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Non-spectrality of self-affine measures on the spatial Sierpinski gasket

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ABSTRACT

Let $\mu_{M,D}$ be the self-affine measure corresponding to a diagonal matrix M with entries $p_1, p_2, p_3 \in \mathbb{Z}\backslash\{0, \pm 1\}$ and $D = \{0, e_1, e_2, e_3\}$ in the space $\mathbb{R}^3$, where e_1, e_2, e_3 are the standard basis of unit column vectors in $\mathbb{R}^3$. Such a measure is supported on the spatial Sierpinski gasket. In this paper, we prove the non-spectrality of $\mu_{M,D}$. By characterizing the zero set $Z(\hat{\mu}_{M,D})$ of the Fourier transform $\hat{\mu}_{M,D}$, we obtain that if $p_1 \in 2\mathbb{Z}$ and $p_2, p_3 \in 2\mathbb{Z} + 1$, then $\mu_{M,D}$ is a non-spectral measure, and there are at most a finite number of orthogonal exponential functions in $L^2(\mu_{M,D})$. This completely solves the problem on the finiteness or infiniteness of orthogonal exponentials in the Hilbert space $L^2(\mu_{M,D})$.

1. Introduction

Let $M \in M_n(\mathbb{Z})$ be an expanding integer matrix and $D \subset \mathbb{Z}^n$ be a finite digit set of the cardinality $|D|$. Associated with M and D, it is known [7] that here exists a unique probability measure $\mu := \mu_{M,D}$ satisfying

$$\mu = \frac{1}{|D|} \sum_{d \in D} \mu \circ \phi_d^{-1}, \tag{1.1}$$

such a measure is called *self-affine measure* and is supported on the compact set $T \subset \mathbb{R}^n$, where $T := T(M, D)$ is the *attractor* (or *invariant set*) of the affine iterated function system (IFS) $\{\phi_d(x) = M^{-1}(x + d)\}_{d \in D}$. The measure $\mu_{M,D}$ is called *spectral* if there exists a set $\Lambda \subset \mathbb{R}^n$ such that $E(\Lambda) := \{e_\lambda(x) = e^{2\pi i \langle \lambda, x\rangle} : \lambda \in \Lambda\}$ forms an orthogonal basis (Fourier basis) for the Hilbert space $L^2(\mu_{M,D})$. The set Λ is then called a *spectrum* for $\mu_{M,D}$; we also say that $(\mu_{M,D}, \Lambda)$ is a *spectral pair*. The question we are concerned is the spectrality or non-spectrality of $\mu_{M,D}$. This question has its origin in analysis and geometry. It was initiated by Fuglede [6] who investigated which subsets of $\mathbb{R}^n$ with the Lebesgue measures are spectral. In the same paper, Fuglede proposed his famous conjecture on the relationship between the

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spectral set and the translation tile of $\mathbb{R}^n$. This conjecture and its related problems have received much attention in the last few decades (see [10]). In particular, the operator-theoretic approach on these problems initiated by Jorgensen and Pedersen lead the research into the realm of fractals. In terms of fractal measures, it is started with the work of Jorgensen and Pedersen [8,9] who showed that for certain M and D, the measure $\mu_{M,D}$ may be spectral, while for another M and D, the measure $\mu_{M,D}$ may be non-spectral. Subsequently, there are many researches on this question (see [1–3,5,14,15,18,19] and the references cited therein). The previous researches illustrate that the spectrality of $\mu_{M,D}$ requires strict conditions on the Fourier transform $\hat{\mu}_{M,D}$, it has close relation with the problem of finiteness or infiniteness of orthogonal exponentials in the Hilbert space $L^2(\mu_{M,D})$. And for some pairs (M, D), the non-spectrality of $\mu_{M,D}$ is due to the fact that there are at most a finite number of orthogonal exponential functions in the Hilbert space $L^2(\mu_{M,D})$. The present paper will follow the paper [13] to further proving the non-spectrality of self-affine measure $\mu_{M,D}$ on the typical fractal: the spatial Sierpinski gasket $T(M, D)$, where

$$M = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \quad (p_1, p_2, p_3 \in \mathbb{Z} \setminus \{0, \pm 1\}) \quad \text{and}$$
$$D = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}. \tag{1.2}$$

For such a pair (M, D) in (1.2), the spectrality or non-spectrality of $\mu_{M,D}$ can be summarized as the following Theorem A.

Theorem A. *For the self-affine measure $\mu_{M,D}$ corresponding to (1.2), the following spectrality and non-spectrality hold:*

(*i*) *If $p_1 = p_2 = p_3 = p$ and $p \in 2\mathbb{Z} \setminus \{0\}$, then $\mu_{M,D}$ is a spectral measure;*
(*ii*) *If $p_j \in 2\mathbb{Z} \setminus \{0, 2\}$ for $j = 1, 2, 3$, then $\mu_{M,D}$ is a spectral measure;*
(*iii*) *If $p_j \in (2\mathbb{Z} + 1) \setminus \{\pm 1\}$ for $j = 1, 2, 3$, then $\mu_{M,D}$ is a non-spectral measure, and there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, where the number 4 is the best upper bound.*

See [8], [9, Example 7.1], [16], [17, Example 2.9(e)], [4, Theorem 5.1(iii)], [11, Theorem 1], [12]. Also there are two problems on the spectrality of such self-affine measure $\mu_{M,D}$:

Question 1. How about the spectrality of $\mu_{M,D}$ if $p_j \in 2\mathbb{Z} \setminus \{0\}$ $(j = 1, 2, 3)$ and one or two of the three numbers p_1, p_2, p_3 can take the value 2?

Question 2. How about the spectrality of $\mu_{M,D}$ if p_j $(j = 1, 2, 3)$ have different parity?

In a recent paper [13], we settled Question 1 except for the case that two of the three numbers p_1, p_2, p_3 are 2 and the other number is -2, or except for the case that two of the three numbers p_1, p_2, p_3 are -2 and the other number is 2. The spectrality of $\mu_{M,D}$ in the case when $p_j \in 2\mathbb{Z} \setminus \{0\}$ $(j = 1, 2, 3)$ can also be obtained by applying a recent result of [3] and [5]. The answer is that $\mu_{M,D}$ is a spectral measure. So the remaining problem relating to this case is to determine all the spectra for such a measure $\mu_{M,D}$. However, for Question 2, only a little result is known. We summarize the known result as the following Theorem B.

Theorem B. *(See [13].) (i) If M and D are given by (1.2) with $p_1 \in 2\mathbb{Z}$ and $p_2 = p_3 \in 2\mathbb{Z} + 1$, then $\mu_{M,D}$ is a non-spectral measure, and there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, where the number 4 is the best upper bound; (ii) If M and D are given by (1.2) with any two of the three* *numbers p_1, p_2, p_3 being in the set $2\mathbb{Z}$, then there are infinite families of orthogonal exponentials $E(\Lambda)$ in $L^2(\mu_{M,D})$ with $\Lambda \subseteq \mathbb{Z}^3$.*

In the present paper, we shall further consider Question 2 by relaxing the condition $p_2 = p_3 \in 2\mathbb{Z}+1$ of Theorem B(i). The main result shows that if two of the three numbers p_1, p_2, p_3 are in the set $2\mathbb{Z}+1$ and the other one of the three numbers p_1, p_2, p_3 is in the set $2\mathbb{Z}$, then $\mu_{M,D}$ is a non-spectral measure, and there are at most a finite number of orthogonal exponential functions in $L^2(\mu_{M,D})$. Therefore, combined with Theorem A(iii) and Theorem B(ii), we know that for the spatial Sierpinski gasket (1.2), if any two of the three numbers p_1, p_2, p_3 are in the set $2\mathbb{Z}$, then there are infinite families of orthogonal exponentials $E(\Lambda)$ in $L^2(\mu_{M,D})$ with $\Lambda \subseteq \mathbb{Z}^3$; if any two of the three numbers p_1, p_2, p_3 are in the set $2\mathbb{Z}+1$, then there are at most a finite number of orthogonal exponential functions in $L^2(\mu_{M,D})$. This completely solves the problem of how to determine the $L^2(\mu_{M,D})$-space has finite or infinite orthogonal exponentials on the spatial Sierpinski gasket (1.2).

2. Main result and its proof

The main result which generalizes Theorem B(i) is contained in the following.

Theorem 2.1. *If M and D are given by (1.2) with $p_1 \in 2\mathbb{Z}$ and $p_2, p_3 \in 2\mathbb{Z}+1$, then $\mu_{M,D}$ is a non-spectral measure, and there are at most a finite number of orthogonal exponential functions in $L^2(\mu_{M,D})$.*

Proof. Since the case $p_2 = p_3$ is contained in Theorem B(i), we mainly deal with the case $p_2 \neq p_3$ in the following discussion. From (1.1), the Fourier transform $\hat{\mu}_{M,D}(\xi)$ of the measure $\mu_{M,D}$ is given by

$$\hat{\mu}_{M,D}(\xi) = \int e^{2\pi i\langle x,\xi\rangle} d\mu_{M,D}(x) = \prod_{j=1}^{\infty} m_D(M^{*-j}\xi) \quad (\xi \in \mathbb{R}^n)$$

where M^* denotes the transposed conjugate of M (in fact, $M^* = M^T$) and

$$m_D(x) = \frac{1}{|D|}\sum_{d\in D} e^{2\pi i\langle d,x\rangle} \quad (x \in \mathbb{R}^n).$$

For the pair (M, D) given by (1.2), it is known [13] that the zero set $Z(\hat{\mu}_{M,D}(\xi))$ of the Fourier transform $\hat{\mu}_{M,D}(\xi)$ is

$$Z(\hat{\mu}_{M,D}(\xi)) = \bigcup_{j=1}^{\infty} M^{*j} Z(m_D(\xi)) := B_1 \cup B_2 \cup B_3, \tag{2.1}$$

where

$$B_1 = \bigcup_{j=1}^{\infty}\left\{\begin{pmatrix}(1/2+k_1)p_1^j\\(a+k_2)p_2^j\\(1/2+a+k_3)p_3^j\end{pmatrix} : a \in \mathbb{R}, k_1, k_2, k_3 \in \mathbb{Z}\right\} \subset \mathbb{R}^3, \tag{2.2}$$

$$B_2 = \bigcup_{j=1}^{\infty}\left\{\begin{pmatrix}(1/2+a+k_1)p_1^j\\(1/2+k_2)p_2^j\\(a+k_3)p_3^j\end{pmatrix} : a \in \mathbb{R}, k_1, k_2, k_3 \in \mathbb{Z}\right\} \subset \mathbb{R}^3, \tag{2.3}$$

$$B_3 = \bigcup_{j=1}^{\infty}\left\{\begin{pmatrix}(a+k_1)p_1^j\\(1/2+a+k_2)p_2^j\\(1/2+k_3)p_3^j\end{pmatrix} : a \in \mathbb{R}, k_1, k_2, k_3 \in \mathbb{Z}\right\} \subset \mathbb{R}^3. \tag{2.4}$$

From $p_1 \in 2\mathbb{Z} \setminus \{0\}$ and $p_2, p_3 \in (2\mathbb{Z}+1) \setminus \{\pm 1\}$, we can verify that the following two lemmas hold.

Lemma 2.1. *Let $\xi = (\xi_1, \xi_2, \xi_3)^T \in Z(\hat{\mu}_{M,D}(\xi)) = B_1 \cup B_2 \cup B_3$, where the sets B_j $(j = 1, 2, 3)$ are given by (2.2), (2.3) and (2.4) respectively. Then the following statements hold:*

(a) $\xi \in B_j \Longleftrightarrow -\xi \in B_j$ $(j = 1, 2, 3)$;
(b) $Z(\hat{\mu}_{M,D}(\xi)) \bigcap \mathbb{Z}^3 = Z(\hat{\mu}_{M,D}(\xi)) \bigcap ((1/2, 1/2, 1/2)^T + \mathbb{Z}^3) = \emptyset$;
(c) *If* $\xi \in B_j$, *then* $\xi_j \in \frac{1}{2} + \mathbb{Z}$, *where* $j = 2, 3$;
(d) *If* $\xi = (\xi_1, \xi_2, \xi_3)^T \in B_2 \pm B_2$, *then* $\xi_2 \in \mathbb{Z}$;
(e) *If* $\xi = (\xi_1, \xi_2, \xi_3)^T \in B_3 \pm B_3$, *then* $\xi_3 \in \mathbb{Z}$.
(f) *If* $\xi_2 \in \mathbb{Z}$, *then* $\xi \in B_1 \cup B_3$ *and* $\xi_3 \notin \mathbb{Z}$;
(g) *If* $\xi_3 \in \mathbb{Z}$, *then* $\xi \in B_1 \cup B_2$ *and* $\xi_2 \notin \mathbb{Z}$.

The conclusions of Lemma 2.1(f), (g) illustrate that the last two coordinates ξ_2, ξ_3 of ξ cannot be integers simultaneously.

Lemma 2.2. *Let $\xi = (\xi_1, \xi_2, \xi_3)^T \in Z(\hat{\mu}_{M,D}(\xi)) = B_1 \cup B_2 \cup B_3$. In the case when $\xi \in B_1$, $\xi_1 \in \mathbb{Z}$ and the following statements hold:*

(i) *If* $\xi_2 \in \mathbb{Z}$, *then there exist* $j \in \mathbb{N} = \{1, 2, 3, \ldots\}$ *and* $\hat{k}_1, \hat{k}_2 \in \mathbb{Z}$ *such that*

$$\xi_3 = \frac{1}{2} + \hat{k}_1 \frac{p_3^j}{p_2^j} + \hat{k}_2; \tag{2.5}$$

(ii) *If* $\xi_3 \in \mathbb{Z}$, *then there exist* $j \in \mathbb{N}$ *and* $\hat{k}_1, \hat{k}_2 \in \mathbb{Z}$ *such that*

$$\xi_2 = (2\hat{k}_1 - 1)\frac{p_2^j}{(2p_3^j)} + \hat{k}_2. \tag{2.6}$$

Secondly, assume that $\lambda_j \in \mathbb{R}^3$ $(j = 1, 2, 3, \ldots)$ are such that the infinite family of functions

$$e^{2\pi i\langle\lambda_1, x\rangle}, \quad e^{2\pi i\langle\lambda_2, x\rangle}, \quad e^{2\pi i\langle\lambda_3, x\rangle}, \quad \cdots$$

are mutually orthogonal in $L^2(\mu_{M,D})$, then

$$\lambda_j - \lambda_k \in Z(\hat{\mu}_{M,D}(\xi)) = B_1 \cup B_2 \cup B_3 \quad (j, k \geq 1,\ j \neq k). \tag{2.7}$$

We shall apply the above two lemmas as well as the other properties established for B_1 to look for a contradiction.

Now, from (2.7), the following infinite differences:

$$\begin{array}{r} \lambda_2 - \lambda_1,\ \ \lambda_3 - \lambda_1,\ \ \lambda_4 - \lambda_1,\ \ \lambda_5 - \lambda_1, \ldots, \lambda_n - \lambda_1, \ldots; \\ \lambda_3 - \lambda_2,\ \ \lambda_4 - \lambda_2,\ \ \lambda_5 - \lambda_2, \ldots, \lambda_n - \lambda_2, \ldots; \\ \lambda_4 - \lambda_3,\ \ \lambda_5 - \lambda_3, \ldots, \lambda_n - \lambda_3, \ldots; \\ \lambda_5 - \lambda_4, \ldots, \lambda_n - \lambda_4, \ldots; \\ \cdots \quad \cdots \quad \cdots \end{array} \tag{2.8}$$

belong to the union of the three sets B_1, B_2 and B_3. Let Row1 denote the first row of (2.8), that is,

$$\text{Row1} = \{\lambda_2 - \lambda_1,\ \lambda_3 - \lambda_1,\ \lambda_4 - \lambda_1,\ \lambda_5 - \lambda_1, \ldots, \lambda_n - \lambda_1, \ldots\}.$$

Row2 denotes the second row of (2.8), and so on. We also denote the difference by

$$\lambda_j - \lambda_k = (x_{j,k}, y_{j,k}, z_{j,k})^T \in \mathbb{R}^3 \quad \text{for } j, k \geq 1 \text{ and } j \neq k.$$

From the first row of (2.8) and Row1 $\subseteq B_1 \cup B_2 \cup B_3$, we only need to consider the following three cases:

Case 1: The set B_1 contains infinite elements of the set Row1;
Case 2: The set B_2 contains infinite elements of the set Row1;
Case 3: The set B_3 contains infinite elements of the set Row1.

2.1. Proof of Case 2 and Case 3

We first prove Case 2. Since B_2 contains infinite differences of Row1, the method of proving Case 2 is to consider these differences and to write the differences of these differences as (2.8), then we apply the above Lemmas 2.1 and 2.2 to deduce a contradiction.

Without loss of generality, we may assume that Case 2 is

$$\lambda_2 - \lambda_1,\ \lambda_3 - \lambda_1,\ \lambda_4 - \lambda_1,\ \lambda_5 - \lambda_1, \ldots, \lambda_n - \lambda_1, \ldots \in B_2, \tag{2.9}$$

that is, Row1 $\subseteq B_2$. Then, applying Lemma 2.1(d), (f), we have

$$\begin{aligned} \lambda_3 - \lambda_2,\ \lambda_4 - \lambda_2,\ \lambda_5 - \lambda_2, \ldots, \lambda_n - \lambda_2, \ldots &\in B_2 - B_2 \subseteq B_1 \cup B_3; \\ \lambda_4 - \lambda_3,\ \lambda_5 - \lambda_3, \ldots, \lambda_n - \lambda_3, \ldots &\in B_2 - B_2 \subseteq B_1 \cup B_3; \\ \lambda_5 - \lambda_4, \ldots, \lambda_n - \lambda_4, \ldots &\in B_2 - B_2 \subseteq B_1 \cup B_3; \\ \cdots \qquad \cdots \qquad & \cdots, \end{aligned} \tag{2.10}$$

that is,

$$\text{Row2, Row3, Row4}, \ldots \subseteq B_2 - B_2 \subseteq B_1 \cup B_3, \tag{2.11}$$

and the second coordinate of each difference in (2.10) is in $\mathbb{Z}$.

From Lemma 2.1(f) and (2.11), we observe that for any $j = 2, 3, 4, \ldots$, B_3 cannot contain any two elements of the set Row$j$. For example, if B_3 contains two elements of Row2, say $\lambda_3 - \lambda_2$ and $\lambda_4 - \lambda_2$, then $\lambda_4 - \lambda_3 = (\lambda_4 - \lambda_2) - (\lambda_3 - \lambda_2) \in B_3 - B_3$, and $z_{4,3} \in \mathbb{Z}$, a contradiction of the fact that $y_{4,3} \in \mathbb{Z}$. Hence, according to the first row of (2.10), B_3 contains at most one element of the set Row2, and the discussion leads to the following two cases:

Case 2.1: $\lambda_3 - \lambda_2 \in B_3$ and Row2 $\setminus \{\lambda_3 - \lambda_2\} \subseteq B_1$;
Case 2.2: Row2 $\subseteq B_1$.

Case 2.1 denotes that B_3 contains one element of the set Row2, say $\lambda_3 - \lambda_2$. Case 2.2 denotes that B_3 contains no element of the set Row2.

In Case 2.1, we have

$$\lambda_3 - \lambda_2 = (x_{3,2}, y_{3,2}, z_{3,2})^T \in B_3 \quad \text{and} \quad x_{3,2} \in \mathbb{R},\ y_{3,2} \in \mathbb{Z},\ z_{3,2} \in \frac{1}{2} + \mathbb{Z}; \tag{2.12}$$

$$\lambda_n - \lambda_2 = (x_{n,2}, y_{n,2}, z_{n,2})^T \in B_1 \quad \text{and} \quad x_{n,2}, y_{n,2} \in \mathbb{Z},\ z_{n,2} \in \mathbb{R}\ (n = 4, 5, 6, \ldots), \tag{2.13}$$

where $z_{n,2}$ $(n = 4, 5, 6, \ldots)$ can be represented by (2.5).

It follows from Lemma 2.1, (2.11) and (2.12) that

$$\text{Row3} \subseteq B_1. \tag{2.14}$$

That is, B_3 contains no element of the set Row3, for if B_3 contains one element of the set Row3, say $\lambda_4 - \lambda_3$, then $\lambda_4 - \lambda_2 = (\lambda_4 - \lambda_3) + (\lambda_3 - \lambda_2) \in B_3 + B_3$, and $z_{4,2} \in \mathbb{Z}$, a contradiction of the fact that $y_{4,2} \in \mathbb{Z}$.

Now, consider the elements of the sets Row2 and Row3. From (2.14) and Row2 $\setminus \{\lambda_3 - \lambda_2\} \subseteq B_1$, we see that the difference $\lambda_3 - \lambda_2$ can be written as

$$\lambda_3 - \lambda_2 = (\lambda_n - \lambda_2) - (\lambda_n - \lambda_3) \quad \text{for each } n = 4, 5, 6, \ldots,$$

which yields

$$z_{3,2} = z_{n,2} - z_{n,3} \quad \text{for a given } n \in \{4, 5, 6, \ldots\}, \tag{2.15}$$

where (by Lemma 2.2(i))

$$z_{n,2} = \frac{1}{2} + \hat{k}_1 \frac{p_3^j}{p_2^j} + \hat{k}_2 \quad \text{and} \quad z_{n,3} = \frac{1}{2} + \hat{k}_3 \frac{p_3^{j_1}}{p_2^{j_1}} + \hat{k}_4, \tag{2.16}$$

for some $j, j_1 \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $\hat{k}_1, \hat{k}_2, \hat{k}_3, \hat{k}_4 \in \mathbb{Z}$. From (2.12), we write $z_{3,2} = \frac{1}{2} + \hat{k}$ $(\hat{k} \in \mathbb{Z})$. Then, (2.15) and (2.16) yield

$$\left(\frac{1}{2} + \hat{k}_1 \frac{p_3^j}{p_2^j} + \hat{k}_2\right) - \left(\frac{1}{2} + \hat{k}_3 \frac{p_3^{j_1}}{p_2^{j_1}} + \hat{k}_4\right) = \frac{1}{2} + \hat{k}. \tag{2.17}$$

Assume that $j \geq j_1$, we obtain

$$\hat{k}_1 p_3^j - \hat{k}_3 p_3^{j_1} p_2^{j-j_1} + (\hat{k}_2 - \hat{k}_4 - \hat{k}) p_2^j = \frac{1}{2} p_2^j. \tag{2.18}$$

Since the left-hand side of (2.18) is in $\mathbb{Z}$ and the right-hand side of (2.18) is in $\frac{1}{2} + \mathbb{Z}$, we thus get a contradiction. Case 2.1 is proved.

In Case 2.2, we consider the elements of the set Row3 $\subseteq B_2 - B_2 \subseteq B_1 \cup B_3$. With the same method as above, B_3 contains at most one element of the set Row3. So the discussion leads to the following two cases:

Case 2.2.1: Row2 $\subseteq B_1$, $\lambda_4 - \lambda_3 \in B_3$ and Row3 $\setminus \{\lambda_4 - \lambda_3\} \subseteq B_1$;
Case 2.2.2: Row2 $\subseteq B_1$ and Row3 $\subseteq B_1$.

Case 2.2.1 denotes that B_3 contains one element of the set Row3, say $\lambda_4 - \lambda_3$. Case 2.2.2 denotes that B_3 contains no element of the set Row3. The proof of Case 2.2.1 is similar to the proof of Case 2.1, and the proof of Case 2.2.2 is easier than the proof of Case 2.2.1. In fact, in both Case 2.2.1 and Case 2.2.2, there always exist three differences, say $\lambda_3 - \lambda_2$, $\lambda_5 - \lambda_2$ and $\lambda_5 - \lambda_3$ in B_1 such that

$$(\lambda_5 - \lambda_2) - (\lambda_5 - \lambda_3) = \lambda_3 - \lambda_2.$$

There are many such relations in Case 2.2.1 and in Case 2.2.2 respectively. This yields

$$z_{3,2} = z_{5,2} - z_{5,3}, \tag{2.19}$$

where (by Lemma 2.2(i))

$$z_{3,2} = \frac{1}{2} + \hat{k}_1 \frac{p_3^{j_1}}{p_2^{j_1}} + \hat{k}_2, \quad z_{5,2} = \frac{1}{2} + \hat{k}_3 \frac{p_3^{j_2}}{p_2^{j_2}} + \hat{k}_4 \quad \text{and} \quad z_{5,3} = \frac{1}{2} + \hat{k}_5 \frac{p_3^{j_3}}{p_2^{j_3}} + \hat{k}_6, \tag{2.20}$$

for some $j_1, j_2, j_3 \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $\hat{k}_1, \hat{k}_2, \hat{k}_3, \hat{k}_4, \hat{k}_5, \hat{k}_6 \in \mathbb{Z}$. Let $j_0 = \max\{j_1, j_2, j_3\}$. Then, from (2.19) and (2.20), we obtain

$$\frac{1}{2} p_2^{j_0} = -\hat{k}_1 p_3^{j_1} p_2^{j_0 - j_1} + \hat{k}_3 p_3^{j_2} p_2^{j_0 - j_2} - \hat{k}_5 p_3^{j_3} p_2^{j_0 - j_3} + (\hat{k}_4 - \hat{k}_6 - \hat{k}_2) p_2^{j_0}. \tag{2.21}$$

Since the left-hand side of (2.21) is in $\frac{1}{2} + \mathbb{Z}$ and the right-hand side of (2.21) is in $\mathbb{Z}$, we thus get a contradiction. Case 2.2 is proved. Therefore, the proof of Case 2 is completed.

The proof of Case 3 is similar to the proof of Case 2, so we omit the proof of Case 3.

2.2. Proof of Case 1

Without loss of generality, we may assume that Case 1 is

$$\lambda_2 - \lambda_1,\ \lambda_3 - \lambda_1,\ \lambda_4 - \lambda_1,\ \lambda_5 - \lambda_1, \ldots, \lambda_n - \lambda_1, \ldots \in B_1, \tag{2.22}$$

that is, Row1 $\subseteq B_1$. Then, we have

$$\begin{aligned} \lambda_3 - \lambda_2,\ \lambda_4 - \lambda_2,\ \lambda_5 - \lambda_2, \ldots, \lambda_n - \lambda_2, \ldots &\in B_1 - B_1; \\ \lambda_4 - \lambda_3,\ \lambda_5 - \lambda_3, \ldots, \lambda_n - \lambda_3, \ldots &\in B_1 - B_1; \\ \lambda_5 - \lambda_4, \ldots, \lambda_n - \lambda_4, \ldots &\in B_1 - B_1; \\ \cdots \qquad \cdots \qquad &\cdots, \end{aligned} \tag{2.23}$$

that is, Row2, Row3, Row4, $\ldots \subseteq B_1 - B_1$, and the first coordinate of each difference in (2.22) and (2.23) is in $\mathbb{Z}$.

Now, we consider (2.23) instead of (2.8). There are three cases:

Case $\hat{\mathbf{1}}$: The set B_1 contains infinite elements of the set Row2;
Case $\hat{\mathbf{2}}$: The set B_2 contains infinite elements of the set Row2;
Case $\hat{\mathbf{3}}$: The set B_3 contains infinite elements of the set Row2.

With the same method as in Subsection 2.1, we know that Case $\hat{2}$ and Case $\hat{3}$ are impossible. That is, B_2 and B_3 contain only finite elements of the set Row2, while the set B_1 contains infinite elements of the set Row2. Equivalently, except the finite elements of the set Row2, all other infinite elements of the set Row2 are in B_1.

Similarly, we know that B_2 and B_3 contain only finite elements of the set Row3, while the set B_1 contains infinite elements of the set Row3. That is, for each $j = 2, 3, 4, \ldots$, except the finite elements of the set Row$j$, all other infinite elements of the set Row$j$ are in B_1. Therefore, without loss of generality, from (2.22) and (2.23), we may assume that

$$\begin{aligned} \lambda_2 - \lambda_1,\ \lambda_3 - \lambda_1,\ \lambda_4 - \lambda_1,\ \lambda_5 - \lambda_1, \ldots, \lambda_n - \lambda_1, \ldots &\in B_1; \\ \lambda_3 - \lambda_2,\ \lambda_4 - \lambda_2,\ \lambda_5 - \lambda_2, \ldots, \lambda_n - \lambda_2, \ldots &\in B_1 \cap (B_1 - B_1); \\ \lambda_4 - \lambda_3,\ \lambda_5 - \lambda_3, \ldots, \lambda_n - \lambda_3, \ldots &\in B_1 \cap (B_1 - B_1); \\ \lambda_5 - \lambda_4, \ldots, \lambda_n - \lambda_4, \ldots &\in B_1 \cap (B_1 - B_1); \\ \cdots \qquad \cdots \qquad &\cdots \end{aligned} \tag{2.24}$$

From (2.2), we let $B_1 = B_{1,1} \cup B_{1,2} \cup B_{1,3} \cup \cdots = \bigcup_{j=1}^{\infty} B_{1,j}$, where

$$B_{1,j} = \left\{ \begin{pmatrix} (1/2 + k_1)p_1^j \\ (a + k_2)p_2^j \\ (1/2 + a + k_3)p_3^j \end{pmatrix} : a \in \mathbb{R}, k_1, k_2, k_3 \in \mathbb{Z} \right\} \subset \mathbb{R}^3. \tag{2.25}$$

By analyzing the first coordinate of the elements of B_1, we have the following Lemma 2.3.

Lemma 2.3.

(i) *For each $j \in \mathbb{N}$ and an element $\xi \in (B_{1,j} \pm B_{1,j})$, if $\xi \in B_{1,\hat{j}}$ for some integer $\hat{j} \in \mathbb{N}$, then $\hat{j} > j$;*
(ii) *Let $j, \tilde{j} \in \mathbb{N}$ and $j \neq \tilde{j}$. For each element $\xi \in (B_{1,j} \pm B_{1,\tilde{j}})$, if $\xi \in B_{1,\hat{j}}$ for some integer $\hat{j} \in \mathbb{N}$, then $\hat{j} = \min\{j, \tilde{j}\}$.*

Also, the following Lemma 2.4 is fundamental.

Lemma 2.4. *Let $p_2, p_3 \in (2\mathbb{Z}+1)\setminus\{\pm 1\}$ and $p_2 \neq p_3$. If $\alpha, \beta \in \mathbb{N}$ have different parity, then for any $k, \tilde{k} \in \mathbb{Z}$,*

$$(2k + 1)(p_3^\alpha - p_2^\alpha) \neq (2\tilde{k} + 1)(p_3^\beta - p_2^\beta). \tag{2.26}$$

Proof. For any $n \in \mathbb{N}$, we have

$$p_3^n - p_2^n = (p_3 - p_2)(p_3^{n-1}p_2^0 + p_3^{n-2}p_2^1 + p_3^{n-3}p_2^2 + \cdots + p_3^1 p_2^{n-2} + p_3^0 p_2^{n-1}),$$

which yields

$$\frac{p_3^n - p_2^n}{p_3 - p_2} = \begin{cases} \text{even number}, & \text{if } n \text{ is even;} \\ \text{odd number}, & \text{if } n \text{ is odd.} \end{cases} \tag{2.27}$$

Hence, if α is odd, then β is even, and for any $k, \tilde{k} \in \mathbb{Z}$,

$$(2k + 1)\frac{p_3^\alpha - p_2^\alpha}{p_3 - p_2} \text{ is odd number} \quad \text{and} \quad (2\tilde{k} + 1)\frac{p_3^\beta - p_2^\beta}{p_3 - p_2} \text{ is even number,} \tag{2.28}$$

and thus (2.26) holds. Similar result holds if α is even and β is odd. This proves Lemma 2.4. □

Note that, for the larger j, $\hat{j}$ and $j > \hat{j}$, the differences $\lambda_j - \lambda_{\hat{j}}$ in (2.24) can be represented by the other differences in the following form: $\lambda_j - \lambda_{\hat{j}} = (\lambda_j - \lambda_1) - (\lambda_{\hat{j}} - \lambda_1) = (\lambda_j - \lambda_2) - (\lambda_{\hat{j}} - \lambda_2) = \cdots = (\lambda_j - \lambda_{\hat{j}-1}) - (\lambda_{\hat{j}} - \lambda_{\hat{j}-1})$. These representations readily deduce a contradiction from Lemma 2.3 and Lemma 2.4. In the following discussion, we mainly illustrate this method.

We write $B_1 = B_{odd} \bigcup B_{even}$, where $B_{odd} = \cup_{j=1}^{\infty} B_{1,2j-1}$ and $B_{even} = \cup_{j=1}^{\infty} B_{1,2j}$. From (2.24), we only need to consider the two cases: $\lambda_2 - \lambda_1 \in B_{odd}$ and $\lambda_2 - \lambda_1 \in B_{even}$.

(I) Case $\tilde{\mathbf{1}}$: $\lambda_2 - \lambda_1 \in B_{odd}$.

We may assume that $\lambda_2 - \lambda_1 \in B_{1,1}$. Since the other differences of (2.24) are in the set $B_{odd} \cup B_{even}$, we begin with the case that Row1 or Row2 has an element belonging to B_{even}. Then, we consider the case that all elements of Row1 or Row2 are in B_{odd}. Each case concludes with a contradiction.

Step 1. In Row1, let $\lambda_{k_1} - \lambda_1 \in B_{even}$ for some $k_1 > 2$. More precisely, let $\lambda_{k_1} - \lambda_1 \in B_{1,\tilde{k}_1}$ for some even number $\tilde{k}_1 > 1$. Then, for any $\tilde{k}_2 \geq \tilde{k}_1$ and for any element $\lambda_{k_2} - \lambda_1 \in B_{1,\tilde{k}_2}$, we have (by Lemma 2.3)

$$\lambda_{k_1} - \lambda_2 = (\lambda_{k_1} - \lambda_1) - (\lambda_2 - \lambda_1) \in B_{1,1}; \tag{2.29}$$

$$\lambda_{k_2} - \lambda_2 = (\lambda_{k_2} - \lambda_1) - (\lambda_2 - \lambda_1) \in B_{1,1}; \tag{2.30}$$

$$\lambda_{k_2} - \lambda_{k_1} = (\lambda_{k_2} - \lambda_1) - (\lambda_{k_1} - \lambda_1) \in B_{1,\tilde{k}_1}, \quad \text{if } \tilde{k}_2 > \tilde{k}_1; \tag{2.31}$$

$$\lambda_{k_2} - \lambda_{k_1} = (\lambda_{k_2} - \lambda_1) - (\lambda_{k_1} - \lambda_1) \in B_{1,\tilde{k}}, \quad \text{if } \tilde{k}_2 = \tilde{k}_1, \tag{2.32}$$

where $\tilde{k} > \tilde{k}_2 = \tilde{k}_1$. Also, from (2.29), (2.30) and Lemma 2.3(i), we have

$$\lambda_{k_2} - \lambda_{k_1} = (\lambda_{k_2} - \lambda_2) - (\lambda_{k_1} - \lambda_2) \in B_{1,\tilde{k}_3}, \quad \text{for some } \tilde{k}_3 > 1. \tag{2.33}$$

From (2.25), we can write the differences in (2.29)–(2.33) as follows:

$$\lambda_2 - \lambda_1 = \begin{pmatrix} (1/2 + k_{211})p_1 \\ (a_{21} + k_{212})p_2 \\ (1/2 + a_{21} + k_{213})p_3 \end{pmatrix}, \quad a_{21} \in \mathbb{R},\ k_{211}, k_{212}, k_{213} \in \mathbb{Z}; \tag{2.34}$$

$$\lambda_{k_1} - \lambda_1 = \begin{pmatrix} (1/2 + k_{k_1 11})p_1^{\tilde{k}_1} \\ (a_{k_1 1} + k_{k_1 12})p_2^{\tilde{k}_1} \\ (1/2 + a_{k_1 1} + k_{k_1 13})p_3^{\tilde{k}_1} \end{pmatrix}, \quad a_{k_1 1} \in \mathbb{R},\ k_{k_1 11}, k_{k_1 12}, k_{k_1 13} \in \mathbb{Z}; \tag{2.35}$$

$$\lambda_{k_2} - \lambda_1 = \begin{pmatrix} (1/2 + k_{k_2 11})p_1^{\tilde{k}_2} \\ (a_{k_2 1} + k_{k_2 12})p_2^{\tilde{k}_2} \\ (1/2 + a_{k_2 1} + k_{k_2 13})p_3^{\tilde{k}_2} \end{pmatrix}, \quad a_{k_2 1} \in \mathbb{R},\ k_{k_2 11}, k_{k_2 12}, k_{k_2 13} \in \mathbb{Z}, \tag{2.36}$$

and write $\lambda_{k_1} - \lambda_2$, $\lambda_{k_2} - \lambda_2$, $\lambda_{k_2} - \lambda_{k_1}$ in the same form as above. Then, from (2.29), we have

$$\begin{pmatrix} (\frac{1}{2} + k_{k_1 21})p_1 \\ (a_{k_1 2} + k_{k_1 22})p_2 \\ (\frac{1}{2} + a_{k_1 2} + k_{k_1 23})p_3 \end{pmatrix} = \begin{pmatrix} (\frac{1}{2} + k_{k_1 11})p_1^{\tilde{k}_1} \\ (a_{k_1 1} + k_{k_1 12})p_2^{\tilde{k}_1} \\ (\frac{1}{2} + a_{k_1 1} + k_{k_1 13})p_3^{\tilde{k}_1} \end{pmatrix} - \begin{pmatrix} (\frac{1}{2} + k_{211})p_1 \\ (a_{21} + k_{212})p_2 \\ (\frac{1}{2} + a_{21} + k_{213})p_3 \end{pmatrix}. \tag{2.37}$$

The comparison of the second coordinate and the third coordinate in (2.37) shows that $p_3^{\tilde{k}_1 - 1} \neq p_2^{\tilde{k}_1 - 1}$ and

$$a_{k_1 1} = \frac{2\hat{k} + 1}{2(p_3^{\tilde{k}_1 - 1} - p_2^{\tilde{k}_1 - 1})} \quad \text{for some } \hat{k} \in \mathbb{Z}. \tag{2.38}$$

Similarly, from (2.30), we have

$$\begin{pmatrix} (\frac{1}{2} + k_{k_2 21})p_1 \\ (a_{k_2 2} + k_{k_2 22})p_2 \\ (\frac{1}{2} + a_{k_2 2} + k_{k_2 23})p_3 \end{pmatrix} = \begin{pmatrix} (\frac{1}{2} + k_{k_2 11})p_1^{\tilde{k}_2} \\ (a_{k_2 1} + k_{k_2 12})p_2^{\tilde{k}_2} \\ (\frac{1}{2} + a_{k_2 1} + k_{k_2 13})p_3^{\tilde{k}_2} \end{pmatrix} - \begin{pmatrix} (\frac{1}{2} + k_{211})p_1 \\ (a_{21} + k_{212})p_2 \\ (\frac{1}{2} + a_{21} + k_{213})p_3 \end{pmatrix}. \tag{2.39}$$

The comparison of the second coordinate and the third coordinate in (2.39) shows that $p_3^{\tilde{k}_2 - 1} \neq p_2^{\tilde{k}_2 - 1}$ and

$$a_{k_2 1} = \frac{2\hat{k}_1 + 1}{2(p_3^{\tilde{k}_2 - 1} - p_2^{\tilde{k}_2 - 1})} \quad \text{for some } \hat{k}_1 \in \mathbb{Z}. \tag{2.40}$$

From (2.31), we have

$$\begin{pmatrix} (\frac{1}{2}+k_{k_2k_11})p_1^{\tilde{k}_1} \\ (a_{k_2k_1}+k_{k_2k_12})p_2^{\tilde{k}_1} \\ (\frac{1}{2}+a_{k_2k_1}+k_{k_2k_13})p_3^{\tilde{k}_1} \end{pmatrix} = \begin{pmatrix} (\frac{1}{2}+k_{k_211})p_1^{\tilde{k}_2} \\ (a_{k_21}+k_{k_212})p_2^{\tilde{k}_2} \\ (\frac{1}{2}+a_{k_21}+k_{k_213})p_3^{\tilde{k}_2} \end{pmatrix} - \begin{pmatrix} (\frac{1}{2}+k_{k_111})p_1^{\tilde{k}_1} \\ (a_{k_11}+k_{k_112})p_2^{\tilde{k}_1} \\ (\frac{1}{2}+a_{k_11}+k_{k_113})p_3^{\tilde{k}_1} \end{pmatrix}, \tag{2.41}$$

which yields $p_3^{\tilde{k}_2-\tilde{k}_1} \neq p_2^{\tilde{k}_2-\tilde{k}_1}$ and

$$a_{k_21} = \frac{2\hat{k}_2+1}{2(p_3^{\tilde{k}_2-\tilde{k}_1}-p_2^{\tilde{k}_2-\tilde{k}_1})} \quad \text{for some } \hat{k}_2 \in \mathbb{Z}, \quad \text{if } \tilde{k}_2 > \tilde{k}_1. \tag{2.42}$$

(i) In the case when $\tilde{k}_2 > \tilde{k}_1$, it follows from (2.40) and (2.42) that

$$(2\hat{k}_1+1)(p_3^{\tilde{k}_2-\tilde{k}_1}-p_2^{\tilde{k}_2-\tilde{k}_1}) = (2\hat{k}_2+1)(p_3^{\tilde{k}_2-1}-p_2^{\tilde{k}_2-1}). \tag{2.43}$$

Since $\tilde{k}_1$ is an even number, $\tilde{k}_2-1$ and $\tilde{k}_2-\tilde{k}_1$ have different parity for any $\tilde{k}_2 \in \mathbb{N}$. So (2.43) is a contradiction of (2.26).

(ii) In the case when $\tilde{k}_2 = \tilde{k}_1$, from (2.32), we get $p_3^{\tilde{k}-\tilde{k}_1} \neq p_2^{\tilde{k}-\tilde{k}_1}$ and

$$a_{k_2k_1} = \frac{2\hat{k}_3+1}{2(p_3^{\tilde{k}-\tilde{k}_1}-p_2^{\tilde{k}-\tilde{k}_1})} = \frac{2\hat{k}_3+1}{2(p_3^{\tilde{k}-\tilde{k}_2}-p_2^{\tilde{k}-\tilde{k}_2})} \quad \text{for some } \hat{k}_3 \in \mathbb{Z}. \tag{2.44}$$

Observe from (2.32) and (2.33) that in the case when $\tilde{k}_2 = \tilde{k}_1$, we can take $\tilde{k}_3 = \tilde{k}$ in (2.33). Hence, from (2.33), we get $p_3^{\tilde{k}-1} \neq p_2^{\tilde{k}-1}$ and

$$a_{k_2k_1} = \frac{2\hat{k}_4+1}{2(p_3^{\tilde{k}-1}-p_2^{\tilde{k}-1})} \quad \text{for some } \hat{k}_4 \in \mathbb{Z}. \tag{2.45}$$

It follows from (2.44) and (2.45) that

$$(2\hat{k}_4+1)(p_3^{\tilde{k}-\tilde{k}_1}-p_2^{\tilde{k}-\tilde{k}_1}) = (2\hat{k}_3+1)(p_3^{\tilde{k}-1}-p_2^{\tilde{k}-1}). \tag{2.46}$$

Since $\tilde{k}_1$ is an even number, $\tilde{k}-1$ and $\tilde{k}-\tilde{k}_1$ have different parity for any $\tilde{k} \in \mathbb{N}$. So (2.46) is a contradiction of (2.26).

Step 2. In Row2, let $\lambda_{k_1} - \lambda_2 \in B_{even}$ for some $k_1 > 2$. More precisely, let $\lambda_{k_1} - \lambda_2 \in B_{1,\tilde{k}_1}$ for some even number $\tilde{k}_1 > 1$. Then, for any $\tilde{k}_2 \geq \tilde{k}_1$ and for any element $\lambda_{k_2} - \lambda_2 \in B_{1,\tilde{k}_2}$, we have (by Lemma 2.3)

$$\lambda_{k_1} - \lambda_1 = (\lambda_{k_1} - \lambda_2) + (\lambda_2 - \lambda_1) \in B_{1,1}; \tag{2.47}$$

$$\lambda_{k_2} - \lambda_1 = (\lambda_{k_2} - \lambda_2) + (\lambda_2 - \lambda_1) \in B_{1,1}; \tag{2.48}$$

$$\lambda_{k_2} - \lambda_{k_1} = (\lambda_{k_2} - \lambda_2) - (\lambda_{k_1} - \lambda_2) \in B_{1,\tilde{k}_1}, \quad \text{if } \tilde{k}_2 > \tilde{k}_1; \tag{2.49}$$

$$\lambda_{k_2} - \lambda_{k_1} = (\lambda_{k_2} - \lambda_2) - (\lambda_{k_1} - \lambda_2) \in B_{1,\tilde{k}}, \quad \text{if } \tilde{k}_2 = \tilde{k}_1, \tag{2.50}$$

where $\tilde{k} > \tilde{k}_2 = \tilde{k}_1$. Also, from (2.47), (2.48) and Lemma 2.3(i), we have

$$\lambda_{k_2} - \lambda_{k_1} = (\lambda_{k_2} - \lambda_1) - (\lambda_{k_1} - \lambda_1) \in B_{1,\tilde{k}_3}, \quad \text{for some } \tilde{k}_3 > 1, \tag{2.51}$$

which is analogous to (2.29)–(2.33) respectively. The same method will yield a contradiction of (2.26).

Step 3. In the case when Row1 $\subseteq B_{odd}$, we can choose a larger $n \in \mathbb{N}$, and let $\lambda_{k_1} - \lambda_1 \in$ Row1 $\cap B_{1,2n+1}$ for some $k_1 > 2$. Then, by Lemma 2.3, $\lambda_{k_1} - \lambda_2 = (\lambda_{k_1} - \lambda_1) - (\lambda_2 - \lambda_1) \in B_{1,1}$ gives

$$\begin{pmatrix} \frac{1}{2} + k_{k_1 21} \\ a_{k_1 2} + k_{k_1 22} \\ \frac{1}{2} + a_{k_1 2} + k_{k_1 23} \end{pmatrix} = \begin{pmatrix} (\frac{1}{2} + k_{k_1 11})p_1^{2n} \\ (a_{k_1 1} + k_{k_1 12})p_2^{2n} \\ (\frac{1}{2} + a_{k_1 1} + k_{k_1 13})p_3^{2n} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} + k_{211} \\ a_{21} + k_{212} \\ \frac{1}{2} + a_{21} + k_{213} \end{pmatrix}. \tag{2.52}$$

The comparison of the second coordinate and the third coordinate in (2.52) shows that $p_3^{2n} \neq p_2^{2n}$ and

$$a_{k_1 1} = \frac{2\hat{k} + 1}{2(p_3^{2n} - p_2^{2n})} = \frac{2\hat{k} + 1}{2((p_3^2)^n - (p_2^2)^n)} \quad \text{for some } \hat{k} \in \mathbb{Z}. \tag{2.53}$$

For any $\lambda_{k_2} - \lambda_1 \in B_{1,2n_1+1}$ with $k_2 \neq k_1$ and odd number $n_1 < n$, the equality $\lambda_{k_2} - \lambda_{k_1} = (\lambda_{k_2} - \lambda_1) - (\lambda_{k_1} - \lambda_1) \in B_{1,2n_1+1}$ gives $p_3^{(2n-2n_1)} \neq p_2^{(2n-2n_1)}$ and

$$a_{k_1 1} = \frac{2\hat{k}_1 + 1}{2(p_3^{(2n-2n_1)} - p_2^{(2n-2n_1)})} = \frac{2\hat{k}_1 + 1}{2((p_3^2)^{(n-n_1)} - (p_2^2)^{(n-n_1)})} \quad \text{for some } \hat{k}_1 \in \mathbb{Z}. \tag{2.54}$$

It follows from (2.53) and (2.54) that

$$(2\hat{k} + 1)((p_3^2)^{(n-n_1)} - (p_2^2)^{(n-n_1)}) = (2\hat{k}_1 + 1)((p_3^2)^n - (p_2^2)^n). \tag{2.55}$$

Since n_1 is an odd number, n and $n - n_1$ have different parity for any $n \in \mathbb{N}$. So (2.55) is a contradiction of (2.26).

Step 4. Similar to Step 3, in the case when Row2 $\subseteq B_{odd}$, we can also choose a larger $n \in \mathbb{N}$, and let $\lambda_{k_1} - \lambda_2 \in$ Row2 $\cap B_{1,2n+1}$ for some $k_1 > 2$. Then, by Lemma 2.3, $\lambda_{k_1} - \lambda_1 = (\lambda_{k_1} - \lambda_2) + (\lambda_2 - \lambda_1) \in B_{1,1}$ gives

$$\begin{pmatrix} \frac{1}{2} + k_{k_1 11} \\ a_{k_1 1} + k_{k_1 12} \\ \frac{1}{2} + a_{k_1 1} + k_{k_1 13} \end{pmatrix} = \begin{pmatrix} (\frac{1}{2} + k_{k_1 21})p_1^{2n} \\ (a_{k_1 2} + k_{k_1 22})p_2^{2n} \\ (\frac{1}{2} + a_{k_1 2} + k_{k_1 23})p_3^{2n} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} + k_{211} \\ a_{21} + k_{212} \\ \frac{1}{2} + a_{21} + k_{213} \end{pmatrix}. \tag{2.56}$$

The comparison of the second coordinate and the third coordinate in (2.56) shows that $p_3^{2n} \neq p_2^{2n}$ and

$$a_{k_1 2} = \frac{2\hat{k} + 1}{2(p_3^{2n} - p_2^{2n})} = \frac{2\hat{k} + 1}{2((p_3^2)^n - (p_2^2)^n)} \quad \text{for some } \hat{k} \in \mathbb{Z}. \tag{2.57}$$

For any $\lambda_{k_2} - \lambda_2 \in B_{1,2n_1+1}$ with $k_2 \neq k_1$ and odd number $n_1 < n$, the equality $\lambda_{k_2} - \lambda_{k_1} = (\lambda_{k_2} - \lambda_2) - (\lambda_{k_1} - \lambda_2) \in B_{1,2n_1+1}$ gives $p_3^{(2n-2n_1)} \neq p_2^{(2n-2n_1)}$ and

$$a_{k_1 2} = \frac{2\hat{k}_1 + 1}{2(p_3^{(2n-2n_1)} - p_2^{(2n-2n_1)})} = \frac{2\hat{k}_1 + 1}{2((p_3^2)^{(n-n_1)} - (p_2^2)^{(n-n_1)})} \quad \text{for some } \hat{k}_1 \in \mathbb{Z}. \tag{2.58}$$

It follows from (2.57) and (2.58) that

$$(2\hat{k} + 1)((p_3^2)^{(n-n_1)} - (p_2^2)^{(n-n_1)}) = (2\hat{k}_1 + 1)((p_3^2)^n - (p_2^2)^n). \tag{2.59}$$

Since n and $n - n_1$ have different parity for any $n \in \mathbb{N}$, (2.59) is a contradiction of (2.26).

Note that $\lambda_2 - \lambda_1 \in B_{1,1}$, if Row1 $\subseteq B_{1,1}$, then by Lemma 2.3(i), Row2 will have elements belonging to $B_{1,j}$ for some $j > 1$. Consider these elements and choose a larger j, then we can use the above-mentioned steps and method to provide a contradiction.

(II) Case $\tilde{\mathbf{2}}$: $\lambda_2 - \lambda_1 \in B_{even}$.

In the case when $\lambda_2 - \lambda_1 \in B_{even}$, without loss of generality, we may assume that $\lambda_2 - \lambda_1 \in B_{1,2}$. Then, consider the other differences of (2.24), the same method as above can be applied to show that if Row1 or Row2 has an element belonging to B_{odd}, then a contradiction can be obtained. Also, if Row1$\cup$Row2 $\subseteq B_{even}$ happens, we can consider the other sets Row$j$ $(j = 3, 4, \ldots)$, and apply the above-mentioned method to get a contradiction. This completes the proof of Theorem 2.1. $\square$

In the end of this paper, we would like to point out that the finiteness of $\mu_{M,D}$-orthogonal exponentials implies that $\mu_{M,D}$ is a non-spectral measure. Theorem 2.1 extends Theorem B(i) in the non-spectrality of self-affine measure, and the method of proving Theorem 2.1 is different from the known [12,13] method. On the other hand, the conclusion of Theorem B(i) includes the precise number "4". This is the best upper bound for the number of mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. In any case, the Hilbert space $L^2(\mu_{M,D})$ corresponding to (1.2) always contains four-element mutually orthogonal exponentials $E(M^*(\Lambda_1)) = E(M(\Lambda_1))$ with

$$\Lambda_1 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 + k_1 \\ 1/2 + k_2 \\ k_3 \end{pmatrix}, \begin{pmatrix} 1/2 + \alpha + k_4 \\ \alpha + k_5 \\ 1/2 + k_6 \end{pmatrix}, \begin{pmatrix} \alpha + k_7 \\ 1/2 + \alpha + k_8 \\ 1/2 + k_9 \end{pmatrix} \right\} \tag{2.60}$$

or

$$\Lambda_1 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 + k_1 \\ k_2 \\ 1/2 + k_3 \end{pmatrix}, \begin{pmatrix} 1/2 + \beta + k_4 \\ 1/2 + k_5 \\ \beta + k_6 \end{pmatrix}, \begin{pmatrix} \beta + k_7 \\ 1/2 + k_8 \\ 1/2 + \beta + k_9 \end{pmatrix} \right\} \tag{2.61}$$

or

$$\Lambda_1 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} k_1 \\ 1/2 + k_2 \\ 1/2 + k_3 \end{pmatrix}, \begin{pmatrix} 1/2 + k_4 \\ 1/2 + \gamma + k_5 \\ \gamma + k_6 \end{pmatrix}, \begin{pmatrix} 1/2 + k_7 \\ \gamma + k_8 \\ 1/2 + \gamma + k_9 \end{pmatrix} \right\}, \tag{2.62}$$

where $k_1, k_2, \ldots, k_9 \in \mathbb{Z}$ and $\alpha, \beta, \gamma \in \mathbb{R}$. However, to reach the precise number "4" in Theorem 2.1, the above method combined with the known method will play an essential role. There are many complicated cases to deal with. We conjecture that the conclusion of Theorem B(i) is also suitable to Theorem 2.1.

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