Doctopic: Partial Differential Equations ARTICLE IN PRESS YJMAA:18708

J. Math. Anal. Appl. • • • (• • • •) • • •–• • •

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Note

Sharp constant and extremal function for weighted Trudinger–Moser type inequalities in $\mathbb{R}^2$

Francisco S.B. Albuquerque

Universidade Estadual da Paraíba, Centro de Ciências Exatas e Sociais Aplicadas, 58700-070, Patos, PB, Brazil

A R T I C L E I N F O

Article history:
Received 9 March 2014
Available online xxxx
Submitted by M. del Pino

Keywords:
Trudinger–Moser inequality
Extremal function
Weighted Sobolev spaces

A B S T R A C T

In this note, we prove the sharpness and the existence of extremal function for a Trudinger–Moser type inequality in weighted Sobolev spaces established by Albuquerque–Alves–Medeiros in 2014 (see [6, Theorem 1.1]).

1. Introduction and main notation

We recall that if Ω is a bounded domain in $\mathbb{R}^2$, the classical Trudinger–Moser inequality (cf. [11,16]) asserts that $e^{\alpha u^2} \in L^1(\Omega)$ for all $u \in H_0^1(\Omega)$ and $\alpha > 0$. Moreover, there exists a constant $C = C(\Omega) > 0$ such that

$$\sup_{\|u\|_{H_0^1(\Omega)} \leq 1} \int_\Omega e^{\alpha u^2} dx \leq C, \quad \text{if } \alpha \leq 4\pi, \tag{1.1}$$

where $\|u\|_{H_0^1(\Omega)} = (\int_\Omega |\nabla u|^2 dx)^{1/2}$. Furthermore, (1.1) is sharp in the sense that if $\alpha > 4\pi$ the supremum (1.1) is $+\infty$. Related inequalities for unbounded domains have been proposed by Cao [7] and Ruf [12] (and by Adachi–Tanaka [1], do Ó [8] and Li–Ruf [10] in general dimension). However in [1,7,8] they assumed a growth $e^{\alpha u^2}$ with $\alpha < 4\pi$, i.e. with subcritical growth. See also Adams [2]. In [12], the author proved that there exists a constant $d > 0$ such that for any domain $\Omega \subset \mathbb{R}^2$,

$$\sup_{\|u\|_S \leq 1} \int_\Omega \left(e^{4\pi u^2} - 1\right) dx \leq d, \tag{1.2}$$

E-mail address: siberio_ba@uepb.edu.br.

http://dx.doi.org/10.1016/j.jmaa.2014.07.035
0022-247X/

Please cite this article in press as: F.S.B. Albuquerque, Sharp constant and extremal function for weighted Trudinger–Moser type inequalities in $\mathbb{R}^2$, J. Math. Anal. Appl. (2014), http://dx.doi.org/10.1016/j.jmaa.2014.07.035

where $\|u\|_S = (\int_\Omega (|\nabla u|^2 + |u|^2)dx)^{1/2}$. Moreover, the inequality (1.2) is sharp in the sense that for any growth $e^{\alpha u^2}$ with $\alpha > 4\pi$ the supremum (1.2) is $+\infty$. Furthermore, he proved that the supremum (1.2) is attained whenever it is finite. On the other hand, Adimurthi–Sandeep [3] extended the Trudinger–Moser inequality (1.1) for singular weights. More precisely, they proved that if Ω is a bounded domain in $\mathbb{R}^2$ containing the origin, $u \in H_0^1(\Omega)$ and $\beta \in [0, 2)$, then

$$\sup_{\|u\|_{H_0^1(\Omega)} \leq 1} \int_\Omega \frac{e^{\alpha u^2} - 1}{|x|^\beta} dx < +\infty \quad \Leftrightarrow \quad 0 < \alpha \leq 4\pi(1 - \beta/2). \tag{1.3}$$

Later, do Ó–de Souza in [9] investigated the Trudinger–Moser type inequality also with a singular weight for any domain $\Omega \subset \mathbb{R}^2$ containing the origin as well as some applications. More precisely, they proved that if $\alpha > 0$, $\beta \in [0, 2)$ is such that $\alpha/4\pi + \beta/2 < 1$ and $\|u\|_{L^2(\Omega)} \leq M$, then there exists a constant $C = C(\alpha, M) > 0$ (independent of Ω) such that

$$\sup_{\|\nabla u\|_{L^2(\Omega)} \leq 1} \int_\Omega \frac{e^{\alpha u^2} - 1}{|x|^\beta} dx \leq C(\alpha, M) \tag{1.4}$$

and the above inequality does not hold if $\alpha/4\pi + \beta/2 > 1$. We also refer the reader to [4] for a Trudinger–Moser type inequality with a singular weight in high dimensions.

Throughout the note, we consider weight functions $V(|x|)$ and $Q(|x|)$ satisfying the following assumptions:

(V) $V \in C(0, \infty)$, $V(r) > 0$ and there exist $a, a_0 > -2$ such that

$$\limsup_{r \to 0} \frac{V(r)}{r^{a_0}} < \infty \quad \text{and} \quad \liminf_{r \to +\infty} \frac{V(r)}{r^a} > 0.$$

(Q) $Q \in C(0, \infty)$, $Q(r) > 0$ and there exist $b < (a - 2)/2$ and $-2 < b_0 \leq 0$ such that

$$0 < \liminf_{r \to 0} \frac{Q(r)}{r^{b_0}} \leq \limsup_{r \to 0} \frac{Q(r)}{r^{b_0}} < \infty \quad \text{and} \quad \limsup_{r \to +\infty} \frac{Q(r)}{r^b} < \infty.$$

Remark 1.1.

1) Singular weight functions of the form

$$V(x) = |x|^\alpha \quad \text{and} \quad Q(x) = |x|^\beta$$

with $2(\beta + 1) < \alpha < 0$ are simple examples of functions that satisfy (V) and (Q), respectively. Indeed, just take $a = a_0 = \alpha$ and $b = b_0 = \beta$.

2) Notice that the condition on V at the origin in (V) implies that there exist $r_0 > 0$ and $C_0 > 0$ such that

$$V(|x|) \leq C_0 |x|^{a_0}, \quad \text{for all } 0 < |x| \leq r_0. \tag{1.5}$$

Notation. In order to establish our main results, we need to recall some notation.

- In all the integrals we omit the symbol dx and we use $C, C_0, C_1, C_2, \ldots$ to denote (possibly different) positive constants.
- $B_r \subset \mathbb{R}^2$ denotes the open ball centered at the origin with radius $r > 0$ and $B_R \setminus B_r$ denotes the annulus with interior radius r and exterior radius R. For any set $A \subset \mathbb{R}^2$, A^c denotes the complement of A.

- $C_0^\infty(\mathbb{R}^2)$ denotes the set of smooth functions with compact support.
- $C_{0,\mathrm{rad}}^\infty(\mathbb{R}^2) = \{u \in C_0^\infty(\mathbb{R}^2)\colon\ u \text{ is radial}\}$.
- $D_{\mathrm{rad}}^{1,2}(\mathbb{R}^2)$ denotes $\overline{C_{0,\mathrm{rad}}^\infty(\mathbb{R}^2)}$ under the norm $\|\nabla u\|_{L^2(\mathbb{R}^2)}$.
- If $1 \le p < \infty$ we define

$$L^p\left(\mathbb{R}^2; Q\right) \doteq \left\{u : \mathbb{R}^2 \to \mathbb{R}\colon\ u \text{ is measurable}, \int_{\mathbb{R}^2} Q\left(|x|\right)|u|^p < \infty\right\}.$$

Similarly we define $L^2(\mathbb{R}^2; V)$. Then we set

$$H_{\mathrm{rad}}^1\left(\mathbb{R}^2; V\right) \doteq D_{\mathrm{rad}}^{1,2}\left(\mathbb{R}^2\right) \cap L^2\left(\mathbb{R}^2; V\right),$$

which is a Hilbert space (see [2,15]) with the norm

$$\|u\| \doteq \left(\int_{\mathbb{R}^2} |\nabla u|^2 + V\left(|x|\right)|u|^2\right)^{1/2}.$$

- $H_{\mathrm{rad}}^1(\mathbb{R}^2; V)$ will be denoted by E and its norm by $\|\cdot\|$.
- Let $A \subset \mathbb{R}^2$ and define $H_{\mathrm{rad}}^1(A; V) = \{u_{|A}\colon\ u \in H_{\mathrm{rad}}^1(\mathbb{R}^2; V)\}$.

2. Preliminaries and main result

With the aid of inequalities (1.1), (1.3) and inspired by similar arguments developed in [6,7,12], we obtain what the title of this note states.

Theorem 2.1. *Assume that* (V)–(Q) *hold. Then there holds*

$$S_\alpha = \sup_{u \in E; \|u\| \le 1} \int_{\mathbb{R}^2} Q\left(|x|\right)\left(e^{\alpha u^2} - 1\right) < +\infty \tag{2.6}$$

if and only if $0 < \alpha \le \alpha' \doteq 2\pi(b_0 + 2)$. *Moreover, the supremum* (2.6) *is attained provided* $0 < \alpha < \alpha'$.

Remark 2.2. Theorem 2.1 complements (1.1)–(1.4) and the Trudinger–Moser type inequality obtained in [6].

Remark 2.3. In [5,6], the authors also used estimate (2.6) to study the existence and multiplicity of solutions for some classes of nonlinear Schrödinger elliptic equations (and systems of equations) with unbounded, singular or decaying radial potentials and involving nonlinearities with exponential critical growth of Trudinger–Moser type. In the argument, they combined the inequality (2.6) and variational methods.

Before giving the proof of the main result, we need establish some embeddings from E into the weighted Lebesgue space $L^p(\mathbb{R}^2; Q)$.

Lemma 2.4. *(See [13,14].) Assume that* (V) *holds. Then, there exists* $C > 0$ *such that for all* $u \in E$,

$$\left|u(x)\right| \le C\|u\||x|^{-\frac{a+2}{4}}, \quad |x| \gg 1.$$

Lemma 2.5. *(See [14].) Assume that* (V)–(Q) *hold and let* $1 \le p < \infty$. *For any* $0 < r < R < \infty$, *with* $R \ge 1$,

i) *the embeddings* $H_{\mathrm{rad}}^1(B_R \setminus B_r; V) \hookrightarrow L^p(B_R \setminus B_r; Q)$ *are compact;*
ii) *the embedding* $H_{\mathrm{rad}}^1(B_R; V) \hookrightarrow H^1(B_R)$ *is continuous.*

In particular, as a consequence of ii) we have that $H^1_{\text{rad}}(B_R;V)$ is compactly embedded in $L^q(B_R)$ for all $1 \le q < \infty$. If we assume that (V)–(Q) hold, by using Lemmas 2.4 and 2.5, a Hardy inequality with remainder terms (see [17]) and the same ideas from [14] we have:

Lemma 2.6. *Assume that (V)–(Q) hold. Then the embeddings $E \hookrightarrow L^p(\mathbb{R}^2;Q)$ are compact for all $2 \le p < \infty$.*

In order to use similar arguments developed in [12] we need the following version of the Radial Lemma for functions in $L^2(\mathbb{R}^2;V)$.

Lemma 2.7. *Assume that (V) holds. If $u \in L^2(\mathbb{R}^2;V)$ is a radial non-increasing function (i.e. $0 \le u(x) \le u(y)$ if $|x| \ge |y|$), then one has*

$$\left|u(x)\right| \le C\|u\|_{L^2(\mathbb{R}^2;V)}|x|^{-\frac{a+2}{2}}, \quad |x| \gg 1.$$

Proof. It follows from (V) that there exists $R_0 > 0$ such that $V(|x|) \ge C_0|x|^a$, for $|x| \ge R_0$. Then for $\rho > 0$ such that $\rho/2 > R_0$, we have (setting $\rho = |x|$)

$$\|u\|^2_{L^2(\mathbb{R}^2;V)} \ge 2\pi \int_{\rho/2}^{\rho} V(s)u^2(s)s\,ds \ge 2\pi C_0 u^2(\rho) \int_{\rho/2}^{\rho} s^{a+1}ds = C\rho^{a+2}u^2(\rho).$$

Thus we conclude that

$$\left|u(x)\right| \le C\|u\|_{L^2(\mathbb{R}^2;V)}|x|^{-\frac{a+2}{2}}, \quad \forall |x| > R_0.$$

Hence, the lemma is proved. □

In order to prove the sharpness of (2.6), recall the Moser's function sequence (see [11]):

$$\widetilde{M_n}(x,r) = \frac{1}{(2\pi)^{1/2}} \begin{cases} (\log n)^{1/2}, & |x| \le r/n, \\ \frac{\log \frac{r}{|x|}}{(\log n)^{1/2}}, & r/n < |x| \le r, \\ 0, & |x| > r, \end{cases}$$

with $0 < r \le r_0$ fixed and r_0 given in (1.5). We have the following estimate for $\|\widetilde{M_n}\|$:

Lemma 2.8. *Under the hypothesis (V),*

$$\|\widetilde{M_n}\|^2 \le 1 + \frac{m(r)}{\log n}\big(1 + o_n(1)\big),$$

where $m(r) = 2C_0 r^{a_0+2}/(a_0+2)^3$.

Proof. It is easy to compute

$$\int_{\mathbb{R}^2} |\nabla \widetilde{M_n}|^2 = \frac{1}{2\pi} \int_{r/n \le |x| \le r} \frac{1}{|x|^2 \log n} = 1.$$

On the other hand, (1.5) and integration by parts give

$$\begin{aligned}\int_{\mathbb{R}^2} V(|x|)|\widetilde{M_n}|^2 &\le \frac{C_0}{2\pi}\int_{|x|\le r/n} |x|^{a_0}\log n + \frac{C_0}{2\pi}\int_{r/n\le|x|\le r} |x|^{a_0}\frac{(\log\frac{r}{|x|})^2}{\log n} \\ &= -\frac{2C_0 r^{a_0+2}}{(a_0+2)^2}\left(\frac{1}{n}\right)^{a_0+2} + \frac{2C_0 r^{a_0+2}}{(a_0+2)^3}\frac{1}{\log n} - \frac{2C_0 r^{a_0+2}}{(a_0+2)^3}\frac{1}{\log n}\left(\frac{1}{n}\right)^{a_0+2} \\ &= \frac{2C_0 r^{a_0+2}/(a_0+2)^3}{\log n}\big(1+o_n(1)\big) \\ &= \frac{m(r)}{\log n}\big(1+o_n(1)\big),\end{aligned}$$

and thus

$$\|\widetilde{M_n}\|^2 = \int_{\mathbb{R}^2} |\nabla\widetilde{M_n}|^2 + \int_{\mathbb{R}^2} V(|x|)|\widetilde{M_n}|^2 \le 1 + \frac{m(r)}{\log n}\big(1+o_n(1)\big).$$

Hence, the lemma is proved. □

Proof of Theorem 2.1. By hypothesis (Q), there exist $0 < r_0 < R_0$ such that

$$\begin{aligned} &Q\big(|x|\big) \le C_0|x|^b, \quad \text{for } |x| \ge R_0, \\ &Q\big(|x|\big) \le C_0|x|^{b_0}, \quad \text{for } 0 < |x| \le r_0. \end{aligned} \tag{2.7}$$

Let $R > 0$ be large enough. We write

$$\int_{\mathbb{R}^2} Q\big(e^{\alpha u^2}-1\big) = \int_{B_R} Q\big(e^{\alpha u^2}-1\big) + \int_{B_R^c} Q\big(e^{\alpha u^2}-1\big). \tag{2.8}$$

We are going to estimate each integral in (2.8). For the integral on B_R, we have two cases to consider:

Case 1. $b_0 = 0$. From the second inequality in (2.7) and the continuity of $Q(r)$, there exists $C > 0$ such that

$$\int_{B_R} Q\big(e^{\alpha u^2}-1\big) \le C\int_{B_R} e^{\alpha u^2}. \tag{2.9}$$

As in [11,12], we use Schwarz symmetrization theory by defining the radially symmetric function u^* as follows: for all $s > 0$

$$\big|\big\{x \in B_R\colon\ u^*(x) > s\big\}\big| = \big|\big\{x \in B_R\colon\ u(x) > s\big\}\big|.$$

It follows from the properties of this construction that:

- u^* is a non-increasing function in $|x|$;
- $u^* \in H_0^1(B_R)$ and $\int_{B_R}|\nabla u^*|^2 \le \int_{B_R}|\nabla u|^2$;
- $\int_{B_R} e^{\alpha|u^*|^2} = \int_{B_R} e^{\alpha|u|^2}$.

Thus, we may assume that u in the second integral from (2.9) is non-increasing. Let

$$v(r) = \begin{cases} u(r) - u(R), & \text{if } 0 \le r \le R; \\ 0, & \text{if } r \ge R. \end{cases}$$

By Lemma 2.7,

$$\begin{aligned} u^2(r) &= v^2(r) + 2v(r)u(R) + u^2(R) \\ &\le v^2(r) + Cv^2(r)R^{-(a+2)/2}\|u\|^2_{L^2(\mathbb{R}^2;V)} + 1 + CR^{-(a+2)/2}\|u\|^2_{L^2(\mathbb{R}^2;V)} \\ &= v^2(r)\big[1 + CR^{-(a+2)/2}\|u\|^2_{L^2(\mathbb{R}^2;V)}\big] + d(R). \end{aligned}$$

Hence

$$u(r) \le v(r)\big[1 + CR^{-(a+2)/2}\|u\|^2_{L^2(\mathbb{R}^2;V)}\big]^{1/2} + d^{1/2}(R) \doteq w(r) + d^{1/2}(R).$$

By assumption

$$\int\limits_{B_R} |\nabla v|^2 = \int\limits_{B_R} |\nabla u|^2 \le 1 - \|u\|^2_{L^2(\mathbb{R}^2;V)}$$

and so

$$\begin{aligned} \int\limits_{B_R} |\nabla w|^2 &= \int\limits_{B_R} \big|\nabla v\big[1 + CR^{-(a+2)/2}\|u\|^2_{L^2(\mathbb{R}^2;V)}\big]^{1/2}\big|^2 \\ &= \big[1 + CR^{-(a+2)/2}\|u\|^2_{L^2(\mathbb{R}^2;V)}\big] \int\limits_{B_R} |\nabla v|^2 \\ &\le \big[1 + CR^{-(a+2)/2}\|u\|^2_{L^2(\mathbb{R}^2;V)}\big]\big[1 - \|u\|^2_{L^2(\mathbb{R}^2;V)}\big] \\ &= 1 + CR^{-(a+2)/2}\|u\|^2_{L^2(\mathbb{R}^2;V)} - \|u\|^2_{L^2(\mathbb{R}^2;V)} - CR^{-(a+2)/2}\|u\|^4_{L^2(\mathbb{R}^2;V)} \\ &\le 1. \end{aligned}$$

Since $u^2(r) \le w^2(r) + d(R)$ we get

$$\int\limits_{B_R} Q\big(e^{\alpha u^2} - 1\big) \le C \int\limits_{B_R} e^{\alpha u^2} \le Ce^{\alpha d} \int\limits_{B_R} e^{\alpha w^2}.$$

Taking into account that $w \in H^1_0(B_R)$ and $\|w\|_{H^1_0(B_R)} = \|\nabla w\|_{L^2(B_R)} \le 1$, we conclude that

$$\sup_{u\in E;\|u\|\le 1} \int\limits_{B_R} Q\big(e^{\alpha u^2} - 1\big) < +\infty,$$

by the classical Trudinger–Moser inequality (1.1).

Case 2. $-2 < b_0 < 0$. The estimates in B_R in this case and outside the ball (the second integral in (2.8)) follow by similar computations done in [6, Theorem 1.1]. Therefore, (2.6) holds.

Next we will show that (2.6) does not hold if $\alpha > \alpha'$. Set $M_n(x,r) = \frac{1}{\|\widetilde{M_n}\|}\widetilde{M_n}(x,r)$, then M_n belongs to E with its support in $\overline{B_r(0)}$ and $\|M_n\| = 1$. From Lemma 2.8, when $|x| \le r/n$, we have

$$M_n^2(x) \geq \frac{1}{2\pi} \frac{\log n}{1 + \frac{m(r)}{\log n}(1 + o_n(1))} = (2\pi)^{-1} \log n - (2\pi)^{-1} m(r) + o_n(1).$$

By assumption (Q), $Q(|x|) \geq C_0 |x|^{b_0}$ for $0 < |x| \leq r_0$. Thus, for $0 < r \leq r_0$ we have

$$\begin{aligned}\int_{\mathbb{R}^2} Q\big(e^{\alpha M_n^2} - 1\big) &\geq \int_{B_{r/n}} Q\big(e^{\alpha M_n^2} - 1\big) \\ &\geq C_0 \int_{B_{r/n}} |x|^{b_0} \big(e^{\alpha[(2\pi)^{-1} \log n - (2\pi)^{-1} m(r) + o_n(1)]} - 1\big) \\ &= C n^{\alpha(2\pi)^{-1} - (b_0+2)} e^{o_n(1)} + o_n(1).\end{aligned}$$

Consequently since $-2 < b_0 \leq 0$, then $\alpha > \alpha' \Leftrightarrow \alpha(2\pi)^{-1} - (2 + b_0) > 0$ and so we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} Q\big(e^{\alpha M_n^2} - 1\big) = +\infty,$$

concluding the first part of theorem. For the last part of theorem, we consider $0 < \alpha < \alpha'$. Let $(u_n) \subset E$ be a maximizing sequence, with $\|u_n\| \leq 1$. Then, up to subsequences, we can assume that $u_n \rightharpoonup u_0$ weakly in E and, by Lemma 2.6, $u_n \to u_0$ strongly in $L^p(\mathbb{R}^2; Q)$ for $2 \leq p < \infty$. Using the elementary inequality $|e^x - e^y| \leq |x - y|(e^x + e^y)$, $\forall x, y \in \mathbb{R}$, we estimate

$$\left|\int_{\mathbb{R}^2} Q\big(e^{\alpha u_n^2} - e^{\alpha u_0^2}\big)\right| \leq \alpha \int_{\mathbb{R}^2} Q e^{\alpha u_n^2} \big|u_n^2 - u_0^2\big| + \alpha \int_{\mathbb{R}^2} Q e^{\alpha u_0^2} \big|u_n^2 - u_0^2\big|. \tag{2.10}$$

Writing

$$\int_{\mathbb{R}^2} Q e^{\alpha u_n^2} \big|u_n^2 - u_0^2\big| = \int_{\mathbb{R}^2} Q\big(e^{\alpha u_n^2} - 1\big)\big|u_n^2 - u_0^2\big| + \int_{\mathbb{R}^2} Q\big|u_n^2 - u_0^2\big|$$

and taking $r_1 > 1$ sufficiently close to 1 such that $r_1\alpha \leq \alpha'$ (it is possible because we are assuming $\alpha < \alpha'$) and $r_2 \geq 2$ such that $1/r_1 + 1/r_2 = 1$, Hölder's inequality implies that

$$\begin{aligned}\int_{\mathbb{R}^2} Q e^{\alpha u_n^2} \big|u_n^2 - u_0^2\big| &\leq \Big(\int_{\mathbb{R}^2} Q\big(e^{r_1 \alpha u_n^2} - 1\big)\Big)^{1/r_1} \Big(\int_{\mathbb{R}^2} Q\big|u_n^2 - u_0^2\big|^{r_2}\Big)^{1/r_2} \\ &\quad + \Big(\int_{\mathbb{R}^2} Q|u_n - u_0|^2\Big)^{1/2} \Big(\int_{\mathbb{R}^2} Q|u_n + u_0|^2\Big)^{1/2}\end{aligned}$$

and likewise for the integral in (2.10) containing $e^{\alpha u_0^2}$. Thus, it follows from the first part of theorem and Lemma 2.6 that

$$S_\alpha + o_n(1) = \int_{\mathbb{R}^2} Q\big(e^{\alpha u_n^2} - 1\big) = \int_{\mathbb{R}^2} Q\big(e^{\alpha u_0^2} - 1\big) + o_n(1).$$

Finally, since $\|u_0\| \leq 1$, we see that u_0 is the required extremal function. This completes the proof of the result. □

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Remark 2.9. The maximizer u_0 can be chosen unitary, i.e., $\|u_0\| = 1$. Indeed, since for instance if $\|u_0\| < 1$, then setting $v_0 = u_0/\|u_0\|$, we would have

$$\int_{\mathbb{R}^2} Q\big(e^{\alpha v_0^2} - 1\big) > \int_{\mathbb{R}^2} Q\big(e^{\alpha u_0^2} - 1\big) = S_\alpha,$$

which is a contradiction.

Acknowledgments

The author thanks the anonymous referee for the careful reading, valuable comments and corrections, which provided a significant improvement of the paper. This work is a part of the author's Ph.D. thesis at the UFPB Department of Mathematics and the author is greatly indebted to his thesis adviser Professor Everaldo Souto de Medeiros for many useful discussions, suggestions and comments.

References

[1] S. Adachi, K. Tanaka, Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents, Proc. Amer. Math. Soc. 128 (2000) 2051–2057.
[2] R. Adams, Sobolev Spaces, Academic Press, New York, 1975.
[3] Adimurthi, K. Sandeep, A singular Moser–Trudinger embedding and its applications, Nonlinear Differential Equations Appl. 13 (2007) 585–603.
[4] Adimurthi, Y. Yang, An interpolation of Hardy inequality and Trudinger–Moser inequality in $\mathbb{R}^N$ and its applications, Int. Math. Res. Not. 13 (2010) 2394–2426.
[5] F.S.B. Albuquerque, Nonlinear Schrödinger elliptic system involving exponential critical growth in $\mathbb{R}^2$, Electron. J. Differential Equations 2014 (59) (2014) 1–12.
[6] F.S.B. Albuquerque, C.O. Alves, E.S. Medeiros, Nonlinear Schrödinger equation with unbounded or decaying radial potentials involving exponential critical growth in $\mathbb{R}^2$, J. Math. Anal. Appl. 409 (2014) 1021–1031.
[7] D.M. Cao, Nontrivial solution of semilinear elliptic equation with critical exponent in $\mathbb{R}^2$, Comm. Partial Differential Equations 17 (1992) 407–435.
[8] J.M. do Ó, N-Laplacian equations in $\mathbb{R}^N$ with critical growth, Abstr. Appl. Anal. 2 (1997) 301–315.
[9] J.M. do Ó, M. de Souza, On a class of singular Trudinger–Moser type inequalities and its applications, Math. Nachr. 284 (14–15) (2011) 1754–1776.
[10] Y. Li, B. Ruf, A sharp Trudinger–Moser type inequality for unbounded domains in $\mathbb{R}^n$, Indiana Univ. Math. J. 57 (2008) 451–480.
[11] J. Moser, A sharp form of an inequality by N. Trudinger, Indiana Univ. Math. J. 20 (1971) 1077–1092.
[12] B. Ruf, A sharp Trudinger–Moser type inequality for unbounded domains in $\mathbb{R}^2$, J. Funct. Anal. 219 (2) (2005) 340–367.
[13] W.A. Strauss, Existence of solitary waves in higher dimensions, Comm. Math. Phys. 55 (1977) 149–162.
[14] J. Su, Z.-Q. Wang, M. Willem, Nonlinear Schrödinger equations with unbounded and decaying radial potentials, Commun. Contemp. Math. 9 (2007) 571–583.
[15] J. Su, Z.-Q. Wang, M. Willem, Weighted Sobolev embedding with unbounded and decaying radial potentials, J. Differential Equations 238 (2007) 201–219.
[16] N.S. Trudinger, On the embedding into Orlicz spaces and some applications, J. Math. Mech. 17 (1967) 473–484.
[17] Z.-Q. Wang, M. Willem, Caffarelli–Kohn–Nirenberg inequalities with remainder terms, J. Funct. Anal. 203 (2003) 550–568.

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