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Limit cycles near a homoclinic loop by perturbing a class of integrable systems ☆

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ABSTRACT

In this paper, the problem of limit cycle bifurcation is investigated by perturbing a class of integrable systems with a homoclinic loop. Under the assumption that the homoclinic loop passes through a degenerate singular point at the origin, the asymptotic expansion of the Melnikov function along the level curves of the first integral inside the homoclinic loop is studied near the loop. Meanwhile, the formulas for the first coefficients in the expansion are given, which can be used to study the number of limit cycles near the homoclinic loop. Finally, an example is provided to demonstrate the obtained results.

1. Introduction

As we know, many nonlinear wave phenomena can be modeled by partial differential equations (PDEs). These PDEs have their traveling wave systems given by ordinary differential equations (ODEs). The corresponding solutions of these ODEs are traveling wave trajectories associated with PDEs. For more details, see [1,7] and the references therein. Consider a nonlinear wave equation

$$\frac{\partial^2 u}{\partial\rho\partial t} = \alpha e^{mu} + \beta e^{nu}, \tag{1.1}$$

where α, β are real numbers and m, n are integers. It is called a generalized Tzitzéica–Dodd–Bullough–Mikhailov equation, see [13] and the references cited therein. Make a variable transformation

$$u(\rho, t) = \ln x(\xi), \quad \xi = \rho - ct,$$

where $c > 0$ is the wave speed. Then, (1.1) becomes

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$$c\dot{x}^2 - cx\ddot{x} = \alpha x^{m+2} + \beta x^{n+2},$$

which is equivalent to

$$\dot{x} = y, \quad \dot{y} = \frac{cy^2 - \alpha x^{m+2} - \beta x^{n+2}}{cx} \tag{1.2}$$

as $x \neq 0$. Therefore, the solutions of (1.2) are the traveling wave solutions of (1.1). Clearly, the system (1.2) has the form

$$\dot{x} = y, \quad \dot{y} = \frac{g(x) + r(x)y^2}{f(x)}, \tag{1.3}$$

where r, f and g are C^∞ functions satisfying

$$r(0) \neq 0, \quad f(x) = x\bar{f}(x), \quad \bar{f}(0) \neq 0, \quad g(x) = x^n \bar{g}(x), \quad n \geq 1, \quad \bar{g}(0) \neq 0, \quad n \in \mathbb{N}.$$

In fact, many nonlinear wave equations, such as Kdv equation [3], KP equation and its generalized equation proposed in [16], $K(m, n)$ equation [12] and so on, also have the traveling wave equations of the form (1.3). The system (1.3) has the same phase orbits as the system

$$\dot{x} = f(x)y, \quad \dot{y} = g(x) + r(x)y^2, \tag{1.4}$$

in the region $\{(x, y) \mid f(x) \neq 0\}$ with the same (as $f(x) > 0$) or different (as $f(x) < 0$) orientation. For example, (1.2) has the same phase portraits as the system

$$\dot{x} = cxy, \quad \dot{y} = cy^2 - \alpha x^{m+2} - \beta x^{n+2} \tag{1.5}$$

with the same or different orientation on $x > 0$ or $x < 0$ respectively. All possible phase portraits of (1.5) were showed in Section 2.2.2 of [7] for $n \geq m \geq 1$.

By analyzing phase portraits of system (1.4), one can investigate the type of the traveling wave solutions and even obtain their exact parametric representations, see [7,11,14] and the references therein. On the other hand, one can obtain limit cycles by perturbing system (1.4). This problem is related to Hilbert's 16 problem originated by [6] and there are many papers concerning it, see [4,8–10,15,17–20] and the references quoted therein. Recently, the authors [2] have studied the phase portrait of (1.4) with $n = 1$ near the origin and considered the problem of limit cycle bifurcation for its perturbed system, obtaining some new and interesting results.

This paper is devoted to studying all possible phase portraits of (1.4) in the vicinity of the origin and researching the problem of limit cycle bifurcation by perturbing (1.4) for $n \geq 2$. Obviously, the origin is a degenerate singular point of (1.4). Thus, a fundamental problem in this case is to study the orbital behavior of (1.4) near the point. Further, one assumes that system (1.4) has a family of ovals bounded by a homoclinic loop with a singular point at the origin (see Theorem 2.1) and then study the expansion of the first order Melnikov function near the homoclinic loop (see Theorem 2.2), obtaining some general theory. As an example, we apply our main results to consider a cubic polynomial system, finding 3 limit cycles near a loop (see an example in Section 4).

The rest of the paper is organized as follows. In Section 2, some preliminary lemmas and the main results are presented. In Section 3, the expansion of the first order Melnikov function is investigated near a homoclinic loop to prove the main results. In Section 4, an example is provided as an application of the main results.

2. Preliminaries and the main results

We perturb system (1.4) inside a class of C^∞ planar differential systems. That is to say, we study a system of the form

$$\dot{x} = f(x)y + \varepsilon p(x, y, \delta), \quad \dot{y} = g(x) + r(x)y^2 + \varepsilon q(x, y, \delta), \tag{2.1}$$

where ε is a small parameter, $\delta \in \mathcal{D} \subset \mathbb{R}$ with $\mathcal{D}$ bounded, and f, g, r, p, q are C^∞ functions satisfying

$$r(0) \neq 0, \quad f(x) = x\bar{f}(x), \quad \bar{f}(0) = 1, \quad g(x) = x^n \bar{g}(x), \quad n \geq 2, \quad \bar{g}(0) = 1 \text{ or } -1 \tag{2.2}$$

and n is a positive integer. Here $\mathbb{N}$ stands for the positive integer constant. In (2.2), we have made $\bar{f}(0) = 1$ and $\bar{g}(0) = \pm 1$. In fact, when $\bar{f}(0)\bar{g}(0) \neq 0$, one can make a variable transformation of the form

$$(x_1, y_1) = \begin{cases} \left(\left(\bar{f}(0)\bar{g}(0)\right)^{\frac{1}{n}} x, \bar{f}(0)y\right) & \text{for } \bar{f}(0)\bar{g}(0) > 0, \\ \left(\left(-\bar{f}(0)\bar{g}(0)\right)^{\frac{1}{n}} x, \bar{f}(0)y\right) & \text{for } \bar{f}(0)\bar{g}(0) < 0 \end{cases}$$

such that the resulting system satisfies (2.2). For $\varepsilon = 0$, system (2.1) becomes (1.4), which is a reversible system since $(2.1)|_{\varepsilon=0}$ keeps invariant under the change $(x, y, t) \to (x, -y, -t)$. A general definition for a reversible system can be found in [2].

Under the condition (2.2), system (1.4) has a degenerate singular point at $(0, 0)$ and an invariant straight line $x = 0$. Thus, if system (1.4) has a homoclinic loop, it must lie on the region $\{(x, y) \mid x \geq 0\}$ or $\{(x, y) \mid x \leq 0\}$. For definiteness, we assume that system (1.4) has a homoclinic loop on the region $\{(x, y) \mid x \geq 0\}$.

From [2], (1.4) is an integrable system with an integrating factor of the form

$$\mu(x) = \frac{1}{f(x)} \exp\left(-2\int \frac{r(x)}{f(x)} dx\right). \tag{2.3}$$

Hence, on the region $x > 0$, system (2.1) is equivalent to the following system

$$\begin{aligned} \dot{x} &= \mu(x)f(x)y + \varepsilon\mu(x)p(x, y, \delta), \\ \dot{y} &= \mu(x)g(x) + \mu(x)r(x)y^2 + \varepsilon\mu(x)q(x, y, \delta). \end{aligned} \tag{2.4}$$

Clearly, one can see that $(2.4)|_{\varepsilon=0}$ is a Hamiltonian system with the Hamiltonian function given by

$$H(x, y) = \frac{1}{2}\mu(x)f(x)y^2 + s(x), \tag{2.5}$$

where $s(x)$ is a function satisfying

$$s'(x) = -\mu(x)g(x). \tag{2.6}$$

Let us first discuss the distribution of the level curves of $H(x, y) = h$ near the origin on the right-plane. To do it, we need to obtain an expression of $s(x)$ in (2.5). For the purpose, we have two lemmas below.

Lemma 2.1. *Let (2.2) hold. Then, for $x \geq 0$, there exists a C^∞ function $\bar{\mu}(x)$ with $\bar{\mu}(0) = 1$ such that*

$$\mu(x) = x^{-\alpha}\bar{\mu}(x), \tag{2.7}$$

where $\alpha = 2r(0) + 1 \neq 1$.

Proof. Note that, for $x \geq 0$, $-2\frac{r(x)}{f(x)} = -\frac{2r(0)}{x} - \frac{2}{x}\big(\frac{r(x)}{\bar{f}(x)} - r(0)\big)$. Then, by (2.3), one has

$$\begin{aligned}\mu(x) &= \frac{1}{x\bar{f}(x)} \exp\left[-2\int \frac{r(0)}{x}dx - 2\int \frac{1}{x}\Big(\frac{r(x)}{\bar{f}(x)} - r(0)\Big)dx\right] \\ &= x^{-2r(0)-1}\bar{\mu}(x),\end{aligned}$$

where

$$\bar{\mu}(x) = \frac{1}{\bar{f}(x)} \exp\big(-2\tilde{\mu}(x)\big), \quad \tilde{\mu}(x) = \int \frac{1}{x}\Big(\frac{r(x)}{\bar{f}(x)} - r(0)\Big)dx.$$

It is easy to prove that $\tilde{\mu}(x)$ is well defined at $x = 0$. This implies that $\tilde{\mu}(x)$ is a C^∞ function for $x \geq 0$ with $\tilde{\mu}(0) = 0$. Furthermore, from (2.2), we have $\bar{f}(0) = 1$. Thus, $\bar{\mu}(x)$ is a C^∞ function on $x \geq 0$ satisfying $\bar{\mu}(0) = 1$. This ends the proof. □

For the function $s(x)$ in (2.5), we have

Lemma 2.2. *For $x \geq 0$, one of the expressions of $s(x)$ in (2.5) can be written as*

$$s(x) = \begin{cases} x^{n-\alpha+1}\bar{s}(x), & \alpha \neq n+1, \\ \tilde{s}(x)\ln x, & \alpha = n+1, \end{cases} \tag{2.8}$$

where $\bar{s}(x)$ and $\tilde{s}(x)$ are C^∞ in x with

$$\bar{s}(0) = \frac{\bar{g}(0)}{\alpha - n - 1} \triangleq s_0, \quad \tilde{s}(0) = -\bar{g}(0).$$

Proof. By (2.2) and (2.7), we have

$$\mu(x)g(x) = x^{n-\alpha}\bar{\mu}(x)\bar{g}(x) = \bar{g}(0)x^{n-\alpha} + x^{n-\alpha}\big(\bar{\mu}(x)\bar{g}(x) - \bar{g}(0)\big).$$

Then, in view of (2.6), one achieves

$$s(x) = -\int \mu(x)g(x)dx = -\int \bar{g}(0)x^{n-\alpha}dx - \rho(x), \tag{2.9}$$

where $\rho(x) = \int x^{n-\alpha}\big(\bar{\mu}(x)\bar{g}(x) - \bar{g}(0)\big)dx$.

When $\alpha \neq n+1$, we have from (2.9)

$$s(x) = x^{n-\alpha+1}\Big(\frac{\bar{g}(0)}{\alpha - n - 1} - \frac{\rho(x)}{x^{n-\alpha+1}}\Big).$$

Obviously, $\frac{\rho(x)}{x^{n-\alpha+1}}$ is well defined on $x = 0$, which means that it is a C^∞ function on $x \geq 0$ with $\frac{\rho(x)}{x^{n-\alpha+1}}\big|_{x=0} = 0$.

Similarly, one can prove the case $\alpha = n+1$. The proof of the lemma is completed. □

Then, by Lemmas 2.1 and 2.2, one can get the following theorem, which gives all possible phase portraits of (1.4) near the origin for $x \geq 0$.

Theorem 2.1. *Let (2.2) be satisfied. Then, all possible phase portraits of $(2.1)|_{\varepsilon=0}$ or (1.4) are shown in Figs. 2.1 and 2.2 for $x \geq 0$ sufficiently small.*

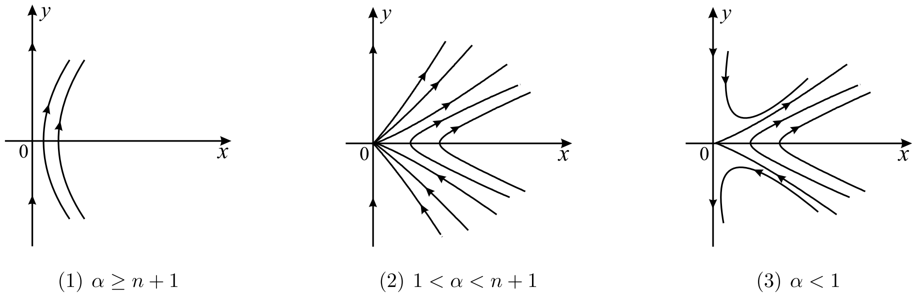


Fig. 2.1. Possible phase portraits of system (2.1)$|_{\varepsilon=0}$ for $0 \le x \ll 1$ with $\bar{g}(0) = 1$.

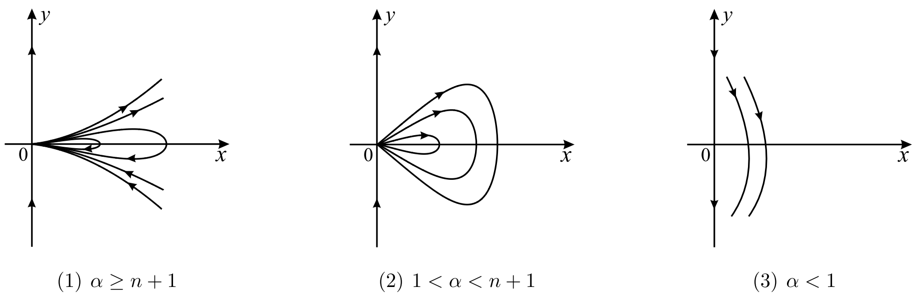


Fig. 2.2. Possible phase portraits of system (2.1)$|_{\varepsilon=0}$ for $0 \le x \ll 1$ with $\bar{g}(0) = -1$.

Proof. For $\alpha \neq n+1$, inserting (2.8) into (2.5), together with (2.2) and (2.7), gives

$$H(x,y) = \frac{1}{2}x^{-\alpha+1}\bar{\mu}(x)\bar{f}(x)y^2 + x^{n-\alpha+1}\bar{s}(x),$$

where $\bar{\mu}(x)\bar{f}(x)$, $\bar{s}(x)$ are C^∞ with $\bar{\mu}(0)\bar{f}(0) = 1$, $\bar{s}(0) = \frac{\bar{g}(0)}{\alpha-n-1}$. Hence, for $x \ge 0$ small, the equation $H(x,y) = h$ becomes

$$y^2 = \frac{2x^{\alpha-1}}{\bar{\mu}(x)\bar{f}(x)}\big(h - x^{n-\alpha+1}\bar{s}(x)\big) = Y_1(x)Y_2(x,h), \tag{2.10}$$

where

$$\begin{aligned} Y_1(x) &= \frac{2x^{\alpha-1}}{\bar{\mu}(x)\bar{f}(x)} = 2x^{\alpha-1}(1+O(x)) > 0, \quad 0 < x \ll 1, \\ Y_2(x,h) &= h - x^{n-\alpha+1}\bar{s}(x) = h - \frac{\bar{g}(0)}{\alpha-n-1}x^{n-\alpha+1}\big(1+O(x)\big). \end{aligned} \tag{2.11}$$

If for $x \in (a,b)$

$$Y_1(x)Y_2(x,h) > 0, \tag{2.12}$$

then by (2.10), we know that the equation $H(x,y) = h$ has exactly two solutions $\bar{y}_1(x,h)$ and $\bar{y}_2(x,h)$, where $\bar{y}_1(x,h) = -\bar{y}_2(x,h) = \big(Y_1(x)Y_2(x,h)\big)^{\frac{1}{2}}$. This implies that for $x \in (a,b)$, there exist two symmetry curves with respect to the x-axis. Furthermore, if one can find the value of the function $Y_1(x)Y_2(x,h)$ at the endpoints a and b, then one easily knows each level curve of $H(x,y) = h$ tends as x approaches to a or b. In this direction, the level curves of $H(x,y) = h$ can be plotted on the plane easily.

From the above discussion, in order to finish the proof, it suffices to do two things below:

(1) Find the interval (a, b) of variable x such that (2.12) holds;
(2) Compute the value of the function y^2 (i.e. $Y_1(x)Y_2(x, h)$) at the endpoints a and b.

Note that the interval (a, b) for $Y_1(x)Y_2(x, h) > 0$ is the same as the interval for $Y_2(x, h) > 0$. Hence, we only need to discuss the interval of x for $Y_2(x, h) > 0$.

For the case $(\alpha - n - 1)h\bar{g}(0) \le 0$, from (2.11), one can find that

$$Y_2(x, h) > 0, \quad 0 < x \ll 1,$$

for $\bar{g}(0) = 1$, $h \ge 0$, $\alpha < n + 1$ and $\alpha \ne 1$ or for $\bar{g}(0) = -1$, $h \ge 0$ and $\alpha > n + 1$.

For the case $(\alpha - n - 1)h\bar{g}(0) > 0$, $Y_2(x, h) = 0$ is equivalent to $\left[\frac{(\alpha-n-1)h}{\bar{g}(0)}\right]^{\frac{1}{n-\alpha+1}} = x(1 + O(x))$. Hence, by the implicit function theorem, a unique function $a(h) = \left[\frac{(\alpha-n-1)h}{\bar{g}(0)}\right]^{\frac{1}{n-\alpha+1}} + O(|h|^{\frac{2}{n-\alpha+1}})$ exists such that $Y_2(a(h), h) = 0$. Then, it follows that for $(\alpha - n - 1)h\bar{g}(0) > 0$

$$Y_2(x, h) > 0 \Leftrightarrow \bar{g}(0)(x - a(h)) > 0, \quad \text{i.e.} \quad Y_2(x, h) > 0 \Leftrightarrow \begin{cases} x > a(h), & \bar{g}(0) = 1, \\ 0 < x < a(h), & \bar{g}(0) = -1. \end{cases}$$

Then, summarizing the above discussion, it is easy to obtain the interval of x such that (2.12) holds. More precisely,

For $\bar{g}(0) = 1$

$$Y_1(x)Y_2(x, h) > 0 \Leftrightarrow \begin{cases} a(h) < x \ll 1, & \text{for} \quad h > 0, \quad \alpha > n + 1, \\ a(h) < x \ll 1, & \text{for} \quad h < 0, \quad \alpha < n + 1, \ \alpha \ne 1, \\ 0 < x \ll 1, & \text{for} \quad h \ge 0, \quad \alpha < n + 1, \ \alpha \ne 1; \end{cases} \tag{2.13}$$

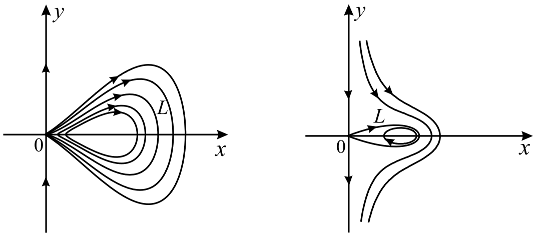
For $\bar{g}(0) = -1$,

$$Y_1(x)Y_2(x, h) > 0 \Leftrightarrow \begin{cases} 0 < x \ll 1, & \text{for} \quad h \ge 0, \quad \alpha > n + 1, \\ 0 < x < a(h), & \text{for} \quad h < 0, \quad \alpha > n + 1, \\ 0 < x < a(h), & \text{for} \quad h > 0, \quad \alpha < n + 1, \ \alpha \ne 1. \end{cases} \tag{2.14}$$

Now, we compute the values of the function $Y_1(x)Y_2(x, h)$ at the end point of the corresponding interval for each case in (2.13) and (2.14). That is,

For $\bar{g}(0) = 1$,

$$\begin{aligned}
&Y_1(a(h))Y_2(a(h), h) = 0, \quad \text{for} \quad h > 0, \quad \alpha > n + 1, \\
&Y_1(a(h))Y_2(a(h), h) = 0, \quad \text{for} \quad h < 0, \quad \alpha < n + 1, \ \alpha \ne 1, \\
&Y_1(0)Y_2(0, h) = 0, \quad \text{for} \quad h \ge 0, \quad 1 < \alpha < n + 1, \\
&Y_1(0)Y_2(0, h) = 0, \quad \text{for} \quad h = 0, \quad \alpha < 1, \\
&Y_1(0)Y_2(0, h) = +\infty, \quad \text{for} \quad h > 0, \quad \alpha < 1;
\end{aligned} \tag{2.15}$$



(1) $1 < \alpha < n+1$ (2) $\alpha < 1$

Fig. 2.3. The homoclinic loop L under **(H1)** and **(H2)**.

For $\bar{g}(0) = -1$,

$$\begin{aligned}
&Y_1(0)Y_2(0,h) = 0, \quad \text{for} \quad h \geq 0, \quad \alpha > n+1,\\
&Y_1(a(h))Y_2(a(h),h) = Y_1(0)Y_2(0,h) = 0, \quad \text{for} \quad h < 0, \quad \alpha > n+1,\\
&Y_1(a(h))Y_2(a(h),h) = Y_1(0)Y_2(0,h) = 0, \quad \text{for} \quad h > 0, \quad 1 < \alpha < n+1,\\
&Y_1(0)Y_2(0,h) = +\infty, \quad Y_1(a(h))Y_2(a(h),h) = 0, \quad \text{for} \quad h > 0, \quad \alpha < 1.
\end{aligned} \tag{2.16}$$

Therefore, by (2.13)–(2.16) and the above discussions, one can easily obtain the phase portrait of $(2.1)|_{\varepsilon=0}$ for the case $\alpha \neq n+1$. By using the same arguments, the case $\alpha = n+1$ can be discussed easily. The proof is then ended. □

In view of Theorem 2.1 and (2.5), we further make the following two assumptions.

(H1) $\bar{g}(0) = 1$ and $\alpha < n+1$, $\alpha \neq 1$;
(H2) There exists a constant $\beta > 0$ such that the equation $H(x,y) = h$, $h \in (-\beta, 0)$ defines a closed curve L_h on the region $x > 0$.

Obviously, if **(H1)** and **(H2)** hold, then as h tends to 0^-, the limit of L_h is a homoclinic loop denoted by L, namely, $\lim\limits_{h\to 0^-} L_h = L$. In fact, for $1 < \alpha < n+1$, there are infinitely many homoclinic loops passing through the origin, while for $\alpha < 1$, there exists a unique homoclinic loop, see Fig. 2.3.

Thus, associated with system (2.4), we have the first order Melnikov function

$$M(h,\delta) = \oint\limits_{L_h} \mu q dx - \mu p dy, \quad h \in (-\beta, 0). \tag{2.17}$$

Then, for $0 < -h \ll 1$, we have the following

Theorem 2.2. *Assume that (2.2),* **(H1)** *and* **(H2)** *are satisfied. Then, there exist constants B_i, $i \geq 0$ and a C^∞ function $\phi(h,\delta)$ such that for $0 < -h \ll 1$*

$$M(h,\delta) = \sum_{i\geq 0, \beta_i \notin \mathbb{N}} B_i |h|^{\beta_i} + \sum_{i \geq 0, \beta_i \in \mathbb{N}} B_i |h|^{\beta_i} \ln|h| + \phi(h,\delta), \tag{2.18}$$

where $\beta_i = \frac{i}{n-\alpha+1} + \frac{n-2\alpha}{2(n-\alpha+1)}$ and $\mathbb{N}$ denotes the positive integers.

The proof of Theorem 2.2 will be given in the next section.

3. The expansion of $M(h,\delta)$

The main task of this section is to understand the asymptotic expansions of $M(h,\delta)$ in (2.17) near the homoclinic loop L. By Green formula or integration by parts, we have from (2.17)

$$M(h,\delta) = \oint_{L_h} \bar{q}(x,y,\delta)dx, \tag{3.1}$$

where

$$\bar{q}(x,y,\delta) = \mu(x)q(x,y,\delta) - \mu(x)q(x,0,\delta) + \int_0^y (\mu p)_x(x,\tau,\delta)d\tau \tag{3.2}$$

satisfying

$$\bar{q}_y(x,y,\delta) = \big[\mu(x)p(x,y,\delta)\big]_x + \big[\mu(x)q(x,y,\delta)\big]_y \ \text{ and } \ \bar{q}(x,0,\delta) = 0. \tag{3.3}$$

Take a small constant $x_0 > 0$ such that for $0 < -h \ll 1$, the oval L_h is split into two parts $L_{h1} = L_h|_{x \le x_0}$, $L_{h2} = L_h|_{x \ge x_0}$ by the line segment $x = x_0$. Then, by (3.1)

$$M(h,\delta) = I_1(h,\delta) + I_2(h,\delta), \tag{3.4}$$

in which

$$I_i(h,\delta) = \int_{L_{hi}} \bar{q}(x,y,\delta)dx, \quad i = 1,2. \tag{3.5}$$

Since $I_2 \in C^\infty$ for $0 < -h \ll 1$, we only need to investigate the expansion of $I_1(h,\delta)$ at the origin. In the light of $\bar{\mu},\ p,\ q \in C^\infty$, then $\bar{\mu}$, p, q can be rewritten as for $|x| + |y| > 0$ small

$$\begin{aligned} \mu(x) &= x^{-\alpha}\bar{\mu}(x) = x^{-\alpha}\sum_{j\ge 0}\mu_j x^j, \quad \mu_0 = 1, \\ p(x,y,\delta) &= \sum_{i+j\ge 0} a_{ij}x^i y^j, \quad q(x,y,\delta) = \sum_{i+j\ge 0} b_{ij}x^i y^j, \end{aligned} \tag{3.6}$$

where μ, $\bar{\mu}$ and p, q are defined in (2.7) and (2.1) respectively. Then, one has

Lemma 3.1. *Let* **(H1)**, **(H2)** *and* (3.6) *hold. Then,* $I_1(h,\delta)$ *in* (3.5) *has the form*

$$I_1(h,\delta) = \sum_{k\ge 1}\int_{L_{h1}} q_k(x)y^k dx,$$

where

$$\begin{aligned} q_{k+1}(x) &= x^{-\alpha-1}\sum_{l\ge 0}\bar{a}_{lk}x^l, \quad k \ge 0, \\ \bar{a}_{0k} &= -\frac{\alpha\mu_0 a_{0k}}{k+1}, \\ \bar{a}_{lk} &= \frac{1}{k+1}\Big[\sum_{i+j=l}(j-\alpha)\mu_j a_{ik} + \sum_{i+j=l-1}\mu_j\big((i+1)a_{i+1,k} + (k+1)b_{i,k+1}\big)\Big], \quad l \ge 1. \end{aligned}$$

Proof. From (3.2) and (3.3), one obtains

$$\bar{q}(x,y,\delta) = \sum_{k\geq 1} q_k(x) y^k, \tag{3.7}$$

in which

$$\begin{aligned} q_{k+1}(x) &= \frac{1}{(k+1)!}\frac{\partial^k}{\partial y^k}\Big(\big[\mu(x)p(x,y,\delta)\big]_x + \big[\mu(x)q(x,y,\delta)\big]_y\Big)\Big|_{y=0} \\ &= \frac{1}{(k+1)!}\Big[\mu'(x)\frac{\partial^k}{\partial y^k}p(x,y,\delta) + \mu(x)\frac{\partial^k}{\partial y^k}(p_x+q_y)(x,y,\delta)\Big]\Big|_{y=0}. \end{aligned} \tag{3.8}$$

Notice that

$$\begin{aligned} \mu'(x) = x^{-\alpha-1}\sum_{j\geq 0}(j-\alpha)\mu_j x^j, \quad \frac{\partial^k p}{\partial y^k} &= \sum_{j\geq k}\frac{j!}{(j-k)!}\sum_{i\geq 0} a_{ij}x^i y^{j-k}, \\ \frac{\partial^k}{\partial y^k}(p_x+q_y) &= \frac{\partial^k}{\partial y^k}\Big(\sum_{i+j\geq 0}\big[(i+1)a_{i+1,j} + (j+1)b_{i,j+1}\big]x^i y^j\Big) \\ &= \sum_{j\geq k}\frac{j!}{(j-k)!}\sum_{i\geq 0}\big[(i+1)a_{i+1,j} + (j+1)b_{i,j+1}\big]x^i y^{j-k}. \end{aligned}$$

Then,

$$\begin{aligned} \mu'(x)\frac{\partial^k p}{\partial y^k}\Big|_{y=0} &= x^{-\alpha-1}\sum_{j\geq 0}(j-\alpha)\mu_j x^j k!\sum_{i\geq 0}a_{ik}x^i = k!x^{-\alpha-1}\sum_{\tau\geq 0}\tilde{a}_{\tau k}x^\tau, \\ \mu(x)\frac{\partial^k}{\partial y^k}(p_x+q_y)\Big|_{y=0} &= x^{-\alpha}\sum_{j\geq 0}\mu_j x^j k!\sum_{i\geq 0}\big[(i+1)a_{i+1,k} + (k+1)b_{i,k+1}\big]x^i \\ &= k!x^{-\alpha}\sum_{\tau\geq 0}\tilde{b}_{\tau k}x^\tau, \end{aligned} \tag{3.9}$$

where

$$\tilde{a}_{\tau k} = \sum_{i+j=\tau}(j-\alpha)\mu_j a_{ik}, \quad \tilde{b}_{\tau k} = \sum_{i+j=\tau}\mu_j\big[(i+1)a_{i+1,k} + (k+1)b_{i,k+1}\big], \quad \tau,\ k\geq 0. \tag{3.10}$$

Thus, by (3.9), we represent (3.8) as

$$\begin{aligned} q_{k+1}(x) &= \frac{1}{k+1}\Big(x^{-\alpha-1}\sum_{\tau\geq 0}\tilde{a}_{\tau k}x^\tau + x^{-\alpha}\sum_{\tau\geq 0}\tilde{b}_{\tau k}x^\tau\Big) \\ &= x^{-\alpha-1}\Big(\frac{1}{k+1}\sum_{\tau\geq 0}\tilde{a}_{\tau k}x^\tau + \frac{1}{k+1}\sum_{\tau\geq 0}\tilde{b}_{\tau k}x^{\tau+1}\Big) \\ &= x^{-\alpha-1}\sum_{l\geq 0}\bar{a}_{lk}x^l, \end{aligned} \tag{3.11}$$

in which

$$\bar{a}_{0k} = \frac{\tilde{a}_{0k}}{k+1}, \quad \bar{a}_{lk} = \frac{\tilde{a}_{lk} + \tilde{b}_{l-1,k}}{k+1}, \quad l\geq 1. \tag{3.12}$$

Then, the conclusion follows from (3.5), (3.7), (3.10)–(3.12). This completes the proof. □

Further, from the proof of Theorem 2.1, it is not hard to obtain the lemma below.

Lemma 3.2. *Suppose* **(H1)** *and* **(H2)** *hold. Then the curve L_{h1} intersects the positive x-axis at a point $A = (a(h), 0)$, where*

$$a(h) = \big[(\alpha - n - 1)h\big]^{\frac{1}{n-\alpha+1}} + O\big(|h|^{\frac{2}{n-\alpha+1}}\big), \quad 0 < -h \ll 1.$$

Furthermore, for $a(h) \le x \le x_0$ with $0 < -h \ll 1$, the equation $H(x, y) = 0$ has exactly two solutions $y_1(x, \omega)$ and $y_2(x, \omega)$, where

$$y_1(x, \omega) = -y_2(x, \omega) = x^{\frac{\alpha-1}{2}} \big(\frac{1}{2}\bar{\mu}(x)\bar{f}(x)\big)^{-\frac{1}{2}} \omega$$

with $\omega = \sqrt{h - x^{n-\alpha+1}\bar{s}(x)}$.

Then, by Lemmas 3.1 and 3.2, one obtains

Lemma 3.3. *Assume that* **(H1)**, **(H2)** *and (3.6) are satisfied. Then, we have*

$$I_1(h, \delta) = \sum_{l+k \ge 0} r_{lk} I_{lk}(h, u_0), \tag{3.13}$$

where

$$I_{lk}(h, u_0) = \int_{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}}^{u_0} u^{k(\alpha-1)-\frac{\alpha+3}{2}+l} (h - s_0 u^{n-\alpha+1})^{k+\frac{1}{2}} du \tag{3.14}$$

and

$$r_{0k} = 2^{k+\frac{3}{2}} \bar{a}_{0,2k}, \quad r_{lk} = 2^{k+\frac{3}{2}} \bar{a}_{l,2k} + O(|\bar{a}_{0,2k}, \bar{a}_{1,2k}, \cdots, \bar{a}_{l-1,2k}|), \quad l \ge 1. \tag{3.15}$$

Proof. From Lemmas 3.1 and 3.2, one can find that

$$\begin{aligned} I_1(h, \delta) &= \sum_{k \ge 1} \Big[\int_{a(h)}^{x_0} q_k(x)\big(y_1(x, \omega)\big)^k + \int_{x_0}^{a(h)} q_k(x)\big(-y_1(x, \omega)\big)^k \Big] \\ &= \sum_{k \ge 1} \int_{a(h)}^{x_0} \big[1 + (-1)^{k+1}\big] q_k(x) x^{\frac{k(\alpha-1)}{2}} \big(\frac{1}{2}\bar{\mu}(x)\bar{f}(x)\big)^{-\frac{k}{2}} \omega^k dx \\ &= \sum_{k \ge 0} \int_{a(h)}^{x_0} \bar{q}_k(x)\big(h - x^{n-\alpha+1}\bar{s}(x)\big)^{k+\frac{1}{2}} dx, \end{aligned} \tag{3.16}$$

where

$$\bar{q}_k(x) = 2^{k+\frac{3}{2}} q_{2k+1}(x) x^{(k+\frac{1}{2})(\alpha-1)} \big(\bar{\mu}(x)\bar{f}(x)\big)^{-k-\frac{1}{2}}. \tag{3.17}$$

Let $u = x\big(s_0^{-1}\bar{s}(x)\big)^{\frac{1}{n-\alpha+1}} \triangleq \varphi(x)$, where s_0 is given in Lemma 2.2. Then, for $x > 0$ small

$$\varphi(x) = x + O(x^2), \quad \varphi^{-1}(u) = u + O(u^2). \tag{3.18}$$

Thus, make a transformation $u = \varphi(x)$ so that (3.16) becomes

$$I_1(h,\delta) = \sum_{k\geq 0} \int\limits_{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}}^{u_0} \tilde{q}_k(u)(h - s_0 u^{n-\alpha+1})^{k+\frac{1}{2}} du, \tag{3.19}$$

where $u_0 = \varphi(x_0)$ and

$$\tilde{q}_k(u) = \frac{\bar{q}_k(x)}{\varphi'(x)}\Big|_{x=\varphi^{-1}(u)}. \tag{3.20}$$

By using (2.2), (3.17), (3.18) and Lemma 3.1, we have from (3.20)

$$\begin{aligned}
\tilde{q}_k(u) &= \frac{2^{k+\frac{3}{2}} x^{k(\alpha-1)-\frac{\alpha+3}{2}} \sum_{l\geq 0} \bar{a}_{l,2k} x^l \left(\bar{\mu}(x)\bar{f}(x)\right)^{-k-\frac{1}{2}}}{1+O(x)}\Big|_{x=u+O(u^2)} \\
&= 2^{k+\frac{3}{2}} x^{k(\alpha-1)-\frac{\alpha+3}{2}} \sum_{l\geq 0} \bar{a}_{l,2k} x^l (1+O(x))\big|_{x=u+O(u^2)} \\
&= 2^{k+\frac{3}{2}} u^{k(\alpha-1)-\frac{\alpha+3}{2}} \sum_{l\geq 0} \bar{a}_{l,2k} u^l (1+O(u)).
\end{aligned}$$

Combining (3.19) and the above gives (3.13)–(3.15). This ends the proof. □

The following lemma gives some information on the expressions of $I_{lk}(h, u_0)$ in (3.13) or (3.14).

Lemma 3.4. *The functions $I_{lk}(h, u_0)$ in (3.14) have the expressions*

$$I_{lk}(h,u_0) = \begin{cases} B_{lk}|h|^{\beta_{nk+l}} + \phi_{lk}(h,u_0), & \text{for } \beta_{nk+l} \notin \mathbb{N}, \\ B_{lk}|h|^{\beta_{nk+l}} \ln|h| + \phi_{lk}(h,u_0), & \text{for } \beta_{nk+l} \in \mathbb{N}, \end{cases} \tag{3.21}$$

where each ϕ_{lk} is a C^∞ function and

$$\begin{aligned}
\beta_{nk+l} &= \frac{nk+l}{n-\alpha+1} + \frac{n-2\alpha}{2(n-\alpha+1)}, \\
B_{lk} &= \begin{cases} \frac{1}{(n-\alpha+1)|s_0|^{\beta_{nk+l}-k-\frac{1}{2}}} \sum\limits_{i\geq 0} \frac{c_{ki}}{i-\beta_{nk+l}}, & \text{for } \beta_{nk+l} \notin \mathbb{N}, \\ \frac{-c_{k,\beta_{nk+l}}}{(n-\alpha+1)|s_0|^{\beta_{nk+l}-k-\frac{1}{2}}}, & \text{for } \beta_{nk+l} \in \mathbb{N} \end{cases}
\end{aligned} \tag{3.22}$$

with

$$c_{k0} = 1, \quad c_{ki} = (-1)^i \frac{\prod_{j=0}^{i-1}(k+\frac{1}{2}-j)}{i!}, \quad i \geq 1. \tag{3.23}$$

Proof. Introduce a variable transformation $v = \frac{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}}{u}$ to (3.14) to get that

$$I_{lk}(h,u_0) = -\int\limits_{1}^{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}} u_0^{-1}} \frac{|\frac{h}{s_0}|^{\frac{k(\alpha-1)-\frac{\alpha+3}{2}+l}{n-\alpha+1}}}{v^{k(\alpha-1)-\frac{\alpha+3}{2}+l}} \left(h - \frac{h}{v^{n-\alpha+1}}\right)^{k+\frac{1}{2}} \frac{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}}{v^2} dv$$

$$= -\frac{|h|^{\frac{k(\alpha-1)-\frac{\alpha+3}{2}+l+1}{n-\alpha+1}}}{|s_0|^{\frac{k(\alpha-1)-\frac{\alpha+3}{2}+l+1}{n-\alpha+1}}}|h|^{k+\frac{1}{2}} \int_1^{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}u_0^{-1}} \frac{(1-v^{n-\alpha+1})^{k+\frac{1}{2}}}{v^{k(\alpha-1)-\frac{\alpha+3}{2}+l+(k+\frac{1}{2})(n-\alpha+1)+2}}dv$$

$$= -\frac{|h|^{\beta_{nk+l}}}{|s_0|^{\beta_{nk+l}-k-\frac{1}{2}}} \int_1^{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}u_0^{-1}} \frac{(1-v^{n-\alpha+1})^{k+\frac{1}{2}}}{v^{1-\alpha+\frac{1}{2}n+kn+l}}dv. \tag{3.24}$$

Further, one has

$$(1-v^{n-\alpha+1})^{k+\frac{1}{2}} = \sum_{i\ge 0} c_{ki} v^{(n-\alpha+1)i},$$

which is uniformly convergent for $v \in [0,1]$, where c_{ki}, $i \ge 0$ are defined in (3.23). Note that $0 < \left|\frac{h}{s_0}\right|^{\frac{1}{n-\alpha+1}} u_0^{-1} < 1$. Thus, from (3.24),

$$I_{lk}(h,u_0) = -\frac{|h|^{\beta_{nk+l}}}{|s_0|^{\beta_{nk+l}-k-\frac{1}{2}}} \sum_{i\ge 0} c_{ki} \int_1^{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}u_0^{-1}} v^{(n-\alpha+1)i-1+\alpha-\frac{1}{2}n-kn-l}dv$$

$$= -\frac{|h|^{\beta_{nk+l}}}{|s_0|^{\beta_{nk+l}-k-\frac{1}{2}}} \sum_{i\ge 0} c_{ki} \int_1^{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}u_0^{-1}} v^{(n-\alpha+1)(i-\beta_{nk+l})-1}dv. \tag{3.25}$$

If $\beta_{nk+l} \notin \mathbb{N}$, then from (3.25), one finds

$$I_{lk}(h,u_0) = -\frac{|h|^{\beta_{nk+l}}}{|s_0|^{\beta_{nk+l}-k-\frac{1}{2}}} \sum_{i\ge 0} \frac{c_{ki}}{(n-\alpha+1)(i-\beta_{nk+l})} v^{(n-\alpha+1)(i-\beta_{nk+l})}\Big|_1^{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}u_0^{-1}}$$

$$= \sum_{i\ge 0} \frac{c_{ki}}{(n-\alpha+1)(i-\beta_{nk+l})} \frac{|h|^{\beta_{nk+l}}}{|s_0|^{\beta_{nk+l}-k-\frac{1}{2}}} - \sum_{i\ge 0} \frac{c_{ki}u_0^{-(n-\alpha+1)(i-\beta_{nk+l})}}{(n-\alpha+1)(i-\beta_{nk+l})} \frac{|h|^i}{|s_0|^{i-k-\frac{1}{2}}}. \tag{3.26}$$

If $\beta_{nk+l} \in \mathbb{N}$, there exist $i_0 \in \mathbb{N}$ such that $\beta_{nk+l} = i_0$. In this case, from (3.25), we have

$$I_{lk}(h,u_0)$$

$$= -\frac{|h|^{i_0}}{|s_0|^{i_0-k-\frac{1}{2}}} \left[\sum_{i\ge 0, i\ne i_0} \frac{c_{ki}}{(n-\alpha+1)(i-i_0)} v^{(n-\alpha+1)(i-i_0)} + c_{k,i_0}\ln|v|\right]\Bigg|_1^{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}u_0^{-1}}$$

$$= -\sum_{i\ge 0, i\ne i_0} \frac{c_{ki}u_0^{-(n-\alpha+1)(i-i_0)}}{(n-\alpha+1)(i-i_0)} \frac{|h|^i}{|s_0|^{i-k-\frac{1}{2}}} + \sum_{i\ge 0, i\ne i_0} \frac{c_{ki}}{(n-\alpha+1)(i-i_0)} \frac{|h|^{i_0}}{|s_0|^{i_0-k-\frac{1}{2}}}$$

$$+ \frac{c_{k,i_0}\left[\ln u_0 + (n-\alpha+1)^{-1}\ln|s_0|\right]}{|s_0|^{i_0-k-\frac{1}{2}}}|h|^{i_0} - \frac{c_{k,i_0}}{(n-\alpha+1)|s_0|^{i_0-k-\frac{1}{2}}}|h|^{i_0}\ln|h|. \tag{3.27}$$

Then, by (3.26) and (3.27), we obtain (3.21)–(3.23). The proof is completed.

Now, by the above discussion, we are able to prove Theorem 2.2.

Proof of Theorem 22. By applying (3.4), (3.13) and (3.21), one derives

$$\begin{aligned}
M(h,\delta) &= \sum_{l+k\geq 0} r_{lk} I_{lk}(h,u_0) + I_2(h,u_0) \\
&= \sum_{l+k\geq 0,\beta_{nk+l}\notin\mathbb{N}} r_{lk}\big(B_{lk}|h|^{\beta_{nk+l}} + \phi_{lk}(h,u_0)\big) \\
&\quad + \sum_{l+k\geq 0,\beta_{nk+l}\in\mathbb{N}} r_{lk}\big(B_{lk}|h|^{\beta_{nk+l}}\ln|h| + \phi_{lk}(h,u_0)\big) + I_2(h,u_0) \\
&= \sum_{l+k\geq 0,\beta_{nk+l}\notin\mathbb{N}} r_{lk}B_{lk}|h|^{\frac{nk+l}{n-\alpha+1}+\frac{n-2\alpha}{2(n-\alpha+1)}} \\
&\quad + \sum_{l+k\geq 0,\beta_{nk+l}\in\mathbb{N}} r_{lk}B_{lk}|h|^{\frac{nk+l}{n-\alpha+1}+\frac{n-2\alpha}{2(n-\alpha+1)}}\ln|h| + \phi(h,\delta) \\
&= |h|^{\frac{n-2\alpha}{2(n-\alpha+1)}}\Bigg(\sum_{i\geq 0,\beta_i\notin\mathbb{N}} B_i|h|^{\frac{i}{n-\alpha+1}} + \sum_{i\geq 0,\beta_i\in\mathbb{N}} B_i|h|^{\frac{i}{n-\alpha+1}}\ln|h|\Bigg) + \phi(h,\delta),
\end{aligned}$$

where

$$\begin{aligned}
B_i &= \sum_{nk+l=i} r_{lk}B_{lk}, \quad \beta_i = \frac{i}{n-\alpha+1} + \frac{n-2\alpha}{2(n-\alpha+1)}, \\
\phi(h,\delta) &= \sum_{l+k\geq 0} r_{lk}\phi_{lk}(h,u_0) + I_2(h,u_0) \in C^\infty.
\end{aligned}$$

This ends the proof. $\square$

Let $\phi(h,u_0) = C_0 + C_1 h + O(h^2)$. Then, using Theorem 2.2, we can further derive the specific form for the function $M(h,\delta)$ in (2.18). In other words, we easily have

Corollary 3.1. *Suppose that* (2.2) *and* **(H1)**, **(H2)** *hold. Then,*

(i) *For* $\alpha = \frac{n}{2}+\kappa$, $\kappa = 0,1,\cdots,[\frac{n+1}{2}]$,

$$\begin{aligned}
M(h,\delta) &= \sum_{i=0}^{\kappa-1} B_i|h|^{\frac{i-\kappa}{\frac{n}{2}+1-\kappa}} + B_\kappa\ln|h| + C_0 + \sum_{i=\kappa+1}^{[\frac{n+1}{2}]} B_i|h|^{\frac{i-\kappa}{\frac{n}{2}+1-\kappa}} \\
&\quad + \Big([\frac{n}{2}]+1-[\frac{n+1}{2}]\Big)B_{[\frac{n}{2}]+1}|h|\ln|h| + C_1 h + \sum_{i=[\frac{n}{2}]+2}^{n-\kappa+1} B_i|h|^{\frac{i-\kappa}{\frac{n}{2}+1-\kappa}} \\
&\quad + B_{n-\kappa+2}|h|^2\ln|h| + O(|h|^2).
\end{aligned}$$

(ii) *For* $\alpha = \frac{n}{2}-\kappa$, $\kappa = 1,2,\cdots$,

$$\begin{aligned}
M(h,\delta) &= C_0 + \sum_{i=0}^{[\frac{n+1}{2}]} B_i|h|^{\frac{i+\kappa}{\frac{n}{2}+1+\kappa}} + \Big([\frac{n}{2}]+1-[\frac{n+1}{2}]\Big)B_{[\frac{n}{2}]+1}|h|\ln|h| \\
&\quad + C_1 h + \sum_{i=[\frac{n}{2}]+2}^{n+\kappa+1} B_i|h|^{\frac{i+\kappa}{\frac{n}{2}+1+\kappa}} + B_{n+\kappa+2}|h|^2\ln|h| + O(|h|^2).
\end{aligned}$$

(iii) *For* $\alpha \in \left(\frac{n}{2}+\kappa, \frac{n}{2}+1+\kappa\right)$, $\kappa = 0, 1, \cdots, [\frac{n+1}{2}]-1$ *or* $\alpha \in \left(\frac{n}{2}+\kappa, n+1\right)$, $\kappa = [\frac{n+1}{2}]$

$$\begin{aligned}M(h,\delta) &= \sum_{i=0}^{\kappa} B_i|h|^{\frac{i}{n-\alpha+1}+\frac{n-2\alpha}{2(n-\alpha+1)}} + C_0 + \sum_{i=\kappa+1}^{[\frac{n+1}{2}]} B_i|h|^{\frac{i}{n-\alpha+1}+\frac{n-2\alpha}{2(n-\alpha+1)}} \\ &\quad + \Big([\frac{n}{2}]+1-[\frac{n+1}{2}]\Big)B_{[\frac{n}{2}]+1}|h|\ln|h| + C_1 h + \sum_{i=[\frac{n}{2}]+2}^{n-\kappa+1} B_i|h|^{\frac{i}{n-\alpha+1}+\frac{n-2\alpha}{2(n-\alpha+1)}} \\ &\quad + O(|h|^2).\end{aligned}$$

(iv) *For* $\alpha \in \left(\frac{n}{2}-\kappa, \frac{n}{2}+1-\kappa\right)$, $\kappa = 1, 2, \cdots$,

$$\begin{aligned}M(h,\delta) &= C_0 + \sum_{i=0}^{[\frac{n+1}{2}]} B_i|h|^{\frac{i}{n-\alpha+1}+\frac{n-2\alpha}{2(n-\alpha+1)}} + \Big([\frac{n}{2}]+1-[\frac{n+1}{2}]\Big)B_{[\frac{n}{2}]+1}|h|\ln|h| \\ &\quad + C_1 h + \sum_{i=[\frac{n}{2}]+2}^{n+\kappa+1} B_i|h|^{\frac{i}{n-\alpha+1}+\frac{n-2\alpha}{2(n-\alpha+1)}} + O(|h|^2).\end{aligned}$$

Here, $\alpha \neq 1$ *and* B_i, $i \geq 0$ *are given in* (3.28).

Proof. From (2.18), it is easy to see that $\beta_i = i_0 \in \mathbb{N}$ if and only if

$$2i - 2i_0(n+1) + n = 2\alpha(1-i_0). \tag{3.28}$$

Apparently, from (3.28) if $i_0 = 1$, then n is even. Otherwise, $i = \frac{n}{2}+1 \notin \mathbb{N}$, this is a contradiction. This is to say, only when n is even, the expansion of $M(h,\delta)$ has the term $|h|\ln|h|$.

From (3.28), if $i_0 = 0$, then

$$\alpha = i + \frac{n}{2}, \quad i = 0, 1, 2\cdots,$$

and if $i_0 = 2$, then

$$\alpha = -i + \frac{3n+4}{2}, \quad i = 0, 1, 2, \cdots.$$

Introduce two sets

$$\begin{aligned}\mathcal{G}_1 &= \Big\{ i + \frac{n}{2} < n+1 \;\Big|\; i = 0, 1, 2, \cdots \Big\} = \Big\{ \frac{n}{2}, \frac{n}{2}+1, \cdots, [\frac{n+1}{2}] + \frac{n}{2} \Big\}, \\ \mathcal{G}_2 &= \Big\{ -i + \frac{3n+4}{2} < n+1 \;\Big|\; i = 0, 1, 2, \cdots \Big\} \\ &= \Big\{ [\frac{n+1}{2}] + \frac{n}{2}, [\frac{n+1}{2}] + \frac{n}{2} - 1, \cdots, \frac{n}{2}, \frac{n}{2} - 1, \cdots \Big\}.\end{aligned}$$

Then

$$\mathcal{G}_1 \cap \mathcal{G}_2 = \Big\{ \frac{n}{2}, \frac{n}{2}+1, \cdots, [\frac{n+1}{2}] + \frac{n}{2} \Big\}.$$

Thus, when $\alpha \in \mathcal{G}_1 \cap \mathcal{G}_2$, the expansion of $M(h,\delta)$ has both of the terms $\ln|h|$, $|h|^2\ln|h|$, when $\alpha \in \mathcal{G}_2 \setminus \mathcal{G}_1$, the expansion of $M(h,\delta)$ has the term $|h|^2\ln|h|$, and when $\alpha \notin \mathcal{G}_2$, the expansion of $M(h,\delta)$ has no both of terms $\ln|h|$, $|h|^2\ln|h|$.

By the above discussion, it is not hard to obtain the conclusions. The proof is then ended. □

From the proof of Lemmas 3.4 and 3.5 in [2], it is easy to verify the following lemma, which gives the formulas for C_0 and C_1 under special conditions.

Lemma 3.5. (i) *If one of the following two cases holds:* (a) $\alpha \in [\frac{n}{2}+\kappa, \frac{n}{2}+1+\kappa)$, $\kappa = 0, 1, \cdots, [\frac{n+1}{2}]-1$ *or* $\alpha \in [\frac{n}{2}+\kappa, n+1)$, $\kappa = [\frac{n+1}{2}]$ *and* $B_0 = B_1 = \cdots = B_\kappa$*;* (b) $\alpha \in [\frac{n}{2}-\kappa, \frac{n}{2}+1-\kappa)$, $\kappa = 1, 2, \cdots$, *then*

$$C_0 = \lim_{h\to 0^-} \oint_{L_h} \mu q dx - \mu p dy.$$

(ii) *For* $\alpha \in (-\infty, n+1) \setminus \{1\}$*, if* $B_0 = B_1 = \cdots = B_{[\frac{n}{2}]+1} = 0$*, then*

$$C_1 = \lim_{h\to 0^-} \oint_{L_h} \left[(\mu p)_x + (\mu q)_y\right] dt.$$

Now, by using the expansion of $M(h, \delta)$ in (2.2), we can study the problem of limit cycle bifurcation near L for system (2.1). To do this, on account of Corollary 3.1, we write

$$\begin{aligned} M(h,\delta) = {} & A_0|h|^{\rho_0} + A_1|h|^{\rho_1} + \cdots + A_{k_0-2}|h|^{\rho_{k_0-2}} + A_{k_0-1}\ln|h| + A_{k_0}|h|^{\rho_{k_0}} \\ & + A_{k_0+1}|h|^{\rho_{k_0+1}} + \cdots + A_{k_1-2}|h|^{\rho_{k_1-2}} + A_{k_1-1}|h|\ln|h| + A_{k_1}|h|^{\rho_{k_1}} \\ & + A_{k_1+1}|h|^{\rho_{k_1+1}} + \cdots + A_{k_2-2}|h|^{\rho_{k_2-2}} + A_{k_2-1}|h|^2\ln|h| + O(|h|^2), \end{aligned} \tag{3.29}$$

where $k_0,\ k_1,\ k_2 \in \mathbb{N}$ and $\rho_{k_j} = j$, $j = 0,\ 1,\ 2$.

Let

$$\bar{A}_0 = A_0, \quad \bar{A}_j = A_j|_{A_i=0,\ i=0,1,\cdots,j-1}, \quad j = 1, 2, \cdots, k_2-1. \tag{3.30}$$

Then, one has

$$A_j = \bar{A}_j + O(|A_0, A_1, \cdots, A_{j-1}|).$$

Applying (3.29) and (3.30), one can get the following theorem similar to Theorem 3.2.3 in [5].

Theorem 3.1. *Let the conditions of Theorem 2.2 hold. If there exist* $0 \le l \le k_2 - 1$ *and* $\delta_0 \in \mathbb{R}^m$ *such that*

$$\bar{A}_j(\delta_0) = 0, \quad j = 0, \cdots, l-1, \quad \bar{A}_l(\delta_0) \ne 0, \quad \mathrm{rank}\frac{\partial(\bar{A}_0, \cdots, \bar{A}_{l-1})}{\partial(\delta_1, \cdots, \delta_m)}(\delta_0) = l,$$

then system (2.1) *can have* l *limit cycles in the neighborhood of* L *for some* (ε, δ) *near* $(0, \delta_0)$*.*

From the proof of Lemma 3.4, we have the formula

$$B_i = \sum_{nk+l=i} r_{lk} B_{lk}, \tag{3.31}$$

where r_{lk} and B_{lk} are given in (3.15) and (3.22) respectively. However, from the formula (3.22), one only finds values B_{lk} with $\beta_i \in \mathbb{N}$. For values of B_{lk} with $\beta_i \notin \mathbb{N}$, we have

Lemma 3.6. (i) *For* $l \le [\frac{n+1}{2}]$,

$$B_{l0} = -\frac{\big(n-\alpha+1\big)\int_0^1 v^{\frac{n}{2}-l}(1-v^{n-\alpha+1})^{-\frac{1}{2}}dv}{\big(2l+n-2\alpha\big)|s_0|^{\frac{-\alpha-1+l}{2(n-\alpha+1)}}}.$$

(ii) *For* $[\frac{n}{2}]+2 \le l < \frac{3}{2}n+2-\alpha$,

$$B_{l0} = \frac{-(n-\alpha+1)}{\big(2l+n-2\alpha\big)|s_0|^{\frac{-\alpha+2l-1}{2(n-\alpha+1)}}}\Big(\int_0^1 \frac{v^{\frac{3n}{2}-l-\alpha+1}}{\sqrt{1-v^{n-\alpha+1}}\big(1+\sqrt{1-v^{n-\alpha+1}}\big)}dv + \frac{2}{n-2l+2}\Big).$$

(iii) *For* $l < \min\{\frac{n}{2}+2-\alpha, \frac{n}{2}+1\}$,

$$B_{l1} = \frac{-1}{|s_0|^{\frac{\alpha+2l-3}{2(n-\alpha+1)}}}\Big(\int_0^1 \frac{\sqrt{1-v^{n-\alpha+1}}-1}{v^{l+\frac{n}{2}}}dv + \frac{2}{2-n-2l}\Big) - s_0 B_{n+l,0}.$$

Proof. From (3.21) and (3.22), one derives

$$\frac{\partial I_{l0}(h,u_0)}{\partial h} = -\frac{2l+n-2\alpha}{2(n-\alpha+1)}B_{l0}|h|^{\frac{2(l-1)-n}{2(n-\alpha+1)}} + \frac{\partial \phi_{l0}(h,u_0)}{\partial h}. \tag{3.32}$$

In view of (3.14), we obtain

$$\begin{aligned}\frac{\partial I_{l0}(h,u_0)}{\partial h} &= \frac{1}{2}\int_{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}}^{u_0} u^{-\frac{\alpha+3}{2}+l}\big(h-s_0u^{n-\alpha+1}\big)^{-\frac{1}{2}}du \quad \Big(\text{Let} \quad v=|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}u^{-1}\Big)\\ &= -\frac{1}{2}\frac{|h|^{\frac{-\frac{\alpha+3}{2}+l+1}{n-\alpha+1}-\frac{1}{2}}}{|s_0|^{\frac{-\frac{\alpha+3}{2}+l+1}{n-\alpha+1}}}\int_1^{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}u_0^{-1}} v^{\frac{n}{2}-l}(1-v^{n-\alpha+1})^{-\frac{1}{2}}dv. \end{aligned}\tag{3.33}$$

For $l \le [\frac{n+1}{2}]$, from (3.33), it follows that

$$\begin{aligned}\frac{\partial I_{l0}(h,u_0)}{\partial h} &= -\frac{1}{2}\frac{|h|^{\frac{-n+2(l-1)}{2(n-\alpha+1)}}}{|s_0|^{\frac{-\alpha-1+l}{n-\alpha+1}}}\Big(\int_1^0 + \int_0^{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}u_0^{-1}}\Big)v^{\frac{n}{2}-l}(1-v^{n-\alpha+1})^{-\frac{1}{2}}dv\\ &= -\frac{1}{2}\frac{|h|^{\frac{-n+2(l-1)}{2(n-\alpha+1)}}}{|s_0|^{\frac{-\alpha-1+l}{n-\alpha+1}}}\Big[\int_1^0 v^{\frac{n}{2}-l}(1-v^{n-\alpha+1})^{-\frac{1}{2}}dv + O\big(|h|^{\frac{n+2(1-l)}{2(n-\alpha+1)}}\big)\Big]\\ &= \frac{1}{2}\frac{\int_0^1 v^{\frac{n}{2}-l}(1-v^{n-\alpha+1})^{-\frac{1}{2}}dv}{|s_0|^{\frac{-\alpha-1+l}{2(n-\alpha+1)}}}|h|^{\frac{-n+2(l-1)}{2(n-\alpha+1)}} + O(1).\end{aligned}$$

In this case, comparing (3.32) and the above formula, we obtain the conclusion (i).

For $[\frac{n}{2}]+2 \le l < \frac{3}{2}n+2-\alpha$, by (3.33), we have

$$\frac{\partial I_{l0}(h,u_0)}{\partial h} = -\frac{1}{2}\frac{|h|^{\frac{-n+2(l-1)}{2(n-\alpha+1)}}}{|s_0|^{\frac{-\alpha-1+l}{n-\alpha+1}}}\Big[\int_1^{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}u_0^{-1}} \frac{v^{\frac{n}{2}-l}\big(1-\sqrt{1-v^{n-\alpha+1}}\big)}{\sqrt{1-v^{n-\alpha+1}}}dv$$

$$+\int_1^{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}u_0^{-1}} v^{\frac{n}{2}-l}dv\Big]$$

$$= -\frac{1}{2}\frac{|h|^{\frac{-n+2(l-1)}{2(n-\alpha+1)}}}{|s_0|^{\frac{-\alpha-1+l}{n-\alpha+1}}}\Big[\Big(\int_1^0 + \int_0^{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}u_0^{-1}}\Big)\frac{v^{\frac{n}{2}-l}\big(1-\sqrt{1-v^{n-\alpha+1}}\big)}{\sqrt{1-v^{n-\alpha+1}}}dv$$

$$+\frac{2}{n-2l+2}\Big|\frac{h}{s_0}\Big|^{\frac{n-2l+2}{2(n-\alpha+1)}}u_0^{l-1-\frac{n}{2}} - \frac{2}{n-2l+2}\Big]$$

$$= -\frac{1}{2}\frac{|h|^{\frac{-n+2(l-1)}{2(n-\alpha+1)}}}{|s_0|^{\frac{-\alpha-1+l}{n-\alpha+1}}}\Big(\int_1^0 \frac{v^{\frac{3n}{2}-l-\alpha+1}}{\sqrt{1-v^{n-\alpha+1}}\big(1+\sqrt{1-v^{n-\alpha+1}}\big)}dv - \frac{2}{n-2l+2}\Big) + O(1),$$

which, together with (3.32), implies the conclusion (ii). Now, we prove the conclusion (iii). From (3.14),

$$\begin{aligned}
I_{l1}(h,u_0) &= \int_{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}}^{u_0} u^{(\alpha-1)-\frac{\alpha+3}{2}+l}(h-s_0u^{n-\alpha+1})(h-s_0u^{n-\alpha+1})^{\frac{1}{2}}du \\
&= h\int_{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}}^{u_0} u^{\frac{\alpha}{2}+l-\frac{5}{2}}(h-s_0u^{n-\alpha+1})^{\frac{1}{2}}du - s_0\int_{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}}^{u_0} u^{n+l-\frac{\alpha+3}{2}}(h-s_0u^{n-\alpha+1})^{\frac{1}{2}}du \\
&= h\bar{I}_{l1}(h,u_0) - s_0I_{n+l,0}.
\end{aligned} \tag{3.34}$$

Perform a transformation $v = |\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}u^{-1}$ to the integral $\bar{I}_{l1}(h,u_0)$ in the above to obtain that

$$\begin{aligned}
\bar{I}_{l1}(h,u_0) &= -\Big|\frac{h}{s_0}\Big|^{\frac{\frac{\alpha}{2}+l-\frac{3}{2}}{n-\alpha+1}}|h|^{\frac{1}{2}}\int_1^{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}u_0^{-1}} v^{-l-\frac{n}{2}}(1-v^{n-\alpha+1})^{\frac{1}{2}}dv \\
&= -\frac{|h|^{\frac{n+2l-2}{2(n-\alpha+1)}}}{|s_0|^{\frac{\alpha+2l-3}{2(n-\alpha+1)}}}\Big[\int_1^{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}u_0^{-1}} \frac{\sqrt{1-v^{n-\alpha+1}}-1}{v^{l+\frac{n}{2}}}dv + \int_1^{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}u_0^{-1}} v^{-l-\frac{n}{2}}dv\Big].
\end{aligned} \tag{3.35}$$

Note that

$$\begin{aligned}
\int_1^{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}u_0^{-1}} \frac{\sqrt{1-v^{n-\alpha+1}}-1}{v^{l+\frac{n}{2}}}dv &= \Big(\int_1^0 + \int_0^{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}u_0^{-1}}\Big)\frac{\sqrt{1-v^{n-\alpha+1}}-1}{v^{l+\frac{n}{2}}}dv \\
&= -\int_0^1 \frac{\sqrt{1-v^{n-\alpha+1}}-1}{v^{l+\frac{n}{2}}}dv + O(|h|^{\frac{n-2\alpha+4-2l}{2(n-\alpha+1)}}),
\end{aligned}$$

$$\int_1^{|\frac{h}{s_0}|^{\frac{1}{n-\alpha+1}}u_0^{-1}} v^{-l-\frac{n}{2}}dv = \frac{2}{2-n-2l}\Big|\frac{h}{s_0}\Big|^{\frac{-2l-n+2}{2(n-\alpha+1)}}u_0^{l+\frac{n}{2}-1} - \frac{2}{2-n-2l}.$$

In view of (3.34), (3.35) and the above, we easily obtain the conclusion from (3.21). This ends the proof. □

4. Applications

In this section, we will use the method established in Section 3 to study limit cycle bifurcation of a cubic system. More precisely, we consider the following cubic system

$$\dot{x} = xy, \quad \dot{y} = x^2 - x^3 + \frac{1}{2}y^2 + \varepsilon \sum_{i=0}^{3} a_i x^i y. \tag{4.1}$$

We easily see that (4.1)$|_{\varepsilon=0}$ has an integrating factor x^{-2} with $\alpha = -1$ and its first integral has the from

$$H(x,y) = \frac{y^2}{2x} - x + \frac{1}{2}x^2.$$

Furthermore, (4.1)$|_{\varepsilon=0}$ has an elementary center at $(1,0)$ and a degenerate singular point at $(0,0)$. The equations

$$H(x,y) = h, \quad h \in \big(-\frac{1}{2}, 0\big)$$

define a family of ovals, which surround the center $(1,0)$ and terminate at a homoclinic loop given by $\{(x,y)|H(x,y) = 0,\ x \geq 0\} \triangleq L$. For $x \geq 0$, the phase portrait of (4.1)$|_{\varepsilon=0}$ is the same as Fig. 2.3(1). By [7], one can easily obtain that for $h \in (-\frac{1}{2}, +\infty)$, the oval $H(x,y) = h$ determines a smooth periodic wave and the oval $H(x,y) = 0$, namely L determines a solitary wave solution.

Further, by Corollary 3.1, in the neighborhood of L, the Melnikov function on (4.1) is

$$M(h,\delta) = B_0|h|^{-1} + B_1 \ln|h| + C_0 + B_2|h|\ln|h| + C_1 h + O(|h|^2 \ln|h|),$$

where, together with (3.31) and Lemma 3.5

$$\begin{aligned}
&B_0 = r_{00}B_{00}, \quad B_1 = r_{10}B_{10}, \quad B_2 = r_{20}B_{20} + r_{01}B_{01},\\
&C_0|_{B_0=B_1=0} = \lim_{h\to 0^-} \oint\limits_{H(x,y)=h} \sum_{i=0}^{3} a_i x^{i-2} y dx,\\
&C_1|_{B_0=B_1=B_2=0} = \lim_{h\to 0^-} \oint\limits_{H(x,y)=h} \sum_{i=0}^{3} a_i x^{i-2} dt.
\end{aligned}$$

Based on the discussion in Section 3 and by the simple computation, we obtain from the above formula

$$\begin{aligned}
&B_0 \equiv 0, \quad B_1 = -2\sqrt{2}a_0, \quad B_2 = \sqrt{2}a_1 + O(|a_0|),\\
&C_0|_{B_0=B_1=0} = 2\lim_{h\to 0^-} \int_{1-\sqrt{1+2h}}^{1+\sqrt{1+2h}} (a_1x^{-1} + a_2 + a_3x)\sqrt{2x(h + x - \frac{1}{2}x^2)}dx\\
&= \int_0^2 (a_1x^{-1} + a_2 + a_3x)\sqrt{2x(x - \frac{1}{2}x^2)}dx\\
&= \frac{8\sqrt{2}}{3}\Big(a_1 + \frac{4}{5}a_2 + \frac{32}{35}a_3\Big),
\end{aligned}$$

$$C_1|_{B_0=B_1=B_2=0} = 2 \lim_{h\to 0^-} \int_{1-\sqrt{1+2h}}^{1+\sqrt{1+2h}} \frac{a_2 x + a_3 x^2}{\sqrt{2x(h+x-\frac{1}{2}x^2)}} dx$$

$$= 2\int_0^2 \frac{a_2 x + a_3 x^2}{\sqrt{2x(x-\frac{1}{2}x^2)}} dx = 4\sqrt{2}(a_2 + \frac{4}{3}a_3).$$

Solving the equations $B_1 = B_2 = C_0 = 0$, we obtain $a_0 = a_1 = 0$, $a_3 = -\frac{7}{8}a_2$. In this case, we further have

$$C_1 = -\frac{2}{3}\sqrt{2}a_2, \quad \det \frac{\partial(B_1, B_2, C_0)}{\partial(a_0, a_1, a_2)} \neq 0,$$

which means that system (4.1) can have three limit cycles near L from Theorem 3.1 for some (a_0, a_1, a_3) near $(0, 0, -\frac{7}{8}a_2)$ if $a_2 \neq 0$.

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