



C^* -algebras associated with Hilbert C^* -quad modules of C^* -textile dynamical systems



Kengo Matsumoto

Department of Mathematics, Joetsu University of Education, Joetsu 943-8512, Japan

ARTICLE INFO

Article history:

Received 5 October 2015

Available online 10 February 2016

Submitted by D. Blecher

Keywords:

C^* -algebras

Hilbert C^* -bimodules tiling

Subshift

Textile system

ABSTRACT

A C^* -textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ consists of a unital C^* -algebra \mathcal{A} , two families of endomorphisms $\{\rho_\alpha\}_{\alpha \in \Sigma^\rho}$ and $\{\eta_a\}_{a \in \Sigma^\eta}$ of \mathcal{A} and certain commutation relations κ among them. It yields a two-dimensional subshift and a multistructure Hilbert C^* -bimodule, which we call a Hilbert C^* -quad module. We introduce a C^* -algebra from the Hilbert C^* -quad module as a two-dimensional analogue of Pimsner's construction of C^* -algebras from Hilbert C^* -bimodules. We study the C^* -algebras defined by the Hilbert C^* -quad modules and prove that they have universal properties subject to certain operator relations. We also present examples arising from commuting matrices.

© 2016 Elsevier Inc. All rights reserved.

1. Introduction

In [12], the author has introduced a notion of λ -graph system as a generalization of finite labeled graphs. The λ -graph systems yield C^* -algebras so that their K-theory groups are related to topological conjugacy invariants of the underlying symbolic dynamical systems. He has extended the notion of λ -graph system to C^* -symbolic dynamical system, which is a generalization of both a λ -graph system and an automorphism of a unital C^* -algebra. It is denoted by $(\mathcal{A}, \rho, \Sigma)$ and consists of a finite family $\{\rho_\alpha\}_{\alpha \in \Sigma}$ of endomorphisms of a unital C^* -algebra \mathcal{A} such that $\rho_\alpha(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}$, $\alpha \in \Sigma$ and $\sum_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1$ where $Z_{\mathcal{A}}$ denotes the center of \mathcal{A} . A λ -graph system \mathfrak{L} yields a C^* -symbolic dynamical system $(\mathcal{A}_{\mathfrak{L}}, \rho^{\mathfrak{L}}, \Sigma)$ such that $\mathcal{A}_{\mathfrak{L}}$ is $C(\Omega_{\mathfrak{L}})$ for some compact Hausdorff space $\Omega_{\mathfrak{L}}$ with $\dim \Omega_{\mathfrak{L}} = 0$. A C^* -symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ provides a subshift Λ_ρ over Σ and a Hilbert C^* -bimodule $\mathcal{H}_{\mathcal{A}}^\rho$ over \mathcal{A} which gives rise to a C^* -algebra \mathcal{O}_ρ as a Cuntz–Pimsner algebra ([14], cf. [7,23]).

G. Robertson and T. Steger [25] have initiated a certain study of higher dimensional analogue of Cuntz–Krieger algebras from the view point of tiling systems of 2-dimensional plane. After their work, A. Kumjian and D. Pask [8] have generalized their construction to introduce the notion of higher rank graphs and

E-mail address: kengo@juen.ac.jp.

their C^* -algebras. Since then, there have been many studies on these C^* -algebras by many authors (see for example [5,6,8,9,24,22,25], etc.).

M. Nasu in [21] has introduced the notion of textile system which is useful in analyzing automorphisms and endomorphisms of topological Markov shifts. A textile system also gives rise to a two-dimensional tiling (cf. [11,26]). LR textile systems have specific properties that consist of two commuting symbolic matrices. In [15], the author has extended the notion of textile systems to λ -graph systems and has defined a notion of textile systems on λ -graph systems, which are called textile λ -graph systems for short. C^* -algebras associated to textile systems have been initiated by V. Deaconu [5].

In [19], the author has extended the notion of C^* -symbolic dynamical system to C^* -textile dynamical system which is a higher dimensional analogue of C^* -symbolic dynamical system. The C^* -textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ consists of two C^* -symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^\eta)$ with a common unital C^* -algebra \mathcal{A} and a commutation relation between ρ and η through a map κ below. Set

$$\Sigma^{\rho\eta} = \{(\alpha, b) \in \Sigma^\rho \times \Sigma^\eta \mid \eta_b \circ \rho_\alpha \neq 0\}, \quad \Sigma^{\eta\rho} = \{(a, \beta) \in \Sigma^\eta \times \Sigma^\rho \mid \rho_\beta \circ \eta_a \neq 0\}.$$

We require that there exists a bijection $\kappa : \Sigma^{\rho\eta} \longrightarrow \Sigma^{\eta\rho}$, which we fix and call a specification. Then the required commutation relations are

$$\eta_b \circ \rho_\alpha = \rho_\beta \circ \eta_a \quad \text{if} \quad \kappa(\alpha, b) = (a, \beta). \quad (1.1)$$

The author has also introduced a C^* -algebra $\mathcal{O}_{\rho, \eta}^\kappa$ from $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ which is realized as the universal C^* -algebra $C^*(x, S_\alpha, T_a; x \in \mathcal{A}, \alpha \in \Sigma^\rho, a \in \Sigma^\eta)$ generated by $x \in \mathcal{A}$ and two families of partial isometries $S_\alpha, \alpha \in \Sigma^\rho, T_a, a \in \Sigma^\eta$ subject to the following relations called $(\rho, \eta; \kappa)$:

$$\sum_{\beta \in \Sigma^\rho} S_\beta S_\beta^* = 1, \quad x S_\alpha S_\alpha^* = S_\alpha S_\alpha^* x, \quad S_\alpha^* x S_\alpha = \rho_\alpha(x), \quad (1.2)$$

$$\sum_{b \in \Sigma^\eta} T_b T_b^* = 1, \quad x T_a T_a^* = T_a T_a^* x, \quad T_a^* x T_a = \eta_a(x), \quad (1.3)$$

$$S_\alpha T_b = T_a S_\beta \quad \text{if} \quad \kappa(\alpha, b) = (a, \beta) \quad (1.4)$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ [19]. This algebra is a generalization of some of higher rank graph algebras.

In the present paper, the author will introduce another kind of C^* -algebras associated with the C^* -textile dynamical systems from the view point of Hilbert C^* -modules. The resulting C^* -algebras $\mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}}$ are different from the above algebras $\mathcal{O}_{\rho, \eta}^\kappa$. A C^* -textile dynamical system provides a two-dimensional subshift and a Hilbert C^* -bimodule that has multi right actions and multi left actions and multi-inner products. We call it a Hilbert C^* -quad module denoted by $\mathcal{H}_\kappa^{\rho, \eta}$. The C^* -algebra $\mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}}$, which we will introduce in the present paper, is constructed in a concrete way from the structure of the Hilbert C^* -quad module $\mathcal{H}_\kappa^{\rho, \eta}$ by a two-dimensional analogue of Pimsner's construction from Hilbert C^* -bimodules. It is generated by the quotient images of creation operators on a two-dimensional analogue of Fock Hilbert module by the compact operators. As a result, we will show that the C^* -algebra has a universal property subject to certain operator relations of generators.

For a C^* -textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$, consider the set of quadruplet of symbols

$$\Sigma_\kappa = \{\omega = (\alpha, b, a, \beta) \in \Sigma^\rho \times \Sigma^\eta \times \Sigma^\eta \times \Sigma^\rho \mid \kappa(\alpha, b) = (a, \beta)\}. \quad (1.5)$$

Each element of Σ_κ is regarded as a tile $\begin{array}{ccc} & \xrightarrow{\alpha} & \\ a \downarrow & & \downarrow b \\ & \xrightarrow{\beta} & \end{array}$ of the associated two-dimensional subshift. Denote by

\mathcal{O}_ρ and by \mathcal{O}_η the C^* -algebras associated with the C^* -symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^\eta)$ respectively. Let $S_\alpha, \alpha \in \Sigma^\rho$ and $T_a, a \in \Sigma^\eta$ be the generating partial isometries of \mathcal{O}_ρ and those of \mathcal{O}_η , which satisfy (1.2) and (1.3) respectively. Denote by \mathcal{B}_ρ the C^* -subalgebra of \mathcal{O}_ρ generated by elements $S_\alpha x S_\alpha^*, \alpha \in \Sigma^\rho, x \in \mathcal{A}$ and by \mathcal{B}_η that of \mathcal{O}_η generated by elements $T_a x T_a^*, a \in \Sigma^\eta, x \in \mathcal{A}$ respectively. The endomorphisms $\rho_\alpha, \alpha \in \Sigma^\rho$ and $\eta_a, a \in \Sigma^\eta$ on \mathcal{A} extend to \mathcal{B}_ρ and to \mathcal{B}_η as $*$ -homomorphisms $\hat{\rho}_\alpha : \mathcal{B}_\rho \rightarrow \mathcal{A}$ and $\hat{\eta}_a : \mathcal{B}_\eta \rightarrow \mathcal{A}$ by

$$\hat{\rho}_\alpha(w) = S_\alpha^* w S_\alpha \in \mathcal{A}, \quad w \in \mathcal{B}_\rho \quad \text{and} \quad \hat{\eta}_a(z) = T_a^* z T_a \in \mathcal{A}, \quad z \in \mathcal{B}_\eta. \quad (1.6)$$

They also extend to \mathcal{B}_η and to \mathcal{B}_ρ as endomorphisms $\hat{\rho}_\alpha^\eta$ on \mathcal{B}_η and $\hat{\eta}_a^\rho$ on \mathcal{B}_ρ by

$$\hat{\rho}_\alpha^\eta(z) = \sum_{\substack{b, a, \beta \\ (\alpha, b, a, \beta) \in \Sigma_\kappa}} T_b \rho_\beta(\hat{\eta}_a(z)) T_b^* \in \mathcal{B}_\eta, \quad z \in \mathcal{B}_\eta, \quad (1.7)$$

$$\hat{\eta}_a^\rho(w) = \sum_{\substack{\alpha, b, \beta \\ (\alpha, b, a, \beta) \in \Sigma_\kappa}} S_\beta \eta_b(\hat{\rho}_\alpha(w)) S_\beta^* \in \mathcal{B}_\rho, \quad w \in \mathcal{B}_\rho. \quad (1.8)$$

For $\omega \in \Sigma_\kappa$, define the projection $E_\omega = \eta_b(\rho_\alpha(1)) (= \rho_\beta(\eta_a(1))) \in \mathcal{A}$. The vector space

$$\mathcal{H}_\kappa^{\rho, \eta} = \sum_{\omega \in \Sigma_\kappa} \mathbb{C} e_\omega \otimes E_\omega \mathcal{A} \quad (1.9)$$

has a natural structure of a Hilbert C^* -right \mathcal{A} -module. In addition to the \mathcal{A} -module structure, $\mathcal{H}_\kappa^{\rho, \eta}$ has a multistructure of Hilbert C^* -bimodules, a Hilbert C^* -bimodule structure over \mathcal{B}_ρ and a Hilbert C^* -bimodule structure over \mathcal{B}_η . We call it Hilbert C^* -quad module over $(\mathcal{A}; \mathcal{B}_\rho, \mathcal{B}_\eta)$. We will construct a C^* -algebra $\mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}}$ in a concrete way from the Hilbert C^* -quad module $\mathcal{H}_\kappa^{\rho, \eta}$ by a two-dimensional analogue of Pimsner's construction of C^* -algebras from Hilbert C^* -bimodules. It is generated by two kinds of creation operators, the horizontal creation operators and the vertical creation operators, on two-dimensional analogue of Fock Hilbert module. Denote by $\iota_\rho : \mathcal{A} \hookrightarrow \mathcal{B}_\rho$ and $\iota_\eta : \mathcal{A} \hookrightarrow \mathcal{B}_\eta$ natural embeddings. We assume that the algebra \mathcal{A} is commutative. The main result of the paper is the following theorem, which states that the algebraic structure of the algebra $\mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}}$ is determined by the behavior of the $*$ -homomorphisms $\hat{\rho}_\alpha, \hat{\eta}_a, \hat{\rho}_\alpha^\eta$ and $\hat{\eta}_a^\rho$.

Theorem 1.1 (*Theorem 5.17*). *For a C^* -textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$, the C^* -algebra $\mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}}$ associated with the Hilbert C^* -quad module $\mathcal{H}_\kappa^{\rho, \eta}$ is realized as the universal concrete C^* -algebra generated by the operators $z \in \mathcal{B}_\eta, w \in \mathcal{B}_\rho$ and partial isometries $u_\alpha, \alpha \in \Sigma^\rho, v_a, a \in \Sigma^\eta$ subject to the relations:*

$$\begin{aligned} \sum_{\beta \in \Sigma^\rho} u_\beta u_\beta^* + \sum_{b \in \Sigma^\eta} v_b v_b^* &= 1, \\ u_\alpha u_\alpha^* w &= w u_\alpha u_\alpha^*, & v_a v_a^* w &= w v_a v_a^*, \\ u_\alpha u_\alpha^* z &= z u_\alpha u_\alpha^*, & v_a v_a^* z &= z v_a v_a^*, \\ \hat{\rho}_\alpha(w) &= u_\alpha^* w u_\alpha, & \hat{\eta}_a(z) &= v_a^* z v_a, \\ \hat{\rho}_\alpha^\eta(z) &= u_\alpha^* z u_\alpha, & \hat{\eta}_a^\rho(w) &= v_a^* w v_a, \\ \iota_\eta(y) &= \iota_\rho(y) \end{aligned}$$

for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta, \alpha \in \Sigma^\rho, a \in \Sigma^\eta, y \in \mathcal{A}$.

Thanks to the above theorem, a simplicity condition of the C^* -algebra $\mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}}$ will be presented ([Corollary 5.18](#)).

Let A, B be two $N \times N$ matrices with entries in nonnegative integers. They yield directed graphs $G_A = (V, E_A)$ and $G_B = (V, E_B)$ with a common vertex set $V = \{v_1, \dots, v_N\}$ and edge sets E_A and E_B respectively, where the edge set E_A consists of $A(i, j)$ -edges from the vertex v_i to the vertex v_j and E_B consists of $B(i, j)$ -edges from the vertex v_i to the vertex v_j . We then have two C^* -symbolic dynamical systems $(\mathcal{A}_N, \rho^A, E_A)$ and $(\mathcal{A}_N, \rho^B, E_B)$ with $\mathcal{A}_N = \mathbb{C}^N$. Denote by $s(e), r(e)$ the source vertex and the range vertex of an edge e . Put

$$\begin{aligned}\Sigma^{AB} &= \{(\alpha, b) \in E_A \times E_B \mid r(\alpha) = s(b)\}, \\ \Sigma^{BA} &= \{(a, \beta) \in E_B \times E_A \mid r(a) = s(\beta)\}.\end{aligned}$$

Assume that the commutation relation

$$AB = BA \tag{1.10}$$

holds. We may take a bijection $\kappa : \Sigma^{AB} \longrightarrow \Sigma^{BA}$ such that $s(\alpha) = s(a), r(b) = r(\beta)$ for $\kappa(\alpha, b) = (a, \beta)$ which we fix. This situation is called an LR-textile system introduced by Nasu [\[21\]](#). We then have a C^* -textile dynamical system $(\mathcal{A}_N, \rho^A, \rho^B, E_A, E_B, \kappa)$. We set

$$\Omega_\kappa = \{(\alpha, a) \in E_A \times E_B \mid s(\alpha) = s(a), \kappa(\alpha, b) = (a, \beta) \text{ for some } \beta \in E_A, b \in E_B\}$$

and define two $|\Omega_\kappa| \times |\Omega_\kappa|$ -matrices A_κ and B_κ with entries in $\{0, 1\}$ by

$$A_\kappa((\alpha, a), (\delta, b)) = \begin{cases} 1 & \text{if there exists } \beta \in E_A \text{ such that } \kappa(\alpha, b) = (a, \beta), \\ 0 & \text{otherwise} \end{cases}$$

for $(\alpha, a), (\delta, b) \in \Omega_\kappa$, and

$$B_\kappa((\alpha, a), (\beta, d)) = \begin{cases} 1 & \text{if there exists } b \in E_B \text{ such that } \kappa(\alpha, b) = (a, \beta), \\ 0 & \text{otherwise} \end{cases}$$

for $(\alpha, a), (\beta, d) \in \Omega_\kappa$ respectively. Denote by $\mathcal{H}_\kappa^{A, B}$ the associated Hilbert C^* -quad module.

Theorem 1.2 ([Theorem 7.10](#)). *The C^* -algebra $\mathcal{O}_{\mathcal{H}_\kappa^{A, B}}$ associated with the Hilbert C^* -quad module $\mathcal{H}_\kappa^{A, B}$ defined by commuting matrices A, B and a specification κ is generated by two families of partial isometries $S_{(\alpha, a)}, T_{(\alpha, a)}$ for $(\alpha, a) \in \Omega_\kappa$ satisfying the relations:*

$$\begin{aligned}\sum_{(\delta, b) \in \Omega_\kappa} S_{(\delta, b)} S_{(\delta, b)}^* + \sum_{(\beta, d) \in \Omega_\kappa} T_{(\beta, d)} T_{(\beta, d)}^* &= 1, \\ S_{(\alpha, a)}^* S_{(\alpha, a)} &= \sum_{(\delta, b) \in \Omega_\kappa} A_\kappa((\alpha, a), (\delta, b)) (S_{(\delta, b)} S_{(\delta, b)}^* + T_{(\delta, b)} T_{(\delta, b)}^*), \\ T_{(\alpha, a)}^* T_{(\alpha, a)} &= \sum_{(\beta, d) \in \Omega_\kappa} B_\kappa((\alpha, a), (\beta, d)) (S_{(\beta, d)} S_{(\beta, d)}^* + T_{(\beta, d)} T_{(\beta, d)}^*)\end{aligned}$$

for $(\alpha, a) \in \Omega_\kappa$. Hence the C^* -algebra $\mathcal{O}_{\mathcal{H}_\kappa^{A, B}}$ is $*$ -isomorphic to the Cuntz–Krieger algebra \mathcal{O}_{H_κ} for the matrix $H_\kappa = \begin{bmatrix} A_\kappa & A_\kappa \\ B_\kappa & B_\kappa \end{bmatrix}$.

The paper is organized in the following way: In Section 2, we will state basic facts on the C^* -symbolic dynamical systems and the C^* -textile dynamical systems. In Section 3, we will introduce Hilbert C^* -quad modules from C^* -textile dynamical systems. In Section 4, we will introduce Fock Hilbert C^* -quad modules which are two-dimensional analogue of Fock Hilbert C^* -bimodules, and study creation operators on the Fock Hilbert C^* -quad modules. In Section 5, we will prove the main result stated as Theorem 1.1. In Section 6, we will state a relationship between the C^* -algebras $\mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}}$ and $\mathcal{O}_{\rho,\eta}^\kappa$ so that the algebra $\mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}}$ is realized as a C^* -subalgebra of the tensor product $\mathcal{O}_{\rho,\eta}^\kappa \otimes \mathcal{O}_2$ in a natural way. In Section 7, we will study the C^* -algebras arising from the Hilbert C^* -quad modules of the C^* -textile dynamical systems defined by commuting matrices and will prove Theorem 1.2.

Throughout the paper, we will denote by \mathbb{Z}_+ the set of nonnegative integers and by \mathbb{N} the set of positive integers.

This paper is a revised version of the paper, arXiv:1111.3091v1.

2. C^* -symbolic dynamical systems and C^* -textile dynamical systems

In this section, we will briefly state basic facts on C^* -symbolic dynamical systems and C^* -textile dynamical systems. Throughout the section, Σ denotes a finite set with its discrete topology, that is called an alphabet. Each element of Σ is called a symbol. Let $\Sigma^\mathbb{Z}$ be the infinite product space $\prod_{i \in \mathbb{Z}} \Sigma_i$, where $\Sigma_i = \Sigma$, endowed with the product topology. The transformation σ on $\Sigma^\mathbb{Z}$ given by $\sigma((x_i)_{i \in \mathbb{Z}}) = (x_{i+1})_{i \in \mathbb{Z}}$ is called the full shift over Σ . Let Λ be a shift invariant closed subset of $\Sigma^\mathbb{Z}$ i.e. $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|_\Lambda)$ is called a two-sided subshift, written as Λ for brevity. Finite directed graphs represent a class of subshifts called shifts of finite type. More generally, finite directed labeled graphs represent a class of subshifts called sofic shifts (see [10] for general theory of symbolic dynamical systems). The author has introduced a notion of λ -graph system as a generalization of finite labeled graphs. The λ -graph systems represent all the subshifts. Furthermore, the author has introduced a notion of C^* -symbolic dynamical system which generalizes λ -graph systems and automorphisms of unital C^* -algebras. C^* -symbolic dynamical systems may be interpreted as a generalization of subshifts to C^* -algebras.

Let \mathcal{A} be a unital C^* -algebra. In what follows, an endomorphism of \mathcal{A} means a $*$ -endomorphism of \mathcal{A} that does not necessarily preserve the unit $1_{\mathcal{A}}$ of \mathcal{A} . The unit $1_{\mathcal{A}}$ is denoted by 1 unless we specify. Denote by $Z_{\mathcal{A}}$ the center of \mathcal{A} . Let $\rho_\alpha, \alpha \in \Sigma$ be a finite family of endomorphisms of \mathcal{A} indexed by symbols of a finite set Σ . We assume that $\rho_\alpha(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}, \alpha \in \Sigma$. The family $\rho_\alpha, \alpha \in \Sigma$ of endomorphisms of \mathcal{A} is said to be essential if $\rho_\alpha(1) \neq 0$ for all $\alpha \in \Sigma$ and $\sum_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1$. It is said to be faithful if for any nonzero $x \in \mathcal{A}$ there exists a symbol $\alpha \in \Sigma$ such that $\rho_\alpha(x) \neq 0$. A C^* -symbolic dynamical system is a triplet $(\mathcal{A}, \rho, \Sigma)$ consisting of a unital C^* -algebra \mathcal{A} and an essential and faithful finite family $\{\rho_\alpha\}_{\alpha \in \Sigma}$ of endomorphisms of \mathcal{A} . In [14,16,17], we have defined a C^* -symbolic dynamical system in a less restrictive way than the above definition. Instead of the above condition $\sum_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1$ with $\rho_\alpha(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}, \alpha \in \Sigma$, we have used the condition in the papers that the closed ideal generated by $\rho_\alpha(1), \alpha \in \Sigma$ coincides with \mathcal{A} . All the examples appeared in the papers [14,16,17] satisfy the condition $\sum_{\alpha \in \Sigma} \rho_\alpha(1) \geq 1$ with $\rho_\alpha(Z_{\mathcal{A}}) \subset Z_{\mathcal{A}}, \alpha \in \Sigma$, and all discussions in the papers work well under the new definition.

A C^* -symbolic dynamical system $(\mathcal{A}, \rho, \Sigma)$ yields a subshift Λ_ρ over Σ such that a word $\alpha_1 \cdots \alpha_k$ of Σ is admissible for Λ_ρ if and only if $(\rho_{\alpha_k} \circ \cdots \circ \rho_{\alpha_1})(1) \neq 0$ [14, Proposition 2.1]. Denote by $B_k(\Lambda_\rho)$ the set of admissible words of Λ_ρ with length k . Put $B_*(\Lambda_\rho) = \bigcup_{k=0}^\infty B_k(\Lambda_\rho)$, where $B_0(\Lambda_\rho)$ consists of the empty word. The C^* -algebra \mathcal{O}_ρ associated with $(\mathcal{A}, \rho, \Sigma)$ has been originally constructed in [14] from an associated Hilbert C^* -bimodule (cf. [23,7] etc.). It is realized as a universal C^* -algebra $C^*(x, S_\alpha; x \in \mathcal{A}, \alpha \in \Sigma)$ generated by $x \in \mathcal{A}$ and partial isometries $S_\alpha, \alpha \in \Sigma$ subject to the following relations called (ρ) :

$$\sum_{\beta \in \Sigma} S_\beta S_\beta^* = 1, \quad x S_\alpha S_\alpha^* = S_\alpha S_\alpha^* x, \quad S_\alpha^* x S_\alpha = \rho_\alpha(x)$$

for all $x \in \mathcal{A}$ and $\alpha \in \Sigma$. The C^* -algebra \mathcal{O}_ρ is a generalization of the C^* -algebra $\mathcal{O}_\mathfrak{L}$ associated with the λ -graph system \mathfrak{L} (cf. [13]).

Let $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ be a C^* -textile dynamical system. It consists of two C^* -symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^\eta)$ with common unital C^* -algebra \mathcal{A} and commutation relations between their endomorphisms $\rho_\alpha, \alpha \in \Sigma^\rho$ and $\eta_a, a \in \Sigma^\eta$ through a bijection κ satisfying (1.1). Let $S_\alpha, \alpha \in \Sigma^\rho$ and $T_a, a \in \Sigma^\eta$ be the generating partial isometries of \mathcal{O}_ρ and of \mathcal{O}_η , which satisfy (1.2) and (1.3) respectively. We set two C^* -algebras

$$\mathcal{B}_\rho = C^*(S_\alpha x S_\alpha^* : \alpha \in \Sigma^\rho, x \in \mathcal{A}), \quad \mathcal{B}_\eta = C^*(T_a x T_a^* : a \in \Sigma^\eta, x \in \mathcal{A}).$$

They are realized concretely as subalgebras of \mathcal{O}_ρ and of \mathcal{O}_η respectively. Both the algebras \mathcal{B}_ρ and \mathcal{B}_η contain the algebra \mathcal{A} through the identifications

$$x = \sum_{\alpha \in \Sigma^\rho} S_\alpha \rho_\alpha(x) S_\alpha^* = \sum_{a \in \Sigma^\eta} T_a \eta_a(x) T_a^*, \quad x \in \mathcal{A}. \quad (2.1)$$

We define the projections

$$P_\alpha = \rho_\alpha(1) \text{ for } \alpha \in \Sigma^\rho, \quad Q_a = \eta_a(1) \text{ for } a \in \Sigma^\eta.$$

Elements $w \in \mathcal{B}_\rho$ and $z \in \mathcal{B}_\eta$ are uniquely written in the following way:

$$w = \sum_{\alpha \in \Sigma^\rho} S_\alpha w_\alpha S_\alpha^* \text{ such that } w_\alpha = P_\alpha w_\alpha P_\alpha \in \mathcal{A} \text{ for } \alpha \in \Sigma^\rho, \quad (2.2)$$

$$z = \sum_{a \in \Sigma^\eta} T_a z_a T_a^* \text{ such that } z_a = Q_a z_a Q_a \in \mathcal{A} \text{ for } a \in \Sigma^\eta. \quad (2.3)$$

Define an alphabet set Σ_κ as in (1.5). For $\omega = (\alpha, b, a, \beta) \in \Sigma_\kappa$, we set

$$\alpha = t(\omega) \in \Sigma^\rho, \quad b = r(\omega) \in \Sigma^\eta, \quad a = l(\omega) \in \Sigma^\eta, \quad \beta = b(\omega) \in \Sigma^\rho,$$

which stand for: top, right, left, bottom respectively as in the following figure.

$$\begin{array}{ccc} \cdot & \xrightarrow{\alpha=t(\omega)} & \cdot \\ a=l(\omega) \downarrow & & \downarrow b=r(\omega) \\ \cdot & \xrightarrow{\beta=b(\omega)} & \cdot \end{array}$$

Define $*$ -homomorphisms $\hat{\rho}_\alpha : \mathcal{B}_\rho \longrightarrow \mathcal{A}$ for $\alpha \in \Sigma^\rho$ and $\hat{\eta}_a : \mathcal{B}_\eta \longrightarrow \mathcal{A}$ for $a \in \Sigma^\eta$ by (1.6) which satisfy the equalities

$$\hat{\rho}_\alpha(w) = P_\alpha w_\alpha P_\alpha \quad \text{and} \quad \hat{\eta}_a(z) = Q_a z_a Q_a$$

for $w = \sum_{\beta \in \Sigma^\rho} S_\beta w_\beta S_\beta^* \in \mathcal{B}_\rho$ as in (2.2) and $z = \sum_{b \in \Sigma^\eta} T_b z_b T_b^* \in \mathcal{B}_\eta$ as in (2.3). Their restrictions to \mathcal{A} coincide with ρ_α and η_a respectively.

Lemma 2.1. *Keep the above notations.*

(i) For $\alpha \in \Sigma^\rho$ and $z = \sum_{b \in \Sigma^\eta} T_b z_b T_b^* \in \mathcal{B}_\eta$ as in (2.3), put

$$\widehat{\rho}_\alpha^\eta(z) = \sum_{\substack{b,a,\beta \\ (\alpha,b,a,\beta) \in \Sigma_\kappa}} T_b \rho_\beta(z_a) T_b^* \in \mathcal{B}_\eta. \quad (2.4)$$

Then $\widehat{\rho}_\alpha^\eta : \mathcal{B}_\eta \rightarrow \mathcal{B}_\eta$ is a $*$ -homomorphism such that $\widehat{\rho}_\alpha^\eta(y) = \rho_\alpha(y)$ for $y \in \mathcal{A}$.

(ii) For $a \in \Sigma^\eta$ and $w = \sum_{\beta \in \Sigma^\rho} S_\beta w_\beta S_\beta^* \in \mathcal{B}_\rho$ as in (2.2), put

$$\widehat{\eta}_a^\rho(w) = \sum_{\substack{\alpha,b,\beta \\ (\alpha,b,a,\beta) \in \Sigma_\kappa}} S_\beta \eta_b(w_\alpha) S_\beta^* \in \mathcal{B}_\rho. \quad (2.5)$$

Then $\widehat{\eta}_a^\rho : \mathcal{B}_\rho \rightarrow \mathcal{B}_\rho$ is a $*$ -homomorphism such that $\widehat{\eta}_a^\rho(y) = \eta_a(y)$ for $y \in \mathcal{A}$.

Proof. (i) Since $z_a = \widehat{\eta}_a(z)$, the equality (2.4) becomes

$$\widehat{\rho}_\alpha^\eta(z) = \sum_{\substack{b,a,\beta \\ (\alpha,b,a,\beta) \in \Sigma_\kappa}} T_b \rho_\beta(\widehat{\eta}_a(z)) T_b^*$$

as in (1.7). It is easy to see that $\widehat{\rho}_\alpha^\eta : \mathcal{B}_\eta \rightarrow \mathcal{B}_\eta$ yields a $*$ -homomorphism. If in particular $z = y \in \mathcal{A}$, we have $y = \sum_{b \in \Sigma^\eta} T_b \eta_b(y) T_b^*$ so that

$$\widehat{\rho}_\alpha^\eta(y) = \sum_{\substack{b,a,\beta \\ (\alpha,b,a,\beta) \in \Sigma_\kappa}} T_b \rho_\beta(\eta_a(y)) T_b^* = \sum_{\substack{a,b,\beta \\ (\alpha,b,a,\beta) \in \Sigma_\kappa}} T_b \eta_b(\rho_\alpha(y)) T_b^* = \rho_\alpha(y).$$

(ii) is similar to (i). \square

The commutation relations (1.1) on \mathcal{A} extend to \mathcal{B}_ρ and to \mathcal{B}_η in the following lemma.

Lemma 2.2. For $\omega = (\alpha, b, a, \beta) \in \Sigma_\kappa$, we have

(i) $\eta_b \circ \widehat{\rho}_\alpha(w) = \widehat{\rho}_\beta \circ \widehat{\eta}_a^\rho(w)$ for $w \in \mathcal{B}_\rho$.

(ii) $\rho_\beta \circ \widehat{\eta}_a(z) = \widehat{\eta}_b \circ \widehat{\rho}_\alpha^\eta(z)$ for $z \in \mathcal{B}_\eta$.

Proof. (i) For $w = \sum_{\alpha' \in \Sigma^\rho} S_{\alpha'} w_{\alpha'} S_{\alpha'}^*$ as in (2.2), we have $S_\beta^* \widehat{\eta}_a^\rho(w) S_\beta = S_\beta^* S_\beta \eta_b(w_\alpha) S_\beta^* S_\beta$ so that by (1.1)

$$\begin{aligned} \widehat{\rho}_\beta \circ \widehat{\eta}_a^\rho(w) &= P_\beta \eta_b(w_\alpha) P_\beta \\ &= \rho_\beta(1) \eta_b(\rho_\alpha(1)) \eta_b(w_\alpha) \eta_b(\rho_\alpha(1)) \rho_\beta(1) \\ &= \rho_\beta(\eta_a(1)) \eta_b(w_\alpha) \rho_\beta(\eta_a(1)) \\ &= \eta_b(\rho_\alpha(1) w_\alpha \rho_\alpha(1)) = \eta_b(\widehat{\rho}_\alpha(w)). \end{aligned}$$

(ii) is similar to (i). \square

3. Hilbert C^* -quad modules from C^* -textile dynamical systems

We fix a C^* -textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$. For $\omega = (\alpha, b, a, \beta) \in \Sigma_\kappa$, we define the projection

$$E_\omega = \eta_b(\rho_\alpha(1)) (= \rho_\beta(\eta_a(1))) \in Z_{\mathcal{A}}. \quad (3.1)$$

Let $e_\omega, \omega \in \Sigma_\kappa$ denote the orthogonal basis of the vector space $\mathbb{C}^{|\Sigma_\kappa|}$ where $|\Sigma_\kappa|$ means the cardinal number of the finite set Σ_κ . Define the vector space

$$\mathcal{H}_\kappa^{\rho, \eta} = \sum_{\omega \in \Sigma_\kappa} \mathbb{C} e_\omega \otimes E_\omega \mathcal{A} \quad (3.2)$$

which is naturally isomorphic to the vector space $\bigoplus_{\omega \in \Sigma_\kappa} E_\omega \mathcal{A}$.

We first endow $\mathcal{H}_\kappa^{\rho, \eta}$ with right \mathcal{A} -module structure and \mathcal{A} -valued inner product as follows: For $\xi = \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega x_\omega, \xi' = \sum_{\omega' \in \Sigma_\kappa} e_{\omega'} \otimes E_{\omega'} x'_{\omega'}$ with $x_\omega, x'_{\omega'} \in \mathcal{A}$ and $y \in \mathcal{A}$, set

$$\begin{aligned} \xi \varphi_{\mathcal{A}}(y) &:= \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega x_\omega y \in \mathcal{H}_\kappa^{\rho, \eta}, \\ \langle \xi \mid \xi' \rangle_{\mathcal{A}} &:= \sum_{\omega \in \Sigma_\kappa} x_\omega^* E_\omega x'_{\omega} \in \mathcal{A} \end{aligned}$$

which satisfy the relations:

$$\langle \xi \mid \xi' \varphi_{\mathcal{A}}(y) \rangle_{\mathcal{A}} = \langle \xi \mid \xi' \rangle_{\mathcal{A}} \cdot y, \quad \langle \xi \mid \xi' \rangle_{\mathcal{A}}^* = \langle \xi' \mid \xi \rangle_{\mathcal{A}}.$$

We will further endow $\mathcal{H}_\kappa^{\rho, \eta}$ with two other Hilbert C^* -bimodule structures. Such a system will be called a Hilbert C^* -quad module. Let $\xi = \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega x_\omega, \xi' = \sum_{\omega' \in \Sigma_\kappa} e_{\omega'} \otimes E_{\omega'} x'_{\omega'} \in \mathcal{H}_\kappa^{\rho, \eta}$ with $x_\omega, x'_{\omega'} \in \mathcal{A}$, and $w = \sum_{\alpha' \in \Sigma_\rho} S_{\alpha'} w_{\alpha'} S_{\alpha'}^* \in \mathcal{B}_\rho$ as in (2.2), $z = \sum_{a' \in \Sigma_\eta} T_{a'} z_{a'} T_{a'}^* \in \mathcal{B}_\eta$ as in (2.3). We define:

1. The right \mathcal{B}_ρ -action φ_ρ and the right \mathcal{B}_η -action φ_η :

$$\xi \varphi_\rho(w) := \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega x_\omega w_{b(\omega)}, \quad \xi \varphi_\eta(z) := \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega x_\omega z_{r(\omega)}.$$

2. The left \mathcal{B}_ρ -action ϕ_ρ and the left \mathcal{B}_η -action ϕ_η :

$$\phi_\rho(w) \xi := \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega \eta_{r(\omega)}(w_{t(\omega)}) x_\omega, \quad \phi_\eta(z) \xi := \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega \rho_{b(\omega)}(z_{l(\omega)}) x_\omega.$$

3. The right \mathcal{B}_ρ -valued inner product $\langle \cdot \mid \cdot \rangle_\rho$ and the right \mathcal{B}_η -valued inner product $\langle \cdot \mid \cdot \rangle_\eta$:

$$\langle \xi \mid \xi' \rangle_\rho := \sum_{\omega \in \Sigma_\kappa} S_{b(\omega)} x_\omega^* E_\omega x'_{\omega} S_{b(\omega)}^*, \quad \langle \xi \mid \xi' \rangle_\eta := \sum_{\omega \in \Sigma_\kappa} T_{r(\omega)} x_\omega^* E_\omega x'_{\omega} T_{r(\omega)}^*.$$

The following lemma is straightforward.

Lemma 3.1. For $\xi \in \mathcal{H}_\kappa^{\rho, \eta}$ and $w, w' \in \mathcal{B}_\rho, z, z' \in \mathcal{B}_\eta$, we have

$$\begin{aligned} (\xi \varphi_\rho(w)) \varphi_\rho(w') &= \xi \varphi_\rho(w w'), & (\xi \varphi_\eta(z)) \varphi_\eta(z') &= \xi \varphi_\eta(z z'), \\ \phi_\rho(w) (\phi_\rho(w') \xi) &= \phi_\rho(w w') \xi, & \phi_\eta(z) (\phi_\eta(z') \xi) &= \phi_\eta(z z') \xi, \\ \phi_\rho(w) (\xi \varphi_\rho(w')) &= (\phi_\rho(w) \xi) \varphi_\rho(w'), & \phi_\eta(z) (\xi \varphi_\eta(z')) &= (\phi_\eta(z) \xi) \varphi_\eta(z'). \end{aligned}$$

Lemma 3.2. For $\xi, \xi' \in \mathcal{H}_\kappa^{\rho, \eta}$ and $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$, we have

$$\begin{aligned} \langle \xi \mid \xi' \varphi_\rho(w) \rangle_\rho &= \langle \xi \mid \xi' \rangle_\rho \cdot w, & \langle \xi \mid \xi' \varphi_\eta(z) \rangle_\eta &= \langle \xi \mid \xi' \rangle_\eta \cdot z, \\ \langle \xi \varphi_\rho(w) \mid \xi' \rangle_\rho &= w^* \cdot \langle \xi \mid \xi' \rangle_\rho, & \langle \xi \varphi_\eta(z) \mid \xi' \rangle_\eta &= z^* \cdot \langle \xi \mid \xi' \rangle_\eta, \\ \langle \phi_\rho(w) \xi \mid \xi' \rangle_\rho &= \langle \xi \mid \phi_\rho(w^*) \xi' \rangle_\rho, & \langle \phi_\eta(z) \xi \mid \xi' \rangle_\eta &= \langle \xi \mid \phi_\eta(z^*) \xi' \rangle_\eta. \end{aligned}$$

Proof. We will show the equalities

$$\langle \xi \mid \xi' \varphi_\rho(w) \rangle_\rho = \langle \xi \mid \xi' \rangle_\rho \cdot w, \quad \langle \phi_\rho(w) \xi \mid \xi' \rangle_\rho = \langle \xi \mid \phi_\rho(w^*) \xi' \rangle_\rho.$$

For $\xi = \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega x_\omega$, $\xi' = \sum_{\omega' \in \Sigma_\kappa} e_{\omega'} \otimes E_{\omega'} x'_{\omega'} \in \mathcal{H}_\kappa^{\rho, \eta}$ with $x_\omega, x'_{\omega'} \in \mathcal{A}$, and $w = \sum_{\gamma \in \Sigma^\eta} S_\gamma w_\gamma S_\gamma^* \in \mathcal{B}_\rho$ as in (2.2), we have

$$\begin{aligned} \langle \xi \mid \xi' \varphi_\rho(w) \rangle_\rho &= \sum_{\omega, \omega' \in \Sigma_\kappa} \langle e_\omega \otimes E_\omega x_\omega \mid e_{\omega'} \otimes E_{\omega'} x'_{\omega'} w_{b(\omega')} \rangle_\rho \\ &= \sum_{\omega \in \Sigma_\kappa} S_{b(\omega)} x_\omega^* E_\omega x'_{\omega'} w_{b(\omega)} S_{b(\omega)}^* \\ &= \left(\sum_{\omega \in \Sigma_\kappa} S_{b(\omega)} x_\omega^* E_\omega x'_{\omega'} S_{b(\omega)}^* \right) \cdot \left(\sum_{\gamma \in \Sigma^\rho} S_\gamma w_\gamma S_\gamma^* \right) \\ &= \langle \xi \mid \xi' \rangle_\rho \cdot w. \end{aligned}$$

We also have

$$\begin{aligned} \langle \phi_\rho(w) \xi \mid \xi' \rangle_\rho &= \sum_{\omega, \omega' \in \Sigma_\kappa} \sum_{\gamma \in \Sigma^\rho} \langle \phi_\rho(S_\gamma w_\gamma S_\gamma^*)(e_\omega \otimes E_\omega x_\omega) \mid e_{\omega'} \otimes E_{\omega'} x'_{\omega'} \rangle_\rho \\ &= \sum_{\omega, \omega' \in \Sigma_\kappa} \langle e_\omega \otimes E_\omega \eta_{r(\omega)}(w_{t(\omega)}) x_\omega \mid e_{\omega'} \otimes E_{\omega'} x'_{\omega'} \rangle_\rho \\ &= \sum_{\omega \in \Sigma_\kappa} S_{b(\omega)} x_\omega^* E_\omega \eta_{r(\omega)}(w_{t(\omega)}^*) x'_{\omega'} S_{b(\omega)}^* \\ &= \sum_{\omega, \omega' \in \Sigma_\kappa} \langle e_\omega \otimes E_\omega x_\omega \mid e_{\omega'} \otimes E_{\omega'} \eta_{r(\omega')}(w_{t(\omega')}^*) x'_{\omega'} \rangle_\rho \\ &= \sum_{\omega, \omega' \in \Sigma_\kappa} \sum_{\gamma \in \Sigma^\rho} \langle e_\omega \otimes E_\omega x_\omega \mid \phi_\rho(S_\gamma w_\gamma^* S_\gamma^*)(e_{\omega'} \otimes E_{\omega'} x'_{\omega'}) \rangle_\rho \\ &= \langle \xi \mid \phi_\rho(w^*) \xi' \rangle_\rho. \end{aligned}$$

The three equalities on the right hand side are shown similarly to the above equalities. \square

Hence we have

$$\begin{aligned} \phi_\rho(w^*) &= \phi_\rho(w)^* : \text{the adjoint with respect to the inner product } \langle \cdot \mid \cdot \rangle_\rho, \\ \phi_\eta(z^*) &= \phi_\eta(z)^* : \text{the adjoint with respect to the inner product } \langle \cdot \mid \cdot \rangle_\eta. \end{aligned}$$

The following lemma is direct and shows that the two module structures are compatible to each other.

Lemma 3.3. For $w \in \mathcal{B}_\rho$, $z \in \mathcal{B}_\eta$ and $\xi \in \mathcal{H}_\kappa^{\rho, \eta}$, we have

- (i) $(\phi_\rho(w) \xi) \varphi_\eta(z) = \phi_\rho(w)(\xi \varphi_\eta(z)).$
- (ii) $(\phi_\eta(z) \xi) \varphi_\rho(w) = \phi_\eta(z)(\xi \varphi_\rho(w)).$

Then we have the following proposition

Proposition 3.4. Keep the above notations.

- (i) $(\mathcal{H}_\kappa^{\rho,\eta}, \varphi_\rho)$ is a right \mathcal{B}_ρ -module with right \mathcal{B}_ρ -valued inner product $\langle \cdot | \cdot \rangle_\rho$ and left \mathcal{B}_ρ -action by ϕ_ρ . Hence $\mathcal{H}_\kappa^{\rho,\eta}$ is a Hilbert C^* -bimodule over \mathcal{B}_ρ .
- (ii) $(\mathcal{H}_\kappa^{\rho,\eta}, \varphi_\eta)$ is a right \mathcal{B}_η -module with right \mathcal{B}_η -valued inner product $\langle \cdot | \cdot \rangle_\eta$ and left \mathcal{B}_η -action by ϕ_η . Hence $\mathcal{H}_\kappa^{\rho,\eta}$ is a Hilbert C^* -bimodule over \mathcal{B}_η .

Therefore $\mathcal{H}_\kappa^{\rho,\eta}$ has a multistructure of Hilbert C^* -bimodules, which are compatible to each other.

$\mathcal{H}_\kappa^{\rho,\eta}$ is originally a Hilbert C^* -right module over \mathcal{A} , which is also compatible to the two left actions ϕ_ρ of \mathcal{B}_ρ and ϕ_η of \mathcal{B}_η as in the following lemma. Its proof is straightforward.

Lemma 3.5. For $\xi \in \mathcal{H}_\kappa^{\rho,\eta}$ and $y \in \mathcal{A}$, we have

- (i) $\phi_\rho(w)(\xi\varphi_\mathcal{A}(y)) = (\phi_\rho(w)\xi)\varphi_\mathcal{A}(y)$ for $w \in \mathcal{B}_\rho$.
- (ii) $\phi_\eta(z)(\xi\varphi_\mathcal{A}(y)) = (\phi_\eta(z)\xi)\varphi_\mathcal{A}(y)$ for $z \in \mathcal{B}_\eta$.

Hence both $\phi_\eta(z)$ and $\phi_\rho(w)$ are right \mathcal{A} -module maps.

Define positive maps $\psi_\rho : \mathcal{A} \longrightarrow \mathcal{B}_\rho$ and $\psi_\eta : \mathcal{A} \longrightarrow \mathcal{B}_\eta$ by

$$\psi_\rho(y) = \sum_{\alpha \in \Sigma^\rho} S_\alpha y S_\alpha^* \in \mathcal{B}_\rho, \quad \psi_\eta(y) = \sum_{a \in \Sigma^\eta} T_a y T_a^* \in \mathcal{B}_\eta \quad (3.3)$$

for $y \in \mathcal{A}$. Then we have

Lemma 3.6. For $\xi \in \mathcal{H}_\kappa^{\rho,\eta}$ and $y \in \mathcal{A}$, we have

- (i) $\xi\varphi_\rho(w\psi_\rho(y)) = (\xi\varphi_\rho(w))\varphi_\mathcal{A}(y)$ for $w \in \mathcal{B}_\rho$.
- (ii) $\xi\varphi_\eta(z\psi_\eta(y)) = (\xi\varphi_\eta(z))\varphi_\mathcal{A}(y)$ for $z \in \mathcal{B}_\eta$.

Hence we have

$$\xi\varphi_\rho(\psi_\rho(y)) = \xi\varphi_\eta(\psi_\eta(y)) = \xi\varphi_\mathcal{A}(y).$$

Proof. (i) For $\xi = \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega x_\omega \in \mathcal{H}_\kappa^{\rho,\eta}$ with $x_\omega \in \mathcal{A}$, and $w = \sum_{\alpha' \in \Sigma^\rho} S_{\alpha'} w_{\alpha'} S_{\alpha'}^* \in \mathcal{B}_\rho$ as in (2.2), we have

$$w\psi_\rho(y) = \sum_{\alpha' \in \Sigma^\rho} S_{\alpha'} w_{\alpha'} S_{\alpha'}^* \sum_{\beta' \in \Sigma^\rho} S_{\beta'} y S_{\beta'}^* = \sum_{\alpha' \in \Sigma^\rho} S_{\alpha'} w_{\alpha'} y S_{\alpha'}^*$$

so that

$$\begin{aligned} \xi\varphi_\rho(w\psi_\rho(y)) &= \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega x_\omega w_{b(\omega)} y \\ &= \sum_{\omega \in \Sigma_\kappa} [(e_\omega \otimes E_\omega x_\omega)\varphi_\rho(w)]\varphi_\mathcal{A}(y) = [\xi\varphi_\rho(w)]\varphi_\mathcal{A}(y). \end{aligned}$$

(ii) is similar to (i). \square

Lemma 3.7. For $y \in \mathcal{A}$ and $\xi \in \mathcal{H}_\kappa^{\rho,\eta}$, we have $\phi_\rho(y)\xi = \phi_\eta(y)\xi$.

Proof. Since the identities $y = \sum_{\alpha \in \Sigma^\rho} S_\alpha \rho_\alpha(y) S_\alpha^* = \sum_{a \in \Sigma^\eta} T_a \eta_a(y) T_a^*$ hold, we have for $\xi = \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega x_\omega$ with $x_\omega \in \mathcal{A}$,

$$\phi_\rho(y)\xi = \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega \eta_{r(\omega)}(\rho_{t(\omega)}(y))x_\omega.$$

On the other hand, we have

$$\phi_\eta(y)\xi = \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega \rho_{b(\omega)}(\eta_{l(\omega)}(y))x_\omega.$$

As $\eta_{r(\omega)}(\rho_{t(\omega)}(y)) = \rho_{b(\omega)}(\eta_{l(\omega)}(y))$, we obtain the desired equality. \square

By the above lemma, we may define the left action ϕ of \mathcal{A} on $\mathcal{H}_\kappa^{\rho,\eta}$ by

$$\phi(y)\xi := \phi_\rho(y)\xi = \phi_\eta(y)\xi, \quad y \in \mathcal{A}, \xi \in \mathcal{H}_\kappa^{\rho,\eta}$$

so that $\mathcal{H}_\kappa^{\rho,\eta}$ has a structure of a Hilbert C^* -bimodule over \mathcal{A} . We note the following lemma.

Lemma 3.8. *If the algebra \mathcal{A} is commutative, we have*

$$\phi_\rho(w)\phi_\eta(z) = \phi_\eta(z)\phi_\rho(w), \quad w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta.$$

Proof. For $w = \sum_{\alpha \in \Sigma^\rho} S_\alpha w_\alpha S_\alpha^*$ as in (2.2), $z = \sum_{a' \in \Sigma^\eta} T_{a'} z_{a'} T_{a'}^*$ as in (2.3) and $\xi = \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega x_\omega$ with $x_\omega \in \mathcal{A}$, we have

$$\begin{aligned} \phi_\rho(w)\phi_\eta(z)\xi &= \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega \eta_{r(\omega)}(w_{t(\omega)})\rho_{b(\omega)}(z_{l(\omega)})x_\omega \\ &= \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega \rho_{b(\omega)}(z_{l(\omega)})\eta_{r(\omega)}(w_{t(\omega)})x_\omega \\ &= \phi_\eta(z)\phi_\rho(w)\xi. \quad \square \end{aligned}$$

Put for $\alpha \in \Sigma^\rho$ and $a \in \Sigma^\eta$

$$u_\alpha = \sum_{\omega \in \Sigma_\kappa, \alpha=t(\omega)} e_\omega \otimes E_\omega \in \mathcal{H}_\kappa^{\rho,\eta}, \quad v_a = \sum_{\omega \in \Sigma_\kappa, a=l(\omega)} e_\omega \otimes E_\omega \in \mathcal{H}_\kappa^{\rho,\eta}.$$

Lemma 3.9. *Keep the above notations.*

- (i) $\{u_\alpha\}_{\alpha \in \Sigma^\rho}$ forms an essential orthogonal finite basis of $\mathcal{H}_\kappa^{\rho,\eta}$ with respect to the \mathcal{B}_η -valued inner product $\langle \cdot | \cdot \rangle_\eta$ as right \mathcal{B}_η -module through φ_η .
- (ii) $\{v_a\}_{a \in \Sigma^\eta}$ forms an essential orthogonal finite basis of $\mathcal{H}_\kappa^{\rho,\eta}$ with respect to the \mathcal{B}_ρ -valued inner product $\langle \cdot | \cdot \rangle_\rho$ as right \mathcal{B}_ρ -module through φ_ρ .

Hence we have

$$\xi = \sum_{\alpha \in \Sigma^\rho} u_\alpha \varphi_\eta(\langle u_\alpha | \xi \rangle_\eta) = \sum_{a \in \Sigma^\eta} v_a \varphi_\rho(\langle v_a | \xi \rangle_\rho), \quad \xi \in \mathcal{H}_\kappa^{\rho,\eta}.$$

Proof. (i) For $\alpha, \beta \in \Sigma^\rho$, we have

$$\begin{aligned}\langle u_\alpha | u_\beta \rangle_\eta &= \left\langle \sum_{\omega \in \Sigma_\kappa, \alpha=t(\omega)} e_\omega \otimes E_\omega \mid \sum_{\omega' \in \Sigma_\kappa, \beta=t(\omega')} e_{\omega'} \otimes E_{\omega'} \right\rangle_\eta \\ &= \begin{cases} \sum_{\omega \in \Sigma_\kappa, \alpha=t(\omega)} \langle e_\omega \otimes E_\omega \mid e_\omega \otimes E_\omega \rangle_\eta & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}\end{aligned}$$

Since

$$\begin{aligned}\sum_{\omega \in \Sigma_\kappa, \alpha=t(\omega)} \langle e_\omega \otimes E_\omega \mid e_\omega \otimes E_\omega \rangle_\eta &= \sum_{\omega \in \Sigma_\kappa, \alpha=t(\omega)} T_{r(\omega)} E_\omega T_{r(\omega)}^* \\ &= \sum_{\omega \in \Sigma_\kappa, \alpha=t(\omega)} T_{r(\omega)} T_{r(\omega)}^* \rho_\alpha(1) T_{r(\omega)} T_{r(\omega)}^* = P_\alpha,\end{aligned}$$

we see

$$\langle u_\alpha | u_\beta \rangle_\eta = \begin{cases} P_\alpha & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

Hence we have

$$\sum_{\alpha \in \Sigma^\rho} \langle u_\alpha | u_\alpha \rangle_\eta = \sum_{\alpha \in \Sigma^\rho} P_\alpha \geq 1.$$

For $\xi = \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega x_\omega \in \mathcal{H}_\kappa^{\rho, \eta}$ with $x_\omega \in \mathcal{A}$, we have

$$\langle u_\alpha | \xi \rangle_\eta = \sum_{\omega \in \Sigma_\kappa, \alpha=t(\omega)} T_{r(\omega)} E_\omega x_\omega T_{r(\omega)}^*.$$

It then follows that

$$\begin{aligned}u_\alpha \varphi_\eta(\langle u_\alpha | \xi \rangle_\eta) &= \left(\sum_{\omega \in \Sigma_\kappa, \alpha=t(\omega)} e_\omega \otimes E_\omega \right) \varphi_\eta \left(\sum_{\omega' \in \Sigma_\kappa, \alpha=t(\omega')} T_{r(\omega')} E_{\omega'} x_{\omega'} T_{r(\omega')}^* \right) \\ &= \sum_{\omega, \omega' \in \Sigma_\kappa, \alpha=t(\omega)=t(\omega')} (e_\omega \otimes E_\omega) \varphi_\eta(T_{r(\omega')} E_{\omega'} x_{\omega'} T_{r(\omega')}^*) \\ &= \sum_{\omega \in \Sigma_\kappa, \alpha=t(\omega)} e_\omega \otimes E_\omega x_\omega\end{aligned}$$

so that

$$\sum_{\alpha \in \Sigma^\rho} u_\alpha \varphi_\eta(\langle u_\alpha | \xi \rangle_\eta) = \sum_{\alpha \in \Sigma^\rho} \sum_{\omega \in \Sigma_\kappa, \alpha=t(\omega)} e_\omega \otimes E_\omega x_\omega = \sum_{\omega \in \mathcal{H}_\kappa^{\rho, \eta}} e_\omega \otimes E_\omega x_\omega = \xi.$$

(ii) is similar to (i). \square

As $\langle u_\alpha | u_\alpha \rangle_\eta = P_\alpha$ and $\langle v_a | v_a \rangle_\rho = Q_a$, we note that the equality

$$\eta_b(\langle u_\alpha | u_\alpha \rangle_\eta) = \rho_\beta(\langle v_a | v_a \rangle_\rho) = E_\omega$$

holds for $\omega = (\alpha, b, a, \beta) \in \Sigma_\kappa$.

Lemma 3.10. For $\alpha \in \Sigma^\rho$, $a \in \Sigma^\eta$ and $y \in \mathcal{A}$, we have

- (i) $\phi(y)u_\alpha = u_\alpha\varphi_\eta(\rho_\alpha(y))$ and hence $\rho_\alpha(y) = \langle u_\alpha | \phi(y)u_\alpha \rangle_\eta$.
(ii) $\phi(y)v_a = v_a\varphi_\rho(\eta_a(y))$ and hence $\eta_a(y) = \langle v_a | \phi(y)v_a \rangle_\rho$.

Therefore the commutation relations (1.1) are rephrased as the equality

$$\langle v_b | \phi(\langle u_\alpha | \phi(y)u_\alpha \rangle_\eta)v_b \rangle_\rho = \langle u_\beta | \phi(\langle v_a | \phi(y)v_a \rangle_\rho)u_\beta \rangle_\eta \quad (3.4)$$

for $\omega = (\alpha, b, a, \beta) \in \Sigma_\kappa$ and $y \in \mathcal{A}$.

Proof. (i) It follows that

$$\begin{aligned} \phi(y)u_\alpha &= \phi_\eta(y)u_\alpha = \phi_\eta\left(\sum_{a' \in \Sigma^\eta} T_{a'}\eta_a(y)T_{a'}^*\right)\left(\sum_{\omega \in \Sigma_\kappa, \alpha=t(\omega)} e_\omega \otimes E_\omega\right) \\ &= \sum_{\omega \in \Sigma_\kappa, \alpha=t(\omega)} e_\omega \otimes E_\omega \rho_{b(\omega)}(\eta_{l(\omega)}(y)) \\ &= \sum_{\omega \in \Sigma_\kappa, \alpha=t(\omega)} e_\omega \otimes E_\omega \eta_{r(\omega)}(\rho_\alpha(y)) \\ &= \sum_{\omega \in \Sigma_\kappa, \alpha=t(\omega)} [(e_\omega \otimes E_\omega)\varphi_\eta\left(\sum_{b' \in \Sigma^\eta} T_{b'}\eta_{b'}(\rho_\alpha(y))T_{b'}^*\right)] \\ &= u_\alpha\varphi_\eta(\rho_\alpha(y)), \end{aligned}$$

so that

$$\langle u_\alpha | \phi(y)u_\alpha \rangle_\eta = \langle u_\alpha | u_\alpha\varphi_\eta(\rho_\alpha(y)) \rangle_\eta = \langle u_\alpha | u_\alpha \rangle_\eta \cdot \rho_\alpha(y) = \rho_\alpha(y).$$

(ii) is similar to (i). \square

More generally we have

Lemma 3.11. For $\alpha \in \Sigma^\rho$, $a \in \Sigma^\eta$ and $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$, we have

- (i) $\phi_\rho(w)u_\alpha = u_\alpha\varphi_\eta(\langle u_\alpha | \phi_\rho(w)u_\alpha \rangle_\eta)$, $\phi_\rho(w)v_a = v_a\varphi_\rho(\langle v_a | \phi_\rho(w)v_a \rangle_\rho)$.
(ii) $\phi_\eta(z)u_\alpha = u_\alpha\varphi_\eta(\langle u_\alpha | \phi_\eta(z)u_\alpha \rangle_\eta)$, $\phi_\eta(z)v_a = v_a\varphi_\rho(\langle v_a | \phi_\eta(z)v_a \rangle_\rho)$.

Proof. (i) For $\xi \in \mathcal{H}_\kappa^{\rho, \eta}$, we have $\xi = \sum_{\alpha' \in \Sigma^\rho} u_{\alpha'}\varphi_\eta(\langle u_{\alpha'} | \xi \rangle_\eta)$. For $\alpha \neq \alpha'$, we have $\langle u_\alpha | \phi_\rho(w)u_{\alpha'} \rangle_\eta = 0$ so that

$$\phi_\rho(w)u_\alpha = \sum_{\alpha' \in \Sigma^\rho} u_{\alpha'}\varphi_\eta(\langle u_{\alpha'} | \phi_\rho(w)u_\alpha \rangle_\eta) = u_\alpha\varphi_\eta(\langle u_\alpha | \phi_\rho(w)u_\alpha \rangle_\eta).$$

Similarly for $a \neq a'$, we have $\langle v_{a'} | \phi_\rho(w)v_a \rangle_\rho = 0$ so that

$$\phi_\rho(w)v_a = \sum_{a' \in \Sigma^\eta} v_{a'}\varphi_\rho(\langle v_{a'} | \phi_\rho(w)v_a \rangle_\rho) = v_a\varphi_\rho(\langle v_a | \phi_\rho(w)v_a \rangle_\rho).$$

(ii) is similar to (i). \square

The following lemma states that the $*$ -homomorphisms $\hat{\rho}_\alpha, \hat{\eta}_a^\rho$ on \mathcal{B}_ρ and $\hat{\eta}_a, \hat{\rho}_\alpha^\eta$ on \mathcal{B}_η are given by inner products.

Lemma 3.12. For $\alpha \in \Sigma^\rho$, $a \in \Sigma^\eta$ and $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$, we have

- (i) $\hat{\rho}_\alpha(w) = \langle u_\alpha \mid \phi_\rho(w)u_\alpha \rangle_\eta$, $\hat{\eta}_a(z) = \langle v_a \mid \phi_\eta(z)v_a \rangle_\rho$.
- (ii) $\hat{\rho}_\alpha^\eta(z) = \langle u_\alpha \mid \phi_\eta(z)u_\alpha \rangle_\eta$, $\hat{\eta}_a^\rho(w) = \langle v_a \mid \phi_\rho(w)v_a \rangle_\rho$.

Proof. (i) For $w = \sum_{\alpha' \in \Sigma^\rho} S_{\alpha'} w_{\alpha'} S_{\alpha'}^* \in \mathcal{B}_\rho$ as in (2.2), we have

$$\phi_\rho(w)u_\alpha = \sum_{\omega \in \Sigma_\kappa, t(\omega)=\alpha} e_\omega \otimes E_\omega \eta_{r(\omega)}(w_\alpha)$$

so that

$$\begin{aligned} \langle u_\alpha \mid \phi_\rho(w)u_\alpha \rangle_\eta &= \left\langle \sum_{\omega' \in \Sigma_\kappa, t(\omega')=\alpha} e_{\omega'} \otimes E_{\omega'} \mid \sum_{\omega \in \Sigma_\kappa, t(\omega)=\alpha} e_\omega \otimes E_\omega \eta_{r(\omega)}(w_\alpha) \right\rangle_\eta \\ &= \sum_{\omega \in \Sigma_\kappa, t(\omega)=\alpha} T_{r(\omega)} E_\omega \eta_{r(\omega)}(w_\alpha) T_{r(\omega)}^* \\ &= \sum_{\substack{b \\ (\alpha, b) \in \Sigma^{\rho\eta}}} T_b T_b^* w_\alpha T_b T_b^* \\ &= \sum_{b \in \Sigma^\eta} T_b T_b^* S_\alpha^* S_\alpha w_\alpha S_\alpha^* S_\alpha T_b T_b^* = P_\alpha w_\alpha P_\alpha = \hat{\rho}_\alpha(w). \end{aligned}$$

The other equality for $\hat{\eta}_a(z)$ is shown similarly to the above equalities.

(ii) For $z = \sum_{a \in \Sigma^\eta} T_a z_a T_a^* \in \mathcal{B}_\eta$ as in (2.3), we have

$$\phi_\eta(z)u_\alpha = \sum_{\omega \in \Sigma_\kappa, t(\omega)=\alpha} e_\omega \otimes E_\omega \rho_{b(\omega)}(z_{l(\omega)})$$

so that

$$\langle u_\alpha \mid \phi_\eta(z)u_\alpha \rangle_\eta = \sum_{\omega \in \Sigma_\kappa, t(\omega)=\alpha} T_{r(\omega)} \rho_{b(\omega)}(z_{l(\omega)}) T_{r(\omega)}^* = \hat{\rho}_\alpha^\eta(z).$$

The other equality for $\hat{\eta}_a^\rho(w)$ is shown similarly to the above equalities. \square

We will next study the norms on $\mathcal{H}_\kappa^{\rho, \eta}$ induced by the two inner products $\langle \cdot \mid \cdot \rangle_\eta$ and $\langle \cdot \mid \cdot \rangle_\rho$

Lemma 3.13. For $\xi = \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega x_\omega \in \mathcal{H}_\kappa^{\rho, \eta}$ with $x_\omega \in \mathcal{A}$, we have

- (i) $\|\langle \xi \mid \xi \rangle_\rho\| = \max_{\beta \in \Sigma^\rho} \|\sum_{a \in \Sigma^\eta} x_{a, \beta}^* x_{a, \beta}\|$,
- (ii) $\|\langle \xi \mid \xi \rangle_\eta\| = \max_{b \in \Sigma^\eta} \|\sum_{\alpha \in \Sigma^\rho} x_{\alpha, b}^* x_{\alpha, b}\|$,

where $E_\omega x_\omega = x_{\alpha, b} = x_{a, \beta}$ for $\omega = (\alpha, b, a, \beta) \in \Sigma_\kappa$.

Proof. (i) We have

$$\begin{aligned} \|\langle \xi \mid \xi \rangle_\rho\| &= \left\| \sum_{\omega \in \Sigma_\kappa} S_{b(\omega)} x_\omega^* E_\omega x_\omega S_{b(\omega)}^* \right\| = \left\| \sum_{(a, \beta) \in \Sigma^{\eta\rho}} S_\beta x_{a, \beta}^* x_{a, \beta} S_\beta^* \right\| \\ &= \left\| \sum_{\beta \in \Sigma^\rho} S_\beta \left(\sum_{a \in \Sigma^\eta} x_{a, \beta}^* x_{a, \beta} \right) S_\beta^* \right\| = \max_{\beta \in \Sigma^\rho} \left\| S_\beta \left(\sum_{a \in \Sigma^\eta} x_{a, \beta}^* x_{a, \beta} \right) S_\beta^* \right\|. \end{aligned}$$

Since $x_{a,\beta} = x_{a,\beta}P_\beta$, we have for $\beta \in \Sigma^\rho$

$$\|S_\beta(\sum_{a \in \Sigma^\eta} x_{a,\beta}^* x_{a,\beta}) S_\beta^*\| \leq \|\sum_{a \in \Sigma^\eta} x_{a,\beta}^* x_{a,\beta}\| \leq \|\sum_{a \in \Sigma^\eta} P_\beta x_{a,\beta}^* x_{a,\beta} P_\beta\|$$

one has

$$\|S_\beta(\sum_{a \in \Sigma^\eta} x_{a,\beta}^* x_{a,\beta}) S_\beta^*\| = \|\sum_{a \in \Sigma^\eta} x_{a,\beta}^* x_{a,\beta}\|.$$

Therefore we have

$$\|\langle \xi | \xi \rangle_\rho\| = \max_{\beta \in \Sigma^\rho} \|\sum_{a \in \Sigma^\eta} x_{a,\beta}^* x_{a,\beta}\|.$$

(ii) is similar to (i). \square

Define positive maps $\lambda_\rho : \mathcal{B}_\rho \rightarrow \mathcal{A}$ and $\lambda_\eta : \mathcal{B}_\eta \rightarrow \mathcal{A}$ by

$$\lambda_\rho(w) = \sum_{\alpha \in \Sigma^\rho} \hat{\rho}_\alpha(w), \quad w \in \mathcal{B}_\rho, \quad \lambda_\eta(z) = \sum_{a \in \Sigma^\eta} \hat{\eta}_a(z), \quad z \in \mathcal{B}_\eta. \quad (3.5)$$

Then we have for $\xi, \xi' \in \mathcal{H}_\kappa^{\rho,\eta}$

$$\langle \xi | \xi' \rangle_{\mathcal{A}} = \lambda_\rho(\langle \xi | \xi' \rangle_\rho) = \lambda_\eta(\langle \xi | \xi' \rangle_\eta). \quad (3.6)$$

Put $C_\rho = \|\lambda_\rho(1)\|$, $C_\eta = \|\lambda_\eta(1)\|$. As $\lambda_\rho(1) = \sum_{\alpha \in \Sigma^\rho} \rho_\alpha(1) \geq 1$, one sees $C_\rho \geq 1$ and similarly $C_\eta \geq 1$. Define the three norms for $\xi \in \mathcal{H}_\kappa^{\rho,\eta}$

$$\|\xi\|_{\mathcal{A}} = \|\langle \xi | \xi \rangle_{\mathcal{A}}\|^{\frac{1}{2}}, \quad \|\xi\|_\rho = \|\langle \xi | \xi \rangle_\rho\|^{\frac{1}{2}}, \quad \|\xi\|_\eta = \|\langle \xi | \xi \rangle_\eta\|^{\frac{1}{2}}. \quad (3.7)$$

Lemma 3.14. *The following inequalities hold for $\xi \in \mathcal{H}_\kappa^{\rho,\eta}$:*

$$\|\xi\|_\rho \leq \|\xi\|_{\mathcal{A}} \leq C_\rho^{\frac{1}{2}} \|\xi\|_\rho \quad \text{and} \quad \|\xi\|_\eta \leq \|\xi\|_{\mathcal{A}} \leq C_\eta^{\frac{1}{2}} \|\xi\|_\eta.$$

Hence the three norms $\|\xi\|_{\mathcal{A}}$, $\|\xi\|_\eta$, $\|\xi\|_\rho$ are equivalent to each other.

Proof. For $\xi = \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega x_\omega \in \mathcal{H}_\kappa^{\rho,\eta}$ with $x_\omega \in \mathcal{A}$, where $E_\omega x_\omega = x_{\alpha,b} = x_{a,\beta}$ for $\omega = (\alpha, b, a, \beta) \in \Sigma_\kappa$, we have

$$\|\langle \xi | \xi \rangle_\rho\| = \max_{\beta \in \Sigma^\rho} \|\sum_{a \in \Sigma^\eta} x_{a,\beta}^* x_{a,\beta}\|.$$

We then have

$$\|\xi\|_{\mathcal{A}} = \|\langle \xi | \xi \rangle_{\mathcal{A}}\|^{\frac{1}{2}} = \|\sum_{\omega \in \Sigma_\kappa} x_\omega^* x_\omega\|^{\frac{1}{2}} = \|\sum_{\beta \in \Sigma^\rho} \sum_{a \in \Sigma^\eta} x_{a,\beta}^* x_{a,\beta}\|^{\frac{1}{2}}.$$

Since

$$\|\sum_{a \in \Sigma^\eta} x_{a,\beta}^* x_{a,\beta}\| \leq \|\sum_{\beta \in \Sigma^\rho} \sum_{a \in \Sigma^\eta} x_{a,\beta}^* x_{a,\beta}\|,$$

we have

$$\|\langle \xi \mid \xi \rangle_\rho\| \leq \|\xi\|_{\mathcal{A}}.$$

On the other hand, by the equality $\langle \xi \mid \xi \rangle_{\mathcal{A}} = \lambda_\rho(\langle \xi \mid \xi \rangle_\rho)$, we have

$$\|\langle \xi \mid \xi \rangle_{\mathcal{A}}\| \leq \|\lambda_\rho\| \|\langle \xi \mid \xi \rangle_\rho\| = \|\lambda_\rho(1)\| \|\langle \xi \mid \xi \rangle_\rho\| = C_\rho \|\langle \xi \mid \xi \rangle_\rho\|.$$

Therefore we have

$$\|\xi\|_\rho \leq \|\xi\|_{\mathcal{A}} \leq C_\rho^{\frac{1}{2}} \|\xi\|_\rho \quad \text{and similarly} \quad \|\xi\|_\eta \leq \|\xi\|_{\mathcal{A}} \leq C_\rho^{\frac{1}{2}} \|\xi\|_\eta. \quad \square$$

4. Fock Hilbert C^* -quad modules and creation operators

In this section, we will consider relative tensor products of Hilbert C^* -quad modules and introduce the Fock space of Hilbert C^* -quad modules which is a two-dimensional analogue of the Fock space of Hilbert C^* -bimodules. The Hilbert C^* -module $\mathcal{H}_\kappa^{\rho,\eta}$ is originally a Hilbert C^* -right module $(\mathcal{H}_\kappa^{\rho,\eta}, \varphi_{\mathcal{A}})$ over \mathcal{A} with \mathcal{A} -valued inner product $\langle \cdot \mid \cdot \rangle_{\mathcal{A}}$. It has two other multistructures of Hilbert C^* -bimodules. The Hilbert C^* -bimodule $(\phi_\rho, \mathcal{H}_\kappa^{\rho,\eta}, \varphi_\rho)$ over \mathcal{B}_ρ and the Hilbert C^* -bimodule $(\phi_\eta, \mathcal{H}_\kappa^{\rho,\eta}, \varphi_\eta)$ over \mathcal{B}_η . This situation is written as in the figure:

$$\begin{array}{ccccc} & & \mathcal{B}_\rho & & \\ & & \downarrow \phi_\rho & & \\ \mathcal{B}_\eta & \xrightarrow{\phi_\eta} & \mathcal{H}_\kappa^{\rho,\eta} & \xleftarrow{\varphi_\eta} & \mathcal{B}_\eta \\ & & \uparrow \varphi_\rho & & \\ & & \mathcal{B}_\rho & & \end{array}$$

There exist faithful completely positive maps $\lambda_\rho : \mathcal{B}_\rho \longrightarrow \mathcal{A}$ and $\lambda_\eta : \mathcal{B}_\eta \longrightarrow \mathcal{A}$ satisfying (3.6) so that the three norms induced by their respective inner products $\langle \cdot \mid \cdot \rangle_{\mathcal{A}}, \langle \cdot \mid \cdot \rangle_\rho, \langle \cdot \mid \cdot \rangle_\eta$ are equivalent to each other. The Hilbert C^* -right module $(\mathcal{H}_\kappa^{\rho,\eta}, \varphi_{\mathcal{A}})$ over \mathcal{A} with multistructure Hilbert C^* -bimodules $(\phi_\rho, \mathcal{H}_\kappa^{\rho,\eta}, \varphi_\rho)$ over \mathcal{B}_ρ and $(\phi_\eta, \mathcal{H}_\kappa^{\rho,\eta}, \varphi_\eta)$ over \mathcal{B}_η is called a *Hilbert C^* -quad module over $(\mathcal{A}; \mathcal{B}_\rho, \mathcal{B}_\eta)$* . We will define two kinds of relative tensor products

$$\mathcal{H}_\kappa^{\rho,\eta} \otimes_\eta \mathcal{H}_\kappa^{\rho,\eta}, \quad \mathcal{H}_\kappa^{\rho,\eta} \otimes_\rho \mathcal{H}_\kappa^{\rho,\eta}$$

as Hilbert C^* -quad modules over $(\mathcal{A}; \mathcal{B}_\rho, \mathcal{B}_\eta)$. The latter one should be written vertically as

$$\begin{array}{c} \mathcal{H}_\kappa^{\rho,\eta} \\ \otimes_\rho \\ \mathcal{H}_\kappa^{\rho,\eta} \end{array}$$

rather than horizontally $\mathcal{H}_\kappa^{\rho,\eta} \otimes_\rho \mathcal{H}_\kappa^{\rho,\eta}$. The first relative tensor product is defined as

$$\mathcal{H}_\kappa^{\rho,\eta} \otimes_\eta \mathcal{H}_\kappa^{\rho,\eta} := \mathcal{H}_\kappa^{\rho,\eta} \otimes_{\mathcal{B}_\eta} \mathcal{H}_\kappa^{\rho,\eta}$$

the relative tensor product as Hilbert C^* -modules over \mathcal{B}_η , the left $\mathcal{H}_\kappa^{\rho,\eta}$ is a right \mathcal{B}_η -module through φ_η and the right $\mathcal{H}_\kappa^{\rho,\eta}$ is a left \mathcal{B}_η -module through ϕ_η . It has a right \mathcal{B}_η -valued inner product and a right \mathcal{B}_ρ -valued inner product defined by

$$\begin{aligned}\langle \xi \otimes_{\eta} \zeta \mid \xi' \otimes_{\eta} \zeta' \rangle_{\eta} &:= \langle \zeta \mid \phi_{\eta}(\langle \xi \mid \xi' \rangle_{\eta}) \zeta' \rangle_{\eta}, \\ \langle \xi \otimes_{\eta} \zeta \mid \xi' \otimes_{\eta} \zeta' \rangle_{\rho} &:= \langle \zeta \mid \phi_{\eta}(\langle \xi \mid \xi' \rangle_{\eta}) \zeta' \rangle_{\rho}\end{aligned}$$

respectively. It has two right actions, $\text{id} \otimes \varphi_{\eta}$ from \mathcal{B}_{η} and $\text{id} \otimes \varphi_{\rho}$ from \mathcal{B}_{ρ} . It also has two left actions, $\phi_{\eta} \otimes \text{id}$ from \mathcal{B}_{η} and $\phi_{\rho} \otimes \text{id}$ from \mathcal{B}_{ρ} . By these operations $\mathcal{H}_{\kappa}^{\rho, \eta} \otimes_{\eta} \mathcal{H}_{\kappa}^{\rho, \eta}$ is a Hilbert C^* -bimodule over \mathcal{B}_{η} and also is a Hilbert C^* -bimodule over \mathcal{B}_{ρ} . It also has a right \mathcal{A} -valued inner product defined by

$$\langle \xi \otimes_{\eta} \zeta \mid \xi' \otimes_{\eta} \zeta' \rangle_{\mathcal{A}} := \lambda_{\eta}(\langle \xi \otimes_{\eta} \zeta \mid \xi' \otimes_{\eta} \zeta' \rangle_{\eta}) (= \lambda_{\rho}(\langle \xi \otimes_{\eta} \zeta \mid \xi' \otimes_{\eta} \zeta' \rangle_{\rho}))$$

and a right \mathcal{A} -action $\text{id} \otimes \varphi_{\mathcal{A}}$ and a left \mathcal{A} -action $\phi \otimes \text{id}$. By these structures $\mathcal{H}_{\kappa}^{\rho, \eta} \otimes_{\eta} \mathcal{H}_{\kappa}^{\rho, \eta}$ is a Hilbert C^* -quad module over $(\mathcal{A}; \mathcal{B}_{\rho}, \mathcal{B}_{\eta})$.

$$\begin{array}{ccccc} & & \mathcal{B}_{\rho} & & \\ & & \downarrow \phi_{\rho} \otimes \text{id} & & \\ \mathcal{B}_{\eta} & \xrightarrow{\phi_{\eta} \otimes \text{id}} & \mathcal{H}_{\kappa}^{\rho, \eta} \otimes_{\eta} \mathcal{H}_{\kappa}^{\rho, \eta} & \xleftarrow{\text{id} \otimes \varphi_{\eta}} & \mathcal{B}_{\eta} \\ & & \uparrow \text{id} \otimes \varphi_{\rho} & & \\ & & \mathcal{B}_{\rho} & & \end{array}$$

We denote the above operations $\phi_{\rho} \otimes \text{id}, \phi_{\eta} \otimes \text{id}, \text{id} \otimes \varphi_{\rho}, \text{id} \otimes \varphi_{\eta}$ still by $\phi_{\rho}, \phi_{\eta}, \varphi_{\rho}, \varphi_{\eta}$ respectively. Similarly we consider the other relative tensor product defined by

$$\mathcal{H}_{\kappa}^{\rho, \eta} \otimes_{\rho} \mathcal{H}_{\kappa}^{\rho, \eta} := \mathcal{H}_{\kappa}^{\rho, \eta} \otimes_{\mathcal{B}_{\rho}} \mathcal{H}_{\kappa}^{\rho, \eta}$$

the relative tensor product as Hilbert C^* -modules over \mathcal{B}_{ρ} , the left $\mathcal{H}_{\kappa}^{\rho, \eta}$ is a right \mathcal{B}_{ρ} -module through φ_{ρ} and the right $\mathcal{H}_{\kappa}^{\rho, \eta}$ is a left \mathcal{B}_{ρ} -module through ϕ_{ρ} . By a symmetric discussion to the above, $\mathcal{H}_{\kappa}^{\rho, \eta} \otimes_{\rho} \mathcal{H}_{\kappa}^{\rho, \eta}$ is a Hilbert C^* -quad module over $(\mathcal{A}; \mathcal{B}_{\rho}, \mathcal{B}_{\eta})$. The following lemma is routine.

Lemma 4.1. *Let $\mathcal{H}_i = \mathcal{H}_{\kappa}^{\rho, \eta}, i = 1, 2, 3$. The correspondences*

$$\begin{aligned}(\xi_1 \otimes_{\eta} \xi_2) \otimes_{\rho} \xi_3 &\in (\mathcal{H}_1 \otimes_{\eta} \mathcal{H}_2) \otimes_{\rho} \mathcal{H}_3 \longrightarrow \xi_1 \otimes_{\eta} (\xi_2 \otimes_{\rho} \xi_3) \in \mathcal{H}_1 \otimes_{\eta} (\mathcal{H}_2 \otimes_{\rho} \mathcal{H}_3), \\ (\xi_1 \otimes_{\rho} \xi_2) \otimes_{\eta} \xi_3 &\in (\mathcal{H}_1 \otimes_{\rho} \mathcal{H}_2) \otimes_{\eta} \mathcal{H}_3 \longrightarrow \xi_1 \otimes_{\rho} (\xi_2 \otimes_{\eta} \xi_3) \in \mathcal{H}_1 \otimes_{\rho} (\mathcal{H}_2 \otimes_{\eta} \mathcal{H}_3)\end{aligned}$$

yield isomorphisms of Hilbert C^ -quad modules respectively.*

We write the isomorphism class of the former Hilbert C^* -quad modules as $\mathcal{H}_1 \otimes_{\eta} \mathcal{H}_2 \otimes_{\rho} \mathcal{H}_3$ and that of the latter ones as $\mathcal{H}_1 \otimes_{\rho} \mathcal{H}_2 \otimes_{\eta} \mathcal{H}_3$ respectively.

We note that the direct sum $\mathcal{B}_{\eta} \oplus \mathcal{B}_{\rho}$ has a structure of a Hilbert C^* -quad module by the following operations: For $b_1 \oplus b_2, b'_1 \oplus b'_2 \in \mathcal{B}_{\eta} \oplus \mathcal{B}_{\rho}$ and $y \in \mathcal{A}$, set

$$\begin{aligned}(b_1 \oplus b_2) \varphi_{\mathcal{A}}(y) &:= b_1 \psi_{\eta}(y) \oplus b_2 \psi_{\rho}(y) \in \mathcal{B}_{\eta} \oplus \mathcal{B}_{\rho}, \\ \langle b_1 \oplus b_2 \mid b'_1 \oplus b'_2 \rangle_{\mathcal{A}} &:= \lambda_{\eta}(b_1^* b'_1) + \lambda_{\rho}(b_2^* b'_2) \in \mathcal{A}.\end{aligned}$$

It is direct to see

$$\langle b_1 \oplus b_2 \mid (b'_1 \oplus b'_2) \varphi_{\mathcal{A}}(y) \rangle_{\mathcal{A}} = \langle b_1 \oplus b_2 \mid b'_1 \oplus b'_2 \rangle_{\mathcal{A}} \cdot y$$

so that $\mathcal{B}_\eta \oplus \mathcal{B}_\rho$ is a Hilbert C^* -right module over \mathcal{A} . Its C^* -bimodule structures over \mathcal{B}_ρ and over \mathcal{B}_η are defined as follows: For $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$, set:

1. The right \mathcal{B}_ρ -action φ_ρ and the right \mathcal{B}_η -action φ_η :

$$(b_1 \oplus b_2)\varphi_\rho(w) = b_2w, \quad (b_1 \oplus b_2)\varphi_\eta(z) = b_1z.$$

2. The left \mathcal{B}_ρ -action ϕ_ρ and the left \mathcal{B}_η -action ϕ_η :

$$\phi_\rho(w)(b_1 \oplus b_2) = wb_2, \quad \phi_\eta(z)(b_1 \oplus b_2) = zb_1.$$

Let us define the Fock Hilbert C^* -quad module as a two-dimensional analogue of the Fock space of Hilbert C^* -bimodules. Put $\Gamma_0 = \{\emptyset\}$, $\Gamma_n = \{(\pi_1, \dots, \pi_n) \mid \pi_i = \eta, \rho\}, n = 1, 2, \dots$. We set

$$\begin{aligned} F_0(\kappa) &= \mathcal{B}_\eta \oplus \mathcal{B}_\rho, & F_1(\kappa) &= \mathcal{H}_\kappa^{\rho, \eta}, \\ F_2(\kappa) &= (\mathcal{H}_\kappa^{\rho, \eta} \otimes_\eta \mathcal{H}_\kappa^{\rho, \eta}) \oplus (\mathcal{H}_\kappa^{\rho, \eta} \otimes_\rho \mathcal{H}_\kappa^{\rho, \eta}), \\ F_3(\kappa) &= (\mathcal{H}_\kappa^{\rho, \eta} \otimes_\eta \mathcal{H}_\kappa^{\rho, \eta} \otimes_\eta \mathcal{H}_\kappa^{\rho, \eta}) \oplus (\mathcal{H}_\kappa^{\rho, \eta} \otimes_\eta \mathcal{H}_\kappa^{\rho, \eta} \otimes_\rho \mathcal{H}_\kappa^{\rho, \eta}) \\ &\quad \oplus (\mathcal{H}_\kappa^{\rho, \eta} \otimes_\rho \mathcal{H}_\kappa^{\rho, \eta} \otimes_\eta \mathcal{H}_\kappa^{\rho, \eta}) \oplus (\mathcal{H}_\kappa^{\rho, \eta} \otimes_\rho \mathcal{H}_\kappa^{\rho, \eta} \otimes_\rho \mathcal{H}_\kappa^{\rho, \eta}), \\ &\dots\dots\dots \\ F_n(\kappa) &= \bigoplus_{(\pi_1, \dots, \pi_{n-1}) \in \Gamma_{n-1}} \mathcal{H}_\kappa^{\rho, \eta} \otimes_{\pi_1} \mathcal{H}_\kappa^{\rho, \eta} \otimes_{\pi_2} \dots \otimes_{\pi_{n-1}} \mathcal{H}_\kappa^{\rho, \eta} \\ &\dots\dots\dots \end{aligned}$$

as Hilbert C^* -bimodules over \mathcal{A} . We will define the Fock Hilbert C^* -bimodule F_κ over \mathcal{A} by setting

$$F_\kappa := \overline{\bigoplus_{n=0}^\infty F_n(\kappa)}$$

which is the completion of the algebraic direct sum $\bigoplus_{n=0}^\infty F_n(\kappa)$ of the Hilbert C^* -bimodules over \mathcal{A} under the norm $\|\xi\|_\mathcal{A}$ on $\bigoplus_{n=0}^\infty F_n(\kappa)$ induced by the \mathcal{A} -valued inner product $\langle \cdot \mid \cdot \rangle_\mathcal{A}$ on $F_n(\kappa), n = 0, 1, 2, \dots$.

For $\xi \in \mathcal{H}_\kappa^{\rho, \eta}$, we define operators s_ξ and t_ξ from $F_0(\kappa)$ to $F_1(\kappa)$ by

$$s_\xi(b_1 \oplus b_2) = \xi\varphi_\eta(b_1), \quad t_\xi(b_1 \oplus b_2) = \xi\varphi_\rho(b_2)$$

for $b_1 \oplus b_2 \in \mathcal{B}_\eta \oplus \mathcal{B}_\rho$.

Lemma 4.2. *For $\xi \in \mathcal{H}_\kappa^{\rho, \eta}$, we have*

- (i) s_ξ is a right \mathcal{B}_η -module map from $F_0(\kappa)$ to $F_1(\kappa)$.
- (ii) t_ξ is a right \mathcal{B}_ρ -module map from $F_0(\kappa)$ to $F_1(\kappa)$.
- (iii) Both the maps $s_\xi, t_\xi : F_0(\kappa) \longrightarrow F_1(\kappa)$ are right \mathcal{A} -module maps.

Proof. For $b_1 \oplus b_2 \in \mathcal{B}_\eta \oplus \mathcal{B}_\rho$ and $z \in \mathcal{B}_\eta$, we have

$$s_\xi((b_1 \oplus b_2)\varphi_\eta(z)) = s_\xi(b_1z) = \xi\varphi_\eta(b_1z) = (s_\xi(b_1 \oplus b_2))\varphi_\eta(z).$$

Hence s_ξ is a right \mathcal{B}_η -module map and similarly t_ξ is a right \mathcal{B}_ρ -module map. For $y \in \mathcal{A}$, by [Lemma 3.6](#), we have

$$s_\xi((b_1 \oplus b_2)\varphi_\mathcal{A}(y)) = \xi\varphi_\eta(b_1\psi_\eta(y)) = (\xi\varphi_\eta(b_1))\varphi_\mathcal{A}(y) = (s_\xi(b_1 \oplus b_2))\varphi_\mathcal{A}(y).$$

Hence s_ξ and similarly t_ξ are right \mathcal{A} -module maps. \square

For $\xi \in \mathcal{H}_\kappa^{\rho,\eta}$, denote by $s_\xi^*, t_\xi^* : F_1(\kappa) \longrightarrow F_0(\kappa)$ the adjoints of $s_\xi, t_\xi : F_0(\kappa) \longrightarrow F_1(\kappa)$ with respect to the right \mathcal{A} -valued inner products on $F_0(\kappa)$ and $F_1(\kappa)$.

Lemma 4.3. For $\xi, \xi' \in \mathcal{H}_\kappa^{\rho,\eta}$, we have

- (i) $s_\xi^* \xi' = \langle \xi \mid \xi' \rangle_\eta \oplus 0$ in $\mathcal{B}_\eta \oplus \mathcal{B}_\rho$.
- (ii) $t_\xi^* \xi' = 0 \oplus \langle \xi \mid \xi' \rangle_\rho$ in $\mathcal{B}_\eta \oplus \mathcal{B}_\rho$.

Proof. For $b_1 \oplus b_2 \in \mathcal{B}_\eta \oplus \mathcal{B}_\rho$, we have

$$\begin{aligned} & \langle b_1 \oplus b_2 \mid s_\xi^* \xi' \rangle_{\mathcal{A}} \\ &= \langle \xi \varphi_\eta(b_1) \mid \xi' \rangle_{\mathcal{A}} = \lambda_\eta(b_1^* \langle \xi \mid \xi' \rangle_\eta) = \langle b_1 \oplus b_2 \mid \langle \xi \mid \xi' \rangle_\eta \oplus 0 \rangle_{\mathcal{A}} \end{aligned}$$

so that $s_\xi^* \xi' = \langle \xi \mid \xi' \rangle_\eta \oplus 0$.

(ii) is similar to (i). \square

For $\xi \in \mathcal{H}_\kappa^{\rho,\eta}$ and $\xi_1 \otimes_{\pi_1} \cdots \otimes_{\pi_{n-1}} \xi_n \in F_n(\kappa)$ with $(\pi_1, \dots, \pi_{n-1}) \in \Gamma_{n-1}, n = 1, 2, \dots$, set

$$s_\xi(\xi_1 \otimes_{\pi_1} \cdots \otimes_{\pi_{n-1}} \xi_n) = \xi \otimes_\eta \xi_1 \otimes_{\pi_1} \cdots \otimes_{\pi_{n-1}} \xi_n, \quad (4.1)$$

$$t_\xi(\xi_1 \otimes_{\pi_1} \cdots \otimes_{\pi_{n-1}} \xi_n) = \xi \otimes_\rho \xi_1 \otimes_{\pi_1} \cdots \otimes_{\pi_{n-1}} \xi_n. \quad (4.2)$$

The following lemma is direct.

Lemma 4.4. For $\xi \in \mathcal{H}_\kappa^{\rho,\eta}$ and $n = 1, 2, \dots$, we have

- (i) s_ξ is a right \mathcal{B}_η -module map from $F_n(\kappa)$ to $F_{n+1}(\kappa)$.
- (ii) t_ξ is a right \mathcal{B}_ρ -module map from $F_n(\kappa)$ to $F_{n+1}(\kappa)$.
- (iii) Both the maps $s_\xi, t_\xi : F_n(\kappa) \longrightarrow F_{n+1}(\kappa)$ are right \mathcal{A} -module maps.

Denote by $s_\xi^*, t_\xi^* : F_{n+1}(\kappa) \longrightarrow F_n(\kappa)$ the adjoints of $s_\xi, t_\xi : F_n(\kappa) \longrightarrow F_{n+1}(\kappa)$ with respect to the right \mathcal{A} -valued inner products on $F_n(\kappa)$ and $F_{n+1}(\kappa)$ with $n = 1, 2, \dots$.

Lemma 4.5. For $\xi \in \mathcal{H}_\kappa^{\rho,\eta}$ and $\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_n} \xi_{n+1} \in F_{n+1}(\kappa)$ with $n = 1, 2, \dots$, we have

- (i) $s_\xi^*(\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_n} \xi_{n+1}) = \begin{cases} \phi_\eta(\langle \xi \mid \xi_1 \rangle_\eta) \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_n} \xi_{n+1} & \text{if } \pi_1 = \eta, \\ 0 & \text{if } \pi_1 = \rho, \end{cases}$
- (ii) $t_\xi^*(\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_n} \xi_{n+1}) = \begin{cases} \phi_\rho(\langle \xi \mid \xi_1 \rangle_\rho) \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_n} \xi_{n+1} & \text{if } \pi_1 = \rho, \\ 0 & \text{if } \pi_1 = \eta. \end{cases}$

Proof. (i) Let $\gamma = \eta$ or ρ . For $\zeta_1 \otimes_{\theta_1} \zeta_2 \otimes_{\theta_2} \cdots \otimes_{\theta_{n-1}} \zeta_n \in F_n(\kappa)$ with $n = 1, 2, \dots$, we have

$$\begin{aligned} & \langle \zeta_1 \otimes_{\theta_1} \zeta_2 \otimes_{\theta_2} \cdots \otimes_{\theta_{n-1}} \zeta_n \mid s_\xi^*(\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_n} \xi_{n+1}) \rangle_\gamma \\ &= \langle s_\xi(\zeta_1 \otimes_{\theta_1} \zeta_2 \otimes_{\theta_2} \cdots \otimes_{\theta_{n-1}} \zeta_n) \mid \xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_n} \xi_{n+1} \rangle_\gamma \\ &= \langle \xi \otimes_\eta \zeta_1 \otimes_{\theta_1} \zeta_2 \otimes_{\theta_2} \cdots \otimes_{\theta_{n-1}} \zeta_n \mid \xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_n} \xi_{n+1} \rangle_\gamma \\ &= \begin{cases} \langle \zeta_1 \otimes_{\theta_1} \zeta_2 \otimes_{\theta_2} \cdots \otimes_{\theta_{n-1}} \zeta_n \mid \phi_\eta(\langle \xi \mid \xi_1 \rangle_\eta) \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_n} \xi_{n+1} \rangle_\gamma & \text{if } \pi_1 = \eta, \\ 0 & \text{if } \pi_1 = \rho. \end{cases} \end{aligned}$$

Hence the desired formulae hold with respect to the inner products $\langle \cdot | \cdot \rangle_\eta, \langle \cdot | \cdot \rangle_\rho$ on $F_n(\kappa)$, $n = 1, 2, \dots$ and hence to the \mathcal{A} -valued inner product $\langle \cdot | \cdot \rangle_{\mathcal{A}}$ because of the equality

$$\langle \cdot | \cdot \rangle_{\mathcal{A}} = \lambda_\eta(\langle \cdot | \cdot \rangle_\eta) = \lambda_\rho(\langle \cdot | \cdot \rangle_\rho). \quad \square$$

We have shown in the proof of the above lemma that the adjoints of $s_\xi, t_\xi : F_n(\kappa) \longrightarrow F_{n+1}(\kappa)$ with respect to the other two inner products $\langle \cdot | \cdot \rangle_\eta, \langle \cdot | \cdot \rangle_\rho$ have the same form as above. We denote by $\bar{\varphi}_\rho, \bar{\varphi}_\eta, \bar{\phi}_\rho, \bar{\phi}_\eta$ the right \mathcal{B}_ρ -action, the right \mathcal{B}_η -action, the left \mathcal{B}_ρ -action, the left \mathcal{B}_η -action on $F_n(\kappa)$ and hence on F_κ respectively. The left actions $\bar{\phi}_\rho$ of \mathcal{B}_ρ and $\bar{\phi}_\eta$ of \mathcal{B}_η satisfy the following equalities

$$\begin{aligned} \bar{\phi}_\rho(w)(b_1 \oplus b_2) &= wb_2, & \bar{\phi}_\eta(z)(b_1 \oplus b_2) &= zb_1, \\ \bar{\phi}_\rho(w)(\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n) &= (\phi_\rho(w)\xi_1) \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n, \\ \bar{\phi}_\eta(z)(\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n) &= (\phi_\eta(z)\xi_1) \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n \end{aligned}$$

for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta, b_1 \oplus b_2 \in \mathcal{B}_\eta \oplus \mathcal{B}_\rho$ and $\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n \in F_n(\kappa)$. The following lemma is direct.

Lemma 4.6. *For $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$, we have for $n = 0, 1, \dots$*

- (i) $\bar{\phi}_\rho(w)$ is a right \mathcal{B}_ρ -module map from $F_n(\kappa)$ to $F_n(\kappa)$.
- (ii) $\bar{\phi}_\eta(z)$ is a right \mathcal{B}_η -module map from $F_n(\kappa)$ to $F_n(\kappa)$.
- (iii) Both the maps $\bar{\phi}_\rho(w), \bar{\phi}_\eta(z)$ are right \mathcal{A} -module maps on F_κ .

Lemma 4.7. *For $\xi \in \mathcal{H}_\kappa^{\rho, \eta}, w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$, we have*

- (i) $t_{\xi\varphi_\rho(w)} = t_\xi \bar{\phi}_\rho(w)$ and hence $t_\xi = \sum_{a \in \Sigma^\eta} t_{v_a} \bar{\phi}_\rho(\langle v_a | \xi \rangle_\rho)$.
- (ii) $s_{\xi\varphi_\eta(z)} = s_\xi \bar{\phi}_\eta(z)$ and hence $s_\xi = \sum_{\alpha \in \Sigma^\rho} s_{u_\alpha} \bar{\phi}_\eta(\langle u_\alpha | \xi \rangle_\eta)$.

Proof. (i) We have

$$\begin{aligned} t_{\xi\varphi_\rho(w)}(\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n) &= \xi\varphi_\rho(w) \otimes_\rho \xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n \\ &= \xi \otimes_\rho (\phi_\rho(w)\xi_1) \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n \\ &= (t_\xi \bar{\phi}_\rho(w))\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n. \end{aligned}$$

(ii) is similar to (i). \square

By Lemma 3.6, we have $\varphi_\eta(\psi_\eta(y)) = \varphi_\rho(\psi_\rho(y)) = \varphi_{\mathcal{A}}(y)$ for $y \in \mathcal{A}$, the above lemma implies the equalities:

$$t_{\xi\varphi_{\mathcal{A}}(y)} = t_\xi \bar{\phi}_\rho(\psi_\rho(y)) \quad \text{and} \quad s_{\xi\varphi_{\mathcal{A}}(y)} = s_\xi \bar{\phi}_\eta(\psi_\eta(y)) \quad \text{for } y \in \mathcal{A}. \quad (4.3)$$

The following lemma is immediate.

Lemma 4.8. *For $\xi \in \mathcal{H}_\kappa^{\rho, \eta}, w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$, we have*

- (i) $\bar{\phi}_\rho(w)s_\xi = s_{\phi_\rho(w)\xi}$ and $\bar{\phi}_\rho(w)t_\xi = t_{\phi_\rho(w)\xi}$.
- (ii) $\bar{\phi}_\eta(z)s_\xi = s_{\phi_\eta(z)\xi}$ and $\bar{\phi}_\eta(z)t_\xi = t_{\phi_\eta(z)\xi}$.

We set

$$s_\alpha = s_{u_\alpha} \quad \text{for } \alpha \in \Sigma^\rho \quad \text{and} \quad t_a = t_{v_a} \quad \text{for } a \in \Sigma^\eta. \quad (4.4)$$

By Lemma 3.9 and Lemma 4.7, we have for $\xi \in \mathcal{H}_\kappa^{\rho,\eta}$

$$s_\xi = \sum_{\alpha \in \Sigma^\rho} s_\alpha \bar{\phi}_\eta(\langle u_\alpha | \xi \rangle_\eta), \quad t_\xi = \sum_{a \in \Sigma^\eta} t_a \bar{\phi}_\rho(\langle v_a | \xi \rangle_\rho). \quad (4.5)$$

Let P_n be the projection on F_κ onto $F_n(\kappa)$ for $n = 0, 1, \dots$. We also define two projections on F_κ by

$$P_\rho = \text{the projection onto } \sum_{n=0}^{\infty} \sum_{(\pi_1, \dots, \pi_n) \in \Gamma_n} \mathcal{H}_\kappa^{\rho,\eta} \otimes_\eta \mathcal{H}_\kappa^{\rho,\eta} \otimes_{\pi_1} \mathcal{H}_\kappa^{\rho,\eta} \otimes_{\pi_2} \cdots \otimes_{\pi_n} \mathcal{H}_\kappa^{\rho,\eta},$$

$$P_\eta = \text{the projection onto } \sum_{n=0}^{\infty} \sum_{(\pi_1, \dots, \pi_n) \in \Gamma_n} \mathcal{H}_\kappa^{\rho,\eta} \otimes_\rho \mathcal{H}_\kappa^{\rho,\eta} \otimes_{\pi_1} \mathcal{H}_\kappa^{\rho,\eta} \otimes_{\pi_2} \cdots \otimes_{\pi_n} \mathcal{H}_\kappa^{\rho,\eta}.$$

Lemma 4.9. *Keep the above notations.*

$$\sum_{\alpha \in \Sigma^\rho} s_\alpha s_\alpha^* = P_1 + P_\rho \quad \text{and} \quad \sum_{a \in \Sigma^\eta} t_a t_a^* = P_1 + P_\eta.$$

Hence

$$\sum_{\alpha \in \Sigma^\rho} s_\alpha s_\alpha^* + \sum_{a \in \Sigma^\eta} t_a t_a^* + P_0 = 1_{F_\kappa} + P_1. \quad (4.6)$$

Proof. For $\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n \in F_n(\kappa)$ with $2 \leq n \in \mathbb{N}$, we have

$$s_\alpha s_\alpha^*(\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n) = \begin{cases} u_\alpha \otimes_\eta \phi_\eta(\langle u_\alpha | \xi_1 \rangle_\eta) \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n & \text{if } \pi_1 = \eta, \\ 0 & \text{if } \pi_1 = \rho. \end{cases}$$

As $u_\alpha \otimes_\eta \phi_\eta(\langle u_\alpha | \xi_1 \rangle_\eta) \xi_2 = u_\alpha \varphi_\eta(\langle u_\alpha | \xi_1 \rangle_\eta) \otimes_\eta \xi_2$ and $\sum_{\alpha \in \Sigma^\rho} u_\alpha \varphi_\eta(\langle u_\alpha | \xi_1 \rangle_\eta) = \xi_1$, we have

$$\sum_{\alpha \in \Sigma^\rho} s_\alpha s_\alpha^*(\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n) = \begin{cases} \xi_1 \otimes_\eta \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n & \text{if } \pi_1 = \eta, \\ 0 & \text{if } \pi_1 = \rho. \end{cases}$$

Hence we have

$$\sum_{\alpha \in \Sigma^\rho} s_\alpha s_\alpha^*|_{\oplus_{n=2}^\infty F_n(\kappa)} = P_\rho|_{\oplus_{n=2}^\infty F_n(\kappa)}.$$

For $\xi \in F_1(\kappa) = \mathcal{H}_\kappa^{\rho,\eta}$, we have $s_\alpha s_\alpha^* \xi = s_\alpha(\langle u_\alpha | \xi \rangle_\eta \oplus 0) = u_\alpha \varphi_\eta(\langle u_\alpha | \xi \rangle_\eta)$ so that

$$\sum_{\alpha \in \Sigma^\rho} s_\alpha s_\alpha^* \xi = \sum_{\alpha \in \Sigma^\rho} u_\alpha \varphi_\eta(\langle u_\alpha | \xi \rangle_\eta) = \xi.$$

Hence we have

$$\sum_{\alpha \in \Sigma^\rho} s_\alpha s_\alpha^*|_{F_1(\kappa)} = 1_{F_1(\kappa)}.$$

As $s_\alpha s_\alpha^*(b_1 \oplus b_2) = 0$ for $b_1 \oplus b_2 \in \mathcal{B}_\eta \oplus \mathcal{B}_\rho$, we have

$$\sum_{\alpha \in \Sigma^\rho} s_\alpha s_\alpha^*|_{F_0(\kappa)} = 0.$$

Therefore we conclude that

$$\sum_{\alpha \in \Sigma^\rho} s_\alpha s_\alpha^* = P_\rho + P_1 \quad \text{and similarly} \quad \sum_{a \in \Sigma^\eta} t_a t_a^* = P_\eta + P_1.$$

As $P_\eta + P_\rho + P_0 + P_1 = 1_{F_\kappa}$, one has

$$\sum_{\alpha \in \Sigma^\rho} s_\alpha s_\alpha^* + \sum_{a \in \Sigma^\eta} t_a t_a^* + P_0 = 1_{F_\kappa} + P_1. \quad \square$$

Lemma 4.10. $s_\zeta^* s_\xi = \bar{\phi}_\eta(\langle \zeta | \xi \rangle_\eta)$ and $t_\zeta^* t_\xi = \bar{\phi}_\rho(\langle \zeta | \xi \rangle_\rho)$ for $\zeta, \xi \in \mathcal{H}_\kappa^{\rho, \eta}$.

Proof. The equalities for $\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n \in F_n(\kappa)$, $n = 1, 2, \dots$

$$\begin{aligned} s_\zeta^* s_\xi(\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n) &= s_\zeta^*(\xi \otimes_\eta \xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n) \\ &= \phi_\eta(\langle \zeta | \xi \rangle_\eta) \xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n \end{aligned}$$

hold so that $s_\zeta^* s_\xi = \bar{\phi}_\eta(\langle \zeta | \xi \rangle_\eta)$ on $\bigoplus_{n=1}^\infty F_n(\kappa)$. As

$$\begin{aligned} s_\zeta^* s_\xi(b_1 \oplus b_2) &= s_\zeta^*(\xi \varphi_\eta(b_1)) = \langle \zeta | \xi \varphi_\eta(b_1) \rangle_\eta \oplus 0 \\ &= \langle \zeta | \xi \rangle_\eta b_1 \oplus 0 = \phi_\eta(\langle \zeta | \xi \rangle_\eta)(b_1 \oplus b_2), \end{aligned}$$

we have $s_\zeta^* s_\xi = \phi_\eta(\langle \zeta | \xi \rangle_\eta)$ on $F_0(\kappa)$. Hence $s_\zeta^* s_\xi = \bar{\phi}_\eta(\langle \zeta | \xi \rangle_\eta)$ on F_κ and similarly $t_\zeta^* t_\xi = \bar{\phi}_\rho(\langle \zeta | \xi \rangle_\rho)$. \square

As $\bar{\phi}_\rho(y) = \bar{\phi}_\eta(y)$ on $F_n(\kappa)$, $n = 1, 2, 3, \dots$ for $y \in \mathcal{A}$, we write $\bar{\phi}_\rho(y) (= \bar{\phi}_\eta(y))$ as $\bar{\phi}(y)$ on $F_n(\kappa)$, $n = 1, 2, 3, \dots$ for $y \in \mathcal{A}$.

Lemma 4.11. For $\alpha \in \Sigma^\rho$, $a \in \Sigma^\eta$ and $y \in \mathcal{A}$, we have

$$\begin{aligned} s_\alpha \bar{\phi}(y) s_\alpha &= \bar{\phi}_\eta(\rho_\alpha(y)), & t_a \bar{\phi}(y) t_a &= \bar{\phi}_\rho(\eta_a(y)), \\ s_\alpha s_\alpha^* \bar{\phi}(y) &= \bar{\phi}(y) s_\alpha s_\alpha^*, & t_a t_a^* \bar{\phi}(y) &= \bar{\phi}(y) t_a t_a^*. \end{aligned}$$

Proof. The equalities on $F_n(\kappa)$, $n = 1, 2, \dots$

$$\begin{aligned} s_\alpha \bar{\phi}(y) s_\alpha(\xi_1 \otimes_{\pi_1} \cdots \otimes_{\pi_{n-1}} \xi_n) &= s_\alpha^*(\phi(y) u_\alpha \otimes_\eta \xi_1 \otimes_{\pi_1} \cdots \otimes_{\pi_{n-1}} \xi_n) \\ &= \phi_\eta(\langle u_\alpha | \phi(y) u_\alpha \rangle_\eta) \xi_1 \otimes_{\pi_1} \cdots \otimes_{\pi_{n-1}} \xi_n \\ &= \bar{\phi}_\eta(\rho_\alpha(y))(\xi_1 \otimes_{\pi_1} \cdots \otimes_{\pi_{n-1}} \xi_n) \end{aligned}$$

imply $s_\alpha \bar{\phi}(y) s_\alpha = \bar{\phi}_\eta(\rho_\alpha(y))$ on $F_n(\kappa)$, $n = 1, 2, \dots$. We have on $F_0(\kappa)$

$$\begin{aligned} s_\alpha \bar{\phi}(y) s_\alpha(b_1 \oplus b_2) &= \langle u_\alpha | \phi(y) u_\alpha \varphi_\eta(b_1) \rangle_\eta \oplus 0 \\ &= \langle u_\alpha | \varphi_\eta(\rho_\alpha(y) b_1) \rangle_\eta \oplus 0 \\ &= \langle u_\alpha | u_\alpha \rangle_\eta \rho_\alpha(y) b_1 \oplus 0 \\ &= \rho_\alpha(y) b_1 \oplus 0 \\ &= \phi_\eta(\rho_\alpha(y))(b_1 \oplus b_2) \end{aligned}$$

so that $s_\alpha^* \bar{\phi}(y) s_\alpha = \phi_\eta(\rho_\alpha(y))$ on $F_0(\kappa)$. Thus we have $s_\alpha^* \bar{\phi}(y) s_\alpha = \bar{\phi}_\eta(\rho_\alpha(y))$ on F_κ . We similarly have $t_a^* \bar{\phi}(y) t_a = \bar{\phi}_\rho(\eta_a(y))$.

We also have

$$\begin{aligned} & s_\alpha s_\alpha^* \bar{\phi}(y) (\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n) \\ &= \begin{cases} u_\alpha \varphi_\eta(\langle u_\alpha | \phi(y) \xi_1 \rangle_\eta) \otimes_\eta \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n & \text{if } \pi_1 = \eta, \\ 0 & \text{if } \pi_1 \neq \eta. \end{cases} \end{aligned}$$

Since we have

$$\begin{aligned} & u_\alpha \varphi_\eta(\langle u_\alpha | \phi(y) \xi_1 \rangle_\eta) \\ &= u_\alpha \varphi_\eta(\langle u_\alpha \varphi_\eta(\rho_\alpha(y^*)) | \xi_1 \rangle_\eta) = u_\alpha \varphi_\eta(\rho_\alpha(y) \langle u_\alpha | \xi_1 \rangle_\eta) = \phi(y) u_\alpha \varphi_\eta(\langle u_\alpha | \xi_1 \rangle_\eta), \end{aligned}$$

the equality $s_\alpha s_\alpha^* \bar{\phi}(y) = \bar{\phi}(y) s_\alpha s_\alpha^*$ on $F_n(\kappa)$, $n = 1, 2, \dots$ holds. For $b_1 \oplus b_2 \in F_0(\kappa)$, the equality $s_\alpha s_\alpha^* \bar{\phi}_\eta(y)(b_1 \oplus b_2) = \bar{\phi}(y) s_\alpha s_\alpha^*(b_1 \oplus b_2) = 0$ holds so that we conclude $s_\alpha s_\alpha^* \bar{\phi}_\eta(y) = \bar{\phi}(y) s_\alpha s_\alpha^*$ on F_κ and similarly $t_a t_a^* \bar{\phi}_\rho(y) = \bar{\phi}(y) t_a t_a^*$.

Lemma 4.12. For $\omega = (\alpha, b, a, \beta) \in \Sigma_\kappa$ and $y \in \mathcal{A}$, we have

$$t_a s_\beta t_b^* s_\alpha^* \bar{\phi}(y) = \bar{\phi}(y) t_a s_\beta t_b^* s_\alpha^*.$$

Proof. By the preceding lemma, we have

$$\begin{aligned} t_a s_\beta t_b^* s_\alpha^* \bar{\phi}(y) &= t_a s_\beta t_b^* t_b t_b^* s_\alpha^* \bar{\phi}(y) s_\alpha s_\alpha^* = t_a s_\beta \bar{\phi}(\eta_b(\rho_\alpha(y))) t_b^* s_\alpha^*, \\ \bar{\phi}(y) t_a s_\beta t_b^* s_\alpha^* &= t_a t_a^* \bar{\phi}(y) t_a s_\beta s_\beta^* t_b^* s_\alpha^* = t_a s_\beta \bar{\phi}(\rho_\beta(\eta_a(y))) t_b^* s_\alpha^*. \end{aligned}$$

The desired equality holds by (1.1). \square

Lemma 4.13. For $\alpha \in \Sigma^\rho$, $a \in \Sigma^\eta$ and $y \in \mathcal{A}$, we have

$$\bar{\phi}_\rho(S_\alpha y S_\alpha^*) P_\rho = s_\alpha \bar{\phi}_\rho(y) s_\alpha^*, \quad \bar{\phi}_\eta(T_a y T_a^*) P_\eta = t_a \bar{\phi}_\eta(y) t_a^*.$$

Proof. For $\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n \in F_n(\kappa)$, $n = 2, 3, \dots$, we have

$$\begin{aligned} & \bar{\phi}_\rho(S_\alpha y S_\alpha^*) P_\rho (\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n) \\ &= \begin{cases} (\phi_\rho(S_\alpha y S_\alpha^*) \xi_1) \otimes_\eta \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n & \text{if } \pi_1 = \eta, \\ 0 & \text{if } \pi_1 = \rho. \end{cases} \end{aligned}$$

For $\xi_1 = \sum_{\omega \in \Sigma_\kappa} e_\omega \otimes E_\omega x_\omega$ with $x_\omega \in \mathcal{A}$, we have

$$\phi_\rho(S_\alpha y S_\alpha^*) \xi_1 = \sum_{\omega \in \Sigma_\kappa, t(\omega) = \alpha} e_\omega \otimes E_\omega \eta_{r(\omega)}(y) x_\omega.$$

On the other hand, we have

$$\begin{aligned} & s_\alpha \bar{\phi}_\rho(y) s_\alpha^* (\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n) \\ &= \begin{cases} s_\alpha (\phi(y) \phi_\eta(\langle u_\alpha | \xi_1 \rangle_\eta) \xi_2) \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n & \text{if } \pi_1 = \eta, \\ 0 & \text{if } \pi_1 = \rho. \end{cases} \end{aligned}$$

As $\langle u_\alpha \mid \xi_1 \rangle_\eta = \sum_{\omega \in \Sigma_\kappa, \alpha=t(\omega)} T_{r(\omega)} E_\omega x_\omega T_{r(\omega)}^*$, we have

$$\begin{aligned} s_\alpha(\phi(y)(\phi_\eta(\langle u_\alpha \mid \xi_1 \rangle_\eta) \xi_2)) &= s_\alpha \phi_\eta(y \langle u_\alpha \mid \xi_1 \rangle_\eta) \xi_2 \\ &= \sum_{\omega \in \Sigma_\kappa, t(\omega)=\alpha} u_\alpha \otimes_\eta \phi_\eta(T_{r(\omega)} \eta_{r(\omega)}(y) x_\omega T_{r(\omega)}^*) \xi_2 \\ &= \sum_{\omega \in \Sigma_\kappa, t(\omega)=\alpha} u_\alpha \varphi_\eta(T_{r(\omega)} \eta_{r(\omega)}(y) x_\omega T_{r(\omega)}^*) \otimes_\eta \xi_2 \\ &= \sum_{\omega \in \Sigma_\kappa, t(\omega)=\alpha} (e_\omega \otimes E_\omega \eta_{r(\omega)}(y) x_\omega) \otimes_\eta \xi_2. \end{aligned}$$

Hence we have $\bar{\phi}_\rho(S_\alpha y S_\alpha^*) P_\rho = s_\alpha \bar{\phi}_\rho(y) s_\alpha^*$ on $F_n(\kappa)$, $n = 2, 3, \dots$

For $\xi \in \mathcal{H}_\kappa^{\rho, \eta}$, we have $\bar{\phi}_\rho(y) s_\alpha^* \xi = \bar{\phi}_\rho(y)(\langle u_\alpha \mid \xi \rangle_\eta \oplus 0) = 0$ so that

$$s_\alpha \bar{\phi}_\rho(y) s_\alpha^* \xi = \bar{\phi}_\rho(S_\alpha y S_\alpha^*) P_\rho \xi = 0.$$

Therefore we have $\bar{\phi}_\rho(S_\alpha y S_\alpha^*) P_\rho = s_\alpha \bar{\phi}_\rho(y) s_\alpha^*$ on F_κ .

The other equality $\bar{\phi}_\eta(T_a y T_a^*) P_\eta = t_a \bar{\phi}_\eta(y) t_a^*$ is similarly shown. \square

Lemma 4.14. For $w \in \mathcal{B}_\rho$, $z \in \mathcal{B}_\eta$ and $\alpha \in \Sigma^\rho$, $a \in \Sigma^\eta$, we have

$$\begin{aligned} s_\alpha s_\alpha^* \bar{\phi}_\rho(w) &= \bar{\phi}_\rho(w) s_\alpha s_\alpha^*, & t_a t_a^* \bar{\phi}_\rho(w) &= \bar{\phi}_\rho(w) t_a t_a^*, \\ s_\alpha s_\alpha^* \bar{\phi}_\eta(z) &= \bar{\phi}_\eta(z) s_\alpha s_\alpha^*, & t_a t_a^* \bar{\phi}_\eta(z) &= \bar{\phi}_\eta(z) t_a t_a^* \end{aligned}$$

and hence

$$\begin{aligned} P_\rho \bar{\phi}_\rho(w) &= \bar{\phi}_\rho(w) P_\rho, & P_\eta \bar{\phi}_\rho(w) &= \bar{\phi}_\rho(w) P_\eta, \\ P_\rho \bar{\phi}_\eta(z) &= \bar{\phi}_\eta(z) P_\rho, & P_\eta \bar{\phi}_\eta(z) &= \bar{\phi}_\eta(z) P_\eta. \end{aligned}$$

Proof. For $\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n \in F_n(\kappa)$, we have

$$\begin{aligned} &\bar{\phi}_\rho(w) s_\alpha s_\alpha^* (\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n) \\ &= \begin{cases} \phi_\rho(w) u_\alpha \varphi_\eta(\langle u_\alpha \mid \xi_1 \rangle_\eta) \otimes_\eta \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n & \text{if } \pi_1 = \eta, \\ 0 & \text{if } \pi_1 = \rho \end{cases} \end{aligned}$$

and

$$\begin{aligned} &s_\alpha s_\alpha^* \bar{\phi}_\rho(w) (\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n) \\ &= \begin{cases} u_\alpha \varphi_\eta(\langle u_\alpha \mid \phi_\rho(w) \xi_1 \rangle_\eta) \otimes_\eta \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n & \text{if } \pi_1 = \eta, \\ 0 & \text{if } \pi_1 = \rho. \end{cases} \end{aligned}$$

Since $\phi_\rho(w) u_\alpha = u_\alpha \varphi_\eta(\langle u_\alpha \mid \phi_\rho(w) u_\alpha \rangle_\eta)$, we have

$$\begin{aligned} \phi_\rho(w) u_\alpha \varphi_\eta(\langle u_\alpha \mid \xi_1 \rangle_\eta) &= u_\alpha \varphi_\eta(\langle u_\alpha \mid \phi_\rho(w) u_\alpha \varphi_\eta(\langle u_\alpha \mid \xi_1 \rangle_\eta) \rangle_\eta) \\ &= u_\alpha \varphi_\eta(\langle u_\alpha \mid \phi_\rho(w) \sum_{\beta \in \Sigma^\rho} u_\beta \varphi_\eta(\langle u_\beta \mid \xi_1 \rangle_\eta) \rangle_\eta) \\ &= u_\alpha \varphi_\eta(\langle u_\alpha \mid \phi_\rho(w) \xi_1 \rangle_\eta). \end{aligned}$$

Hence $s_\alpha s_\alpha^* \bar{\phi}_\rho(w) = \bar{\phi}_\rho(w) s_\alpha s_\alpha^*$ holds. Similarly by $\phi_\eta(z) u_\alpha = u_\alpha \varphi_\eta(\langle u_\alpha \mid \phi_\eta(z) u_\alpha \rangle_\eta)$, the equality $\bar{\phi}_\eta(z) s_\alpha s_\alpha^* = s_\alpha s_\alpha^* \bar{\phi}_\eta(z)$ holds. As $P_\rho = \sum_{\alpha \in \Sigma^\rho} s_\alpha s_\alpha^*$ on $\oplus_{n=2}^\infty F_n(\kappa)$, we see that P_ρ commutes with $\bar{\phi}_\rho(w)$ and $\bar{\phi}_\eta(z)$.

The other four equalities on the right hand side are similarly shown. \square

Let us denote by $\mathcal{L}_\mathcal{A}(\mathcal{H}_\kappa^{\rho,\eta})$ and $\mathcal{L}_\mathcal{A}(F_\kappa)$ the C^* -algebras of all bounded adjointable right \mathcal{A} -module maps on $\mathcal{H}_\kappa^{\rho,\eta}$ and on F_κ with respect to the right \mathcal{A} -valued inner products respectively. For $L \in \mathcal{L}_\mathcal{A}(\mathcal{H}_\kappa^{\rho,\eta})$, define $\bar{L} \in \mathcal{L}_\mathcal{A}(F_\kappa)$ by

$$\begin{aligned} \bar{L}(b_1 \otimes b_2) &= 0 \quad \text{for } b_1 \oplus b_2 \in \mathcal{B}_\eta \oplus \mathcal{B}_\rho = F_0(\kappa), \\ \bar{L}(\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n) &= (L\xi_1) \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n \end{aligned}$$

for $\xi_1 \otimes_{\pi_1} \xi_2 \otimes_{\pi_2} \cdots \otimes_{\pi_{n-1}} \xi_n \in F_n(\kappa)$, $n = 1, 2, \dots$.

Lemma 4.15. For $L \in \mathcal{L}_\mathcal{A}(\mathcal{H}_\kappa^{\rho,\eta})$ and $\alpha \in \Sigma^\rho$, $a \in \Sigma^\eta$, we have

$$s_\alpha^* \bar{L} s_\alpha = \bar{\phi}_\eta(\langle u_\alpha \mid Lu_\alpha \rangle_\eta), \quad t_a^* \bar{L} t_a = \bar{\phi}_\rho(\langle v_a \mid Lv_a \rangle_\rho).$$

Proof. For $\xi_1 \otimes_{\pi_1} \cdots \otimes_{\pi_{n-1}} \xi_n \in F_n(\kappa)$, $n = 1, 2, \dots$, we have

$$\begin{aligned} s_\alpha^* \bar{L} s_\alpha (\xi_1 \otimes_{\pi_1} \cdots \otimes_{\pi_{n-1}} \xi_n) &= s_\alpha^* (Lu_\alpha \otimes_\eta \xi_1 \otimes_{\pi_1} \cdots \otimes_{\pi_{n-1}} \xi_n) \\ &= \bar{\phi}_\eta(\langle u_\alpha \mid Lu_\alpha \rangle_\eta) \xi_1 \otimes_{\pi_1} \cdots \otimes_{\pi_{n-1}} \xi_n. \end{aligned}$$

For $b_1 \oplus b_2 \in \mathcal{B}_\eta \oplus \mathcal{B}_\rho$, we have

$$s_\alpha^* \bar{L} s_\alpha (b_1 \oplus b_2) = s_\alpha^* L(u_\alpha \varphi_\eta(b_1)) = \langle u_\alpha \mid Lu_\alpha \varphi_\eta(b_1) \rangle_\eta \oplus 0 = \langle u_\alpha \mid Lu_\alpha \rangle_\eta b_1 \oplus 0$$

Since $\langle u_\alpha \mid Lu_\alpha \rangle_\eta b_1 \oplus 0 = \bar{\phi}_\eta(\langle u_\alpha \mid Lu_\alpha \rangle_\eta)(b_1 \oplus b_2)$ we have

$$s_\alpha^* \bar{L} s_\alpha = \bar{\phi}_\eta(\langle u_\alpha \mid Lu_\alpha \rangle_\eta) \quad \text{on } F_\kappa.$$

The other equality for $t_a^* \bar{L} t_a$ is similarly shown. \square

Lemma 4.16. For $w \in \mathcal{B}_\rho$, $z \in \mathcal{B}_\eta$ and $\alpha \in \Sigma^\rho$, $a \in \Sigma^\eta$, we have

$$\begin{aligned} s_\alpha^* \bar{\phi}_\rho(w) s_\alpha &= \bar{\phi}_\eta(\hat{\rho}_\alpha(w)) & t_a^* \bar{\phi}_\eta(z) t_a &= \bar{\phi}_\rho(\hat{\eta}_a(z)), \\ s_\alpha^* \bar{\phi}_\eta(z) s_\alpha &= \bar{\phi}_\eta(\hat{\rho}_\alpha^\eta(z)), & t_a^* \bar{\phi}_\rho(w) t_a &= \bar{\phi}_\rho(\hat{\eta}_a^\rho(w)). \end{aligned}$$

Proof. By Lemma 3.12, we have

$$\hat{\rho}_\alpha(w) = \langle u_\alpha \mid \phi_\rho(w) u_\alpha \rangle_\eta, \quad \hat{\rho}_\alpha^\eta(z) = \langle u_\alpha \mid \phi_\eta(z) u_\alpha \rangle_\eta.$$

Hence the preceding lemma implies

$$\begin{aligned} s_\alpha^* \bar{\phi}_\rho(w) s_\alpha &= \bar{\phi}_\eta(\langle u_\alpha \mid \phi_\rho(w) u_\alpha \rangle_\eta) = \bar{\phi}_\eta(\hat{\rho}_\alpha(w)), \\ s_\alpha^* \bar{\phi}_\eta(z) s_\alpha &= \bar{\phi}_\eta(\langle u_\alpha \mid \phi_\eta(z) u_\alpha \rangle_\eta) = \bar{\phi}_\eta(\hat{\rho}_\alpha^\eta(z)). \end{aligned}$$

The other two equalities on the right hand side are similarly shown. \square

Corollary 4.17. For $w = \sum_{\alpha \in \Sigma^\rho} S_\alpha w_\alpha S_\alpha^*$ as in (2.2) and $z = \sum_{a \in \Sigma^\eta} T_a z_a T_a^*$ as in (2.3), we have

$$\bar{\phi}_\rho(w) = \sum_{\alpha \in \Sigma^\rho} s_\alpha \bar{\phi}_\eta(w_\alpha) s_\alpha^* + P_\eta \bar{\phi}_\rho(w) P_\eta + P_0 \bar{\phi}_\rho(w) P_0, \quad (4.7)$$

$$\bar{\phi}_\eta(z) = \sum_{a \in \Sigma^\eta} t_a \bar{\phi}_\rho(z_a) t_a^* + P_\rho \bar{\phi}_\eta(z) P_\rho + P_0 \bar{\phi}_\eta(z) P_0. \quad (4.8)$$

Proof. As $P_\rho + P_\eta + P_1 + P_0 = 1$ on F_κ and $\sum_{\alpha \in \Sigma^\rho} s_\alpha s_\alpha^* = P_\rho + P_1$, one has

$$\bar{\phi}_\rho(w) = \sum_{\alpha \in \Sigma^\rho} s_\alpha s_\alpha^* \bar{\phi}_\rho(w) s_\alpha s_\alpha^* + \bar{\phi}_\rho(w) P_\eta + \bar{\phi}_\rho(w) P_0.$$

Since $s_\alpha^* \bar{\phi}_\rho(w) s_\alpha = \bar{\phi}_\eta(\hat{\rho}_\alpha(w)) = \bar{\phi}_\eta(w_\alpha)$, we have the equality (4.7). The equality (4.8) is similarly shown. \square

Lemma 4.18.

- (i) $\bar{\phi}_\rho : \mathcal{B}_\rho \longrightarrow \mathcal{L}_\mathcal{A}(F_\kappa)$ is a faithful $*$ -homomorphism.
- (ii) $\bar{\phi}_\eta : \mathcal{B}_\eta \longrightarrow \mathcal{L}_\mathcal{A}(F_\kappa)$ is a faithful $*$ -homomorphism.

Proof. (i) It is enough to show that $\phi_\rho : \mathcal{B}_\rho \longrightarrow \mathcal{L}_\mathcal{A}(\mathcal{H}_\kappa^{\rho,\eta})$ is injective. For $w = \sum_{\alpha \in \Sigma^\rho} S_\alpha w_\alpha S_\alpha^*$ as in (2.2), suppose that $\phi_\rho(w) = 0$ on $\mathcal{H}_\kappa^{\rho,\eta}$. By Lemma 3.12, we have $\hat{\rho}_\alpha(w) = 0$ for all $\alpha \in \Sigma^\rho$ so that $w_\alpha = 0$ for all $\alpha \in \Sigma^\rho$, which shows $w = 0$. (ii) is similar to (i). \square

5. The C^* -algebras associated to the Hilbert C^* -quad modules

In this section, we will study the C^* -algebras generated by the operators s_ξ, t_ξ for $\xi \in \mathcal{H}_\kappa^{\rho,\eta}$. For $\xi, \zeta \in F_\kappa$, denote by $\theta_{\xi,\zeta}$ the rank one operator on F_κ defined by

$$\theta_{\xi,\zeta}(\gamma) = \xi \varphi_\mathcal{A}(\langle \zeta | \gamma \rangle_\mathcal{A}) \quad \text{for } \gamma \in F_\kappa.$$

It is immediate to see that the operators $\theta_{\xi,\zeta}$ for $\xi, \zeta \in F_\kappa$ are \mathcal{A} -module maps through $\varphi_\mathcal{A}$. Let us denote by $\mathcal{K}_\mathcal{A}(F_\kappa)$ the C^* -subalgebra of $\mathcal{L}_\mathcal{A}(F_\kappa)$ generated by the rank one operators $\theta_{\xi,\zeta}$ for $\xi, \zeta \in F_\kappa$. Put the projections for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$

$$p_\alpha = S_\alpha S_\alpha^* \in \mathcal{B}_\rho \subset F_0(\kappa), \quad q_a = T_a T_a^* \in \mathcal{B}_\eta \subset F_0(\kappa).$$

They are regarded as vectors in F_κ .

Lemma 5.1. $\sum_{\alpha \in \Sigma^\rho} \theta_{p_\alpha, p_\alpha} + \sum_{a \in \Sigma^\eta} \theta_{q_a, q_a} = P_0$: the projection on F_κ onto $F_0(\kappa)$.

Proof. For $\alpha \in \Sigma^\rho$ and $b_2 \in \mathcal{B}_\rho$, we have

$$\begin{aligned} \theta_{p_\alpha, p_\alpha}(b_2) &= p_\alpha \varphi_\mathcal{A}(\langle p_\alpha | b_2 \rangle_\mathcal{A}) = p_\alpha \psi_\eta(\lambda_\eta(p_\alpha^* b_2)) \\ &= p_\alpha \sum_{\alpha', \beta' \in \Sigma^\rho} S_{\alpha'} S_{\beta'}^* p_\alpha b_2 S_{\beta'} S_{\alpha'}^* = p_\alpha b_2 \end{aligned}$$

so that $\sum_{\alpha \in \Sigma^\rho} \theta_{p_\alpha, p_\alpha}(b_2) = b_2$ for $b_2 \in \mathcal{B}_\rho$. Similarly we have $\theta_{q_a, q_a}(b_1) = q_a b_1$ for $q_a \in \mathcal{B}_\eta$ so that $\sum_{a \in \Sigma^\eta} \theta_{q_a, q_a}(b_1) = b_1$. As $\theta_{q_a, q_a}(b_2) = 0$ for $b_2 \in \mathcal{B}_\rho$, $\theta_{p_\alpha, p_\alpha}(b_1) = 0$ for $b_1 \in \mathcal{B}_\eta$ and

$$\theta_{p_\alpha, p_\alpha}(\xi) = \theta_{q_a, q_a}(\xi) = 0 \quad \text{for } \xi \in F_n(\kappa), n = 1, 2, \dots,$$

the operator $\sum_{\alpha \in \Sigma^\rho} \theta_{p_\alpha, p_\alpha} + \sum_{a \in \Sigma^\eta} \theta_{q_a, q_a}$ is the projection on F_κ onto $F_0(\kappa)$. \square

Put $\epsilon_\omega := e_\omega \otimes E_\omega \in \mathcal{H}_\kappa^{\rho,\eta}$ for $\omega \in \Sigma_\kappa$. Then we see

$$\langle \epsilon_\omega \mid \epsilon_{\omega'} \rangle_{\mathcal{A}} = \begin{cases} E_\omega & \text{if } \omega = \omega', \\ 0 & \text{if } \omega \neq \omega'. \end{cases}$$

Lemma 5.2. $\{\epsilon_\omega\}_{\omega \in \Sigma_\kappa}$ forms an orthogonal basis of $\mathcal{H}_\kappa^{\rho,\eta}$ with respect to the \mathcal{A} -valued inner product $\langle \cdot \mid \cdot \rangle_{\mathcal{A}}$ as a right \mathcal{A} -module through $\varphi_{\mathcal{A}}$.

Proof. For $\xi = \sum_{\omega' \in \Sigma_\kappa} e_{\omega'} \otimes E_{\omega'} x_{\omega'}$ with $x_{\omega'} \in \mathcal{A}$, one has

$$\langle \epsilon_\omega \mid \xi \rangle_{\mathcal{A}} = \sum_{\omega' \in \Sigma_\kappa} \langle \epsilon_\omega \mid \epsilon_{\omega'} \rangle_{\mathcal{A}} x_{\omega'} = E_\omega x_\omega$$

so that

$$\xi = \sum_{\omega \in \Sigma_\kappa} \epsilon_\omega \varphi_{\mathcal{A}}(E_\omega x_\omega) = \sum_{\omega \in \Sigma_\kappa} \epsilon_\omega \varphi_{\mathcal{A}}(\langle \epsilon_\omega \mid \xi \rangle_{\mathcal{A}}). \quad \square \quad (5.1)$$

Lemma 5.3. $\sum_{\omega \in \Sigma_\kappa} \theta_{\epsilon_\omega, \epsilon_\omega} = P_1$: the projection on F_κ onto $F_1(\kappa)$.

Proof. By (5.1), we have $\xi = \sum_{\omega \in \Sigma_\kappa} \theta_{\epsilon_\omega, \epsilon_\omega}(\xi)$ for $\xi \in \mathcal{H}_\kappa^{\rho,\eta}$. Since $\theta_{\epsilon_\omega, \epsilon_\omega}(\xi') = 0$ for $\xi' \in F_n(\kappa)$ with $n \neq 1$, we have $\sum_{\omega \in \Sigma_\kappa} \theta_{\epsilon_\omega, \epsilon_\omega} = P_1$. \square

By the preceding lemmas, we have

Corollary 5.4. $P_0, P_1 \in \mathcal{K}_{\mathcal{A}}(F_\kappa)$.

The C^* -subalgebra of $\mathcal{L}_{\mathcal{A}}(F_\kappa)$ generated by the operators s_ξ, t_ξ for $\xi \in \mathcal{H}_\kappa^{\rho,\eta}$ is denoted by $\mathcal{T}_{\mathcal{H}_\kappa^{\rho,\eta}}$ and is called the Toeplitz quad module algebra.

Definition. The C^* -algebra $\mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}}$ associated with the Hilbert C^* -quad module $\mathcal{H}_\kappa^{\rho,\eta}$ is defined as the quotient C^* -algebra of $\mathcal{T}_{\mathcal{H}_\kappa^{\rho,\eta}}$ by the ideal $\mathcal{T}_{\mathcal{H}_\kappa^{\rho,\eta}} \cap \mathcal{K}_{\mathcal{A}}(F_\kappa)$.

We set the quotients of the operators s_α, t_a in $\mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}}$ for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ as

$$U_\alpha := [s_\alpha] \in \mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}}, \quad V_a := [t_a] \in \mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}}.$$

Since

$$\sum_{\alpha \in \Sigma^\rho} s_\alpha s_\alpha^* + \sum_{a \in \Sigma^\eta} t_a t_a^* + P_0 = 1 + P_1$$

and $P_0, P_1 \in \mathcal{K}_{\mathcal{A}}(F_\kappa)$, we have

$$\sum_{\alpha \in \Sigma^\rho} U_\alpha U_\alpha^* + \sum_{a \in \Sigma^\eta} V_a V_a^* = 1. \quad (5.2)$$

By (4.6), we have for $w \in \mathcal{B}_\rho$

$$\bar{\phi}_\rho(w) = \sum_{\alpha \in \Sigma^\rho} \bar{\phi}_\rho(w) s_\alpha s_\alpha^* + \sum_{a \in \Sigma^\eta} \bar{\phi}_\rho(w) t_a t_a^* + \bar{\phi}_\rho(w) P_0 - \bar{\phi}_\rho(w) P_1.$$

As $\bar{\phi}_\rho(w) s_\alpha = s_{\phi_\rho(w) u_\alpha}$ and $\bar{\phi}_\rho(w) t_a = t_{\phi_\rho(w) v_a}$ by Lemma 4.8, the operators $\bar{\phi}_\rho(w)$ for $w \in \mathcal{B}_\rho$ and similarly $\bar{\phi}_\eta(z)$ for $z \in \mathcal{B}_\eta$ belong to $\mathcal{T}_{\mathcal{H}_\kappa^{\rho,\eta}}$ modulo $\mathcal{K}_{\mathcal{A}}(F_\kappa)$. Let $\Phi_\rho(w), \Phi_\eta(z) \in \mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}}$ denote the quotient images of

$\bar{\phi}_\rho(w), \bar{\phi}_\eta(z)$ for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$ in the quotient $\mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}} = \mathcal{T}_{\mathcal{H}_\kappa^{\rho,\eta}} / (\mathcal{T}_{\mathcal{H}_\kappa^{\rho,\eta}} \cap \mathcal{K}_{\mathcal{A}}(F_\kappa))$. The following lemma is clear by (4.5).

Lemma 5.5. *The C^* -algebra $\mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}}$ is generated by the partial isometries U_α, V_a for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ and the elements $\Phi_\rho(w), \Phi_\eta(z)$ for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$.*

We will show that the C^* -algebra $\mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}}$ has a universal property subject to the operator relations inherited from Lemma 4.14 and Lemma 4.16. The following proposition is direct.

Proposition 5.6. *The operators $\Phi_\rho(w), \Phi_\eta(z), U_\alpha, V_a$ for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta, \alpha \in \Sigma^\rho, a \in \Sigma^\eta$ satisfy the relations:*

$$\begin{aligned} \sum_{\alpha \in \Sigma^\rho} U_\alpha U_\alpha^* + \sum_{a \in \Sigma^\eta} V_a V_a^* &= 1, \\ U_\alpha U_\alpha^* \Phi_\rho(w) &= \Phi_\rho(w) U_\alpha U_\alpha^*, & V_a V_a^* \Phi_\rho(w) &= \Phi_\rho(w) V_a V_a^*, \\ U_\alpha U_\alpha^* \Phi_\eta(z) &= \Phi_\eta(z) U_\alpha U_\alpha^*, & V_a V_a^* \Phi_\eta(z) &= \Phi_\eta(z) V_a V_a^*, \\ \Phi_\rho(\widehat{\rho}_\alpha(w)) &= U_\alpha^* \Phi_\rho(w) U_\alpha, & \Phi_\eta(\widehat{\eta}_a(z)) &= V_a^* \Phi_\eta(z) V_a, \\ \Phi_\eta(\widehat{\rho}_\alpha^\eta(z)) &= U_\alpha^* \Phi_\eta(z) U_\alpha, & \Phi_\rho(\widehat{\eta}_a^\rho(w)) &= V_a^* \Phi_\rho(w) V_a, \\ \Phi_\rho(y) &= \Phi_\eta(y) \end{aligned}$$

for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta, \alpha \in \Sigma^\rho, a \in \Sigma^\eta, y \in \mathcal{A}$.

The ten relations above are called the relations $(\mathcal{H}_\kappa^{\rho,\eta})$. We will henceforth prove that the C^* -algebra $\mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}}$ has the universal property subject to the relations $(\mathcal{H}_\kappa^{\rho,\eta})$. Let \mathcal{B}_κ be the C^* -subalgebra of $\mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}}$ generated by the operators $\Phi_\rho(w), \Phi_\eta(z)$ for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$.

Lemma 5.7. *Assume that the algebra \mathcal{A} is commutative. Then \mathcal{B}_κ is commutative by the relations $(\mathcal{H}_\kappa^{\rho,\eta})$.*

Proof. As the algebra \mathcal{A} is commutative, the algebras \mathcal{B}_ρ and \mathcal{B}_η are both commutative by Lemma 3.8. Hence it is enough to prove that $\Phi_\rho(w)$ commutes with $\Phi_\eta(z)$ for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$. For $\alpha \in \Sigma^\rho$, it follows that

$$\Phi_\eta(z) \Phi_\rho(w) U_\alpha U_\alpha^* = U_\alpha \Phi_\eta(\widehat{\rho}_\alpha^\eta(z)) \Phi_\rho(\widehat{\rho}_\alpha(w)) U_\alpha^*.$$

As $\widehat{\rho}_\alpha(w) \in \mathcal{A}$, we have $\Phi_\rho(\widehat{\rho}_\alpha^\eta(w)) = \Phi_\eta(\widehat{\rho}_\alpha^\eta(w))$ so that

$$\begin{aligned} \Phi_\eta(\widehat{\rho}_\alpha^\eta(z)) \Phi_\rho(\widehat{\rho}_\alpha(w)) &= \Phi_\eta(\widehat{\rho}_\alpha^\eta(z) \widehat{\rho}_\alpha(w)) \\ \Phi_\rho(\widehat{\rho}_\alpha(w)) \Phi_\eta(\widehat{\rho}_\alpha^\eta(z)) &= \Phi_\eta(\widehat{\rho}_\alpha(w) \widehat{\rho}_\alpha^\eta(z)). \end{aligned}$$

Both the elements $\widehat{\rho}_\alpha^\eta(z), \widehat{\rho}_\alpha(w)$ belong to the commutative algebra \mathcal{B}_η , so that

$$\Phi_\eta(z) \Phi_\rho(w) U_\alpha U_\alpha^* = U_\alpha \Phi_\rho(\widehat{\rho}_\alpha(w)) \Phi_\eta(\widehat{\rho}_\alpha^\eta(z)) U_\alpha^* = \Phi_\rho(w) \Phi_\eta(z) U_\alpha U_\alpha^*.$$

Similarly we have

$$\Phi_\eta(z) \Phi_\rho(w) V_a V_a^* = \Phi_\rho(w) \Phi_\eta(z) V_a V_a^*$$

for $a \in \Sigma^\eta$. As $\sum_{\alpha \in \Sigma^\rho} U_\alpha U_\alpha^* + \sum_{a \in \Sigma^\eta} V_a V_a^* = 1$, one concludes that

$$\Phi_\eta(z) \Phi_\rho(w) = \Phi_\rho(w) \Phi_\eta(z). \quad \square$$

Put $\Sigma^{\rho \cup \eta} = \Sigma^\rho \cup \Sigma^\eta$. We set for $\gamma \in \Sigma^{\rho \cup \eta}$ and $X \in \mathcal{B}_\kappa$

$$\rho_\gamma^\kappa(X) = W_\gamma^* X W_\gamma \quad \text{where} \quad W_\gamma = \begin{cases} U_\alpha & \text{if } \gamma = \alpha \in \Sigma^\rho, \\ V_a & \text{if } \gamma = a \in \Sigma^\eta. \end{cases}$$

Since $U_\alpha^* \mathcal{B}_\kappa U_\alpha \subset \mathcal{B}_\kappa$ and $V_a^* \mathcal{B}_\kappa V_a \subset \mathcal{B}_\kappa$, we have

$$\rho_\gamma^\kappa(\mathcal{B}_\kappa) \subset \mathcal{B}_\kappa,$$

so that we have a family of endomorphisms $\rho_\gamma^\kappa, \gamma \in \Sigma^{\rho \cup \eta}$ on \mathcal{B}_κ . In what follows, we assume that the algebra \mathcal{A} is commutative, so that the algebras $\mathcal{B}_\rho, \mathcal{B}_\eta$ and \mathcal{B}_κ are all commutative.

Lemma 5.8. *The triplet $(\mathcal{B}_\kappa, \rho^\kappa, \Sigma^{\rho \cup \eta})$ is a C^* -symbolic dynamical system.*

Proof. Since $\sum_{\gamma \in \Sigma^{\rho \cup \eta}} W_\gamma W_\gamma^* = \sum_{\alpha \in \Sigma^\rho} U_\alpha U_\alpha^* + \sum_{a \in \Sigma^\eta} V_a V_a^* = 1$ and $W_\gamma W_\gamma^*$ commutes with \mathcal{B}_κ , the family $\rho_\gamma^\kappa, \gamma \in \Sigma^{\rho \cup \eta}$ yields endomorphisms on \mathcal{B}_κ . We have

$$\begin{aligned} \sum_{\gamma \in \Sigma^{\rho \cup \eta}} \rho_\gamma^\kappa(1) &= \sum_{\alpha \in \Sigma^\rho} U_\alpha^* U_\alpha + \sum_{a \in \Sigma^\eta} V_a^* V_a \\ &= \sum_{\alpha \in \Sigma^\rho} \Phi_\rho(\rho_\alpha(1)) + \sum_{a \in \Sigma^\eta} \Phi_\eta(\eta_a(1)) \\ &\geq \Phi_\rho(1) + \Phi_\eta(1) \geq 2. \end{aligned}$$

Hence $(\mathcal{B}_\kappa, \rho^\kappa, \Sigma^{\rho \cup \eta})$ is a C^* -symbolic dynamical system. \square

For $\mu = \mu_1 \cdots \mu_n \in B_n(\Sigma^{\rho \cup \eta})$ where $\mu_1, \dots, \mu_n \in \Sigma^{\rho \cup \eta}$, denote by

$$\begin{aligned} |\mu|_\rho &= \text{the number of symbols of } \Sigma^\rho \text{ appearing in the word } \mu_1 \cdots \mu_n, \\ |\mu|_\eta &= \text{the number of symbols of } \Sigma^\eta \text{ appearing in the word } \mu_1 \cdots \mu_n. \end{aligned}$$

Hence $|\mu|_\rho + |\mu|_\eta = n$. For $n \in \mathbb{Z}_+$, denote by \mathcal{F}_n the C^* -subalgebra of $\mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}}$ generated by the operators $W_{\gamma_1 \cdots \gamma_n} b W_{\gamma'_1 \cdots \gamma'_n}^*$ for $b \in \mathcal{B}_\kappa$ and $\gamma_1, \dots, \gamma_n, \gamma'_1, \dots, \gamma'_n \in \Sigma^{\rho \cup \eta}$ such that $|\gamma_1 \cdots \gamma_n|_\rho = |\gamma'_1 \cdots \gamma'_n|_\rho$ and $|\gamma_1 \cdots \gamma_n|_\eta = |\gamma'_1 \cdots \gamma'_n|_\eta$. Since $\sum_{\gamma \in \Sigma^{\rho \cup \eta}} W_\gamma W_\gamma^* = 1$ and $W_\gamma W_\gamma^*$ commutes with \mathcal{B}_κ , the equality for $b \in \mathcal{F}_n$

$$W_{\gamma_1 \cdots \gamma_n} b W_{\gamma'_1 \cdots \gamma'_n}^* = \sum_{\gamma_{n+1} \in \Sigma^{\rho \cup \eta}} W_{\gamma_1 \cdots \gamma_n} W_{\gamma_{n+1}} \rho_{\gamma_{n+1}}^\kappa(b) W_{\gamma_{n+1}}^* W_{\gamma'_1 \cdots \gamma'_n}^*$$

gives rise to an embedding $\mathcal{F}_n \hookrightarrow \mathcal{F}_{n+1}, n \in \mathbb{Z}_+$. Let $\mathcal{F}_{\mathcal{H}_\kappa^{\rho, \eta}}$ be the C^* -subalgebra of $\mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}}$ generated by $\cup_{n=0}^\infty \mathcal{F}_n$.

We will define a unitary $u_{(r_1, r_2)}$ for $(r_1, r_2) \in (\mathbb{R}/\mathbb{Z})^2 = \mathbb{T}^2$ on F_κ which yields gauge action g on $\mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}}$. For $(r_1, r_2) \in \mathbb{T}^2$, put

$$u_{(r_1, r_2)}(b_1 \oplus b_2) = e^{-2\pi i r_1} b_1 \oplus e^{-2\pi i r_2} b_2 \quad \text{for } b_1 \oplus b_2 \in \mathcal{B}_\eta \oplus \mathcal{B}_\rho = F_0(\kappa)$$

and

$$u_{(r_1, r_2)} \xi = \xi \quad \text{for } \xi \in \mathcal{H}_\kappa^{\rho, \eta} = F_1(\kappa).$$

For $(\pi_1, \dots, \pi_{n-1}) \in \Gamma_{n-1}$, put

$$k = |(\pi_1, \dots, \pi_{n-1})|_\eta, \quad l = |(\pi_1, \dots, \pi_{n-1})|_\rho.$$

Define a unitary $u_{(r_1, r_2)}$ on $\mathcal{H}_\kappa^{\rho, \eta} \otimes_{\pi_1} \cdots \otimes_{\pi_{n-1}} \mathcal{H}_\kappa^{\rho, \eta}$ for $(r_1, r_2) \in \mathbb{T}^2$ by

$$u_{(r_1, r_2)}(\xi_1 \otimes_{\pi_1} \cdots \otimes_{\pi_{n-1}} \xi_n) = e^{2\pi i(kr_1 + lr_2)}(\xi_1 \otimes_{\pi_1} \cdots \otimes_{\pi_{n-1}} \xi_n).$$

We extend $u_{(r_1, r_2)}$ on $F_n(\kappa)$ and F_κ . Define

$$g_{(r_1, r_2)} = \text{Ad}(u_{(r_1, r_2)}) \text{ on } F_\kappa \text{ for } (r_1, r_2) \in \mathbb{T}^2.$$

The following lemma is straightforward.

Lemma 5.9. For $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta, \alpha \in \Sigma^\rho, a \in \Sigma^\eta$ and $(r_1, r_2) \in \mathbb{T}^2$, we have

$$\begin{aligned} g_{(r_1, r_2)}(\bar{\phi}_\rho(w)) &= \bar{\phi}_\rho(w), & g_{(r_1, r_2)}(\bar{\phi}_\eta(z)) &= \bar{\phi}_\eta(z), \\ g_{(r_1, r_2)}(s_\alpha) &= e^{2\pi i r_1} s_\alpha, & g_{(r_1, r_2)}(t_a) &= e^{2\pi i r_2} t_a. \end{aligned}$$

It is easy to see that $g_{(r_1, r_2)}(\mathcal{K}_\mathcal{A}(F_\kappa)) = \mathcal{K}_\mathcal{A}(F_\kappa)$ so that $g_{(r_1, r_2)}$ defines an automorphism on $\mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}}$ for $(r_1, r_2) \in \mathbb{T}^2$, which is still denoted by $g_{(r_1, r_2)}$. The automorphisms $g_{(r_1, r_2)}, (r_1, r_2) \in \mathbb{T}^2$ define an action

$$g : (r_1, r_2) \in \mathbb{T}^2 \longrightarrow g_{(r_1, r_2)} \in \text{Aut}(\mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}})$$

of \mathbb{T}^2 , called the gauge action on $\mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}}$. Define a faithful conditional expectation $\mathcal{E}_{\mathcal{H}_\kappa^{\rho, \eta}}$ from $\mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}}$ onto the fixed point algebra $(\mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}})^g$ by setting

$$\mathcal{E}_{\mathcal{H}_\kappa^{\rho, \eta}}(X) = \int_{(r_1, r_2) \in \mathbb{T}^2} g_{(r_1, r_2)}(X) dr_1 dr_2, \quad X \in \mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}}.$$

Then the following lemma holds. Its proof is routine.

Lemma 5.10. The fixed point algebra $(\mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}})^g$ of $\mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}}$ under the action g of \mathbb{T}^2 coincides with $\mathcal{F}_{\mathcal{H}_\kappa^{\rho, \eta}}$.

We will prove that the algebra $\mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}}$ has a universal property subject to the relations $(\mathcal{H}_\kappa^{\rho, \eta})$. Let us denote by $\tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}}$ the universal C^* -algebra generated by the operators $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$ and partial isometries $u_\alpha, \alpha \in \Sigma^\rho, v_a, a \in \Sigma^\eta$ satisfying the following operator relations:

$$\begin{aligned} \sum_{\beta \in \Sigma^\rho} u_\beta u_\beta^* + \sum_{b \in \Sigma^\eta} v_b v_b^* &= 1, \\ u_\alpha u_\alpha^* w &= w u_\alpha u_\alpha^*, & v_a v_a^* w &= w v_a v_a^*, \\ u_\alpha u_\alpha^* z &= z u_\alpha u_\alpha^*, & v_a v_a^* z &= z v_a v_a^*, \\ \hat{\rho}_\alpha(w) &= u_\alpha^* w u_\alpha, & \hat{\eta}_a(z) &= v_a^* z v_a, \\ \hat{\rho}_\alpha^\eta(z) &= u_\alpha^* z u_\alpha, & \hat{\eta}_a^\rho(w) &= v_a^* w v_a, \\ \iota_\eta(y) &= \iota_\rho(y) \end{aligned}$$

for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta, \alpha \in \Sigma^\rho, a \in \Sigma^\eta, y \in \mathcal{A}$ where $\iota_\eta : \mathcal{A} \hookrightarrow \mathcal{B}_\eta$ and $\iota_\rho : \mathcal{A} \hookrightarrow \mathcal{B}_\rho$ are natural embeddings. The above ten relations of operators are also called the relations $(\mathcal{H}_\kappa^{\rho, \eta})$. Denote by $\tilde{\mathcal{B}}_\kappa^{\rho, \eta}$ the C^* -subalgebra of $\tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}}$ generated by the elements $w \in \mathcal{B}_\rho$ and $z \in \mathcal{B}_\eta$. We set for $\gamma \in \Sigma^\rho \cup \Sigma^\eta$ and $x \in \tilde{\mathcal{B}}_\kappa^{\rho, \eta}$

$$\tilde{\rho}_\gamma(x) = w_\gamma^* x w_\gamma \quad \text{where} \quad w_\gamma = \begin{cases} u_\alpha & \text{if } \gamma = \alpha \in \Sigma^\rho, \\ v_a & \text{if } \gamma = a \in \Sigma^\eta. \end{cases}$$

Since $u_\alpha^* \tilde{\mathcal{B}}_\kappa^{\rho, \eta} u_\alpha \subset \tilde{\mathcal{B}}_\kappa^{\rho, \eta}$ and $v_a^* \tilde{\mathcal{B}}_\kappa^{\rho, \eta} v_a \subset \tilde{\mathcal{B}}_\kappa^{\rho, \eta}$, we have

$$\tilde{\rho}_\gamma(\tilde{\mathcal{B}}_\kappa^{\rho, \eta}) \subset \tilde{\mathcal{B}}_\kappa^{\rho, \eta},$$

so that we have a family of endomorphisms $\tilde{\rho}_\gamma, \gamma \in \Sigma^{\rho \cup \eta}$ on $\tilde{\mathcal{B}}_\kappa^{\rho, \eta}$. Similarly as in the preceding lemma, we have

Lemma 5.11. *The triplet $(\tilde{\mathcal{B}}_\kappa^{\rho, \eta}, \tilde{\rho}, \Sigma^{\rho \cup \eta})$ is a C^* -symbolic dynamical system.*

For $n \in \mathbb{Z}_+$, denote by $\tilde{\mathcal{F}}_n$ the C^* -subalgebra of $\tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}}$ generated by the operators $w_{\gamma_1 \dots \gamma_n} b w_{\gamma'_1 \dots \gamma'_n}^*$ for $b \in \tilde{\mathcal{B}}_\kappa^{\rho, \eta}$ and $\gamma_1, \dots, \gamma_n, \gamma'_1, \dots, \gamma'_n \in \Sigma^{\rho \cup \eta}$ such that $|\gamma_1 \dots \gamma_n|_\rho = |\gamma'_1 \dots \gamma'_n|_\rho$ and $|\gamma_1 \dots \gamma_n|_\eta = |\gamma'_1 \dots \gamma'_n|_\eta$. Since $\sum_{\gamma \in \Sigma^{\rho \cup \eta}} w_\gamma w_\gamma^* = 1$ and $w_\gamma w_\gamma^*$ commutes with $\tilde{\mathcal{B}}_\kappa^{\rho, \eta}$, the equality for $b \in \tilde{\mathcal{F}}_n$

$$w_{\gamma_1 \dots \gamma_n} b w_{\gamma'_1 \dots \gamma'_n}^* = \sum_{\gamma_{n+1} \in \Sigma^{\rho \cup \eta}} w_{\gamma_1 \dots \gamma_n} w_{\gamma_{n+1}} \tilde{\rho}_{\gamma_{n+1}}(b) w_{\gamma_{n+1}}^* w_{\gamma'_1 \dots \gamma'_n}^*$$

gives rise to an embedding $\tilde{\mathcal{F}}_n \hookrightarrow \tilde{\mathcal{F}}_{n+1}, n \in \mathbb{Z}_+$. Let $\tilde{\mathcal{F}}_{\mathcal{H}_\kappa^{\rho, \eta}}$ be the C^* -subalgebra of $\tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}}$ generated by $\bigcup_{n=0}^\infty \tilde{\mathcal{F}}_n$. By the universality of the algebra $\tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}}$ subject to the relations $(\mathcal{H}_\kappa^{\rho, \eta})$, the correspondences for each $(r_1, r_2) \in \mathbb{T}^2$

$$\begin{aligned} w &\in \tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}} \longrightarrow w \in \tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}}, & z &\in \tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}} \longrightarrow z \in \tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}}, \\ u_\alpha &\in \tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}} \longrightarrow e^{2\pi i r_1} u_\alpha \in \tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}}, & v_a &\in \tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}} \longrightarrow e^{2\pi i r_2} v_a \in \tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}}, \end{aligned}$$

give rise to an automorphism of $\tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}}$, which we denote by $\tilde{g}_{(r_1, r_2)}$. Similarly to the preceding discussions, \tilde{g} yields an action of \mathbb{T}^2 on $\tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}}$, called the gauge action on $\tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}}$. Define similarly to the preceding discussions a faithful conditional expectation $\tilde{\mathcal{E}}_{\mathcal{H}_\kappa^{\rho, \eta}}$ from $\tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}}$ onto the fixed point algebra $(\tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}})^{\tilde{g}}$ by setting

$$\tilde{\mathcal{E}}_{\mathcal{H}_\kappa^{\rho, \eta}}(X) = \int_{(r_1, r_2) \in \mathbb{T}^2} \tilde{g}_{(r_1, r_2)}(X) dr_1 dr_2, \quad X \in \tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}}.$$

Similarly to the previous discussions, we have

Lemma 5.12. *The fixed point algebra $(\tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}})^{\tilde{g}}$ of $\tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}}$ under the gauge action \tilde{g} of \mathbb{T}^2 coincides with $\tilde{\mathcal{F}}_{\mathcal{H}_\kappa^{\rho, \eta}}$.*

By the universality of the algebra $\tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}}$ subject to the relations $(\mathcal{H}_\kappa^{\rho, \eta})$, there exists a surjective $*$ -homomorphism $\Psi : \tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}} \longrightarrow \mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}}$ satisfying

$$\begin{aligned} \Psi(w) &= \Phi_\rho(w), & \Psi(z) &= \Phi_\eta(z), \\ \Psi(u_\alpha) &= U_\alpha, & \Psi(v_a) &= V_a \end{aligned}$$

for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta, \alpha \in \Sigma^\rho, a \in \Sigma^\eta$. We will prove that there exists a $*$ -homomorphism $\pi_\kappa : \mathcal{B}_\kappa \longrightarrow \tilde{\mathcal{B}}_\kappa^{\rho, \eta}$ such that $\pi_\kappa(\Phi_\rho(w)) = w, \pi_\kappa(\Phi_\eta(z)) = z$.

Lemma 5.13. *For $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$, we have:*

- (i) *The correspondence $U_\alpha \Phi_\eta(z) U_\alpha^* \in U_\alpha \Phi_\eta(\mathcal{B}_\eta) U_\alpha^* \longrightarrow u_\alpha z u_\alpha^* \in u_\alpha \mathcal{B}_\eta u_\alpha^*$ for $z \in \mathcal{B}_\eta$ yields a $*$ -homomorphism.*

- (ii) The correspondence $V_a \Phi_\rho(w) V_a^* \in V_a \Phi_\rho(\mathcal{B}_\rho) V_a^* \longrightarrow v_a w v_a^* \in v_a \mathcal{B}_\rho v_a^*$ for $w \in \mathcal{B}_\rho$ yields a $*$ -homomorphism.

Proof. (i) As $U_\alpha^* U_\alpha = \Phi_\eta(\rho_\alpha(1)) = \Phi_\eta(P_\alpha)$ and $u_\alpha P_\alpha = u_\alpha \hat{\rho}_\alpha(1) = u_\alpha$, the maps

$$\begin{aligned} Ad(U_\alpha^*) : U_\alpha \Phi_\eta(z) U_\alpha^* &\in U_\alpha \Phi_\eta(\mathcal{B}_\eta) U_\alpha^* \longrightarrow U_\alpha^* U_\alpha \Phi_\eta(z) U_\alpha^* U_\alpha \in \Phi_\eta(P_\alpha \mathcal{B}_\eta P_\alpha), \\ Ad(u_\alpha) : P_\alpha z P_\alpha &\in P_\alpha \mathcal{B}_\eta P_\alpha (\subset \mathcal{B}_\eta) \longrightarrow u_\alpha z u_\alpha^* \in u_\alpha \mathcal{B}_\eta u_\alpha^* \end{aligned}$$

are $*$ -homomorphisms. As $\Phi_\eta : \mathcal{B}_\eta \longrightarrow \Phi_\eta(\mathcal{B}_\eta)$ is a $*$ -isomorphism, the desired map

$$U_\alpha \Phi_\eta(z) U_\alpha^* \in U_\alpha \Phi_\eta(\mathcal{B}_\eta) U_\alpha^* \longrightarrow u_\alpha z u_\alpha^* \in u_\alpha \mathcal{B}_\eta u_\alpha^*$$

for $z \in \mathcal{B}_\eta$ yields a $*$ -homomorphism.

(ii) is similar to (i). \square

Lemma 5.14.

- (i) For $\alpha \in \Sigma^\rho$, the correspondence:

$$\Phi_{\gamma_1}(x_{j_1}) \Phi_{\gamma_2}(x_{j_2}) \cdots \Phi_{\gamma_n}(x_{j_n}) U_\alpha U_\alpha^* \in \mathcal{B}_\kappa U_\alpha U_\alpha^* \longrightarrow x_{j_1} x_{j_2} \cdots x_{j_n} u_\alpha u_\alpha^* \in \tilde{\mathcal{B}}_\kappa^{\rho, \eta} u_\alpha u_\alpha^*$$

for $x_{j_k} \in \mathcal{B}_\rho(\gamma_k = \rho)$ and $x_{j_k} \in \mathcal{B}_\eta(\gamma_k = \eta)$ gives rise to a $*$ -homomorphism from $\mathcal{B}_\kappa U_\alpha U_\alpha^*$ to $\tilde{\mathcal{B}}_\kappa^{\rho, \eta} u_\alpha u_\alpha^*$.

- (ii) For $a \in \Sigma^\eta$, the correspondence:

$$\Phi_{\gamma_1}(x_{j_1}) \Phi_{\gamma_2}(x_{j_2}) \cdots \Phi_{\gamma_n}(x_{j_n}) V_a V_a^* \in \mathcal{B}_\kappa V_a V_a^* \longrightarrow x_{j_1} x_{j_2} \cdots x_{j_n} v_a v_a^* \in \tilde{\mathcal{B}}_\kappa^{\rho, \eta} v_a v_a^*$$

for $x_{j_k} \in \mathcal{B}_\rho(\gamma_k = \rho)$ and $x_{j_k} \in \mathcal{B}_\eta(\gamma_k = \eta)$ gives rise to a $*$ -homomorphism from $\mathcal{B}_\kappa V_a V_a^*$ to $\tilde{\mathcal{B}}_\kappa^{\rho, \eta} v_a v_a^*$.

Proof. (i) Since $\hat{\rho}_\alpha(w) \in \mathcal{A} \subset \mathcal{B}_\eta$ for $w \in \mathcal{B}_\rho$, we see $\Phi_\rho(\hat{\rho}_\alpha(w)) = \Phi_\eta(\hat{\rho}_\alpha(w))$ so that

$$U_\alpha^* \Phi_\rho(w) U_\alpha = \Phi_\rho(\hat{\rho}_\alpha(w)) = \Phi_\eta(\hat{\rho}_\alpha(w)), \quad U_\alpha^* \Phi_\eta(z) U_\alpha = \Phi_\eta(\hat{\rho}_\alpha(z))$$

for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$. For $x_{j_k} \in \mathcal{B}_\eta$ or $\in \mathcal{B}_\rho$, put

$$\hat{x}_{j_k} := \begin{cases} \hat{\rho}_\alpha(x_{j_k}) & \text{if } x_{j_k} \in \mathcal{B}_\rho, \\ \hat{\rho}_\alpha^\eta(x_{j_k}) & \text{if } x_{j_k} \in \mathcal{B}_\eta. \end{cases}$$

We then have $\hat{x}_{j_k} \in \mathcal{B}_\eta$ so that

$$\begin{aligned} &\Phi_{\gamma_1}(x_{j_1}) \Phi_{\gamma_2}(x_{j_2}) \cdots \Phi_{\gamma_n}(x_{j_n}) U_\alpha U_\alpha^* \\ &= U_\alpha U_\alpha^* \Phi_{\gamma_1}(x_{j_1}) U_\alpha U_\alpha^* \Phi_{\gamma_2}(x_{j_2}) U_\alpha U_\alpha^* \cdots U_\alpha U_\alpha^* \Phi_{\gamma_n}(x_{j_n}) U_\alpha U_\alpha^* \\ &= U_\alpha \Phi_\eta(\hat{x}_{j_1}) \Phi_\eta(\hat{x}_{j_2}) \cdots \Phi_\eta(\hat{x}_{j_n}) U_\alpha^* \\ &= U_\alpha \Phi_\eta(\hat{x}_{j_1} \hat{x}_{j_2} \cdots \hat{x}_{j_n}) U_\alpha^*. \end{aligned}$$

By the preceding lemma, the correspondence

$$U_\alpha \Phi_\eta(\hat{x}_{j_1} \hat{x}_{j_2} \cdots \hat{x}_{j_n}) U_\alpha^* \in U_\alpha \Phi_\eta(\mathcal{B}_\eta) U_\alpha^* \longrightarrow u_\alpha \hat{x}_{j_1} \hat{x}_{j_2} \cdots \hat{x}_{j_n} u_\alpha^* \in u_\alpha \mathcal{B}_\eta u_\alpha^*$$

gives rise to a $*$ -homomorphism from $U_\alpha \Phi_\eta(\mathcal{B}_\eta) U_\alpha^*$ to $u_\alpha \mathcal{B}_\eta u_\alpha^*$. Since we have

$$\begin{aligned} u_\alpha \widehat{x}_{j_1} \widehat{x}_{j_2} \cdots \widehat{x}_{j_n} u_\alpha^* &= u_\alpha u_\alpha^* x_{j_1} u_\alpha u_\alpha^* x_{j_2} u_\alpha u_\alpha^* \cdots u_\alpha u_\alpha^* x_{j_n} u_\alpha u_\alpha^* \\ &= x_{j_1} x_{j_2} \cdots x_{j_n} u_\alpha u_\alpha^*, \end{aligned}$$

we have a desired $*$ -homomorphism from $\mathcal{B}_\kappa U_\alpha U_\alpha^*$ to $\widetilde{\mathcal{B}}_\kappa^{\rho, \eta} u_\alpha u_\alpha^*$.

(ii) is similar to (i). \square

The above $*$ -homomorphisms of (i) and of (ii) are denoted by

$$\begin{aligned} \pi_\alpha : \Phi_{\gamma_1}(x_{j_1}) \Phi_{\gamma_2}(x_{j_2}) \cdots \Phi_{\gamma_n}(x_{j_n}) U_\alpha U_\alpha^* &\in \mathcal{B}_\kappa U_\alpha U_\alpha^* \longrightarrow x_{j_1} x_{j_2} \cdots x_{j_n} u_\alpha u_\alpha^* \in \widetilde{\mathcal{B}}_\kappa^{\rho, \eta} u_\alpha u_\alpha^* \\ \pi_a : \Phi_{\gamma_1}(x_{j_1}) \Phi_{\gamma_2}(x_{j_2}) \cdots \Phi_{\gamma_n}(x_{j_n}) V_a V_a^* &\in \mathcal{B}_\kappa V_a V_a^* \longrightarrow x_{j_1} x_{j_2} \cdots x_{j_n} v_a v_a^* \in \widetilde{\mathcal{B}}_\kappa^{\rho, \eta} v_a v_a^* \end{aligned}$$

Lemma 5.15. *There exists a $*$ -homomorphism*

$$\pi_\kappa : \mathcal{B}_\kappa \longrightarrow \widetilde{\mathcal{B}}_\kappa^{\rho, \eta}$$

such that $\pi_\kappa(\Phi_\rho(w)) = w$ for $w \in \mathcal{B}_\rho$ and $\pi_\kappa(\Phi_\eta(z)) = z$ for $z \in \mathcal{B}_\eta$.

Proof. Since $\sum_{\alpha \in \Sigma^\rho} U_\alpha U_\alpha^* + \sum_{a \in \Sigma^\eta} V_a V_a^* = 1$ and $\sum_{\alpha \in \Sigma^\rho} u_\alpha u_\alpha^* + \sum_{a \in \Sigma^\eta} v_a v_a^* = 1$ by putting

$$\pi_\kappa(X) := \sum_{\alpha \in \Sigma^\rho} \pi_\alpha(X U_\alpha U_\alpha^*) u_\alpha u_\alpha^* + \sum_{a \in \Sigma^\eta} \pi_a(X V_a V_a^*) v_a v_a^*$$

for $X \in \mathcal{B}_\kappa$, we have a desired $*$ -homomorphism from \mathcal{B}_κ to $\widetilde{\mathcal{B}}_\kappa^{\rho, \eta}$. \square

We will consider the $*$ -homomorphism $\Psi : \widetilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}} \longrightarrow \mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}}$ again. By the above discussions we have

Lemma 5.16. *The restriction $\Psi|_{\widetilde{\mathcal{B}}_\kappa^{\rho, \eta}} : \widetilde{\mathcal{B}}_\kappa^{\rho, \eta} \longrightarrow \mathcal{B}_\kappa$ of Ψ to the subalgebra $\widetilde{\mathcal{B}}_\kappa^{\rho, \eta}$ is the inverse of $\pi_\kappa : \mathcal{B}_\kappa \longrightarrow \widetilde{\mathcal{B}}_\kappa^{\rho, \eta}$. Hence $\Psi|_{\widetilde{\mathcal{B}}_\kappa^{\rho, \eta}} : \widetilde{\mathcal{B}}_\kappa^{\rho, \eta} \longrightarrow \mathcal{B}_\kappa$ is a $*$ -isomorphism.*

Therefore we reach the main result of the paper:

Theorem 5.17. *The C^* -algebra $\mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}}$ associated with the Hilbert C^* -quad module $\mathcal{H}_\kappa^{\rho, \eta}$ is canonically $*$ -isomorphic to the universal C^* -algebra $\widetilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho, \eta}}$ generated by the operators $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$ and partial isometries $u_\alpha, \alpha \in \Sigma^\rho, v_a, a \in \Sigma^\eta$ satisfying the operator relations:*

$$\sum_{\beta \in \Sigma^\rho} u_\beta u_\beta^* + \sum_{b \in \Sigma^\eta} v_b v_b^* = 1, \quad (5.3)$$

$$u_\alpha u_\alpha^* w = w u_\alpha u_\alpha^*, \quad v_a v_a^* w = w v_a v_a^*, \quad (5.4)$$

$$u_\alpha u_\alpha^* z = z u_\alpha u_\alpha^*, \quad v_a v_a^* z = z v_a v_a^*, \quad (5.5)$$

$$\widehat{\rho}_\alpha(w) = u_\alpha^* w u_\alpha, \quad \widehat{\eta}_a(z) = v_a^* z v_a, \quad (5.6)$$

$$\widehat{\rho}_\alpha^\eta(z) = u_\alpha^* z u_\alpha, \quad \widehat{\eta}_a^\rho(w) = v_a^* w v_a, \quad (5.7)$$

$$\iota_\eta(y) = \iota_\rho(y) \quad (5.8)$$

for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta, \alpha \in \Sigma^\rho, a \in \Sigma^\eta, y \in \mathcal{A}$ where $\iota_\rho : \mathcal{A} \hookrightarrow \mathcal{B}_\rho$ and $\iota_\eta : \mathcal{A} \hookrightarrow \mathcal{B}_\eta$ are natural embeddings.

Proof. The triplets $(\tilde{\mathcal{B}}_\kappa^{\rho,\eta}, \tilde{\rho}, \Sigma^{\rho \cup \eta})$ and $(\mathcal{B}_\kappa, \rho^\kappa, \Sigma^{\rho \cup \eta})$ are both the C^* -symbolic dynamical systems. As in the discussions of the proof of [17, Lemma 3.2], the above lemma implies that the restriction $\Psi|_{\tilde{\mathcal{F}}_{\mathcal{H}_\kappa^{\rho,\eta}}} : \tilde{\mathcal{F}}_{\mathcal{H}_\kappa^{\rho,\eta}} \longrightarrow \mathcal{F}_{\mathcal{H}_\kappa^{\rho,\eta}}$ of Ψ to the subalgebra $\tilde{\mathcal{F}}_{\mathcal{H}_\kappa^{\rho,\eta}}$ is a $*$ -isomorphism. The diagram:

$$\begin{array}{ccc} \tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho,\eta}} & \xrightarrow{\Psi} & \mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}} \\ \tilde{\mathcal{E}}_{\mathcal{H}_\kappa^{\rho,\eta}} \downarrow & & \downarrow \mathcal{E}_{\mathcal{H}_\kappa^{\rho,\eta}} \\ \tilde{\mathcal{F}}_{\mathcal{H}_\kappa^{\rho,\eta}} & \xrightarrow{\Psi|_{\tilde{\mathcal{F}}_{\mathcal{H}_\kappa^{\rho,\eta}}}} & \mathcal{F}_{\mathcal{H}_\kappa^{\rho,\eta}} \end{array}$$

is commutative. Since the conditional expectation $\tilde{\mathcal{E}}_{\mathcal{H}_\kappa^{\rho,\eta}}$ is faithful and the restriction $\Psi|_{\tilde{\mathcal{F}}_{\mathcal{H}_\kappa^{\rho,\eta}}}$ of Ψ to $\tilde{\mathcal{F}}_{\mathcal{H}_\kappa^{\rho,\eta}}$ is $*$ -isomorphic, one concludes that $\Psi : \tilde{\mathcal{O}}_{\mathcal{H}_\kappa^{\rho,\eta}} \longrightarrow \mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}}$ is a $*$ -isomorphism by a routine argument as in [4, Proposition 2.9]. \square

The above theorem implies the following: Suppose that there exist two families of partial isometries $\hat{u}_\alpha, \alpha \in \Sigma^\rho, \hat{v}_a, a \in \Sigma^\eta$ in a unital C^* -algebra \mathcal{D} and two $*$ -homomorphisms $\pi_\rho : \mathcal{B}_\rho \longrightarrow \mathcal{D}, \pi_\eta : \mathcal{B}_\eta \longrightarrow \mathcal{D}$ satisfying the relations:

$$\begin{aligned} \sum_{\beta \in \Sigma^\rho} \hat{u}_\beta \hat{u}_\beta^* + \sum_{b \in \Sigma^\eta} \hat{v}_b \hat{v}_b^* &= 1, \\ \hat{u}_\alpha \hat{u}_\alpha^* \pi_\rho(w) &= \pi_\rho(w) \hat{u}_\alpha \hat{u}_\alpha^*, & \hat{v}_a \hat{v}_a^* \pi_\eta(z) &= \pi_\eta(z) \hat{v}_a \hat{v}_a^*, \\ \hat{u}_\alpha \hat{u}_\alpha^* \pi_\eta(z) &= \pi_\eta(z) \hat{u}_\alpha \hat{u}_\alpha^*, & \hat{v}_a \hat{v}_a^* \pi_\rho(w) &= \pi_\rho(w) \hat{v}_a \hat{v}_a^*, \\ \pi_\rho(\hat{\rho}_\alpha(w)) &= \hat{u}_\alpha^* \pi_\rho(w) \hat{u}_\alpha, & \pi_\eta(\hat{\eta}_a(z)) &= \hat{v}_a^* \pi_\eta(z) \hat{v}_a, \\ \pi_\eta(\hat{\rho}_\alpha^\eta(z)) &= \hat{u}_\alpha^* \pi_\eta(z) \hat{u}_\alpha, & \pi_\rho(\hat{\eta}_a^\rho(w)) &= \hat{v}_a^* \pi_\rho(w) \hat{v}_a, \\ \pi_\rho(y) &= \pi_\eta(y) \end{aligned}$$

for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta, \alpha \in \Sigma^\rho, a \in \Sigma^\eta, y \in \mathcal{A}$. Then there exists a $*$ -homomorphism π from $\mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}}$ onto the C^* -algebra generated by $\pi_\rho(w), \pi_\eta(z), \hat{u}_\alpha, \hat{v}_a$ such that $\pi(w) = \pi_\rho(w), \pi(z) = \pi_\eta(z), \pi(U_\alpha) = \hat{u}_\alpha, \pi(V_a) = \hat{v}_a$ for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta, \alpha \in \Sigma^\rho, a \in \Sigma^\eta$.

The notions of condition (I) and irreducibility for C^* -symbolic dynamical systems have been defined in [16] (cf. [17]). The former guarantees a uniqueness of the resulting C^* -algebras under operator relations among generators. The latter does a simplicity of them.

Definition. We say that $\mathcal{H}_\kappa^{\rho,\eta}$ satisfies *condition (I)* if the C^* -symbolic dynamical system $(\mathcal{B}_\kappa, \rho^\kappa, \Sigma^{\rho \cup \eta})$ satisfies condition (I). We also say that $\mathcal{H}_\kappa^{\rho,\eta}$ is *irreducible* if the C^* -symbolic dynamical system $(\mathcal{B}_\kappa, \rho^\kappa, \Sigma^{\rho \cup \eta})$ is irreducible.

Corollary 5.18. Suppose that \mathcal{A} is commutative.

- (i) If $\mathcal{H}_\kappa^{\rho,\eta}$ satisfies condition (I), then the C^* -algebra $\mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}}$ is the unique C^* -algebra subject to the relations $(\mathcal{H}_\kappa^{\rho,\eta})$.
- (ii) If in addition $\mathcal{H}_\kappa^{\rho,\eta}$ is irreducible, the C^* -algebra $\mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}}$ is simple.

In the rest of this section, we will consider a C^* -subalgebra of $\mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}}$ generated by U_α, V_a for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ and elements $\Phi_\rho(x), \Phi_\eta(x)$ for $x \in \mathcal{A}$. The subalgebra is denoted by $\mathcal{O}_{\mathcal{A},\kappa}$. Since the operators $\tilde{\phi}_\rho(x)$ and

$\bar{\phi}_\eta(x)$ for $x \in \mathcal{A}$ are different only on $F_0(\kappa)$, we have $\Phi_\rho(x) = \Phi_\eta(x)$ for $x \in \mathcal{A}$, which we denote by $\Phi(x)$. Hence the following relations hold:

$$\sum_{\beta \in \Sigma^\rho} U_\beta U_\beta^* + \sum_{b \in \Sigma^\eta} V_b V_b^* = 1, \quad (5.9)$$

$$U_\alpha U_\alpha^* \Phi(x) = \Phi(x) U_\alpha U_\alpha^*, \quad V_a V_a^* \Phi(x) = \Phi(x) V_a V_a^*, \quad (5.10)$$

$$U_\alpha^* \Phi(x) U_\alpha = \Phi(\rho_\alpha(x)), \quad V_a^* \Phi(x) V_a = \Phi(\eta_a(x)) \quad (5.11)$$

for $x \in \mathcal{A}$, $\alpha \in \Sigma^\rho$, $a \in \Sigma^\eta$. For $\omega = (\alpha, b, a, \beta) \in \Sigma_\kappa$, define the operators in $\mathcal{O}_{\mathcal{A}, \kappa}$ by

$$C_\omega = V_a U_\beta V_b^* U_\alpha^*, \quad (5.12)$$

$$P_{r, \omega} = V_a U_\beta U_\beta^* V_a^*, \quad P_{s, \omega} = U_\alpha V_b V_b^* U_\alpha^*. \quad (5.13)$$

Lemma 5.19. For $\omega = (\alpha, b, a, \beta) \in \Sigma_\kappa$, we have

(i) C_ω is a partial isometry in $\mathcal{O}_{\mathcal{H}_\kappa^{\rho, \eta}}$ satisfying

$$C_\omega \Phi(x) = \Phi(x) C_\omega, \quad x \in \mathcal{A}, \quad (5.14)$$

$$V_a U_\beta = C_\omega U_\alpha V_b, \quad U_\alpha V_b = C_\omega^* V_a U_\beta. \quad (5.15)$$

(ii) $P_{r, \omega}, P_{s, \omega}$ are projections satisfying

$$P_{r, \omega} = C_\omega C_\omega^*, \quad P_{s, \omega} = C_\omega^* C_\omega. \quad (5.16)$$

(iii) $C_\omega \notin \mathcal{B}_\kappa$.

Proof. (i) By (5.10) and (5.11), we have for $x \in \mathcal{A}$,

$$\begin{aligned} C_\omega \Phi(x) &= V_a U_\beta V_b^* \Phi(\rho_\alpha(x)) U_\alpha^* = V_a U_\beta \Phi(\eta_b(\rho_\alpha(x))) V_b^* U_\alpha^* \\ \Phi(x) C_\omega &= V_a \Phi(\eta_a(x)) U_\beta V_b^* U_\alpha^* = V_a U_\beta \Phi(\rho_\beta(\eta_a(x))) V_b^* U_\alpha^*. \end{aligned}$$

By (1.1), we have (5.14). We also have

$$C_\omega U_\alpha V_b = V_a U_\beta \Phi(\eta_b(\rho_\alpha(1))) = V_a U_\beta \Phi(\rho_\beta(\eta_a(1))) = V_a U_\beta U_\beta^* V_a^* V_a U_\beta = V_a U_\beta$$

and similarly $C_\omega^* V_a U_\beta = U_\alpha V_b$.

(ii) The equalities (5.16) are straightforward.

(iii) Suppose that $C_\omega \in \mathcal{B}_\kappa$. Since $U_\alpha U_\alpha^*$ commutes with $\Phi_\rho(w), \Phi_\eta(z)$ for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$, it commutes with \mathcal{B}_κ and hence with C_ω . Hence we have

$$V_a U_\beta = C_\omega U_\alpha V_b = U_\alpha U_\alpha^* C_\omega U_\alpha V_b.$$

As $V_a V_a^* U_\alpha U_\alpha^* = 0$, one has $V_a U_\beta = 0$. Since $\Phi(\rho_\beta(\eta_a(y))) = U_\beta^* V_a^* y V_a U_\beta = 0$, one has $\rho_\beta(\eta_a(y)) = 0$ for all $y \in \mathcal{A}$, a contradiction. \square

For the two C^* -symbolic dynamical systems $(\mathcal{A}, \rho, \Sigma^\rho)$ and $(\mathcal{A}, \eta, \Sigma^\eta)$ in the C^* -textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$, we define their union $(\mathcal{A}, \rho \cup \eta, \Sigma^{\rho \cup \eta})$ as the following way, where $\Sigma^{\rho \cup \eta} = \Sigma^\rho \cup \Sigma^\eta$. For $\gamma \in \Sigma^{\rho \cup \eta}$, define an endomorphism $(\rho \cup \eta)_\gamma$ on \mathcal{A} by setting

$$(\rho \cup \eta)_\gamma = \begin{cases} \rho_\gamma & \text{if } \gamma \in \Sigma^\rho, \\ \eta_\gamma & \text{if } \gamma \in \Sigma^\eta. \end{cases}$$

It is easy to see that the triplet $(\mathcal{A}, \rho \cup \eta, \Sigma^{\rho \cup \eta})$ is a C^* -symbolic dynamical system. Hence we have a C^* -algebra $\mathcal{O}_{\rho \cup \eta}$ from $(\mathcal{A}, \rho \cup \eta, \Sigma^{\rho \cup \eta})$. Denote by $\mathbf{S}_\alpha, \alpha \in \Sigma^\rho$ and $\mathbf{S}_a, a \in \Sigma^\eta$ its generating partial isometries satisfying the relations:

$$\sum_{\beta \in \Sigma^\rho} \mathbf{S}_\beta \mathbf{S}_\beta^* + \sum_{b \in \Sigma^\eta} \mathbf{S}_b \mathbf{S}_b^* = 1, \quad (5.17)$$

$$\mathbf{S}_\alpha \mathbf{S}_\alpha^* x = x \mathbf{S}_\alpha \mathbf{S}_\alpha^*, \quad \mathbf{S}_a \mathbf{S}_a^* x = x \mathbf{S}_a \mathbf{S}_a^*, \quad (5.18)$$

$$\mathbf{S}_\alpha^* x \mathbf{S}_\alpha = \rho_\alpha(x), \quad \mathbf{S}_a^* x \mathbf{S}_a = \eta_a(x) \quad (5.19)$$

for $x \in \mathcal{A}, \alpha \in \Sigma^\rho, a \in \Sigma^\eta$. Then we have

Proposition 5.20.

(i) *The correspondences:*

$$\mathbf{S}_\alpha \longleftrightarrow U_\alpha \quad \mathbf{S}_a \longleftrightarrow V_a, \quad x \longleftrightarrow \Phi(x)$$

for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta, x \in \mathcal{A}$ yield a $*$ -isomorphism between the C^* -algebras $\mathcal{O}_{\rho \cup \eta}$ and $\mathcal{O}_{\mathcal{A}, \kappa}$. Therefore we have

$$\mathcal{O}_{\mathcal{A}, \kappa} \cong \mathcal{O}_{\rho \cup \eta}, \quad (5.20)$$

(ii) *Hence the C^* -algebra $\mathcal{O}_{\mathcal{A}, \kappa}$ is realized as the universal C^* -algebra generated by partial isometries U_α, V_a for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ and elements $x \in \mathcal{A}$ subject to the relations:*

$$\begin{aligned} \sum_{\beta \in \Sigma^\rho} U_\beta U_\beta^* + \sum_{b \in \Sigma^\eta} V_b V_b^* &= 1, \\ U_\alpha U_\alpha^* \Phi(x) &= \Phi(x) U_\alpha U_\alpha^*, \quad V_a V_a^* \Phi(x) = \Phi(x) V_a V_a^*, \\ U_\alpha^* \Phi(x) U_\alpha &= \Phi(\rho_\alpha(x)), \quad V_a^* \Phi(x) V_a = \Phi(\eta_a(x)). \end{aligned}$$

Proof. By the universality of the algebra $\mathcal{O}_{\rho \cup \eta}$ subject to the relations (5.17), (5.18), (5.19), the correspondences

$$\mathbf{S}_\alpha \longrightarrow U_\alpha \quad \mathbf{S}_a \longrightarrow V_a, \quad x \longrightarrow \Phi(x)$$

for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta, x \in \mathcal{A}$ yield a $*$ -homomorphism from $\mathcal{O}_{\rho \cup \eta}$ to $\mathcal{O}_{\mathcal{A}, \kappa}$ which we denote by Ψ . Since the action $g_{(r_1, r_2)}, (r_1, r_2) \in \mathbb{T}^2$ preserves $\mathcal{O}_{\mathcal{A}, \kappa}$, the automorphisms $g_{(t, t)}$ for $t \in \mathbb{T}$ give rise to an action on $\mathcal{O}_{\mathcal{A}, \kappa}$ which we denote by $g_t^{\mathcal{A}}$. It satisfies

$$g_t^{\mathcal{A}}(\Phi(x)) = \Phi(x), \quad g_t^{\mathcal{A}}(U_\alpha) = e^{2\pi i t} U_\alpha, \quad g_t^{\mathcal{A}}(V_a) = e^{2\pi i t} V_a$$

for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta, x \in \mathcal{A}$ and $t \in \mathbb{T}$. Let \hat{g} be the gauge action on $\mathcal{O}_{\rho \cup \eta}$ which satisfies

$$\hat{g}_t(x) = x, \quad \hat{g}_t(\mathbf{S}_\alpha) = e^{2\pi i t} \mathbf{S}_\alpha, \quad \hat{g}_t(\mathbf{S}_a) = e^{2\pi i t} \mathbf{S}_a$$

for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta, x \in \mathcal{A}$ and $t \in \mathbb{T}$. Hence we have

$$\Psi \circ \hat{g}_t = g_t^{\mathcal{A}} \circ \Psi \quad \text{for } t \in \mathbb{T}.$$

Let $(\mathcal{O}_{\mathcal{A},\kappa})^{g^{\mathcal{A}}}$ be the fixed point algebra of $\mathcal{O}_{\mathcal{A},\kappa}$ under the action $g^{\mathcal{A}}$. Denote by $\mathcal{E}^{\mathcal{A}} : \mathcal{O}_{\mathcal{A},\kappa} \longrightarrow (\mathcal{O}_{\mathcal{A},\kappa})^{g^{\mathcal{A}}}$ the conditional expectation defined by the formula:

$$\mathcal{E}^{\mathcal{A}}(X) = \int_{\mathbb{T}} g_t^{\mathcal{A}}(X) dt \quad \text{for } X \in \mathcal{O}_{\mathcal{A},\kappa}.$$

Denote by $\mathcal{E}^{\rho \cup \eta} : \mathcal{O}_{\rho \cup \eta} \longrightarrow (\mathcal{O}_{\rho \cup \eta})^{\hat{g}}$ the conditional expectation defined similarly to the above by the gauge action \hat{g} . Since we have

$$\Psi \circ \mathcal{E}^{\rho \cup \eta} = \mathcal{E}^{\mathcal{A}} \circ \Psi$$

and $\mathcal{E}^{\rho \cup \eta}$ is faithful, by a routine argument as in [4, Proposition 2.9], one concludes that Ψ is injective and hence $*$ -isomorphic. \square

6. A relation between $\mathcal{O}_{\mathcal{H}_{\kappa}^{\rho,\eta}}$ and $\mathcal{O}_{\rho,\eta}^{\kappa}$

For a C^* -textile dynamical system $(\mathcal{A}, \rho, \eta, \Sigma^{\rho}, \Sigma^{\eta}, \kappa)$, the author has introduced a C^* -algebra $\mathcal{O}_{\rho,\eta}^{\kappa}$ in [19]. It is realized as the universal C^* -algebra $C^*(x, S_{\alpha}, T_a; x \in \mathcal{A}, \alpha \in \Sigma^{\rho}, a \in \Sigma^{\eta})$ generated by $x \in \mathcal{A}$ and two families of partial isometries $S_{\alpha}, \alpha \in \Sigma^{\rho}, T_a, a \in \Sigma^{\eta}$ subject to the relations (1.2), (1.3) and (1.4) that are called the relations $(\rho, \eta; \kappa)$. In this section we will describe a relationship between the two algebras $\mathcal{O}_{\mathcal{H}_{\kappa}^{\rho,\eta}}$ and $\mathcal{O}_{\rho,\eta}^{\kappa}$. As both the C^* -algebras \mathcal{O}_{ρ} and \mathcal{O}_{η} are naturally regarded as C^* -subalgebras of $\mathcal{O}_{\rho,\eta}^{\kappa}$, the algebras \mathcal{B}_{ρ} and \mathcal{B}_{η} may be realized as the C^* -subalgebra of $\mathcal{O}_{\rho,\eta}^{\kappa}$ generated by $S_{\alpha}xS_{\alpha}^*$ for $x \in \mathcal{A}, \alpha \in \Sigma^{\rho}$ and that of $\mathcal{O}_{\rho,\eta}^{\kappa}$ generated by $T_axT_a^*$ for $x \in \mathcal{A}, a \in \Sigma^{\eta}$ respectively.

Lemma 6.1. *Let $S_{\alpha}, \alpha \in \Sigma^{\rho}$ and $T_a, a \in \Sigma^{\eta}$ be partial isometries in the algebra $\mathcal{O}_{\rho,\eta}^{\kappa}$ satisfying the relations (1.2), (1.3) and (1.4).*

(i) *For $\alpha \in \Sigma^{\rho}$ and $z = \sum_{b \in \Sigma^{\eta}} T_b z_b T_b^* \in \mathcal{B}_{\eta}$ as in (2.3),*

$$S_{\alpha}^* z S_{\alpha} = \sum_{\substack{b,a,\beta \\ (\alpha,b,a,\beta) \in \Sigma_{\kappa}}} T_b \rho_{\beta}(z_a) T_b^* = \hat{\rho}_{\alpha}^{\eta}(z). \quad (6.1)$$

(ii) *For $a \in \Sigma^{\eta}$ and $w = \sum_{\beta \in \Sigma^{\rho}} S_{\beta} w_{\beta} S_{\beta}^* \in \mathcal{B}_{\rho}$ as in (2.2),*

$$T_a^* w T_a = \sum_{\substack{\alpha,b,\beta \\ (\alpha,b,a,\beta) \in \Sigma_{\kappa}}} S_{\beta} \eta_b(w_{\alpha}) S_{\beta}^* = \hat{\eta}_a^{\rho}(w). \quad (6.2)$$

Proof. (i) By [19, Lemma 4.2] the following formulae hold for $\alpha \in \Sigma^{\rho}, a \in \Sigma^{\eta}$,

$$T_a^* S_{\alpha} = \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} S_{\beta} \eta_b(\rho_{\alpha}(1)) T_b^*, \quad S_{\alpha}^* T_a = \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} T_b \rho_{\beta}(\eta_a(1)) S_{\beta}^*.$$

It then follows that

$$\begin{aligned} S_{\alpha}^* z S_{\alpha} &= \sum_{a \in \Sigma^{\eta}} S_{\alpha}^* T_a z_a T_a^* S_{\alpha} \\ &= \sum_{a \in \Sigma^{\eta}} \sum_{\substack{b,\beta \\ \kappa(\alpha,b)=(a,\beta)}} T_b \rho_{\beta}(Q_a) S_{\beta}^* z_a \sum_{\substack{b',\beta' \\ \kappa(\alpha,b')=(a,\beta')}} S_{\beta'} \rho_{\beta'}(Q_a) T_{b'}^* \end{aligned}$$

$$= \sum_{\substack{b,a,\beta \\ (\alpha,b,a,\beta) \in \Sigma_\kappa}} T_b \rho_\beta(Q_a z_a Q_a) T_b^* = \widehat{\rho}_\alpha^\eta(z).$$

(ii) is similar to (i). \square

Let us denote by S_1, S_2 isometries satisfying $S_1 S_1^* + S_2 S_2^* = 1$. The C^* -algebra generated by them is the Cuntz algebra \mathcal{O}_2 of order 2. In the tensor product C^* -algebra $\mathcal{O}_{\rho,\eta}^\kappa \otimes \mathcal{O}_2$, we put

$$\begin{aligned} \widehat{u}_\alpha &= S_\alpha \otimes S_1 \text{ for } \alpha \in \Sigma^\rho, & \widehat{v}_a &= T_a \otimes S_2 \text{ for } a \in \Sigma^\eta, \\ \pi_\rho(w) &= w \otimes 1 \text{ for } w \in \mathcal{B}_\rho, & \pi_\eta(z) &= z \otimes 1 \text{ for } z \in \mathcal{B}_\eta. \end{aligned}$$

By the above lemma, the following lemma is straightforward.

Lemma 6.2. *The operators $\widehat{u}_\alpha, \widehat{v}_a$ for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ and $\pi_\rho(w), \pi_\eta(z)$ for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$ satisfy the relations $(\mathcal{H}_\kappa^{\rho,\eta})$.*

We thus have

Theorem 6.3. *Assume that $(\mathcal{A}, \rho, \eta, \Sigma^\rho, \Sigma^\eta, \kappa)$ satisfies condition (I). Then the correspondences*

$$\begin{aligned} U_\alpha &\in \mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}} \longrightarrow S_\alpha \otimes S_1 \in \mathcal{O}_{\rho,\eta}^\kappa \otimes \mathcal{O}_2, \\ V_a &\in \mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}} \longrightarrow T_a \otimes S_2 \in \mathcal{O}_{\rho,\eta}^\kappa \otimes \mathcal{O}_2, \\ w &\in \mathcal{B}_\rho \longrightarrow w \otimes 1 \in \mathcal{O}_{\rho,\eta}^\kappa \otimes \mathcal{O}_2, \\ z &\in \mathcal{B}_\eta \longrightarrow z \otimes 1 \in \mathcal{O}_{\rho,\eta}^\kappa \otimes \mathcal{O}_2 \end{aligned}$$

for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ give rise to a $*$ -isomorphism from $\mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}}$ to the C^* -subalgebra of $\mathcal{O}_{\rho,\eta}^\kappa \otimes \mathcal{O}_2$ generated by the partial isometries $S_\alpha \otimes S_1, T_a \otimes S_2$ for $\alpha \in \Sigma^\rho, a \in \Sigma^\eta$ and the elements $w \otimes 1, z \otimes 1$ for $w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta$. That is

$$\mathcal{O}_{\mathcal{H}_\kappa^{\rho,\eta}} \cong C^*(S_\alpha \otimes S_1, T_a \otimes S_2, w \otimes 1, z \otimes 1 : \alpha \in \Sigma^\rho, a \in \Sigma^\eta, w \in \mathcal{B}_\rho, z \in \mathcal{B}_\eta).$$

We will present an example. Let α, β be automorphisms of a unital commutative C^* -algebra \mathcal{A} . Put $\Sigma^\rho = \{\alpha\}, \Sigma^\eta = \{\beta\}$ and define $\rho_\alpha = \alpha, \eta_\beta = \beta$. We have two C^* -symbolic dynamical systems $(\mathcal{A}, \alpha, \{\alpha\}), (\mathcal{A}, \beta, \{\beta\})$. Assume that $\alpha \circ \beta = \beta \circ \alpha$. Put $\Sigma^{\alpha\beta} = \{(\alpha, \beta)\}, \Sigma^{\beta\alpha} = \{(\beta, \alpha)\}$. The specification $\kappa : \Sigma^{\alpha\beta} \longrightarrow \Sigma^{\beta\alpha}$ is unique and satisfies $\kappa(\alpha, \beta) = (\beta, \alpha)$. We have a C^* -textile dynamical system $(\mathcal{A}, \alpha, \beta, \{\alpha\}, \{\beta\}, \kappa)$. We denote by $\mathcal{H}_\kappa^{\alpha,\beta}$ the associated Hilbert C^* -quad module. Since $\Sigma_\kappa = \{(\alpha, \beta, \beta, \alpha)\}$ is a singleton and $E_\omega = \beta(\alpha(1)) = 1$ so that $\mathcal{H}_\kappa^{\alpha,\beta} = \mathcal{A}$. As $\alpha \circ \beta = \beta \circ \alpha$, they induce an action of \mathbb{Z}^2 on \mathcal{A} . By the universality subject to the relations (1.2), (1.3) and (1.4), one easily sees that the algebra $\mathcal{O}_{\rho,\eta}^\kappa$ for the C^* -textile dynamical system $(\mathcal{A}, \alpha, \beta, \{\alpha\}, \{\beta\}, \kappa)$ is $*$ -isomorphic to the crossed product $\mathcal{A} \times_{\alpha,\beta} \mathbb{Z}^2$. Take implementing unitaries U, V in $\mathcal{A} \times_{\alpha,\beta} \mathbb{Z}^2$ for the action such that

$$\alpha(x) = U^* x U, \quad \beta(x) = V^* x V \quad \text{for } x \in \mathcal{A}.$$

We have $\alpha^{-1}(x) = U x U^*, \beta^{-1}(x) = V x V^*$ for $x \in \mathcal{A}$ which belong to \mathcal{A} so that

$$\mathcal{B}_\alpha = C^*(U x U^* \mid x \in \mathcal{A}) = \mathcal{A}, \quad \mathcal{B}_\beta = C^*(V x V^* \mid x \in \mathcal{A}) = \mathcal{A}.$$

For $w \in \mathcal{B}_\alpha, z \in \mathcal{B}_\beta$ we have

$$w = U\alpha(w)U^*, \quad z = V\alpha(z)V^*.$$

We will write down the Hilbert C^* -quad module structure for $\mathcal{H}_\kappa^{\alpha,\beta} = \mathcal{A}$ defined in Section 3. For $\xi = x, \xi' = x' \in \mathcal{H}_\kappa^{\alpha,\beta} = \mathcal{A}, y \in \mathcal{A}, w \in \mathcal{B}_\alpha = \mathcal{A}, z \in \mathcal{B}_\beta = \mathcal{A}$:

0. The right \mathcal{A} -module and the right \mathcal{A} -valued inner product $\langle \cdot | \cdot \rangle_{\mathcal{A}}$:

$$\xi\varphi_{\mathcal{A}}(y) = xy \text{ for } y \in \mathcal{A}, \quad \langle \xi | \xi' \rangle_{\mathcal{A}} = x^*x'.$$

1. The right action of \mathcal{B}_α and the right action of \mathcal{B}_β :

$$\xi\varphi_\alpha(w) = x\alpha(w), \quad \xi\varphi_\alpha(z) = x\beta(z).$$

2. The right \mathcal{B}_α -valued inner product and the right \mathcal{B}_β -valued inner product:

$$\langle \xi | \xi' \rangle_\alpha = Ux^*x'U^* = \alpha^{-1}(x^*x'), \quad \langle \xi | \xi' \rangle_\beta = Vx^*x'V^* = \beta^{-1}(x^*x').$$

3. The left action of \mathcal{B}_α and the left action of \mathcal{B}_β :

$$\phi_\alpha(w)\xi = \beta(\alpha(w))x, \quad \phi_\beta(z)\xi = \alpha(\beta(z))x.$$

We then have

$$\widehat{\rho}_\alpha(w) = \alpha(w), \quad \widehat{\eta}_\beta(z) = \beta(z) \quad \text{for } w \in \mathcal{B}_\alpha = \mathcal{A}, z \in \mathcal{B}_\beta = \mathcal{A}.$$

For the $*$ -homomorphisms $\widehat{\rho}_\alpha^\eta : \mathcal{B}_\beta \rightarrow \mathcal{B}_\beta$ and $\widehat{\eta}_\beta^\rho : \mathcal{B}_\alpha \rightarrow \mathcal{B}_\alpha$, we have for $z \in \mathcal{B}_\beta$,

$$\widehat{\rho}_\alpha^\eta(z) = V\alpha(z_a)V^* = V\alpha(V^*zV)V^* = \beta^{-1}(\alpha(\beta(z))) = \alpha(z)$$

and similarly $\widehat{\eta}_\beta^\rho(w) = \beta(w)$ for $w \in \mathcal{B}_\alpha$. Hence the C^* -algebra $\mathcal{O}_{\mathcal{H}_\kappa^{\alpha,\beta}}$ defined by the Hilbert C^* -quad module $\mathcal{H}_\kappa^{\alpha,\beta}$ is the universal C^* -algebra generated by two isometries u, v and elements $x \in \mathcal{A}$ subject to the relations

$$\begin{aligned} uu^* + vv^* &= 1, \\ uu^*x &= xuu^*, \quad vv^*x = xvv^*, \\ \alpha(x) &= u^*xu, \quad \beta(x) = v^*xv \end{aligned}$$

for $x \in \mathcal{A}$. By the universality of the algebra $\mathcal{O}_{\mathcal{H}_\kappa^{\alpha,\beta}}$ the correspondence

$$\begin{aligned} u &\in \mathcal{O}_{\mathcal{H}_\kappa^{\alpha,\beta}} \rightarrow \widehat{u} = U \otimes S_1 \in \mathcal{A} \times_{\alpha,\beta} \mathbb{Z}^2 \otimes \mathcal{O}_2, \\ v &\in \mathcal{O}_{\mathcal{H}_\kappa^{\alpha,\beta}} \rightarrow \widehat{v} = V \otimes S_2 \in \mathcal{A} \times_{\alpha,\beta} \mathbb{Z}^2 \otimes \mathcal{O}_2, \\ x &\in \mathcal{A} \rightarrow \widehat{x} = x \otimes 1 \in \mathcal{A} \times_{\alpha,\beta} \mathbb{Z}^2 \otimes \mathcal{O}_2 \end{aligned}$$

gives rise to a $*$ -homomorphism. If in particular, the action $(n, m) \in \mathbb{Z}^2 \rightarrow \alpha^n \circ \beta^m \in \text{Aut}(\mathcal{A})$ is outer, $(\mathcal{A}, \alpha, \beta, \{\alpha\}, \{\beta\}, \kappa)$ satisfies condition (I) so that the above correspondence gives rise to a $*$ -isomorphism, that is

$$\mathcal{O}_{\mathcal{H}_\kappa^{\alpha,\beta}} \cong C^*(U \otimes S_1, V \otimes S_2, x \mid x \in \mathcal{A}) \subset \mathcal{A} \times_{\alpha,\beta} \mathbb{Z}^2 \otimes \mathcal{O}_2.$$

7. Textile systems of commuting matrices

We will study C^* -algebras associated with the Hilbert C^* -quad modules defined by textile systems of commuting matrices (cf. [21]). Let A be an $N \times N$ matrix with entries in nonnegative integers. We may consider a directed graph $G_A = (V, E_A)$ with vertex set $V = \{v_1, \dots, v_N\}$ and edge set E_A consisting of $A(i, j)$ edges from the vertex v_i to the vertex v_j . Denote by $\Sigma^A = E_A$. Let \mathcal{A} be the N -dimensional commutative C^* -algebra \mathbb{C}^N with minimal projections E_1, \dots, E_N such that

$$\mathcal{A} = \mathbb{C}E_1 \oplus \dots \oplus \mathbb{C}E_N.$$

We set for $\alpha \in \Sigma^A, i, j = 1, \dots, N$

$$\widehat{A}(i, \alpha, j) = \begin{cases} 1 & \text{if } s(\alpha) = v_i, r(\alpha) = v_j, \\ 0 & \text{otherwise} \end{cases}$$

where $s(\alpha), r(\alpha)$ mean the source vertex, the range vertex of an edge α respectively. We define an endomorphism ρ_α^A on \mathcal{A} for $\alpha \in \Sigma^A$:

$$\rho_\alpha^A(E_i) = \sum_{j=1}^N \widehat{A}(i, \alpha, j) E_j, \quad i = 1, \dots, N.$$

Then we have a C^* -symbolic dynamical system $(\mathcal{A}, \rho^A, \Sigma^A)$. Let B be another $N \times N$ matrix with entries in nonnegative integers such that

$$AB = BA. \quad (7.1)$$

Consider the associated directed graph $G_B = (V, E_B)$ and the C^* -symbolic dynamical system $(\mathcal{A}, \rho^B, \Sigma^B)$ for B . Let $S_\alpha, \alpha \in E_A, T_a, a \in E_B$ be the generating partial isometries of the associated C^* -algebras \mathcal{O}_{ρ^A} and \mathcal{O}_{ρ^B} respectively. They satisfy the relations:

$$\begin{aligned} \sum_{\beta \in E_A} S_\beta S_\beta^* &= 1, & x S_\alpha S_\alpha^* &= S_\alpha S_\alpha^* x, & S_\alpha^* x S_\alpha &= \rho_\alpha^A(x), \\ \sum_{b \in E_B} T_b T_b^* &= 1, & x T_a T_a^* &= T_a T_a^* x, & T_a^* x T_a &= \rho_a^B(x), \end{aligned}$$

for all $x \in \mathcal{A}$ and $\alpha \in E_A, a \in E_B$ respectively. The C^* -algebras \mathcal{O}_{ρ^A} and \mathcal{O}_{ρ^B} are isomorphic to the Cuntz–Krieger algebras $\mathcal{O}_{\widehat{A}}$ and $\mathcal{O}_{\widehat{B}}$ respectively. Put subalgebras

$$\begin{aligned} \mathcal{B}_A &= C^*(S_\alpha E_i S_\alpha^* : \alpha \in E_A, i = 1, \dots, N) \subset \mathcal{O}_{\rho^A}, \\ \mathcal{B}_B &= C^*(T_a E_i T_a^* : a \in E_B, i = 1, \dots, N) \subset \mathcal{O}_{\rho^B}. \end{aligned}$$

Since $S_\alpha E_i S_\alpha^* \neq 0$ if and only if $S_\alpha^* S_\alpha = E_i$, which is equivalent to $r(\alpha) = v_i$, hence \mathcal{B}_A is of $|E_A|$ -dimension, and similarly \mathcal{B}_B is of $|E_B|$ -dimension. By the identities

$$E_i = \sum_{\alpha \in E_A} S_\alpha \rho_\alpha^A(E_i) S_\alpha^* = \sum_{\alpha \in E_A} \sum_{j=1}^N \widehat{A}(i, \alpha, j) S_\alpha E_j S_\alpha^*, \quad i = 1, \dots, N,$$

E_i belongs to \mathcal{B}_A so that $\mathcal{A} \subset \mathcal{B}_A$ and similarly $\mathcal{A} \subset \mathcal{B}_B$. The equality (7.1) implies that the cardinal numbers of the sets of the pairs of directed edges

$$\Sigma_{(i,j)}^{AB} = \{(\alpha, b) \in E_A \times E_B \mid s(\alpha) = v_i, r(\alpha) = s(b), r(b) = v_j\},$$

$$\Sigma_{(i,j)}^{BA} = \{(a, \beta) \in E_B \times E_A \mid s(a) = v_i, r(a) = s(\beta), r(\beta) = v_j\}$$

coincide with each other for each v_i and v_j , so that one may take a bijection $\kappa : \Sigma^{AB} = \cup_{i,j=1}^N \Sigma_{(i,j)}^{AB} \longrightarrow \Sigma^{BA} = \cup_{i,j=1}^N \Sigma_{(i,j)}^{BA}$ such that

$$s(\alpha) = s(a), \quad r(b) = r(\beta) \quad \text{if } \kappa(\alpha, b) = (a, \beta).$$

We then have

Lemma 7.1. *For $(\alpha, b) \in \Sigma^{AB}$, $(a, \beta) \in \Sigma^{BA}$ with $\kappa(\alpha, b) = (a, \beta)$, we have*

$$\rho_b^B \circ \rho_\alpha^A(E_i) = \rho_\beta^A \circ \rho_a^B(E_i), \quad i = 1, \dots, N.$$

Proof. We have for $i, k = 1, \dots, N$

$$\rho_b^B \circ \rho_\alpha^A(E_i)E_k = \sum_{j=1}^N \hat{A}(i, \alpha, j) \rho_b^B(E_j)E_k = \sum_{j=1}^N \hat{A}(i, \alpha, j) \hat{B}(j, b, k)E_k.$$

Hence $\rho_b^B \circ \rho_\alpha^A(E_i)E_k = E_k$ if and only if $v_i = s(\alpha), r(\alpha) = s(b), r(b) = v_k$. Similarly for $(a, \beta) \in \Sigma^{BA}$, we have $\rho_\beta^A \circ \rho_a^B(E_i)E_k = E_k$ if and only if $v_i = s(a), r(a) = s(\beta), r(\beta) = v_k$. The condition $\kappa(\alpha, b) = (a, \beta)$ implies $r(\alpha) = s(b), r(a) = s(\beta), s(\alpha) = s(a), r(b) = r(\beta)$. Therefore

$$\rho_b^B \circ \rho_\alpha^A(E_i)E_k = \rho_\beta^A \circ \rho_a^B(E_i)E_k \quad \text{for } i, k = 1, \dots, N$$

and hence $\rho_b^B \circ \rho_\alpha^A = \rho_\beta^A \circ \rho_a^B$ if $\kappa(\alpha, b) = (a, \beta)$. \square

We thus have a C^* -textile dynamical system

$$(\mathcal{A}, \rho^A, \rho^B, E_A, E_B, \kappa).$$

We set

$$E_\kappa = \{(\alpha, b, a, \beta) \in E_A \times E_B \times E_B \times E_A \mid \kappa(\alpha, b) = (a, \beta)\}.$$

For $a \in E_B$ and $\alpha \in E_A$, we will describe the $*$ -homomorphisms

$$\hat{\eta}_a^\rho : \mathcal{B}_{\rho^A}(= \mathcal{B}_A) \longrightarrow \mathcal{B}_{\rho^A}(= \mathcal{B}_A) \quad \text{and} \quad \hat{\rho}_\alpha^\eta : \mathcal{B}_{\rho^B}(= \mathcal{B}_B) \longrightarrow \mathcal{B}_{\rho^B}(= \mathcal{B}_B)$$

which will be denoted by $\hat{\rho}_a^A$ and by $\hat{\rho}_\alpha^B$ respectively. We set

$$E_{AB} = \Sigma^{AB}, \quad E_{BA} = \Sigma^{BA}.$$

For $(a, \beta) \in E_{BA}$ there uniquely exists $(\alpha, b) \in E_{AB}$ such that $\kappa(\alpha, b) = (a, \beta)$. We then define a map for $a \in E_B$

$$\kappa_a : \beta \in \{\beta \in E_A \mid (a, \beta) \in E_{BA}\} \longrightarrow \alpha \in \{\alpha \in E_A \mid \kappa(\alpha, b) = (a, \beta) \text{ for some } b \in E_B\}.$$

Similarly, we then define a map for $\alpha \in E_A$

$$\kappa_\alpha : b \in \{b \in E_B \mid (\alpha, b) \in E_{AB}\} \longrightarrow a \in \{a \in E_B \mid \kappa(\alpha, b) = (a, \beta) \text{ for some } \beta \in E_A\}.$$

We write the projection E_i also as E_{v_i} . Hence $w \in \mathcal{B}_A, z \in \mathcal{B}_B$ are uniquely written as $w = \sum_{\alpha \in E_A} w(\alpha) S_\alpha E_{r(\alpha)} S_\alpha^*, z = \sum_{a \in E_B} z(a) T_a E_{r(a)} T_a^*$ for $w(\alpha), z(a) \in \mathbb{C}$.

Lemma 7.2. *Keep the above notations. We have*

$$\hat{\rho}_a^A(w) = \sum_{\beta \in E_A} w(\kappa_a(\beta)) S_\beta E_{r(\beta)} S_\beta^*, \quad \hat{\rho}_\alpha^B(z) = \sum_{b \in E_B} z(\kappa_\alpha(b)) T_b E_{r(b)} T_b^*.$$

Proof. We have

$$\hat{\rho}_a^A(w) = \sum_{\substack{\beta, b, \alpha \\ (\alpha, b, a, \beta) \in E_\kappa}} S_\beta \rho_b^B(w(\alpha) E_{r(\alpha)}) S_\beta^* = \sum_{\substack{\beta, b, \alpha \\ (\alpha, b, a, \beta) \in E_\kappa}} w(\alpha) S_\beta T_b^* E_{r(\alpha)} T_b S_\beta^*.$$

Since $T_b^* E_{r(\alpha)} T_b = E_{r(b)} = E_{r(\beta)}$ and $\alpha = \kappa_a(\beta)$, we see

$$\hat{\rho}_a^A(w) = \sum_{\beta \in E_A} w(\kappa_a(\beta)) S_\beta E_{r(\beta)} S_\beta^*.$$

We similarly have

$$\hat{\rho}_\alpha^B(z) = \sum_{\substack{b, a, \beta \\ (\alpha, b, a, \beta) \in E_\kappa}} T_b \rho_\beta^A(z(a) E_{r(a)}) T_b^* = \sum_{\substack{b, a, \beta \\ (\alpha, b, a, \beta) \in E_\kappa}} z(a) T_b S_\beta^* E_{r(\beta)} S_\beta T_b^*.$$

Since $S_\beta^* E_{r(a)} S_\beta = E_{r(\beta)} = E_{r(b)}$ and $a = \kappa_\alpha(b)$, we see

$$\hat{\rho}_\alpha^B(z) = \sum_{b \in E_B} z(\kappa_\alpha(b)) T_b E_{r(b)} T_b^*. \quad \square$$

For $\omega = (\alpha, b, a, \beta) \in E_\kappa$, put $v(\omega) = r(b) (= r(\beta)) \in V$ and

$$E_\omega = E_{v(\omega)} (= \rho_b^B(\rho_\alpha^A(1)) = \rho_\beta^A(\rho_a^B(1))).$$

Then the Hilbert C^* -quad module $\mathcal{H}_\kappa^{\rho^A, \rho^B}$ is written as $\mathcal{H}_\kappa^{A, B}$ and regarded as

$$\mathcal{H}_\kappa^{A, B} = \bigoplus_{\omega \in E_\kappa} \mathbb{C} E_\omega.$$

For $\xi = \sum_{\omega \in E_\kappa} \xi(\omega) E_{v(\omega)}, \xi' = \sum_{\omega \in E_\kappa} \xi'(\omega) E_{v(\omega)} \in \mathcal{H}_\kappa^{A, B}$ and $y = \sum_{v \in V} y(v) E_v \in \mathcal{A}$ with $\xi(\omega), \xi'(\omega)$ and $y(v) \in \mathbb{C}$, we see:

0. The right \mathcal{A} -module structure and the right \mathcal{A} -valued inner product are written as follows:

$$\xi \varphi_{\mathcal{A}}(y) = \sum_{\omega \in E_\kappa} \xi(\omega) E_{v(\omega)} y(v(\omega)), \quad \langle \xi | \xi' \rangle_{\mathcal{A}} = \sum_{\omega \in E_\kappa} \overline{\xi(\omega)} \xi'(\omega) E_{v(\omega)}.$$

Furthermore for $w = \sum_{\alpha \in E_A} w(\alpha) S_\alpha E_{r(\alpha)} S_\alpha^*, z = \sum_{a \in E_B} z(a) T_a E_{r(a)} T_a^*$ with $w(\alpha), z(a) \in \mathbb{C}$:

1. The right \mathcal{B}_A -action φ_A and the right \mathcal{B}_B -action φ_B are written as follows:

$$\begin{aligned} \xi \varphi_A(w) &= \sum_{\omega \in E_\kappa} \xi(\omega) E_{v(\omega)} \varphi_A(w) = \sum_{\omega \in E_\kappa} \xi(\omega) w(b(\omega)) E_{v(\omega)}, \\ \xi \varphi_B(z) &= \sum_{\omega \in E_\kappa} \xi(\omega) E_{v(\omega)} \varphi_B(z) = \sum_{\omega \in E_\kappa} \xi(\omega) z(r(\omega)) E_{v(\omega)}. \end{aligned}$$

2. The left \mathcal{B}_A -action ϕ_A and the left \mathcal{B}_B -action ϕ_B are written as follows:

$$\begin{aligned}\phi_A(w)\xi &= \sum_{\omega \in E_\kappa} \xi(\omega) E_{v(\omega)} \phi_A(w) \\ &= \sum_{\omega \in E_\kappa} \xi(\omega) E_{v(\omega)} \rho_{r(\omega)}^B(w(t(\omega)) E_{r(t(\omega))}) \\ &= \sum_{\omega \in E_\kappa} \xi(\omega) w(t(\omega)) E_{v(\omega)} \rho_{r(\omega)}^B(E_{r(t(\omega))}).\end{aligned}$$

As $\rho_{r(\omega)}^B(E_{r(t(\omega))}) = E_{v(\omega)}$, we have

$$\phi_A(w)\xi = \sum_{\omega \in E_\kappa} \xi(\omega) w(t(\omega)) E_{v(\omega)}.$$

We also have

$$\begin{aligned}\phi_B(z)\xi &= \sum_{\omega \in E_\kappa} \xi(\omega) E_{v(\omega)} \phi_B(z) \\ &= \sum_{\omega \in E_\kappa} \xi(\omega) E_{v(\omega)} \rho_{b(\omega)}^A(z(l(\omega)) E_{r(l(\omega))}) \\ &= \sum_{\omega \in E_\kappa} \xi(\omega) z(l(\omega)) E_{v(\omega)} \rho_{b(\omega)}^A(E_{r(l(\omega))}).\end{aligned}$$

As $\rho_{b(\omega)}^A(E_{r(l(\omega))}) = E_{v(\omega)}$, we have

$$\phi_B(z)\xi = \sum_{\omega \in E_\kappa} \xi(\omega) z(l(\omega)) E_{v(\omega)}.$$

3. The right \mathcal{B}_A -valued inner product $\langle \cdot | \cdot \rangle_A$ and the right \mathcal{B}_B -valued inner product $\langle \cdot | \cdot \rangle_B$ are written as follows:

$$\begin{aligned}\langle \xi | \xi' \rangle_A &= \sum_{\omega \in E_\kappa} S_{b(\omega)} \overline{\xi(\omega)} E_{v(\omega)} \xi'(\omega) S_{b(\omega)}^*, \\ \langle \xi | \xi' \rangle_B &= \sum_{\omega \in E_\kappa} T_{r(\omega)} \overline{\xi(\omega)} E_{v(\omega)} \xi'(\omega) T_{r(\omega)}^*.\end{aligned}$$

As $E_{v(\omega)} = S_{b(\omega)}^* S_{b(\omega)} = T_{r(\omega)}^* T_{r(\omega)}$, we have

$$\langle \xi | \xi' \rangle_A = \sum_{\omega \in E_\kappa} \overline{\xi(\omega)} \xi'(\omega) S_{b(\omega)} S_{b(\omega)}^*, \quad \langle \xi | \xi' \rangle_B = \sum_{\omega \in E_\kappa} \overline{\xi(\omega)} \xi'(\omega) T_{r(\omega)} T_{r(\omega)}^*.$$

4. The positive maps $\lambda_A : \mathcal{B}_A \longrightarrow \mathcal{A}$ and $\lambda_B : \mathcal{B}_B \longrightarrow \mathcal{A}$ are written as follows:

$$\lambda_A(w) = \sum_{\alpha \in E_A} w(\alpha) E_{r(\alpha)}, \quad \lambda_B(z) = \sum_{a \in E_B} z(a) E_{r(a)}.$$

Hence we have

$$\lambda_A(\langle \xi | \xi' \rangle_A) = \langle \xi | \xi' \rangle_{\mathcal{A}}, \quad \lambda_B(\langle \xi | \xi' \rangle_B) = \langle \xi | \xi' \rangle_{\mathcal{A}}.$$

Put $p_\alpha = S_\alpha E_{r(\alpha)} S_\alpha^*$ for $\alpha \in E_A$ and $q_a = T_a E_{r(a)} T_a^*$ for $a \in E_B$. Hence

$$\mathcal{B}_A = \sum_{\alpha \in E_A} \mathbb{C} p_\alpha, \quad \mathcal{B}_B = \sum_{a \in E_B} \mathbb{C} q_a.$$

By Lemma 7.2, we have

$$\widehat{\rho}_\alpha^A(w) = \sum_{\beta \in E_A} w(\kappa_\alpha(\beta)) p_\beta \quad \text{for } w = \sum_{\alpha \in E_A} w(\alpha) p_\alpha \in \mathcal{B}_A, \quad (7.2)$$

$$\widehat{\rho}_\alpha^B(z) = \sum_{b \in E_B} z(\kappa_\alpha(b)) q_b \quad \text{for } z = \sum_{a \in E_B} z(a) q_a \in \mathcal{B}_B. \quad (7.3)$$

We define $\kappa_B : E_A \times E_B \times E_A \longrightarrow \{0, 1\}$ and $\kappa_A : E_B \times E_A \times E_B \longrightarrow \{0, 1\}$ by

$$\kappa_B(\alpha, a, \beta) = \begin{cases} 1 & \text{if } \kappa_\alpha(\beta) = a, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \kappa_A(a, \alpha, b) = \begin{cases} 1 & \text{if } \kappa_\alpha(b) = a, \\ 0 & \text{otherwise.} \end{cases}$$

We identify the C^* -algebra $\mathcal{O}_{\mathcal{H}_\kappa^{A,B}}$ with the universal C^* -algebra subject to the relation $(\mathcal{H}_\kappa^{A,B})$, and denote the generating partial isometries by $\mathbf{u}_\alpha, \alpha \in E_A$ and $\mathbf{v}_a, a \in E_B$. Since $\widehat{\rho}_\alpha^A(w) = \mathbf{v}_a^* w \mathbf{v}_a$ and $\widehat{\rho}_\alpha^B(z) = \mathbf{u}_\alpha^* z \mathbf{u}_\alpha$ with (7.2) and (7.3), we have

Lemma 7.3.

$$\mathbf{v}_a^* p_\alpha \mathbf{v}_a = \sum_{\beta \in E_A} \kappa_B(\alpha, a, \beta) p_\beta, \quad \mathbf{u}_\alpha^* q_a \mathbf{u}_\alpha = \sum_{b \in E_B} \kappa_A(a, \alpha, b) q_b.$$

Define $|E_A| \times |E_A|$ -matrix $A^E = [A^E(\alpha, \beta)]_{\alpha, \beta \in E_A}$ and $|E_B| \times |E_B|$ -matrix $B^E = [B^E(a, b)]_{a, b \in E_B}$ by

$$A^E(\alpha, \beta) = \begin{cases} 1 & \text{if } r(\alpha) = s(\beta), \\ 0 & \text{if } r(\alpha) \neq s(\beta), \end{cases} \quad B^E(a, b) = \begin{cases} 1 & \text{if } r(a) = s(b), \\ 0 & \text{if } r(a) \neq s(b) \end{cases}$$

respectively. We then have for $\delta \in E_A$,

$$E_{r(\delta)} = \sum_{\beta \in E_A} S_\beta \rho_\beta^A(E_{r(\delta)}) S_\beta^* = \sum_{\beta \in E_A} S_\beta \widehat{A}(r(\delta), \beta, r(\beta)) E_{r(\beta)} S_\beta^* = \sum_{\beta \in E_A} A^E(\delta, \beta) p_\beta$$

and similarly for $d \in E_B$

$$E_{r(d)} = \sum_{b \in E_B} B^E(d, b) q_b.$$

Since we know that

$$\widehat{\rho}_\alpha^B(p_\delta) = \begin{cases} E_{r(\delta)} & \text{if } \delta = \alpha, \\ 0 & \text{otherwise,} \end{cases} \quad \widehat{\rho}_a^A(q_d) = \begin{cases} E_{r(d)} & \text{if } d = a, \\ 0 & \text{otherwise} \end{cases}$$

and $\sum_{\delta \in E_A} p_\delta = \sum_{d \in E_B} q_d = 1$, we have

$$\mathbf{u}_\alpha^* \mathbf{u}_\alpha = \sum_{\beta \in E_A} A^E(\alpha, \beta) p_\beta, \quad \mathbf{v}_a^* \mathbf{v}_a = \sum_{b \in E_B} B^E(a, b) q_b.$$

Therefore we have

Proposition 7.4. *The C^* -algebra $\mathcal{O}_{\mathcal{H}_\kappa^{A,B}}$ is $*$ -isomorphic to the universal C^* -algebra generated by two families of projections $\{p_\alpha\}_{\alpha \in E_A}$, $\{q_a\}_{a \in E_B}$ and two families of partial isometries $\{u_\alpha\}_{\alpha \in E_A}$, $\{v_a\}_{a \in E_B}$ subject to the relations:*

$$\sum_{\beta \in E_A} p_\beta = \sum_{b \in E_B} q_b = \sum_{\beta \in E_A} u_\beta u_\beta^* + \sum_{b \in E_B} v_b v_b^* = 1, \quad (7.4)$$

$$u_\alpha u_\alpha^* p_\alpha = u_\alpha u_\alpha^*, \quad v_a v_a^* q_a = v_a v_a^*, \quad (7.5)$$

$$u_\alpha u_\alpha^* q_a = q_a u_\alpha u_\alpha^*, \quad v_a v_a^* p_\alpha = p_\alpha v_a v_a^*, \quad (7.6)$$

$$u_\alpha^* u_\alpha = \sum_{\beta \in E_A} A^E(\alpha, \beta) p_\beta, \quad v_a^* v_a = \sum_{b \in E_B} B^E(a, b) q_b, \quad (7.7)$$

$$u_\alpha^* q_a u_\alpha = \sum_{b \in E_B} \kappa_A(a, \alpha, b) q_b, \quad v_a^* p_\alpha v_a = \sum_{\beta \in E_A} \kappa_B(\alpha, a, \beta) p_\beta \quad (7.8)$$

for $\alpha \in E_A$ and $a \in E_B$ where

$$\kappa_A(a, \alpha, b) = \begin{cases} 1 & \text{if } \kappa(\alpha, b) = (a, \beta) \text{ for some } \beta \in E_A, \\ 0 & \text{otherwise,} \end{cases}$$

$$\kappa_B(\alpha, a, \beta) = \begin{cases} 1 & \text{if } \kappa(\alpha, b) = (a, \beta) \text{ for some } b \in E_B, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The relations (7.4), (7.5) and (7.7) imply

$$u_\alpha^* p_\delta u_\alpha = \begin{cases} \sum_{\beta \in E_A} A^E(\alpha, \beta) p_\beta & \text{if } \delta = \alpha, \\ 0 & \text{otherwise} \end{cases}$$

and

$$v_a^* q_d v_a = \begin{cases} \sum_{b \in E_B} B^E(a, b) q_b & \text{if } d = a, \\ 0 & \text{otherwise.} \end{cases}$$

The above two relations are equivalent to (5.5) and (5.6). Since the two C^* -algebras \mathcal{B}_A and \mathcal{B}_B are generated by the projections $\{p_\alpha\}_{\alpha \in E_A}$ and $\{q_a\}_{a \in E_B}$ respectively, we see that the relations (7.4), (7.5), (7.6), (7.7) and (7.8) are equivalent to the relations $(\mathcal{H}_\kappa^{A,B})$. \square

We will further study the above operator relations.

Lemma 7.5. p_α commutes with q_a for all $\alpha \in E_A$, $a \in E_B$.

Proof. For $\alpha \in E_A$, $a \in E_B$, by (7.4), we have

$$p_\alpha = \sum_{\beta \in E_A} u_\beta u_\beta^* p_\alpha + \sum_{b \in E_B} v_b v_b^* p_\alpha$$

By (7.5), we have

$$u_\beta u_\beta^* p_\alpha = \begin{cases} u_\alpha u_\alpha^* & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta, \end{cases} \quad q_a v_b v_b^* = \begin{cases} v_a v_a^* & \text{if } a = b, \\ 0 & \text{if } a \neq b \end{cases}$$

so that $p_\alpha = u_\alpha u_\alpha^* + \sum_{b \in E_B} v_b v_b^* p_\alpha$ and hence $q_a p_\alpha = q_a u_\alpha u_\alpha^* + v_a v_a^* p_\alpha$. Since q_a commutes with $u_\alpha u_\alpha^*$ and p_α commutes with $v_a v_a^*$, we have

$$q_a p_\alpha = u_\alpha u_\alpha^* q_a + p_\alpha v_a v_a^*, \quad (7.9)$$

which is symmetrically equal to $p_\alpha q_a$. \square

Put for $\alpha, \beta \in E_A$ and $a, b \in E_B$

$$\kappa_{AB}(\alpha, b) = \begin{cases} 1 & \text{if } r(\alpha) = s(b), \\ 0 & \text{otherwise,} \end{cases} \quad \kappa_{BA}(a, \beta) = \begin{cases} 1 & \text{if } r(a) = s(\beta), \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 7.6. For $\alpha \in E_A, a \in E_B$, we have

$$u_\alpha^* u_\alpha = \sum_{b \in E_B} \kappa_{AB}(\alpha, b) q_b, \quad v_a^* v_a = \sum_{\beta \in E_A} \kappa_{BA}(a, \beta) p_\beta.$$

Proof. By (7.4) and (7.8) and the equality $\sum_{a \in E_B} \kappa_A(a, \alpha, b) = \kappa_{AB}(\alpha, b)$, we have

$$u_\alpha^* u_\alpha = \sum_{a \in E_B} u_\alpha^* q_a u_\alpha = \sum_{a \in E_B} \sum_{b \in E_B} \kappa_A(a, \alpha, b) q_b = \sum_{b \in E_B} \kappa_{AB}(\alpha, b) q_b.$$

The equality for $v_a^* v_a$ is similarly shown. \square

Lemma 7.7. For $\alpha \in E_A, a \in E_B$, if $r(\alpha) = r(a)$, then $u_\alpha^* u_\alpha = v_a^* v_a$.

Proof. The condition $r(\alpha) = r(a)$ implies $\sum_{b \in E_B} \kappa_{AB}(\alpha, b) q_b = \sum_{b \in E_B} B^E(a, b) q_b$. By the equality for $u_\alpha^* u_\alpha$ in the preceding lemma and the equality for $v_a^* v_a$ in (7.7), we have $u_\alpha^* u_\alpha = v_a^* v_a$. \square

Put

$$\Omega_\kappa = \{(\alpha, a) \in E_A \times E_B \mid s(\alpha) = s(a), \kappa(\alpha, b) = (a, \beta) \text{ for some } \beta \in E_A, b \in E_B\}$$

and $e_{\alpha, a} = p_\alpha q_a$ for $(\alpha, a) \in \Omega_\kappa$.

Lemma 7.8.

- (i) $\sum_{(\alpha, a) \in \Omega_\kappa} e_{(\alpha, a)} = 1$.
- (ii) The C^* -subalgebra \mathcal{B}_κ of $\mathcal{O}_{\mathcal{H}_\kappa^{A, B}}$ generated by the subalgebras \mathcal{B}_p and \mathcal{B}_η is $*$ -isomorphic to the direct sum $\bigoplus_{(\alpha, a) \in \Omega_\kappa} \mathbb{C} e_{(\alpha, a)}$. It is the C^* -algebra of all complex valued continuous functions on Ω_κ .

Proof. By the equality (7.9), one sees that $p_\alpha q_a \neq 0$ if and only if $u_\alpha^* q_a u_\alpha \neq 0$ or $v_a^* p_\alpha v_a \neq 0$. The latter condition is equivalent to the condition that there exists $b \in E_B$ such that $\kappa_A(a, \alpha, b) \neq 0$ or there exists $\beta \in E_A$ such that $\kappa_B(\alpha, a, \beta) \neq 0$, which is also equivalent to the condition that there exist $b \in E_B$ and $\beta \in E_A$ such that $\kappa(\alpha, b) = (a, \beta)$. Hence we have $p_\alpha q_a \neq 0$ if and only if $(\alpha, a) \in \Omega_\kappa$. Therefore we have

$$\sum_{(\alpha, a) \in \Omega_\kappa} e_{(\alpha, a)} = \left(\sum_{\alpha \in E_A} p_\alpha \right) \cdot \left(\sum_{a \in E_B} q_a \right) = 1.$$

As $p_\alpha q_a \cdot p_{\alpha'} q_{a'} = 0$ if $\alpha \neq \alpha'$ or $a \neq a'$, the C^* -algebra \mathcal{B}_κ is $*$ -isomorphic to $\bigoplus_{(\alpha, a) \in \Omega_\kappa} \mathbb{C} e_{(\alpha, a)}$. \square

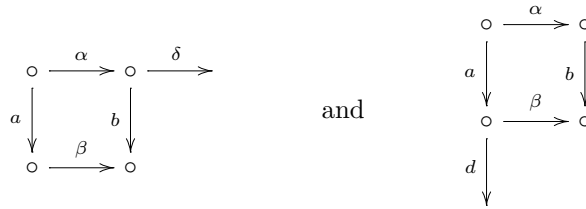
We define two $|\Omega_\kappa| \times |\Omega_\kappa|$ -matrices A_κ and B_κ with entries in $\{0, 1\}$ by

$$A_\kappa((\alpha, a), (\delta, b)) = \begin{cases} 1 & \text{if } \kappa(\alpha, b) = (a, \beta) \text{ for some } \beta \in E_A, \\ 0 & \text{otherwise} \end{cases} \quad (7.10)$$

for $(\alpha, a), (\delta, b) \in \Omega_\kappa$, and

$$B_\kappa((\alpha, a), (\beta, d)) = \begin{cases} 1 & \text{if } \kappa(\alpha, b) = (a, \beta) \text{ for some } b \in E_B, \\ 0 & \text{otherwise} \end{cases} \quad (7.11)$$

for $(\alpha, a), (\beta, d) \in \Omega_\kappa$ respectively. They represent the concatenations of edges as in the following figures respectively:



Proposition 7.9. *The C^* -algebra $\mathcal{O}_{\mathcal{H}_\kappa^{A,B}}$ is $*$ -isomorphic to the universal C^* -algebra generated by a family $\{e_{(\alpha,a)}\}_{(\alpha,a) \in \Omega_\kappa}$ of projections and two families of partial isometries $\{u_\alpha\}_{\alpha \in E_A}$, $\{v_a\}_{a \in E_B}$ subject to the relations:*

$$\sum_{(\alpha,a) \in \Omega_\kappa} e_{(\alpha,a)} = \sum_{\beta \in E_A} u_\beta u_\beta^* + \sum_{b \in E_B} v_b v_b^* = 1, \quad (7.12)$$

$$u_\alpha u_\alpha^* = \sum_{a \in E_B} u_\alpha u_\alpha^* e_{(\alpha,a)} = \sum_{a \in E_B} e_{(\alpha,a)} u_\alpha u_\alpha^*, \quad (7.13)$$

$$v_a v_a^* = \sum_{\alpha \in E_A} v_a v_a^* e_{(\alpha,a)} = \sum_{\alpha \in E_A} e_{(\alpha,a)} v_a v_a^*, \quad (7.14)$$

$$u_\alpha^* e_{(\alpha,a)} u_\alpha = \sum_{(\delta,b) \in \Omega_\kappa} A_\kappa((\alpha,a), (\delta,b)) e_{(\delta,b)}, \quad (7.15)$$

$$v_a^* e_{(\alpha,a)} v_a = \sum_{(\beta,d) \in \Omega_\kappa} B_\kappa((\alpha,a), (\beta,d)) e_{(\beta,d)} \quad (7.16)$$

for $\alpha \in E_A$ and $a \in E_B$.

Proof. Let $u_\alpha, \alpha \in E_A$ and $v_a, a \in E_B$ be the partial isometries as in Proposition 7.4. The equalities (7.13) and (7.14) follow from the equalities (7.4) and (7.5), (7.6). As $u_\alpha^* p_\alpha u_\alpha = u_\alpha u_\alpha^*$ by (7.5), we have by (7.8)

$$\begin{aligned} u_\alpha^* e_{(\alpha,a)} u_\alpha &= u_\alpha^* q_a u_\alpha = \sum_{b \in E_B} \kappa_A(a, \alpha, b) q_b \\ &= \sum_{b \in E_B} \kappa_A(a, \alpha, b) \sum_{\delta \in E_A} p_\delta q_b \\ &= \sum_{(\delta,b) \in \Omega_\kappa} A_\kappa((\alpha,a), (\delta,b)) e_{(\delta,b)}. \end{aligned}$$

The equality (7.16) is similarly shown. Hence the equalities (7.12), ..., (7.16) follow from the equalities (7.4), ..., (7.8). Conversely, from the projections $e_{(\alpha,a)}, (\alpha,a) \in \Omega_\kappa$ by putting

$$p_\alpha = \sum_{a \in E_B} e_{(\alpha,a)}, \quad q_a = \sum_{\alpha \in E_A} e_{(\alpha,a)}$$

the equalities (7.4), ..., (7.8) follow from the equalities (7.12), ..., (7.16) \square

We then see the following theorem:

Theorem 7.10. *The C^* -algebra $\mathcal{O}_{\mathcal{H}_\kappa^{A,B}}$ associated with the Hilbert C^* -quad module $\mathcal{H}_\kappa^{A,B}$ defined by commuting matrices A, B and a specification κ is generated by partial isometries $S_{(\alpha,a)}, T_{(\alpha,a)}$ for $(\alpha,a) \in \Omega_\kappa$ satisfying the relations:*

$$\begin{aligned} \sum_{(\delta,b) \in \Omega_\kappa} S_{(\delta,b)} S_{(\delta,b)}^* + \sum_{(\beta,d) \in \Omega_\kappa} T_{(\beta,d)} T_{(\beta,d)}^* &= 1, \\ S_{(\alpha,a)}^* S_{(\alpha,a)} &= \sum_{(\delta,b) \in \Omega_\kappa} A_\kappa((\alpha,a), (\delta,b)) (S_{(\delta,b)} S_{(\delta,b)}^* + T_{(\delta,b)} T_{(\delta,b)}^*), \\ T_{(\alpha,a)}^* T_{(\alpha,a)} &= \sum_{(\beta,d) \in \Omega_\kappa} B_\kappa((\alpha,a), (\beta,d)) (S_{(\beta,d)} S_{(\beta,d)}^* + T_{(\beta,d)} T_{(\beta,d)}^*) \end{aligned}$$

for $(\alpha, a) \in \Omega_\kappa$.

Proof. The algebra $\mathcal{O}_{\mathcal{H}_\kappa^{A,B}}$ is generated by $\mathbf{u}_\alpha, \alpha \in E_A, \mathbf{v}_a, a \in E_B$ and $e_{(\alpha,a)}, (\alpha, a) \in \Omega_\kappa$ as in the preceding proposition. For $(\alpha, a) \in \Omega_\kappa$, put

$$S_{(\alpha,a)} = e_{(\alpha,a)} \mathbf{u}_\alpha, \quad T_{(\alpha,a)} = e_{(\alpha,a)} \mathbf{v}_a.$$

Denote by $C^*(S_{(\alpha,a)}, T_{(\alpha,a)} : (\alpha, a) \in \Omega_\kappa)$ the C^* -subalgebra of $\mathcal{O}_{\mathcal{H}_\kappa^{A,B}}$ generated by elements $S_{(\alpha,a)}, T_{(\alpha,a)}, (\alpha, a) \in \Omega_\kappa$. We have

$$S_{(\alpha,a)}^* S_{(\alpha,a)} = \mathbf{u}_\alpha^* e_{(\alpha,a)} \mathbf{u}_\alpha = \sum_{(\delta,b) \in \Omega_\kappa} A_\kappa((\alpha,a), (\delta,b)) e_{(\delta,b)}.$$

As $e_{(\alpha,a)} \mathbf{u}_\beta = 0$ for $\beta \neq \alpha$, and $e_{(\alpha,a)} \mathbf{v}_b = 0$ for $b \neq a$, we have

$$\begin{aligned} e_{(\alpha,a)} &= \sum_{\beta \in E_A} e_{(\alpha,a)} \mathbf{u}_\beta \mathbf{u}_\beta^* e_{(\alpha,a)} + \sum_{b \in E_B} e_{(\alpha,a)} \mathbf{v}_b \mathbf{v}_b^* e_{(\alpha,a)} \\ &= e_{(\alpha,a)} \mathbf{u}_\alpha \mathbf{u}_\alpha^* e_{(\alpha,a)} + e_{(\alpha,a)} \mathbf{v}_a \mathbf{v}_a^* e_{(\alpha,a)} \\ &= S_{(\alpha,a)} S_{(\alpha,a)}^* + T_{(\alpha,a)} T_{(\alpha,a)}^* \end{aligned}$$

so that $e_{(\alpha,a)}$ belongs to the algebra $C^*(S_{(\alpha,a)}, T_{(\alpha,a)} : (\alpha, a) \in \Omega_\kappa)$ and the equality

$$S_{(\alpha,a)}^* S_{(\alpha,a)} = \sum_{(\delta,b) \in \Omega_\kappa} A_\kappa((\alpha,a), (\delta,b)) (S_{(\delta,b)} S_{(\delta,b)}^* + T_{(\delta,b)} T_{(\delta,b)}^*)$$

holds. Similarly we have

$$T_{(\alpha,a)}^* T_{(\alpha,a)} = \sum_{(\beta,d) \in \Omega_\kappa} B_\kappa((\alpha,a), (\beta,d)) (S_{(\beta,d)} S_{(\beta,d)}^* + T_{(\beta,d)} T_{(\beta,d)}^*).$$

As $\sum_{(\alpha,a) \in \Omega_\kappa} e_{(\alpha,a)} = 1$ and $e_{(\alpha,a)} \mathbf{u}_\beta = 0$ for $\beta \neq \alpha$, we have

$$\mathbf{u}_\alpha = \sum_{\substack{a \in \Sigma^\eta \\ (\alpha,a) \in \Omega_\kappa}} e_{(\alpha,a)} \mathbf{u}_\alpha = \sum_{(\alpha,a) \in \Omega_\kappa} S_{(\alpha,a)}$$

so that \mathbf{u}_α and similarly \mathbf{v}_a belong to the algebra $C^*(S_{(\alpha,a)}, T_{(\alpha,a)} : (\alpha, a) \in \Omega_\kappa)$. Therefore the C^* -algebra generated by $e_{(\alpha,a)}, \mathbf{u}_\alpha, \mathbf{v}_a$ coincides with the subalgebra $C^*(S_{(\alpha,a)}, T_{(\alpha,a)} : (\alpha, a) \in \Omega_\kappa)$. \square

Put $n = |\Omega_\kappa|$. Define a $2n \times 2n$ -matrix H_κ with entries in $\{0, 1\}$ by the block matrix

$$H_\kappa = \begin{bmatrix} A_\kappa & A_\kappa \\ B_\kappa & B_\kappa \end{bmatrix}.$$

Denote by I_n and I_{2n} the identity matrices of size n and of size $2n$ respectively.

Lemma 7.11.

- (i) $\mathbb{Z}^{2n}/(H_\kappa - I_{2n})\mathbb{Z}^{2n} \cong \mathbb{Z}^n/(A_\kappa + B_\kappa - I_n)\mathbb{Z}^n$.
 (ii) $\text{Ker}(H_\kappa - I_{2n})$ in $\mathbb{Z}^{2n} \cong \text{Ker}(A_\kappa + B_\kappa - I_n)$ in \mathbb{Z}^n .

Proof. (i) Put a $2n \times 2n$ block matrix $\widehat{H}_\kappa = \begin{bmatrix} A_\kappa & I_n \\ B_\kappa & 0 \end{bmatrix}$. Then we easily see

$$\mathbb{Z}^{2n}/(H_\kappa - I_{2n})\mathbb{Z}^{2n} \cong \mathbb{Z}^{2n}/(\widehat{H}_\kappa - I_{2n})\mathbb{Z}^{2n}.$$

Define a map

$$\Psi : (x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbb{Z}^{2n} \longrightarrow (x_1 + y_1, \dots, x_n + y_n) \in \mathbb{Z}^n$$

which is a surjective homomorphism of abelian groups from \mathbb{Z}^{2n} to \mathbb{Z}^n . Since we know $\Psi((\widehat{H}_\kappa - I_{2n})\mathbb{Z}^{2n}) = (A_\kappa + B_\kappa - I_n)\mathbb{Z}^n$, the homomorphism $\Psi : \mathbb{Z}^{2n} \longrightarrow \mathbb{Z}^n$ induces an isomorphism from $\mathbb{Z}^{2n}/(\widehat{H}_\kappa - I_{2n})\mathbb{Z}^{2n}$ to $\mathbb{Z}^n/(A_\kappa + B_\kappa - I_n)\mathbb{Z}^n$. Therefore $\mathbb{Z}^{2n}/(H_\kappa - I_{2n})\mathbb{Z}^{2n}$ is isomorphic to $\mathbb{Z}^n/(A_\kappa + B_\kappa - I_n)\mathbb{Z}^n$.

(ii) The groups $\text{Ker}(H_\kappa - I_{2n})$ in \mathbb{Z}^{2n} and $\text{Ker}(A_\kappa + B_\kappa - I_n)$ in \mathbb{Z}^n are the torsion free part of $\mathbb{Z}^{2n}/(H_\kappa - I_{2n})\mathbb{Z}^{2n}$ and that of $\mathbb{Z}^n/(A_\kappa + B_\kappa - I_n)\mathbb{Z}^n$ respectively, so that they are isomorphic to each other. \square

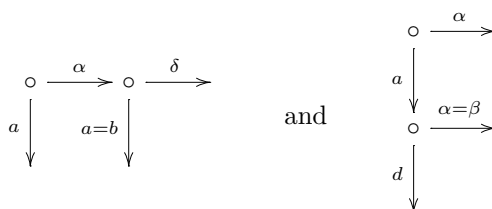
Therefore we reach the following theorem (see [1] for general theory of K-theory for C^* -algebras).

Theorem 7.12. The C^* -algebra $\mathcal{O}_{\mathcal{H}_\kappa^{A,B}}$ associated with the Hilbert C^* -quad module $\mathcal{H}_\kappa^{A,B}$ defined by commuting matrices A, B and a specification κ is isomorphic to the Cuntz–Krieger algebra \mathcal{O}_{H_κ} for the matrix $H_\kappa = \begin{bmatrix} A_\kappa & A_\kappa \\ B_\kappa & B_\kappa \end{bmatrix}$. Its K-groups $K_*(\mathcal{O}_{H_\kappa})$ are computed as

$$\begin{aligned} K_0(\mathcal{O}_{H_\kappa}) &= \mathbb{Z}^n/(A_\kappa + B_\kappa - I_n)\mathbb{Z}^n, \\ K_1(\mathcal{O}_{H_\kappa}) &= \text{Ker}(A_\kappa + B_\kappa - I_n) \text{ in } \mathbb{Z}^n, \end{aligned}$$

where $n = |\Omega_\kappa|$ and the matrices A_κ, B_κ are defined by (7.10) and (7.11).

We will finally present a concrete example. For $1 < N, M \in \mathbb{N}$, let A and B be the 1×1 matrices $[N]$ and $[M]$ respectively. The directed graph G_A associated to the matrix $A = [N]$ is a graph consisting of a vertex denoted by v with N -self directed loops denoted by E_A . Similarly the directed graph G_B consists of the vertex v with M -self directed loops denoted by E_B . We fix a specification $\kappa : E_A \times E_B \longrightarrow E_B \times E_A$ defined by exchanging $\kappa(\alpha, a) = (a, \alpha)$ for $(\alpha, a) \in E_A \times E_B$. Hence $\Omega_\kappa = E_A \times E_B$ so that $|\Omega_\kappa| = |E_A| \times |E_B| = N \times M$. We then know $\kappa_A((\alpha, a), (\delta, b)) = 1$ if and only if $b = a$. And $\kappa_B((\alpha, a), (\beta, d)) = 1$ if and only if $\beta = \alpha$ as in the following figures respectively.



In particular, for the case $N = 2$ and $M = 3$, we write $E_A = \{1, 2\}$, $E_B = \{1, 2, 3\}$ and Ω_κ as

$$\Omega_\kappa = E_A \times E_B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}.$$

By (7.10) and (7.11), the 6×6 matrices A_κ and B_κ are written along the above ordered basis in order as:

$$A_\kappa = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B_\kappa = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

respectively so that we have

$$A_\kappa + B_\kappa - I = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

It is easy to see that

$$\mathbb{Z}^6 / (A_\kappa + B_\kappa - I)\mathbb{Z}^6 \cong \mathbb{Z}/8\mathbb{Z}, \quad \text{Ker}(A_\kappa + B_\kappa - I) \text{ in } \mathbb{Z}^6 \cong \{0\}.$$

Therefore the C^* -algebra $\mathcal{O}_{H_\kappa^{A,B}}$ for $A = [2], B = [3]$ and $\kappa = \text{exchange}$ is a Cuntz–Krieger algebra stably isomorphic to the Cuntz algebra \mathcal{O}_9 ([2,3]), whereas the C^* -algebra $\mathcal{O}_{[2],[3]}^\kappa$ considered in [19] is isomorphic to $\mathcal{O}_2 \otimes \mathcal{O}_3$ which is isomorphic to \mathcal{O}_2 . We further study these C^* -algebras $\mathcal{O}_{H_\kappa^{A,B}}$ for general commuting matrices A, B in [20] (cf. [18]).

Acknowledgments

The author would like to thank the referee for his/her careful reading the first draft of the paper and many useful suggestions. This work was supported by JSPS KAKENHI Grant Numbers 20540215, 23540237 and 15K04896.

References

- [1] B. Blackadar, *K-Theory for Operator Algebras*, Springer-Verlag, 1986.
- [2] J. Cuntz, Simple C^* -algebras generated by isometries, *Comm. Math. Phys.* 57 (1977) 173–185.
- [3] J. Cuntz, K -theory for certain C^* -algebras. II, *J. Operator Theory* 5 (1981) 101–108.
- [4] J. Cuntz, W. Krieger, A class of C^* -algebras and topological Markov chains, *Invent. Math.* 56 (1980) 251–268.
- [5] V. Deaconu, C^* -algebras and Fell bundles associated to a textile system, *J. Math. Anal. Appl.* 372 (2010) 515–524.
- [6] R. Exel, D. Gonçalves, C. Starling, The tiling C^* -algebra viewed as a tight inverse semigroup algebra, *Semigroup Forum* 84 (2012) 229–240.
- [7] T. Kajiwara, C. Pinzari, Y. Watatani, Ideal structure and simplicity of the C^* -algebras generated by Hilbert modules, *J. Funct. Anal.* 159 (1998) 295–322.
- [8] A. Kumjian, D. Pask, Higher rank graph C^* -algebras, *New York J. Math.* 6 (2000) 1–20.
- [9] A. Kumjian, D. Pask, I. Raeburn, J. Renault, Graphs, groupoids and Cuntz–Krieger algebras, *J. Funct. Anal.* 144 (1997) 505–541.
- [10] D. Lind, B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, Cambridge, UK, 1995.
- [11] N.G. Markley, M.E. Paul, Matrix subshifts for \mathbb{Z}' symbolic dynamics, *Proc. London Math.* 43 (1981) 251–272.
- [12] K. Matsumoto, Presentations of subshifts and their topological conjugacy invariants, *Doc. Math.* 4 (1999) 285–340.
- [13] K. Matsumoto, C^* -algebras associated with presentations of subshifts, *Doc. Math.* 7 (2002) 1–30.
- [14] K. Matsumoto, Actions of symbolic dynamical systems on C^* -algebras, *J. Reine Angew. Math.* 605 (2007) 23–49.
- [15] K. Matsumoto, Textile systems on lambda graph systems, *Yokohama Math. J.* 54 (2008) 121–206.
- [16] K. Matsumoto, Orbit equivalence in C^* -algebras defined by actions of symbolic dynamical systems, *Contemp. Math.* 503 (2009) 121–140.

- [17] K. Matsumoto, Actions of symbolic dynamical systems on C^* -algebras II. Simplicity of C^* -symbolic crossed products and some examples, *Math. Z.* 265 (2010) 735–760.
- [18] K. Matsumoto, C^* -algebras associated with Hilbert C^* -quad modules of finite type, *Int. J. Math. Math. Sci.* (2014) 952068.
- [19] K. Matsumoto, C^* -algebras associated with textile dynamical systems, *New York J. Math.* 21 (2015) 1179–1245.
- [20] K. Matsumoto, C^* -algebras associated with Hilbert C^* -quad modules of commuting matrices, *Math. Scand.* 117 (2015) 126–149.
- [21] M. Nasu, Textile systems for endomorphisms and automorphisms of the shift, *Mem. Amer. Math. Soc.* 546 (1995).
- [22] D. Pask, I. Raeburn, N.A. Weaver, A family of 2-graphs arising from two-dimensional subshifts, *Ergodic Theory Dynam. Systems* 29 (2009) 1613–1639.
- [23] M.V. Pimsner, A class of C^* -algebras generalizing both Cuntz–Krieger algebras and crossed products by \mathbb{Z} , in: *Free Probability Theory*, in: *Fields Inst. Commun.*, vol. 12, 1996, pp. 189–212.
- [24] G.A. Pino, J. Clark, A.A. Huff, I. Raeburn, Kumjian–Pask algebras of higher rank graphs, *Trans. Amer. Math. Soc.* 365 (2013) 3613–3641.
- [25] G. Robertson, T. Steger, Affine buildings, tiling systems and higher rank Cuntz–Krieger algebras, *J. Reine Angew. Math.* 513 (1999) 115–144.
- [26] K. Schmidt, *Dynamical Systems of Algebraic Origin*, Birkhäuser, Basel, Boston, Berlin, 1995.