rank } \varphi(\mathcal{I})$.

Proof. In the complete analogy with the argument given in the proof of Lemma 5.5, one can show that

$$\tilde{\ell}_{\Psi,1}^2(\mathbb{N}_0) = \ker \mathcal{K}_\lambda \oplus \ker(\mathcal{K}_\lambda)^\perp = \ker \mathcal{K}_\lambda \oplus \text{ran } \mathcal{K}_\lambda^*,$$

where

$$\text{ran } \mathcal{K}_\lambda^* = \{ \tilde{z} \in \tilde{\ell}_{\Psi,1}^2(\mathbb{N}_0) \mid z = \Phi(\bar{\lambda})\xi, \xi \in \text{ran } \varphi(\mathcal{I}) \},$$

and that

$$\text{ran}(\mathcal{K}_\lambda)^\perp = \ker \mathcal{K}_\lambda^* = \{ \xi \in \mathbb{C}^{2n} \mid \|\Phi(\bar{\lambda})\xi\|_{\Psi} = 0 \} = \{ \xi \in \mathbb{C}^{2n} \mid \Psi_k \Phi_k(\bar{\lambda})\xi = 0, \forall k \in \mathbb{N}_0 \}.$$

Let $\xi_j, j = 1, \dots, m$, denote a basis for $\text{ran } \varphi(\mathcal{I}) = \text{ran } \mathcal{K}_\lambda$, let $u^{[j]} := \Phi(\bar{\lambda})\xi_j$, and thus $\tilde{u}^{[j]} \in \text{ran } \mathcal{K}_\lambda^*$, for $j = 1, \dots, m$. Suppose that $\sum_{j=1}^m \alpha_j \tilde{u}^{[j]} = \tilde{0} \in \tilde{\ell}_{\Psi,1}^2(\mathbb{N}_0)$. Then, $\Psi_k \Phi_k(\bar{\lambda})(\sum_{j=1}^m \alpha_j \xi_j) = 0$, for all $k \in \mathbb{N}_0$. As a consequence,

$$\sum_{j=1}^m \alpha_j \xi_j \in \text{ran } \mathcal{K}_\lambda \cap \text{ran}(\mathcal{K}_\lambda)^\perp = \{0\},$$

and hence, $\alpha_j = 0, j = 1, \dots, m$; thus, $\tilde{u}^{[j]} \in \text{ran } \mathcal{K}_\lambda^*, j = 1, \dots, m$, are independent in $\tilde{\ell}_{\Psi,1}^2(\mathbb{N}_0)$. Hence, $\text{codim}(\ker \mathcal{K}_\lambda) = \dim(\text{ran}(\mathcal{K}_\lambda^*)) = \text{rank } \varphi(\mathcal{I})$. \square

5.3. The minimal relation and its deficiency indices

Before we define the minimal linear relation T_{\min} and establish the fundamental relation between T_{\max} and T_{\min} , we prove two auxiliary lemmas.

Lemma 5.8. *We have $\ker \mathcal{K}_\lambda \subseteq \text{ran}(T_0 - \lambda I)$ for every $\lambda \in \mathbb{C}$.*

Proof. Let $\lambda \in \mathbb{C}$ and $\tilde{f} \in \ker \mathcal{K}_\lambda \subseteq \tilde{\ell}_{\Psi,1}^2(\mathbb{N}_0)$. Then, there is $N \in \mathbb{N}_0$ such that $\Psi_k g_k = 0$ for all $k \geq N$ and all $g \in \tilde{f}$. Moreover, $\mathcal{K}_\lambda(\tilde{f}) = \sum_{k \in \mathbb{N}_0} \Phi_k^*(\bar{\lambda}) \Psi_k g_k = 0$ for all $g \in \tilde{f}$.

For $g \in \tilde{f}$, we note that

$$z_k := -\Phi_k(\lambda) \mathcal{J} \sum_{j=k}^{\infty} \Phi_j^*(\bar{\lambda}) \Psi_j g_j = \begin{cases} -\Phi_k(\lambda) \mathcal{J} \sum_{j=k}^{N-1} \Phi_j^*(\bar{\lambda}) \Psi_j g_j, & k \in [0, N-1]_{\mathbb{Z}}, \\ 0, & k \in [N, \infty)_{\mathbb{Z}}, \end{cases} \tag{5.9}$$

defines a sequence $z \in \ell_{\Psi}^2(\mathbb{N}_0)$ such that $\mathcal{L}(z) = \lambda \Psi z + \Psi g$ (cf. [19, Theorem 3.17]) for which $z_0 = \Phi_0(\lambda) \mathcal{J} \mathcal{K}_\lambda(g) = 0$. Thus, as defined, $z \in \ell_{\Psi,0}^2(\mathbb{N}_0)$ and satisfies $\mathcal{L}(z) - \lambda \Psi z = \Psi g$; thereby showing that $\tilde{f} \in \text{ran}(T_0 - \lambda I)$; cf. (5.3). \square

Lemma 5.9. *We have $\langle \tilde{f}, \tilde{y} \rangle_{\Psi} = \langle \tilde{z}, \tilde{g} \rangle_{\Psi}$ for every $\{\tilde{z}, \tilde{f}\} \in T_0$ and $\{\tilde{y}, \tilde{g}\} \in T_{\max}$.*

Proof. Let $\{\tilde{z}, \tilde{f}\} \in T_0$ and $\{\tilde{y}, \tilde{g}\} \in T_{\max}$. Let $z \in \ell_{\Psi,0}^2(\mathbb{N}_0)$ such that $\mathcal{L}(z) = \Psi f$, and let $y \in \ell_{\Psi}^2(\mathbb{N}_0)$ such that $\mathcal{L}(y) = \Psi g$. Then, by the extended Lagrange identity in (2.8) with $\lambda = \nu = 0$, we see that

$$\langle z, g \rangle_{\Psi, [0,k]_{\mathbb{Z}}} = -z_j^* \mathcal{J} y_j \Big|_0^{k+1} + \sum_{j=0}^k f_j^* \Psi_j y_j = -z_j^* \mathcal{J} y_j \Big|_0^{k+1} + \langle f, y \rangle_{\Psi, [0,k]_{\mathbb{Z}}}.$$

However, $z \in \ell_{\Psi,0}^2(\mathbb{N}_0)$ implies that $z_0 = 0$ and that $z_k = 0$ for sufficiently large $k \in \mathbb{N}_0$. As a consequence we see that $\langle z, g \rangle_{\Psi} = \langle f, y \rangle_{\Psi}$, and hence, $\langle u, g \rangle_{\Psi} = \langle f, w \rangle_{\Psi}$ for all $u \in \tilde{z}$, and $w \in \tilde{y}$. \square

By Lemma 5.9 and (2.14), one obtains

$$T_0 \subseteq T_{\max} \subseteq T_0^*, \tag{5.10}$$

from which we conclude that T_0 is symmetric in $\tilde{\ell}_{\Psi}^2(\mathbb{N}_0)^2$, and hence closable. We then define the minimal operator, T_{\min} , by

$$T_{\min} := \overline{T_0}.$$

Another approach to defining a minimal operator is to let $T_{\min} := T_{\max}^*$; cf. [25, Definition 2.3] and [6, Identity (4.2)]. In the examples cited, this is equivalent to our choice of definition for T_{\min} . The next theorem demonstrates this for the operators at hand. We note also that an alternative demonstration of the next result can be patterned after that presented in [6, pp. 1354–1355] using the results in Section 4 and [6, Proposition A.2].

Theorem 5.10. *For T_{\max} and T_0 as defined in (5.1) and (5.2), respectively,*

$$T_0^* = T_{\min}^* = T_{\max}. \tag{5.11}$$

Proof. By (5.10), note that $T_{\max} \subseteq T_0^* = \overline{T_0}^* = T_{\min}^*$. Thus, it remains to show that $T_0^* \subseteq T_{\max}$; or equivalently, given $\{\tilde{z}, \tilde{f}\} \in T_0^*$, that there is a $v \in \tilde{z} \in \tilde{\ell}_{\Psi}^2(\mathbb{N}_0)$ such that $\mathcal{L}(v) = \Psi f$.

Now, let $\{\tilde{u}, \tilde{g}\} \in T_0$ and note that there exists $y \in \tilde{u}$ such that $y \in \ell_{\Psi,0}^2(\mathbb{N}_0)$ and $\mathcal{L}(y)_k = \Psi_k g_k$ for $k \in \mathbb{N}_0$. Next let $w \in \mathbb{C}(\mathbb{N}_0)^{2n}$ be a sequence such that $\mathcal{L}(w) = \Psi f$, where $\tilde{f} \in T_0^*(\tilde{z})$. Then, by the Lagrange identity (2.8), we see that $\langle w, g \rangle_{\Psi, [0,k]_{\mathbb{Z}}} = w_j^* \mathcal{J} y_j|_0^{k+1} + \langle f, y \rangle_{\Psi, [0,k]_{\mathbb{Z}}}$. Given that $y_0 = 0$ and the $y_k = 0$ for sufficiently large $k \in \mathbb{N}_0$, we see that $\langle w, g \rangle_{\Psi} = \langle f, y \rangle_{\Psi}$. From the definition of T_0^* , note that $\langle \tilde{z}, \tilde{g} \rangle_{\Psi} = \langle \tilde{f}, \tilde{u} \rangle_{\Psi}$. As a consequence, $\langle z - w, g \rangle_{\Psi} = 0$ for any $z \in \tilde{z} \in \text{dom } T_0^*$, and any $g \in \tilde{g} \in \text{ran } T_0$. Recall from Lemma 5.8, with $\lambda = 0$, that $\ker \mathcal{K} \subseteq \text{ran } T_0 \subseteq \tilde{\ell}_{\Psi,1}^2(\mathbb{N}_0)$. Then, for every $g \in \tilde{g} \in \ker \mathcal{K}$ we see that $\langle z - w, g \rangle_{\Psi} = 0$, and that $\sum_{k \in \mathbb{N}} \Phi_k^* \Psi_k g_k = 0$ (i.e., $\Phi_k = \Phi_k(0)$); and as a consequence, for any $\xi \in \mathbb{C}^{2n}$, that

$$\sum_{k=0}^{\infty} (z_k - w_k - \Phi_k \xi)^* \Psi_k g_k = 0. \tag{5.12}$$

Next, let $\mathcal{I} \subset \mathbb{N}_0$ be a finite discrete interval such that $\text{rank } \varphi(\mathcal{I}) = m \leq 2n$ is maximal as noted in Lemma 3.8. Then, by Lemma 5.7, we see that $\tilde{\ell}_{\Psi,1}^2(\mathbb{N}_0) = \ker \mathcal{K} \oplus \ker \mathcal{K}^{\perp}$ and that there is a basis, $\tilde{y}^{[1]}, \dots, \tilde{y}^{[m]} \in \tilde{\ell}_{\Psi,1}^2(\mathbb{N}_0)$, for $\ker \mathcal{K}^{\perp}$, in which $y^{[j]} = \Phi \xi^{[j]}$, and where $\xi^{[j]}$, $j = 1, \dots, m$, forms a basis for $\text{ran } \varphi(\mathcal{I})$. Then, there is discrete finite interval $\hat{\mathcal{I}} = [0, N]_{\mathbb{Z}}$, where $\mathcal{I} \subseteq \hat{\mathcal{I}}$, for which

$$\Psi_k y_k^{[j]} = 0, \quad k \geq N, \quad j = 1, \dots, m. \tag{5.13}$$

Now, suppose that $\tilde{g} \in \ker \mathcal{K}_{\lambda, \hat{\mathcal{I}}}$, then, for $j = 1, \dots, m$,

$$\langle y^{[j]}, g \rangle_{\Psi, \hat{\mathcal{I}}} = \sum_{k \in \hat{\mathcal{I}}} \xi^{[j]*} \Phi_k^* \Psi_k g_k = \xi^{[j]*} \mathcal{K}_{\lambda, \hat{\mathcal{I}}}(\tilde{g}) = 0;$$

and thus, $\tilde{y}^{[j]} \in \ker(\mathcal{K}_{\lambda, \hat{\mathcal{I}}})^{\perp}$, for $j = 1, \dots, m$.

Let $x \in \ell_{\Psi,1}^2(\mathbb{N}_0)$ be defined by

$$x_k = \begin{cases} z_k - w_k, & k \in \hat{\mathcal{I}}, \\ 0, & k \in \mathbb{N}_0 \setminus \hat{\mathcal{I}}, \end{cases}$$

where $\{\tilde{z}, \tilde{f}\} \in T_0^*$ and $w \in \mathbb{C}(\mathbb{N}_0)^{2n}$ with $\mathcal{L}(w) = \Psi f$. Now, $\tilde{x} \in \tilde{\ell}_{\Psi}^2(\mathcal{I})$ and, by Lemma 5.5, there is $\tilde{v} \in \ker(\mathcal{K}_{\lambda, \hat{\mathcal{I}}})^{\perp}$, where $v = \Phi \eta$ for some $\eta \in \text{ran } \varphi(\hat{\mathcal{I}})$, such that $\tilde{x} - \tilde{v} \in \ker \mathcal{K}_{\lambda, \hat{\mathcal{I}}}$. As a consequence $\langle \tilde{x} - \tilde{v}, \tilde{y}^{[j]} \rangle_{\Psi, \hat{\mathcal{I}}} = 0$ for all $j = 1, \dots, m$. Then, by (5.13), we obtain, for $j = 1, \dots, m$, that

$$\sum_{k \in \mathbb{N}_0} (z_k - w_k - \Phi_k \eta)^* \Psi_k y_k^{[j]} = \sum_{k \in \hat{\mathcal{I}}} (z_k - w_k - \Phi_k \eta)^* \Psi_k y_k^{[j]} = \langle \tilde{x} - \tilde{v}, \tilde{y}^{[j]} \rangle_{\Psi, \hat{\mathcal{I}}} = 0. \tag{5.14}$$

Together, (5.12) and (5.14) show that $\langle z - w - \Phi \eta, h \rangle_{\Psi} = 0$ for all $h \in \ell_{\Psi,1}^2(\mathbb{N}_0)$; in particular, for

$$h_k = \begin{cases} z_{k_0} - w_{k_0} - \Phi_{k_0} \eta, & k = k_0, \\ 0, & k \in \mathbb{N}_0 \setminus \{k_0\}. \end{cases}$$

Thus, for any $k_0 \in \mathbb{N}_0$, we see that $\Psi_{k_0}(z_{k_0} - w_{k_0} - \Phi_{k_0} \eta) = 0$, and hence $\|z - (w + \Phi \eta)\|_{\Psi} = 0$. In summary, we get that $w + \Phi \eta \in \tilde{z}$ where $\{\tilde{z}, \tilde{f}\} \in T_0^*$, and further that $\mathcal{L}(w + \Phi \eta) = \mathcal{L}(w) = \Psi f$; hence, showing that $T_0^* \subseteq T_{\max}$. \square

Recalling the definition of deficiency subspaces and indices from Section 2.4 (viz. (2.17), (2.18)), let

$$M_\lambda := \{z \in \ell^2_\Psi(\mathbb{N}_0) \mid \mathcal{L}(z)_k = \lambda \Psi_k z_k, k \in \mathbb{N}_0\}, \quad d_\lambda := \dim M_\lambda,$$

$$\widetilde{M}_\lambda := \{\tilde{z} \in \widetilde{\ell}^2_\Psi(\mathbb{N}_0) \mid \{\tilde{z}, \lambda \tilde{z}\} \in T_{\max} = T_{\min}^*\}, \quad \tilde{d}_\lambda := \dim \widetilde{M}_\lambda.$$

In other words, the subspace $M_\lambda \subseteq \ell^2_\Psi(\mathbb{N}_0)$ consists of all square summable solutions of (S_λ) , while d_λ denotes the number of linearly independent square summable solutions of (S_λ) . Then, $\widetilde{M}_\lambda = \widetilde{M}_\lambda(T_{\min})$ represents the deficiency space, and $\tilde{d}_\lambda = \tilde{d}_\lambda(T_{\min})$ the deficiency index, for T_{\min} and $\lambda \in \mathbb{C}$. It was shown in Theorem 5.10 that T_{\min} is a closed, symmetric linear relation, and hence \tilde{d}_λ is constant on each of the open upper and lower half-planes of \mathbb{C} ; cf. [32, Theorem 2.13]. Thus we let $\tilde{d}_+ := \tilde{d}_i = \tilde{d}_\lambda$, for $\lambda \in \mathbb{C}_+$, and $\tilde{d}_- := \tilde{d}_{-i} = \tilde{d}_\lambda$, for $\lambda \in \mathbb{C}_-$. Then, with the aid of [6, Proposition A.1] we obtain the following von Neumann decomposition of the maximal linear relation T_{\max} in terms of the minimal linear relation T_{\min} and the defect subspaces \widetilde{M}_λ and $\widetilde{M}_{\bar{\lambda}}$, i.e.,

$$T_{\max} = T_{\min} \dot{+} \widetilde{M}_\lambda \dot{+} \widetilde{M}_{\bar{\lambda}},$$

where the direct sum $\dot{+}$ is orthogonal if $\lambda = \pm i$.

Assuming that $\text{rank } \varphi(\mathcal{I})$ is maximal, as described in Lemma 3.8, we next show a relationship between d_λ and \tilde{d}_λ .

Theorem 5.11. *Let $\mathcal{I} \subset \mathbb{N}_0$ be a discrete finite interval such that $\text{rank } \varphi(\mathcal{I})$ is maximal. Then, for any $\lambda \in \mathbb{C}$,*

$$d_\lambda = \tilde{d}_\lambda + 2n - \text{rank } \varphi(\mathcal{I}). \tag{5.15}$$

Proof. Let $\lambda \in \mathbb{C}$, and $\tilde{z} \in \widetilde{M}_\lambda$. Then there exists $z \in \tilde{z}$ which solves system (S_λ) , i.e., $z \in M_\lambda$. With π_1 denoting the restriction of the quotient space map of Section 2.3 given by $\pi_1 := \pi|_{M_\lambda} : M_\lambda \rightarrow \widetilde{M}_\lambda$, we note that

$$\ker \pi_1 = \{z \in \ell^2_\Psi(\mathbb{N}_0) \mid \mathcal{L}(z)_k = \lambda \Psi_k z_k \text{ and } \Psi_k z_k = 0 \text{ for } k \in \mathbb{N}_0\}.$$

Then $z \in \ker \pi_1$ if and only if $z = \Phi \xi$ for some $\xi \in \bigcap_{n \in \mathbb{N}_0} \ker \varphi([0, n]_{\mathbb{Z}}) = \ker \varphi(\mathcal{I})$. Hence $\dim(\ker \pi_1) = \dim(\ker \varphi(\mathcal{I})) = 2n - \text{rank } \varphi(\mathcal{I})$. Since the map π_1 is surjective,

$$\dim M_\lambda = \dim \widetilde{M}_\lambda + \dim(\ker \pi_1),$$

which implies (5.15). \square

The following properties for d_λ and \tilde{d}_λ are a direct consequence of (5.15).

Corollary 5.12. *Let $\lambda \in \mathbb{C}$. Then,*

- (i) $\tilde{d}_\lambda = d_\lambda$ if and only if system (S_λ) is definite on \mathbb{N}_0 ;
- (ii) $d_\lambda - \tilde{d}_\lambda$ is nonnegative and constant for $\lambda \in \mathbb{C}$;
- (iii) d_λ is constant in $\mathbb{C}_+ := \{\lambda \in \mathbb{C} \mid \Im(\lambda) > 0\}$ and in $\mathbb{C}_- := \{\lambda \in \mathbb{C} \mid \Im(\lambda) < 0\}$.

Proof. The first statement follows directly from (5.15) and Theorem 3.9, while the second statement is a consequence of (5.15) and Lemma 3.7. Since the linear relation T_0 is symmetric (cf. (5.10)), the number \tilde{d}_λ is constant in \mathbb{C}_+ and \mathbb{C}_- by [32, Theorem 2.13]. Hence, the value of d_λ is also constant in \mathbb{C}_+ and \mathbb{C}_- by Theorem 5.11, and by the independence of $\text{rank } \varphi(\mathcal{I})$ on the value of λ established in Lemma 3.7. \square

Remark 5.13. The last statement of [Corollary 5.12](#) extends the enumeration and analysis of linearly independent square summable solutions for system (S_λ) found in [\[37\]](#) (viz. [\[37, Theorem 4.9\]](#)) by showing that $r(\lambda)$ defined in [\[37, Identity \(4.1\)\]](#) is constant in \mathbb{C}_+ and \mathbb{C}_- .

Theorem 5.14. For $\lambda \in \mathbb{C}$ we have

$$\ker(T_{\min} - \lambda I) = \{\tilde{0}\}. \quad (5.16)$$

Moreover, for every $\lambda \in \mathbb{R}$ we have $\tilde{d}_\lambda \leq \tilde{d}_\pm$.

Proof. First, note that $\tilde{\ell}_{\Psi,1}^2(\mathbb{N}_0)$ is dense in $\tilde{\ell}_{\Psi}^2(\mathbb{N}_0)$. Then, for $\lambda \in \mathbb{C}$ and $\tilde{f} \in \tilde{\ell}_{\Psi,1}^2(\mathbb{N}_0)$, and with $z \in \ell_{\Psi,1}^2(\mathbb{N}_0)$ as defined in [\(5.9\)](#), we note, by the construction given in [Lemma 5.8](#), that $\{\tilde{z}, \tilde{f}\} \in T_{\max} - \lambda I$. This shows that $\tilde{\ell}_{\Psi,1}^2(\mathbb{N}_0) \subseteq \text{ran}(T_{\max} - \lambda I)$ and hence that $\text{ran}(T_{\max} - \lambda I)$ is dense in $\tilde{\ell}_{\Psi}^2(\mathbb{N}_0)$ for any $\lambda \in \mathbb{C}$. As a consequence of [\(5.11\)](#), we get that $\ker(T_{\min} - \lambda I) = \text{ran}(T_{\max} - \bar{\lambda}I)^\perp = \{\tilde{0}\}$.

The remaining assertion follows from [\[25, Lemma 2.25\]](#) and [\(5.16\)](#), as noted in the comments associated with [\(2.19\)](#) at the end of [Section 2.4](#). \square

By [Corollary 3.10](#), [Corollary 5.12\(i\)](#), and [Theorem 5.14](#) we obtain the following statement.

Corollary 5.15. When system (S_λ) is definite on \mathbb{N}_0 , then $d_\lambda \leq d_\pm$ for any $\lambda \in \mathbb{R}$.

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