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Mixed Bram–Halmos and Agler–Embry conditions

George R. Exner

Department of Mathematics, Bucknell University, Lewisburg, PA 17837, USA

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A B S T R A C T

The subnormality of a Hilbert space operator may be characterized either by the Bram–Halmos conditions (positivity of certain operator matrices) or the Agler–Embry conditions (positivity of certain operator differences). We define and consider mixed conditions involving matrices of operator differences, thus yielding conditions whose extremes are the Bram–Halmos and Agler–Embry conditions. We study these conditions for weighted shifts, showing that they reduce to matrices of differences of the moments of the shifts, and examine these conditions under the perturbation of a single weight of a subnormal shift.

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and $\mathcal{L}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. Recall that T an element of $\mathcal{L}(\mathcal{H})$ is normal if $T^*T = TT^*$, subnormal if T is the restriction of some normal operator N to an invariant subspace, and hyponormal if $T^*T \geq TT^*$ (that is, $[T^*, T] := T^*T - TT^*$ is a positive operator). The study of "weak subnormalities" has been ongoing. The dominant study, of k-hyponormal operators, was introduced by Curto to provide a "bridge" between hyponormality and subnormality; these classes are intermediates that illustrate the distinction between these two notions. Recall that T is k-hyponormal ($k = 1, 2, \ldots$) if

$$\begin{pmatrix} I & T^* & {T^*}^2 & \ldots & {T^*}^k \\ T & T^*T & {T^*}^2T & \ldots & {T^*}^kT \\ T^2 & T^*T^2 & {T^*}^2T^2 & \ldots & {T^*}^kT^2 \\ \vdots & & \vdots & & \vdots \\ T^k & T^*T^k & {T^*}^2T^k & \ldots & {T^*}^kT^k \end{pmatrix} \geq 0. \tag{0.1}$$

When $k = 1$, we call the operator simply hyponormal, and the condition reduces to the familiar $T^*T \geq TT^*$. The consideration of these classes is motivated in part by the Bram–Halmos characterization of subnormality ([5,17]): T is subnormal if and only if it is k-hyponormal for all k. We will occasionally refer to the matrix appearing in (0.1) as a Bram matrix.

E-mail address: exner@bucknell.edu.

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There is an alternative characterization of subnormality (under the mild restriction that the operator is a contraction: $\|T\| \le 1$) which motivates another collection of classes. We say T is n-contractive, $n = 1, 2, \dots$ if

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} T^{*j} T^j \ge 0.$$

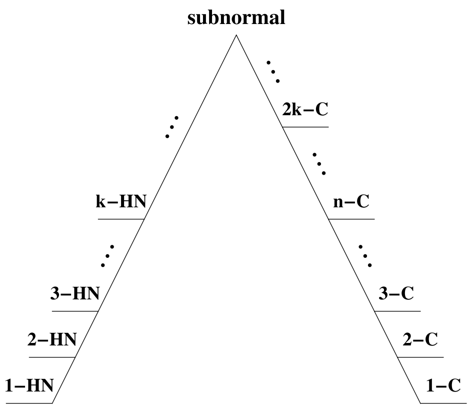
Observe that "1-contractive" is simply "contractive." The Agler–Embry ([2]) characterization of subnormality (stated using the notion of hypercontractivity, which for this theorem is equivalent) is that a contraction T is subnormal if and only if it is n-contractive for all n.

The study of k-hyponormality has been productive since its initiation in [6]. Perhaps the strongest result is the proof that an operator T such that $p(T)$ is hyponormal for every complex polynomial p need not be subnormal [12]. This study and the study of related classes such as "weakly k-hyponormal" (especially quadratically or cubically hyponormal) have been fruitful. (There is a different concept of p-hyponormality – the notational overlap is quite unfortunate – and allied notions such as p-paranormality and absolute p-paranormality, which seem quite unrelated.)

The study of n-contractivity is more recent (more precisely, was re-initiated in [14] about a decade ago after a dormant period following Agler's paper in 1985) and less developed than that of k-hyponormality. This study is related to the study of "n-hyperexpansivity" (a property in some sense apparently dual to n-contractivity; see, for example, [4] and subsequent papers), and to the study of "n-isometries" (see [3] and subsequent papers).

The goal of the current paper is to explore some conditions involving Bram-type matrices of Agler-type differences, for which positivity conditions yield mixtures of these conditions. These conditions include, as special cases in which the size of the matrix or the length of the difference is trivial, the standard ones.

Such exploration is reasonable because the relationships between k-hyponormality and n-contractivity are not well understood. The following diagram is useful.



One cannot expect too much: it is known that if one takes a certain recursively generated weighted shift (yielding a weight sequence of the form $\sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$, where x is regarded as a parameter) – this notation for the two-atomic shift arising from Stampfli's completion ([24]) will be reviewed later – the resulting shift has 1-HN $\Leftarrow$ 2-HN $\Leftrightarrow$ subnormal (a "collapse" of the left hand side); one can show that all the n-C classes are distinct in n as shown by various values of x. On the other hand, there is one general relationship known:

Theorem 0.1. [16] *Suppose T is contractive. If T is k-hyponormal then T is $2k$-contractive.*

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In fact, the weaker condition Embry k-hyponormality is sufficient and what is actually used in the proof (see [13] and the discussion in [16]). This family of conditions is based upon positivity of the Embry matrices,

$$\begin{pmatrix} I & T^*T & T^{*2}T^2 & \dots & T^{*k}T^k \\ T^*T & T^{*2}T^2 & T^{*3}T^3 & \dots & T^{*k+1}T^{k+1} \\ T^{*2}T^2 & T^{*3}T^3 & T^{*4}T^4 & \dots & T^{*k}T^2 \\ \vdots & & \vdots & & \vdots \\ T^{*k}T^k & T^{*k+1}T^{k+1} & \dots & \dots & T^{*2k}T^{2k} \end{pmatrix}. \tag{0.2}$$

See [21] for results concerning the relationships between these positivity conditions; we will consider conditions analogous to the Bram and Embry conditions in the final section.

There is a family of examples for which something can be said in the reverse direction of some n-contractivity condition implying some k-hyponormality. Consider perturbations in the first weight of the j-th Agler shift (recalled more completely below), yielding weight sequence

$$\alpha^j(x) : \sqrt{\frac{x}{j}}, \sqrt{\frac{2}{j+1}}, \sqrt{\frac{3}{j+2}}, \dots,$$

and write $A_j(x)$ for the shift.

Theorem 0.2 *([1]). The operator $A_j(x)$, for some $j \geq 2$, is*

(i) n-contractive iff $x \leq \frac{n+j-1}{n}$, and
(ii) n-hyponormal iff $x \leq \frac{n(n+j)+j-1}{n(n+j)}$.

It follows that $A_j(x)$ is n-hyponormal iff it is $n(n+j)$-contractive.

As Raúl Curto has pointed out, it is nonetheless an enduring puzzle how the Agler conditions (which have all the operators T^* "on the left") can yield even 1-hyponormality, which carries information about a T^* "on the right."

We first set some notation for weighted shifts, the standard testing ground for these various conditions. Consider ℓ^2 with its standard basis $\{e_j\}_{j=0}^{\infty}$. Given a weight sequence

$$\alpha : \sqrt{\alpha_0}, \sqrt{\alpha_1}, \sqrt{\alpha_2}, \dots,$$

we define the weighted shift W_α on ℓ^2 by

$$W_\alpha e_j = \sqrt{\alpha_j} e_{j+1}.$$

The moments of the shift are defined by

$$\gamma_0 = 1 \quad \text{and} \quad \gamma_j = \prod_{i=0}^{j-1} \alpha_i, \quad j \geq 1. \tag{0.3}$$

Some particular shifts we consider are the Agler shifts A_p, $p = 2, 3, \dots$ having weight sequence α^p : $\sqrt{\frac{1}{p}}, \sqrt{\frac{2}{p+1}}, \dots$, and which were used by Agler as model operators for n-contractive operators ([2]). Observe that A_2 is the Bergman shift.

A common device has been to take a weighted shift known to be subnormal and to "perturb" it in some way. For example, one can introduce a parameter into the m-th weight and consider what classes of interest

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result for various values of the parameter. Particularly simple are perturbations in the zeroth weight of some shift W, yielding a weight sequence $\alpha(x) : \sqrt{x\alpha_0}, \sqrt{\alpha_1}, \ldots$; we write $W_{\alpha(x)}$ for the shift. Alternatively, one may form a "back step extension," which is to take the weight sequence for a subnormal shift and prefix one or more weights (parameters) to create a new sequence and new shift. One may then investigate what values of the parameters yield some property of interest.

It is well-known that the test for k-hyponormality simplifies considerably for weighted shifts. A weighted shift is k-hyponormal iff certain Hankel moment matrices are positive for $n = 1, 2, \ldots$ (see [6]):

$$\begin{pmatrix} \gamma_n & \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k} \\ \gamma_{n+1} & \gamma_{n+2} & & \cdots & \gamma_{n+k+1} \\ \gamma_{n+2} & \cdots & & \cdots & \gamma_{n+k+2} \\ \vdots & & \vdots & & \vdots \\ \gamma_{n+k} & \gamma_{n+k+1} & & \cdots & \gamma_{n+2k} \end{pmatrix} \geq 0. \tag{0.4}$$

The test for some n-contractivity is likewise simplified for shifts. We seek positivity of

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} W^{*j} W^j.$$

This is readily seen to be diagonal, so it is enough to consider basis vectors, and the test at e_m yields

$$1 - \binom{n}{1}\alpha_m + \binom{n}{2}\alpha_m\alpha_{m+1} - \ldots.$$

A computation shows that we need

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \gamma_{m+j} \geq 0, \qquad m = 0, 1, 2, \ldots. \tag{0.5}$$

The notions of k-hyponormality and n-contractivity have a fundamental difference: if an operator is k-hyponormal, it is clearly j-hyponormal for all $j \leq k$, as is easy by considering positivity of submatrices. But the Dirichlet shift D (with weights $\sqrt{2/1}, \sqrt{3/2}, \sqrt{4/3}, \ldots$) is 2-contractive (indeed, is the prototypical "2-isometry" for which $I - 2D^*D + D^{*2}D^2 = 0$) but is clearly not a contraction. The notion originally used in [2] is as follows: an operator is n-hypercontractive if it is k-contractive for $k = 1, 2, \ldots, n$. We will mostly be working in the context of contraction operators, and we record the following results, which have been known for some time but have not appeared in print.

Proposition 0.3. *Let W_α be a weighted shift with only finitely many weights in the weight sequence α : $\sqrt{\alpha_0}, \sqrt{\alpha_1}, \ldots$ larger than 1, and let $n \geq 2$ be arbitrary. If W_α is n-contractive, then it is $(n-1)$-contractive (and hence contractive).*

Proof. Given some operator T in $\mathcal{L}(\mathcal{H})$, integer $n \geq 1$, and $h \in \mathcal{H}$, let $\rho_{T,n}(h)$ be the sum

$$\rho_{T,n}(h) := \sum_{p=0}^{n} (-1)^p \binom{n}{p} \|T^p h\|^2.$$

Clearly positivity of $\rho_{T,n}(h)$ is the test at the vector h for positivity of the n-th Agler difference.

Since we have a weighted shift, it is sufficient to check such positivity at the standard basis vectors. Suppose in order to obtain a contradiction that W_α is n-contractive but not $(n-1)$-contractive and let j be the least integer such that

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$$\rho_{W_\alpha,n-1}(e_j) < 0.$$

Citing the easy recursive relationship

$$\rho_{W_\alpha,n}(e_j) = \rho_{W_\alpha,n-1}(e_j) - \alpha_j \rho_{W_\alpha,n-1}(e_{j+1}),$$

and using our assumption that W_α is n-contractive, we have

$$0 \leq \rho_{W_\alpha,n}(e_j) = \rho_{W_\alpha,n-1}(e_j) - \alpha_j \rho_{W_\alpha,n-1}(e_{j+1}), \tag{0.6}$$

and therefore from our assumption that $\rho_{W_\alpha,n-1}(e_{j+1}) < 0$. Continuing, we have that the sequence $\left(\rho_{W_{\alpha,n-1}}(e_k)\right)_{k\geq j}$ has only negative terms.

Further, as soon as k is so large that $\alpha_k \leq 1$, we have from (0.6) that the sequence is weakly decreasing as well as negative.

From [2][Lemma 2.11] we have that if $W_\alpha^n \to 0$ in the strong operator topology, then n-contractivity implies $(n-1)$-contractivity, so by the assumption that W_α is not $(n-1)$-contractive we must have $\lim_{j\to\infty} \alpha_j = 1$. But it is easy to see that in this case (with n fixed) $\lim_{j\to\infty} \rho_{W_\alpha,n}(e_j) = 0$, and this contradicts that the sequence is both negative and decreasing. □

Note that since the Dirichlet shift is a 2-isometry and therefore 2 contractive, but is not contractive, the result may not be trivially improved.

Given an operator T in $\mathcal{L}(\mathcal{H})$ and $h \in \mathcal{H}$ such that $T^n h \neq 0$ $(n = 1, 2, \ldots)$ we may define the weighted shift operator $W_{T,h}$ to be the shift with weight sequence $\left(\frac{\|T^{n+1}h\|}{\|T^n h\|}\right)_{n=0}^{\infty}$ (see, for example, [19]). From that paper we have the relationship

$$\rho_{W_{T,h},m}(e_j) = \frac{\rho_{T,m}(T^j h)}{\|T^j h\|^2}, \qquad m \geq 1; j \geq 0. \tag{0.7}$$

Observe as well that clearly $\|W_{T,h}\| \leq \|T\|$.

The following is [22][Cor. 3], but we present an elementary proof for the convenience of the reader and to anticipate the result in Theorem 1.14 to come.

Theorem 0.4. *Let T be a contraction and n-contractive for some $n \geq 2$. Then T is $(n-1)$-contractive. Therefore, for contractions n-contractivity coincides with n-hypercontractivity.*

Proof. Suppose T is as assumed, and consider the case of some h for which $T^k h \neq 0$ $(k = 1, 2, \ldots)$. Then we may form $W_{T,h}$ and using (0.7) we have $W_{T,h}$ n-contractive and since it is contractive it is $(n-1)$-contractive by the proposition. Then using (0.7) in the other direction and with $j = 0$ we obtain $\rho_{T,n-1}(h) \geq 0$.

Consider now the case of some $h \neq 0$ such that for some k it happens that $T^k h = 0$, and let K be the least such k. Define $\hat{T}$ by

$$\hat{T} = T|_{\bigvee_{j=0}^{K-1} T^j h}.$$

Surely $\hat{T}$ is nilpotent, and therefore $\hat{T}^m$ approaches zero in the strong operator topology. Also, $\hat{T}$ is n-contractive since $\rho_{\hat{T},m}(v) = \rho_{T,m}(v)$ for all $v \in \bigvee_{j=0}^{K-1} T^j h$ and for all m, in particular for $m = n$. By Agler's result [2][Lemma 2.11] $\hat{T}$ is $(n-1)$-contractive, and so

$$\rho_{T,n-1}(h) = \rho_{\hat{T},n-1}(h) \geq 0.$$

This and the previous case together yield the result. □

Remark that the theorem can be weakened to consider operators which one might call "finitely contractive" (but which turn out to be contractive) in that for any h, $\|T^{k+1}h\| > \|T^k h\|$ happens only finitely often in $k = k(h)$.

1. A mixed condition for weighted shifts

The condition "between" the Bram–Halmos and the Agler–Embry conditions (in some sense) involves matrices of differences; it is necessary to assemble some notation. Let W_α be a weighted shift with weight sequence α and recall the moments $(\gamma_j)_{j=0}^\infty$ of the shift are defined in terms of the weights as in (0.3). In what follows we define quantities using superscript α to indicate the underlying weight sequence, but omit the α whenever possible to ease the notation.

So let $\Gamma^\alpha(k \times k, m, j) = \Gamma(k \times k, m, j)$ be the matrix of size $k \times k$, with entries differences of moments of a shift of length m, with γ_j the first term in the $(0,0)$-th entry of the matrix. Thus, for example, $\Gamma(3 \times 3, 2, 1)$ is the matrix

$$\begin{pmatrix} \gamma_1 - 2\gamma_2 + \gamma_3 & \gamma_2 - 2\gamma_3 + \gamma_4 & \gamma_3 - 2\gamma_4 + \gamma_5 \\ \gamma_2 - 2\gamma_3 + \gamma_4 & \gamma_3 - 2\gamma_4 + \gamma_5 & \gamma_4 - 2\gamma_5 + \gamma_6 \\ \gamma_3 - 2\gamma_4 + \gamma_5 & \gamma_4 - 2\gamma_5 + \gamma_6 & \gamma_5 - 2\gamma_6 + \gamma_7 \end{pmatrix}.$$

Observe that the matrices $\Gamma(k+1 \times k+1, 0, j)$, $j = 0, 1, 2, \ldots$, are the matrices relevant to k-hyponormality of the shift; the (one-by-one) matrices $\Gamma(1 \times 1, n, j)$, $j = 0, 1, 2, \ldots$, are the expressions relevant to n-contractivity of the shift. We are concerned with positive (semi-) definiteness of such matrices, so let $P^\alpha(k \times k, n, j) = P(k \times k, n, j)$ denote the condition that $\Gamma(k \times k, n, j) \geq 0$ and $P^\alpha(k \times k, n, *) = P(k \times k, n, *)$ denote $\Gamma(k \times k, n, j) \geq 0$, $j = 0, 1, \ldots$, with similar use of $*$ in other entries. Thus subnormality of the shift is equivalent to either $P(* \times *, 0, *)$ or $P(1 \times 1, *, *)$ by Bram–Halmos and Agler–Embry respectively.

A few results are elementary. First, by looking at the $(0,0)$-th entry of the relevant matrices, we obtain the following.

Proposition 1.1. *For any weighted shift W_α and any k, n, and j, $P(k \times k, n, j)$ implies $P(1 \times 1, n, j)$. It follows that for any k and n, $P(k \times k, n, *)$ implies $P(1 \times 1, n, *)$ which is n-contractivity, and therefore that for any k, $P(k \times k, *, *)$ implies W_α is subnormal.*

Observe that it is not true that (for example) $P(*, 2, *)$ implies subnormality; again the Dirichlet shift, with weights $\sqrt{2/1}, \sqrt{3/2}, \sqrt{4/3}, \ldots$ is a 2-isometry but not even hyponormal. As noted before, if W_α is a contraction, n-contractivity is promoted to n-hypercontractivity and such results do hold.

We turn to some preliminary results, but first need some notation. Consider some contractive subnormal weighted shift with weight sequence $\alpha : \alpha_0, \alpha_1, \ldots$. Recall that W_α has a Berger probability measure μ supported on $[0,1]$ such that the moments $(\gamma_i)_{i=0}^\infty$ of W_α satisfy

$$\gamma_i = \int_0^1 t^i \, d\mu(t), \quad i = 0, 1, \ldots.$$

For each $m = 0, 1, \ldots$, consider the measure $\hat{\mu}_m$ defined by $\hat{\mu}_m(t) = C_m (1-t)^m \mu(t)$ where C_m is the normalizing constant $C_m = \left(\int_0^1 (1-t)^m d\mu(t) \right)^{-1}$. Denote the moments resulting from this measure by $\gamma_n^{(m)}$ and the matrices arising from it by $\Gamma_{(m)}(k \times k, \ell, j)$. It is trivial to verify that some m-length Agler difference for the original shift is a multiple of the appropriate $\gamma_n^{(m)}$:

$$\sum_{j=0}^m (-1)^j \binom{m}{j} \gamma_{n+j} = C_m^{-1} \gamma_n^{(m)}, \qquad n = 0, 1, \ldots.$$

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It follows readily that

$$\Gamma(k\times k,m,j)=C_m^{-k}\,\Gamma_{(m)}(k\times k,0,j),\qquad k=1,2,\ldots;j=0,1,\ldots. \tag{1.1}$$

As one might hope, the following holds:

Theorem 1.2. *If W_α is a contractive subnormal weighted shift, then it satisfies the property $P(*\times *,*,*)$.*

Proof. Suppose that W_α is contractive and subnormal and fix $k\geq 1$, $m\geq 0$, and $j\geq 0$. We have from (1.1) that

$$\Gamma(k\times k,m,j)=C_m^{-k}\,\Gamma_{(m)}(k\times k,0,j).$$

The matrix on the right hand side of this last is non-negative, since it is one of the matrices to test for $(k-1)$-hyponormality of the (subnormal) weighted shift associated with the Berger measure $\hat{\mu}_m$. Therefore the matrices to test for $P^\alpha(k\times k,m,j)$ – that is, positivity of the matrices $\Gamma^\alpha(k\times k,m,j)$ of differences of moments as in (0.3) associated with the weighted shift W_α – are non-negative, and the result follows. □

Consider some perturbation in the 0-th weight of a contractive weighted shift with weight sequence $\alpha:\alpha_0,\alpha_1,\ldots$ to yield a weight sequence $\alpha(x)$ defined by

$$\alpha(x):x\cdot\alpha_0,\alpha_1,\ldots. \tag{1.2}$$

Indicate by superscript x matrices, entries, and positivity conditions relevant to the perturbation. (Strictly speaking this should be indicated by superscripts of the form $\alpha(x)$, but again to ease the notation we will simply use x unless the underling weight sequence is in doubt.) In anticipation of the usual use of the Nested Determinant Test (that is, Sylvester's criterion – see, e.g., [8], pg. 213) to determine matrix positivity, observe that for all $n\geq 2$ and $m\geq 1$,

$$\det\Gamma^x(n\times n,m,0)=(1-x)x^{n-1}\det\Gamma(n-1\times n-1,m,2)+x^n\det\Gamma(n\times n,m,0).$$

(This is easy by adding and subtracting x in the upper-left-most entry and doing an expansion.) Therefore (with $x>0$ assumed)

$$\det\Gamma^x(n\times n,m,0)\geq 0$$
$$\text{if and only if}$$
$$\det\Gamma(n-1\times n-1,m,2)+x(\det\Gamma(n\times n,m,0)-\det\Gamma(n-1\times n-1,m,2))\geq 0. \tag{1.3}$$

We may obtain a result for any 0-th weight perturbations of any (non-recursively generated) subnormal weighted shift, where if $\alpha:\alpha_0,\alpha_1,\ldots$ we take our perturbation to be of the form in (1.2). (Note that some papers simply insert x, instead of $s\alpha_0$, for the zeroth weight.) First we require a number of determinant lemmas. We are greatly indebted to Christian Krattenthaler ([20]) for showing us the approach leading to the proof of Lemma 1.4. For simplification, we adopt some temporary notation: let $H_{n,m}=(g_{i+j+m})_{0\leq i,j\leq n-1}$ and $DH_{n,m}=(g_{i+j+m+1}-g_{i+j+m})_{0\leq i,j\leq n-1}$ (so, with g's instead of γ's, these are just $\Gamma(n\times n,0,m)$ and $-\Gamma(n\times n,1,m)$, respectively). Let $H_{n,m}^{(k)}=(g_{i+j+m+\chi(j\geq k)})_{0\leq i,j\leq n-1}$ (where $\chi(\mathcal{P})$ is 1 if $\mathcal{P}$ is true and 0 if not); this turns out to be the matrix obtained by deleting the last row and k-th column from $H_{n+1,m}$ (recall rows and columns are indexed starting at zero).

We begin with a preliminary lemma (for fuller details see the discussion following line (3.2) of [1]).

Lemma 1.3. *For any collection $\{a_{ij}\}$ and any positive integer n we have*

$$\det(a_{i,j} - a_{i,j+1})_{0\le i,j\le n-1} = \sum_{k=0}^{n}(-1)^k \det(a_{i,j+\chi(j\ge k)})_{0\le i,j\le n-1}.$$

Proof (sketch). The result follows from a determinant expansion using column multilinearity. For an example, consider the determinant of the 2 by 2 matrix

$$M := \begin{pmatrix} a_{00} - a_{01} & a_{01} - a_{02} \\ a_{10} - a_{11} & a_{11} - a_{12} \end{pmatrix}.$$

Surely

$$\det M = \det\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} + \det\begin{pmatrix} a_{00} & -a_{02} \\ a_{10} & -a_{12} \end{pmatrix} + \\ + \det\begin{pmatrix} -a_{01} & a_{01} \\ -a_{11} & a_{11} \end{pmatrix} + \det\begin{pmatrix} -a_{01} & -a_{02} \\ -a_{11} & -a_{12} \end{pmatrix}.$$

The third of these determinants is zero. The first, second, and fourth correspond, respectively, to the values in the sum in the right hand side of the statement arising from $k = 2$, $k = 1$, and $k = 0$. Observe that $n - k$ is the count of the number of second columns used from M (and hence negative signs acquired for the term in the right hand sum). $\square$

We have as well need for a (multi-row) Laplace expansion to compute a determinant. Recall that such an expansion (see, e.g., [23]) is a (signed) sum of terms arising from submatrices: to do a Laplace expansion using the first $n+1$ rows, given a (square) matrix $M = (m_{ij})_{1\le i,j\le q}$ we consider each subset $\mathcal{S}$ of size $n+1$ of $\{1, 2, \ldots, q\}$, say $\mathcal{S} = \{i_1, i_2, \ldots, i_{n+1}\}$ where these are written in increasing order. We form the matrix $M_{\mathcal{S}} = (m_{i_k j})_{1\le k,j\le n+1}$. Letting $\mathcal{S}'$ be the complementary set $\{1, 2, \ldots, q\} \ominus \mathcal{S}$, say $\mathcal{S}' = \{p_1, p_2, \ldots, p_{q-n-1}\}$ again written in increasing order, we form the matrix $M_{\mathcal{S}'} = (m_{p_k j})_{1\le k\le q-n-1, n+2\le j\le q}$ (thus using entries from the last $q - n - 1$ rows of M). We adjust using the sign $\mathrm{Sign}(\mathcal{S})$ of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & \ldots & n+1 & n+2 & n+3 & \ldots & q \\ i_1 & i_2 & i_3 & \ldots & i_{n+1} & p_1 & p_2 & \ldots & p_{q-n-1} \end{pmatrix}. \tag{1.4}$$

We then have that

$$\det M = \sum_{\mathcal{S}\subseteq\{1,2,\ldots,q\}} \mathrm{Sign}(\mathcal{S}) \det M_{\mathcal{S}} \cdot \det M_{\mathcal{S}'}. \tag{1.5}$$

Lemma 1.4. *With $H_{n,m}$, $DH_{n,m}$, and $H^{(k)}_{n,m}$ as defined above, and with any collection $\{g_i\}$, and for any n, we have*

$$\det DH_{n,2} \cdot \det H_{n+1,0} - \det H_{n,2} \cdot \det DH_{n+1,0} = \det DH_{n,1} \cdot \det H_{n+1,1}. \tag{1.6}$$

Proof (sketch). With the use of the previous lemma one obtains

$$\det DH_{n,m} = \sum_{k=0}^{n}(-1)^{n+k} \det H^{(k)}_{n,m}. \tag{1.7}$$

To see this, consider the auxiliary matrix

$$DK_{n,m} := (g_{i+j+m} - g_{i+j+m+1})_{0\le i,j\le n-1}$$

(that is, $DH_{n,m}$ with each column multiplied by -1). We may apply Lemma 1.3 to deduce that

$$\det DK_{n,m} = \sum_{k=0}^{n}(-1)^k \det H_{n,m}^{(k)}.$$

Since $\det DK_{n,m} = (-1)^n \det DH_{n,m}$ we obtain (1.7).

Substituting (1.7) in the left hand side of (1.6), we obtain

$$\begin{aligned}\det DH_{n,2}\cdot\det H_{n+1,0} - \det H_{n,2}\cdot\det DH_{n+1,0} \ &= \\ &= \det H_{n+1,0}\sum_{k=0}^{n}(-1)^{n+k}\det H_{n,2}^{(k)} - \det H_{n,2}\sum_{k=0}^{n+1}(-1)^{n+k+1}\det H_{n+1,0}^{(k)} \\ &= \det H_{n+1,0}\sum_{k=0}^{n-1}(-1)^{n+k}\det H_{n,2}^{(k)} - \det H_{n,2}\sum_{k=0}^{n}(-1)^{n+k+1}\det H_{n+1,0}^{(k)}. \qquad (1.8)\end{aligned}$$

Consider now the auxiliary matrix G defined by

$$G := \begin{pmatrix} g_{n-1} & \cdots & g_1 & g_0 & g_1 & g_2 & \cdots & g_{n+1} \\ g_n & \cdots & g_2 & g_1 & g_2 & g_3 & \cdots & g_{n+2} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ g_{2n-1} & \cdots & g_{n+1} & g_n & g_{n+1} & g_{n+2} & \cdots & g_{2n+1} \\ g_{n+1} & \cdots & g_3 & g_2 & g_2 & g_3 & \cdots & g_{n+2} \\ g_{n+2} & \cdots & g_4 & g_3 & g_3 & g_4 & \cdots & g_{n+3} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ g_{2n} & \cdots & g_{n+2} & g_{n+1} & g_{n+1} & g_{n+2} & \cdots & g_{2n+1} \end{pmatrix}.$$

We will evaluate the determinant of G by a sequence of row and column operations.

Step 1: By subtracting row 1 from row $n+1$ (remembering the row index begins at 0), row 2 from row $n+2$, $\dots$, row n from row $2n$, one sees that $\det G$ is the same as the determinant of the following matrix:

$$G' := \begin{pmatrix} g_{n-1} & \cdots & g_1 & g_0 & g_1 & g_2 & \cdots & g_{n+1} \\ g_n & \cdots & g_2 & g_1 & g_2 & g_3 & \cdots & g_{n+2} \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ g_{2n-1} & \cdots & g_{n+1} & g_n & g_{n+1} & g_{n+2} & \cdots & g_{2n+1} \\ g_{n+1}-g_n & \cdots & g_3-g_2 & g_2-g_1 & 0 & 0 & \cdots & 0 \\ g_{n+2}-g_{n+1} & \cdots & g_4-g_3 & g_3-g_2 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ g_{2n}-g_{2n-1} & \cdots & g_{n+2}-g_{n+1} & g_{n+1}-g_n & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Step 2: We next interchange columns to move the last $n+1$ columns to the front, interchanging the column headed g_{n-1} with the second of the columns headed g_1, the column (not displayed) headed g_{n-2} with the column headed g_2, and so on so as to finish with the (only) column headed g_0 interchanged with the (last) column headed g_{n+1}. (Obviously there is a potential sign change in the determinant which we shall consider shortly.) Observe that the resulting matrix is now block upper triangular with two blocks.

Step 3: We next perform further column interchanges to reverse the order of the (now) final n columns (which, after Step 2, are headed by $g_{n-1}, \dots, g_0$ in that order). (Again there is a possible sign change in the determinant which we will take up soon.) Note that the resulting matrix retains its block upper triangular form.

Step 4: We multiply each of the final n columns by -1 so that the differences in the bottom rows are in the order $g_j - g_{j+1}$.

It is then a matter of counting column interchanges and multiplications by -1 to deduce that

$$(-1)^{s(n)} \det G' = \det DH_{n,1} \cdot \det H_{n+1,1},$$

where $s(n)$ is a parity factor defined by

$$s(n) = \begin{cases} 1 & \text{if } n = 2\ell + 1 \text{ with } \ell \text{ even or } n = 2\ell \text{ with } \ell \text{ odd,} \\ 2 & \text{if } n = 2\ell + 1 \text{ with } \ell \text{ odd or } n = 2\ell \text{ with } \ell \text{ even.} \end{cases}$$

It therefore suffices to show that $(-1)^{s(n)} \det G$ is the same as the expression in the last line of (1.8), that is,

$$(-1)^{s(n)} \det G = \det H_{n+1,0} \sum_{k=0}^{n-1} (-1)^{n+k} \det H_{n,2}^{(k)} - \det H_{n,2} \sum_{k=0}^{n} (-1)^{n+k+1} \det H_{n+1,0}^{(k)}. \tag{1.9}$$

To this end we will perform a Laplace expansion for $\det G$ with respect to the first $n+1$ rows of G. In building terms in the expansion which are not automatically zero because of duplicated columns, it helps to note that G contains columns whose first $n+1$ entries duplicate which are headed (have first entry as) $g_1, \dots, g_{n-1}$ and unduplicated such columns headed by g_0, g_n, and g_{n+1}. It turns out to be convenient to label these as "left" and "right" depending on whether they occur to the left or right of the column headed by g_0 in our presentation of G, and we will call these column headers things like g_1^ℓ and g_1^r. In forming one of the terms in the Laplace expansion as in (1.5), we will call a column (or its header) "used" if it is one whose first $n+1$ rows appear in the submatrix $G_{\mathcal{S}}$ and "unused" if its final n rows appear in the submatrix $G_{\mathcal{S}'}$. It is obvious that to have a term in the Laplace expansion which is not automatically zero we must not have both g_i^ℓ and g_i^r used. (Note that it is possible to have, for some i, neither g_i^ℓ nor g_i^r used and generate a term in the expansion not automatically zero.)

A little consideration shows as well that if some g_i^r $(1 \le i \le n-1)$ is used, then in fact $g_{i+1}^r, g_{i+2}^r, \dots, g_{n-1}^r$ must also be used to avoid a trivial zero (and thus, of course, $g_i^\ell, g_{i+1}^\ell, \dots, g_{n-1}^\ell$ must be unused). For example, if g_1^r is used, then g_1^ℓ must be unused else $\det G_{\mathcal{S}} = 0$. But then if g_2^r is unused, then there are duplicate columns in $G_{\mathcal{S}'}$ and so its determinant will be zero. Thus to achieve a non-zero determinant, if g_1^r is used then we must use g_2^r, and (repeating the argument), $g_3^r, \dots g_{n-1}^r$. Similarly, if g_i^ℓ is unused, g_{i+1}^r and hence $g_{i+1}^r, g_{i+2}^r, \dots, g_{n-1}^r$ must be used.

Various terms not trivially zero may be produced: leave g_0 unused, which turns out to force $g_1^r, g_2^r, \dots, g_{n-1}^r, g_n$, and g_{n+1} used and the others unused; use g_0 but leave both g_1^ℓ and g_1^r unused, which again determines completely the rest of the choices; use g_0, one of the g_1's, and leave both g_2's unused, which turns out to require that it is g_1^ℓ which was used and again determines the rest of the choices; similar things leaving both g_i's unused for some $1 \le i \le n-1$; use g_0 and one of each of the g_i's for $1 \le i \le n-1$ and either g_n or g_{n+1}. We leave to the reader to show that each of these choices yields a term in the final line of (1.8) except possibly for sign and that each such term corresponds to one of the choices (again except possibly for sign). In recording the various possibilities a tree diagram provides a useful record.

We present a sample of the computations needed to show that the signs align. Suppose we consider, for some $1 \le p \le n-1$, the term in which $g_0, g_1^\ell, g_2^\ell, \dots, g_p^\ell$ and $g_{p+1}^r, \dots, g_{n-1}^r, g_n$ are used and (hence) $g_1^r, g_2^r, \dots, g_p^r, g_{p+1}^\ell, \dots, g_{n-1}^\ell, g_{n+1}$ are unused. Consider first $G_{\mathcal{S}}$, which is clearly some column rearrangement of $H_{n+1,0}$. As they occurred in $G_{\mathcal{S}}$ the columns came in the order

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$$g_p^\ell, g_{p-1}^\ell, \ldots, g_1^\ell, g_0, g_{p+1}^r, \ldots, g_{n-1}^r, g_n,$$

so to put them in the order in which they occur in $H_{n+1,0}$ requires p transpositions to move g_0 to first position, then $p-1$ transpositions to place g_1^ℓ, and so on, yielding $1+2+\ldots p = p(p+1)/2$ transpositions (and then all is in order). Thus we have

$$\det G_{\mathcal{S}} = (-1)^{\frac{p(p+1)}{2}} \det H_{n+1,0}. \tag{1.10}$$

Consider now $G_{\mathcal{S}'}$. One may show that the choice of p yields $(n+1)$-st row elements $g_2, g_3, \ldots, g_{2+p-1}, g_{2+p+1}, \ldots, g_{n+2}$ (without regard to order) as the column headers for $G_{\mathcal{S}'}$; this is to say (remembering that we index columns starting at zero) that we have produced some column rearrangement of $H_{n,2}^{(p)}$. It turns out (conveniently) that the permutation to put $g_1^r, g_2^r, \ldots, g_p^r, g_{p+1}^\ell, \ldots, g_{n-1}^\ell, g_{n+1}$ (which are the column headers in G of the columns whose headers in $G_{\mathcal{S}'}$ are $g_2, g_3, \ldots, g_{2+p-1}, g_{2+p+1}, \ldots, g_{n+2}$) in proper order is the same as that required to put the column headers for $G_{\mathcal{S}'}$ in order, so we work with the former. The unused column headers came in the order

$$g_{n-1}^\ell, g_{n-2}^\ell, \ldots, g_{p+1}^\ell, g_1^r, g_2^r, \ldots, g_p^r, g_{n+1},$$

and to put these in proper order requires $n-p-2$ interchanges to move g_{p+1}^ℓ to the front, $n-p-3$ to put g_{p+2}^ℓ in second position, and so on for a total of $1+2+\ldots(n-p-2) = (n-p-2)(n-p-1)/2$ interchanges to order the g_i^ℓ appropriately, and then $n-p-1$ interchanges to move g_1^r to the front, another $n-p-1$ to move g_2^r to second position, and so on, for a total of $p(n-p-1)$ interchanges to complete the ordering. It results that

$$\det G_{\mathcal{S}'} = (-1)^{\frac{(n-p-2)(n-p-1)}{2} + p(n-p-1)} \det H_{n,2}^{(p)}. \tag{1.11}$$

The contribution of the term in question to $\det G$ as in (1.5) also includes the sign of the permutation corresponding to the choice of $\mathcal{S}$ as in (1.4). In the case under consideration, and indexing the columns for convenience starting at 1 instead of 0, this is

$$\left(\begin{matrix} 1 & 2 & \ldots & p+1 & p+2 & \ldots & n & n+1 \\ n-p & n-p+1 & \ldots & n & n+p+1 & \ldots & 2n-1 & 2n \end{matrix} \right.$$
$$\left. \begin{matrix} n+2 & \ldots & 2n-p & 2n-p+1 & \ldots & 2n & 2n+1 \\ 1 & \ldots & n-p-1 & n+1 & \ldots & n+p & 2n+1 \end{matrix} \right).$$

The transpositions required to move "1" in the second row to first position in the second row are obviously $n+1$ in number, and the same is true to move $2, \ldots, n-p-1$ to the front, yielding $(n+1)(n-p-1)$ transpositions. It then takes $n-p-1$ transpositions to move each of $n+1, n+2, \ldots, n+p$ to its position, yielding $p(n-p-1)$ more transpositions. Thus we have that

$$\operatorname{Sign}(\mathcal{S}) = (-1)^{(n+1)(n-p-1)+p(n-p-1)}. \tag{1.12}$$

So we have finally, putting together (1.10), (1.11), and (1.12) with a modest computation, that the contribution of this term in the Laplace expansion toward $\det G$ is

$$(-1)^{-p^2+\frac{3}{2}n(n-1)} \det H_{n+1,0} \det H_{n,2}^{(p)}.$$

This, multiplied by the factor $(-1)^{s(n)}$, is what must be compared to the appropriate term in the sum from (1.8), which is $(-1)^{n+p} \det H_{n+1,0} \det H_{n,2}^{(p)}$ – that is, we must verify a portion of (1.9) – and showing these are equal is a straightforward computation using cases on p even or odd and the cases for n which play into $s(n)$. $\square$

The lemma yields information in terms of the various matrices Γ.

Proposition 1.5. *For any $n \geq 1$, $m \geq 0$, and $j \geq 0$, we have*

$$\begin{aligned}\det \Gamma(n+1 \times n+1, m, j) \det \Gamma(n \times n, m+1, j+2) - \\ - \det \Gamma(n \times n, m, j+2) \det \Gamma(n+1 \times n+1, m+1, j) = \\ = \det \Gamma(n \times n, m+1, j+1) \det \Gamma(n+1 \times n+1, m, j+1).\end{aligned}$$

Proof. This follows immediately from the lemma upon setting g_i to $\sum_{k=0}^{m}(-1)^k\binom{m}{k}\gamma_{i+j+k}$ (the m-th difference beginning at $i+j$) for $0 \leq i$ and upon noticing that the difference of successive m-th differences is the $(m+1)$-st difference. $\square$

We need as well a result concerning determinants of moment matrices for subnormal shifts which are not recursively generated. Recall that a weighted shift is recursively generated if there is a recursion among the moments (see, for example, [8]); it is also known that a shift is recursively generated if and only if it has a finitely atomic Berger measure.

Lemma 1.6. *Let W_α be a subnormal weighted shift which is not recursively generated and let $\Gamma(n \times n, 0, j)$ $(n \geq 2, j \geq 0)$ be the usual moment matrices. Then for any such n,*

$$\det \begin{pmatrix} 0 & \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_{n+1} \\ \vdots & & & & \vdots \\ \gamma_n & \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{2n} \end{pmatrix} = \det \Gamma(n+1 \times n+1, 0, 0) - \det \Gamma(n \times n, 0, 2) < 0.$$

As well, if in addition W_α is contractive, then for any such n and any $m \geq 0$,

$$\det \Gamma(n+1 \times n+1, m, 0) - \det \Gamma(n \times n, m, 2) < 0. \tag{1.13}$$

Proof. The claimed equality is trivial, so suppose the first inequality fails for some such shift and some n. Observe first that each $\Gamma(n \times n, 0, j)$ is both positive and non-singular, where positivity is because W_α is subnormal and non-singularity follows from [9][Prop. 5.13]. Multiplying by $\frac{1}{\alpha_0^{2n}}$, we have

$$0 \leq \det \begin{pmatrix} 0 & \frac{\gamma_1}{\alpha_0^2} & \frac{\gamma_2}{\alpha_0^2} & \cdots & \frac{\gamma_n}{\alpha_0^2} \\ \frac{\gamma_1}{\alpha_0^2} & \frac{\gamma_2}{\alpha_0^2} & \frac{\gamma_3}{\alpha_0^2} & \cdots & \frac{\gamma_{n+1}}{\alpha_0^2} \\ \vdots & & & & \vdots \\ \frac{\gamma_n}{\alpha_0^2} & \frac{\gamma_n+1}{\alpha_0^2} & \frac{\gamma_{n+2}}{\alpha_0^2} & \cdots & \frac{\gamma_{2n}}{\alpha_0^2} \end{pmatrix} = \det \begin{pmatrix} 0 & \gamma_1' & \gamma_2' & \cdots & \gamma_n' \\ \gamma_1' & \gamma_2' & \gamma_3' & \cdots & \gamma_{n+1}' \\ \vdots & & & & \vdots \\ \gamma_n' & \gamma_n'+1 & \gamma_{n+2}' & \cdots & \gamma_{2n}' \end{pmatrix}, \tag{1.14}$$

where the γ_k' denote the moments for the shift with weight sequence $\alpha_1, \alpha_2, \ldots$. (Observe that this shift is both subnormal and non-recursively generated.) Then surely for any $x > 0$,

$$\det \begin{pmatrix} \frac{1}{x^2} & \gamma_1' & \gamma_2' & \cdots & \gamma_n' \\ \gamma_1' & \gamma_2' & \gamma_3' & \cdots & \gamma_{n+1}' \\ \vdots & & & & \vdots \\ \gamma_n' & \gamma_n'+1 & \gamma_{n+2}' & \cdots & \gamma_{2n}' \end{pmatrix} > 0, \tag{1.15}$$

since the left-hand side is the expression in (1.14) plus $\frac{1}{x^2}$ times the determinant of $\Gamma'(n-1\times n-1, 0, 2)$; the latter matrix has strictly positive determinant, noting that this latter is positive and non-singular. But then citing Lemma 3.3 and Proposition 2.2 (v) of [9], we have that the shift with weight sequence $x, \alpha_1, \alpha_2, \ldots$ (a one-length back step extension of $\alpha_1, \alpha_2, \ldots$) is n-hyponormal for any $x > 0$, which contradicts Proposition 3.4 of that same paper.

To show the second claim, fix m, and suppose we consider some (contractive, non-recursively generated) shift with Berger measure μ. Let $C_m = \int_0^1 (1-t)^m \, d\mu(t)$, and consider the shift whose moments are given by the measure $\frac{1}{C_m}(1-t)^m \, d\mu(t)$. Observe that this is subnormal, contractive, and that the two shifts are finitely atomic, or not, together. Denote the matrices relevant to this second shift by $\Gamma^m(n \times n, k, j)$. But then it is easy to see (using $C_m < 1$) that

$$\begin{aligned}
\det \Gamma(n+1 \times n+1, m, 0) - \det \Gamma(n \times n, m, 2) = \\
&= C_m^{n+1} \det \Gamma^m(n+1 \times n+1, 0, 0) - C_m^n \det \Gamma^m(n \times n, 0, 2) \\
&\leq C_m^n \det \Gamma^m(n+1 \times n+1, 0, 0) - C_m^n \det \Gamma^m(n \times n, 0, 2) \\
&< 0,
\end{aligned}$$

by what was just proved. □

Note that the result need not hold if the shift is recursively generated, since its moment matrices of large enough size will have deficient rank and thus be singular (see [9]).

We have finally arrived at the following.

Theorem 1.7. *Let W_α be a subnormal weighted shift not recursively generated, and $W_{\alpha(x)}$ a 0-th weight perturbation of W_α with $x > 0$, where $\alpha(x)$ is as in (1.2). Then for all $n \geq 2$ and all $m \geq 1$, the set of $x > 0$ for which $W_{\alpha(x)}$ has $P^x(n \times n, m, *)$ is a finite half-open interval of the form $(0, x_{n,m}]$ with $x_{n,m}$ strictly positive. The $x_{n,m}$ are decreasing in n and strictly decreasing in m, and this yields that $W_{\alpha(x)}$ has $P^x(n \times n, m, *)$ implies $W_{\alpha(x)}$ has $P^x(n \times n, k, *)$, $0 \leq k \leq m$.*

Proof (sketch). Denote as usual the matrices for the perturbed shift by $\Gamma^x(n \times n, \ell, j)$. Note that we need only check matrices of the form $\Gamma^x(n \times n, k, 0)$ because the $\Gamma^x(n \times n, k, i)$ for $i \geq 1$ are simply scalings, by x^n, of the related matrix $\Gamma(n \times n, k, i)$ for W_α, and these are positive since W_α is subnormal. Note also that for the final conclusion it obviously suffices to show for $W_{\alpha(x)}$ that $P(n \times n, m, *)$ implies $P(n \times n, m-1, *)$.

Consider first the question of when, for some n and m, we have

$$\det \Gamma^x(n \times n, m, 0) \geq 0$$

(which is surely necessary for matrix positivity). Citing (1.3) this happens if and only if

$$\det \Gamma(n-1 \times n-1, m, 2) + x(\det \Gamma(n \times n, m, 0) - \det \Gamma(n-1 \times n-1, m, 2)) \geq 0.$$

Considering the expression just written, we have using (1.13) that this is linear in x and that x has (strictly) negative coefficient, and observe that the constant term is (strictly) positive since W_α is subnormal and not recursively generated. Thus the determinant is non-negative if and only if x falls below the determinant "cutoff" $d_{n,m}$:

$$x \leq d_{n,m} := \frac{-\det \Gamma(n-1 \times n-1, m, 2)}{\det \Gamma(n \times n, m, 0) - \det \Gamma(n-1 \times n-1, m, 2)}.$$

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We may now compare the cutoffs for m and $m-1$; with a little algebra, one may show that

$$d_{n,m-1} > d_{n,m} \tag{1.16}$$

if and only if

$$\begin{aligned} &-\det\Gamma(n-1\times n-1, m-1, 2)\det\Gamma(n\times n, m, 0) + \\ &\qquad + \det\Gamma(n-1\times n-1, m, 2)\det\Gamma(n\times n, m-1, 0) > 0, \end{aligned} \tag{1.17}$$

but the term on the left hand side of (1.17) is

$$\det\Gamma(n-1\times n-1, m, 1)\det\Gamma(n\times n, m-1, 1) \tag{1.18}$$

citing Proposition 1.5, and this last is (strictly) positive since W_α is subnormal and not recursively generated.

We show next that $x_{n,m} = \min\{d_{k,m} : 2 \le k \le n\}$. Since if $\Gamma^x(n\times n, m, 0)$ is positive it surely has non-negative determinant, as do each of its principle submatrices, including each $\Gamma^x(k\times k, m, 0)$, $2 \le k \le n$. Thus $x_{n,m}$ is no larger than the proposed minimum. But if $0 < x < \min\{d_{k,m} : 2 \le k \le n\}$, by what was shown above about the linearity with negative x coefficient of the expressions yielding these quantities, we have that each of the $\det\Gamma^x(k\times k, m, 0)$ is strictly positive, $2 \le k \le n$, so by the nested determinant test we have $\Gamma^x(n\times n, m, 0)$ positive. The interval is closed on the right because matrix positivity is a continuous condition.

It is clear that the $x_{n,m}$ are decreasing in n since positivity of a larger matrix implies positivity of its principle submatrices. To show that the $x_{n,m}$ are decreasing in m, suppose $x < x_{n,m+1}$. It follows that $x < d_{k,m+1}$, $2 \le k \le n$, and arguing as before and using the strict decrease of the $d_{n,j}$ in j (as in (1.16)) we have $x < d_{k,m}$, $2 \le k \le n$. Again using the nested determinant test we have $\Gamma^x(n\times n, m, 0)$ positive. It is then easy to finish the remainder, including the strict decrease in m, again using 1.16. □

Remark that these nested and decreasing intervals are familiar from the cases of k-hyponormality and n-contractivity.

We may obtain additional information for a 0-th weight perturbation of the Agler shifts A_p, $p = 2, 3, \ldots$, to $A_p(x)$ with weight sequence

$$\sqrt{\frac{x}{p}}, \sqrt{\frac{2}{p+1}}, \ldots. \tag{1.19}$$

(Observe this is of the form in (1.2) with the specific weights of the Agler shifts.) It is convenient to rewrite the condition (1.3) in the form

$$\det\Gamma^x(n\times n, m, 0) \ge 0 \quad \text{iff} \quad x \le \frac{1}{1 - \frac{\det\Gamma(k\times k, m, 0)}{\det\Gamma(k-1\times k-1, m, 2)}}. \tag{1.20}$$

Proposition 1.8. *For a 0-th weight perturbation of A_p, $p = 2, 3, \ldots$ to $A_p(x)$ with weight sequence as in (1.19) and for any $k \ge 2$ and $m \ge 0$, the set of $x > 0$ for which $A_p(x)$ satisfies $P(k\times k, m, *)$ is of the form $(0, x_c]$, with*

$$x_c = x_c(p, k, m) = \frac{1}{1 - \frac{(p-1)}{(k+m+p-2)(k)}}.$$

These cutoffs are decreasing in k and m to 1, yielding nested intervals decreasing to $(0, 1]$ for k or m decreasing separately, and it follows that for all p, $k \ge 2$, and m,

$$P(k \times k, m, *) \Rightarrow P(i \times i, n, *), \qquad 1 \le i \le k, 0 \le n \le m.$$

Proof. Fix p in the indicated range, and hold it implicit notationally in what follows. Considering the condition in (1.20), and using (1.1), this is

$$x \le \frac{1}{1 - \frac{\det C_m^k \, \Gamma^{(m)}(k \times k, 0, 0)}{\det C_m^{k-1} \, \Gamma^{(m)}(k-1 \times k-1, 0, 2)}} = \frac{1}{1 - \frac{C_m \det \Gamma^{(m)}(k \times k, 0, 0)}{\det \Gamma^{(m)}(k-1 \times k-1, 0, 2)}}.$$

Since the Berger measure for A_p is $(p-1)(1-t)^{p-2} dt$ we have

$$C_m = \int_0^1 (1-t)^m d\mu(t) = \int_0^1 (1-t)^m (p-1)(1-t)^{p-2} dt = \frac{p-1}{m+p-1}.$$

It remains to obtain the ratios $\frac{\det \Gamma^{(m)}(k \times k, 0, 0)}{\det \Gamma^{(m)}(k-1 \times k-1, 0, 2)}$, and these are simply determinants of the moment matrices for the measure $\hat{\mu}_m(t) = C_m (1-t)^m \mu(t)$, and recalling the implicit dependence upon p, this is for the measure $(1-t)^{m+p-2}(m+p-1)dt$, which is simply the measure for A_{p+m}. These determinant ratios were obtained in [1][Lemma 2.5] (note that what is there is for initial size $(n+1) \times (n+1)$), and what results is

$$\frac{\det \Gamma^{(m)}(k \times k, 0, 0)}{\det \Gamma^{(m)}(k-1 \times k-1, 0, 2)} = \frac{m+p-1}{(k+m+p-2)k}.$$

Putting this all together yields $\det \Gamma^x(k \times k, m, 0) \ge 0$ if and only if $x \le \frac{1}{1 - \frac{(p-1)}{(k+m+p-2)(k)}}$. The rest follows easily. $\square$

Observe that an increase in k is, in an order sense, more powerful to decrease the cutoff than an increase in m. This is reflective of the general fact that the matricial positivity condition is more powerful than the single positivity condition (recall k-hyponormality implies $2k$-contractivity). We remark also that since we have explicit formulas for the determinants one can achieve the result for the $A_p(x)$ directly and without the use of Theorem 1.7.

One may phrase things about these 0-th weight perturbations of the A_p somewhat differently by using the cutoffs above to note when one property forces or coincides with another. For example, we have from a computation the following, which includes [1][Theorem 2.6] as a special case.

Proposition 1.9. *Consider the usual 0-th weight perturbations of A_p for some $p \ge 2$, and any $k \ge 1$ and $m \ge 0$. Then A_p has $P(k \times k, m, *)$ if and only if it is $(pk + k^2 + m + km)$-contractive – i.e., has $P(1 \times 1, pk + k^2 + m + km, *)$.*

Note again the relative "power" of k and m.

As one might expect, equivalence with some k-hyponormality is more complicated. Again computations yield

Proposition 1.10. *Consider the usual 0-th weight perturbations of A_p to A_p^x for some $p \ge 2$, and any $k, k' \ge 1$ and $m \ge 0$. Then A_p^x is k'-hyponormal if it has $P(k \times k, m, *)$ with*

$$k' \le \frac{1}{2}\left(-p + \sqrt{4k^2 + (p-2)^2 + 4k(p-2+m)}\right);$$

alternatively, if it is k'-hyponormal and with the inequality above reversed it has $P(k \times k, m, *)$. These properties coincide supposing the appropriate quantities are equal (and integers). Alternatively, given some A_p, for positivity of matrices of size $k \times k$ using differences of length m to yield positivity of $(k'+1) \times (k'+1)$ moment matrices (no differences) for A_p^x – that is, k'-hyponormality – requires

$$m \geq \frac{(k'+1-k)(k'+1+p-2+k)}{k}$$

(and m integer).

Proof. These are simply a matter of comparing the "cutoffs" in x arising from Proposition 1.8, recalling that j-hyponormality involves matrices of size $j+1$ by $j+1$. $\square$

While some special cases are easy, we know of no "tidy" way to describe the general situation in which integral values are obtained.

We turn next to the case in which the shift to be perturbed is finitely atomic (again we will consider perturbations in the 0-th weight, yielding a weight sequence $x\alpha_0, \alpha_1, \ldots$ as in (1.2)). For a thorough discussion of such shifts see [8] and [9], and we merely indicate what is needed. We will consider (contractive) subnormal shifts whose Berger measure contains r atoms (of course in the interval $[0,1]$); r is the "rank" of the weight sequence. We will denote the weight sequence of such a shift by $(\alpha_0, \alpha_1, \ldots, \alpha_{2r-2})^\wedge$ (as turns out to be appropriate – see [10][Lemma1.2] and surrounding material). It is known that for such a shift one has $\Gamma(r+1 \times r+1, 0, j)$ singular for any j but that the matrices $\Gamma(k \times k, 0, j)$ are non-singular for $1 \leq k \leq r$ (of course all the Γ's are at least non-negative for the original subnormal shift).

It is well known that if $(\alpha_0, \alpha_1, \ldots, \alpha_{2r-2})^\wedge$ is such a weight sequence then (with α_{2r-1} the next weight) one has $(\alpha_0, \alpha_1, \ldots, \alpha_{2r-2})^\wedge = \alpha_0, (\alpha_1, \alpha_2, \ldots, \alpha_{2r-1})^\wedge$. It is therefore essentially equivalent to consider a back step extension of length one or a perturbation of the 0-th weight of such a shift. It then follows that for a 0-th weight perturbation of such a shift r-hyponormality is equivalent to subnormality (see [10][Theorem 1.3]).

It is easy to check that the first result of Lemma 1.6 holds with suitable modification (by comparing expansion about the first row – with the missing "1" – to the expansion about the first row of $\det \Gamma(n+1 \times n+1, 0, 0)$):

Lemma 1.11. *If W_α is a subnormal weighted shift recursively generated of rank r, then for $n \leq r$ we have*

$$\det \begin{pmatrix} 0 & \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ \gamma_1 & \gamma_2 & \gamma_3 & \cdots & \gamma_{n+1} \\ \vdots & & & & \vdots \\ \gamma_n & \gamma_n + 1 & \gamma_{n+2} & \cdots & \gamma_{2n} \end{pmatrix} = \det \Gamma(n+1 \times n+1, 0, 0) - \det \Gamma(n \times n, 0, 2) < 0.$$

We then have the following, whose hypothesis contains two unexpected assumptions we shall address shortly.

Theorem 1.12. *Let W_α be a contractive subnormal weighted shift recursively generated of rank r with no atom at 0 and no atom at 1, and $W_{\alpha(x)}$ the 0-th weight perturbation of W_α with weight sequence $\alpha(x)$ as in (1.2) with $x > 0$. Then for all $2 \leq n \leq r$ and all $m \geq 1$, the set of $x > 0$ for which $W_{\alpha(x)}$ has $P(n \times n, m, *)$ is a finite half-open interval of the form $(0, x_{n,m}]$ with $x_{n,m}$ strictly positive. The set of x for which $W_{\alpha(x)}$ has $P(r+1 \times r+1, m, *)$ is the half-open interval $(0,1]$ for any m. The $x_{n,m}$ are decreasing in n and decreasing in m, and this yields that $W_{\alpha(x)}$ has $P(n \times n, m, *)$ implies $W_{\alpha(x)}$ has $P(n \times n, k, *)$, $2 \leq n \leq r$ and $0 \leq k \leq m$.*

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Proof (sketch). The proofs for the assertions for $n \leq r$ are exactly as in the proof of Theorem 1.7 (no issues of matrix singularities arise). For $n = r + 1$, recall that r-hyponormality, which is $P(r + 1 \times r + 1, 0, *)$, coincides with subnormality (as noted above, this is to be found in [10][Lemma 1.2]), and recall also that a 0-th weight perturbation of a subnormal shift remains subnormal if the 0-th weight remains the same or decreases; these together result in the claim that the set of x for which $W_{\alpha(x)}$ has $P(r + 1 \times r + 1, 0, *)$ is the half-open interval $(0, 1]$.

To consider the set of x for which $W_{\alpha(x)}$ has $P(r + 1 \times r + 1, m, *)$, note that in light of Lemma 1.11 we may consider determinant cutoffs as before; however, when we imitate the computations in (1.16), (1.17), and (1.18) these cutoffs are the same for all m since $\det \Gamma(r + 1 \times r + 1, m - 1, 1) = 0$. The remaining assertions follow easily. □

The result must be modified if there is an atom at 0 or an atom at 1, although neither modification is difficult. Suppose first that there is an atom at 1. If the Berger measure μ for W_α has rank r (is r-atomic) with one of the atoms at 1, the measure for the shifts arising from $(1 - t)^p \, d\mu(t)$ (for $p \geq 1$, which give rise to iterated differences of moments for W_α) is $(r - 1)$-atomic and not r-atomic. Thus the expression in (1.18) for $m = 1$, say, becomes zero not at $n = r + 1$ (from $\det \Gamma(n \times n, m - 1, 1)$ with $n = r + 1$) but at $n = r$ (from $\det \Gamma(n - 1 \times n - 1, m, 1)$).

If there is an atom at 0, the assertion that the set of x for which $W_{\alpha(x)}$ has $P(r + 1 \times r + 1, m, *)$ is "the half-open interval $(0, 1]$ for any m" must be altered to "a half-open interval $(0, c]$ with $c \geq 1$." It is well-known (see, for example, Lemma 5.2 of [6]) that to decrease the zeroth weight of a subnormal weighted shift leaves it subnormal, and that the result of such an operation is to produce a new shift whose Berger measure has a mass at zero. If our initial shift is the result of such an operation, of course the zeroth weight may grow and the shift remain subnormal. A direct computation with finitely atomic measures yields the appropriate constant c; we leave to the interested reader the modifications of the result for these two special cases.

Samples of what results when there is an atom at 1, moved to the case of back step extensions, include the following, whose proofs we omit. Recall first the Stampfli subnormal completion of three increasing weights (see [24]). In that paper, the author showed that given three strictly increasing initial weights, the weight sequence may be completed to the weight sequence of a subnormal shift which is, in fact, two-atomic in general (in special cases, one-atomic), or, equivalently, recursively generated in that there is a finite recursion of length two for the sequence of moments (in special cases, of length one). The weight sequence for such a completion, given the initial weights $\sqrt{a}$, $\sqrt{b}$, and $\sqrt{c}$, is denoted $(\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ with $0 < a < b < c$. We choose $0 < a < b < 1$ and set c to the value

$$c = \frac{a(b^2 - b + 1) - b}{(a - 1)b}, \tag{1.21}$$

which is known to yield norm exactly 1 (equivalently, an atom at 1) by an elementary computation.

Proposition 1.13. *For back-step extensions of $(\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$, with a, b, and c, as just above, we have*

- $\sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ *is subnormal if and only if it is 2-hyponormal (which is $P(3 \times 3, 0, *)$) if and only if it has $P(2 \times 2, 1, *)$;*
- $\sqrt{y}, \sqrt{x}, (\sqrt{a}, \sqrt{b}, \sqrt{c})^\wedge$ *is subnormal if and only if it is 3-hyponormal (which is $P(4 \times 4, 0, *)$) if and only if it has $P(3 \times 3, 1, *)$.*

Efforts to increase the "Agler–Embry" parameter m (the length of the differences) to compensate for further decrease in the "Bram–Halmos" parameter k (the size of the matrix) do not yield subnormality.

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Observe that in this situation we have an equivalence between $P(k \times k, m, *)$ and $P(k' \times k', m', *)$ for $k \neq k'$ and $m \neq m'$.

It turns out that if W_α is a contraction with weights approaching 1 one has a general result relating various properties $P(\cdot \times \cdot, \cdot, \cdot)$. The proof is much the same as for Proposition 0.3.

Theorem 1.14. *Let W_α be a contractive weighted shift with weights $\alpha_j \to 1$. Then for any $k \geq 2$ and $m \geq 0$, $P(k \times k, m+1, *)$ implies $P(k \times k, m, *)$.*

Proof. Suppose that $\Gamma(k \times k, m+1, *) \geq 0$ but there is some (least) j such that $\Gamma(k \times k, m, j) \not\geq 0$. Let v be some non-zero vector such that

$$v^T \Gamma(k \times k, m, j)\, v < 0.$$

It is easy to check that $\Gamma(k \times k, m+1, \ell) = \Gamma(k \times k, m, \ell) - \Gamma(k \times k, m, \ell+1)$ for any ℓ, and therefore that

$$0 \leq v^T \Gamma(k \times k, m+1, j)\, v = v^T \Gamma(k \times k, m, j)\, v - v^T \Gamma(k \times k, m, j+1)\, v.$$

Hence

$$v^T \Gamma(k \times k, m, j+1)\, v \leq v^T \Gamma(k \times k, m, j)\, v$$

and the sequence

$$\left(v^T \Gamma(k \times k, m, \ell)\, v\right)_{\ell \geq j}$$

is negative and weakly decreasing.

However, since k and m are fixed and the weights α_j approach 1, the entries of $\Gamma(k \times k, m, \ell)$ approach 0 as $\ell \to \infty$ (approaching multiples less than one of $(1-1)^m$), and so

$$v^T \Gamma(k \times k, m, \ell) v \to 0 \quad \text{as } \ell \to \infty,$$

which is impossible for a negative weakly decreasing sequence. □

Observe that if W_α is "close enough" to a contraction in having weights approaching 1 the result holds as well and yields that in fact W_α must be a contraction.

The similar result for n-contractivity ($\|W_\alpha\| \leq 1$ and $P(1 \times 1, m+1, *)$ together imply $P(1 \times 1, m, *)$, or, put differently, that for a contractive shift n-contractivity implies n-hypercontractivity) holds even if the weights do not converge to 1, but the result relies on work of Agler [2] using the fact that if $\alpha_j \nrightarrow 1$ then $W_\alpha^n \to 0$ SOT, for which we have not found a substitute.

A general implication (although not equivalence) is available for weighted shifts. It recaptures as a special case the result in Theorem 0.1 that k-hyponormality implies $2k$-contractivity, recalling that k-hyponormality is $P(k+1 \times k+1, 0, *)$. We merely indicate the proof.

Theorem 1.15. *Let W be a weighted shift and $k \geq 1$ and $m \geq 0$ be arbitrary. Then $P(k \times k, m, *)$ implies $P(1 \times 1, 2(k-1)+m, *)$ (which latter is $2(k-1)+m$-contractivity).*

Proof (sketch). An example of the computation is the following, for $k = 3$ and $m = 1$:

$$0 \le (1,-2,1)\begin{pmatrix}\gamma_0-\gamma_1 & \gamma_1-\gamma_2 & \gamma_2-\gamma_3\\ \gamma_1-\gamma_2 & \gamma_2-\gamma_3 & \gamma_3-\gamma_4\\ \gamma_2-\gamma_3 & \gamma_3-\gamma_4 & \gamma_4-\gamma_5\end{pmatrix}\begin{pmatrix}1\\-2\\1\end{pmatrix}$$
$$= \gamma_0 - 5\gamma_1 + 10\gamma_2 - 10\gamma_3 + 5\gamma_4 - \gamma_5.$$

Recalling that for weighted shifts the test for n-contractivity is as in (0.5), we see that this is the test beginning at γ_0 for 5-contractivity (that is, $(2\cdot 2+1)$-contractivity). $\square$

2. Conditions for general operators

What can be said for general operators? Let $D_m := \sum_{i=0}^{m}(-1)^i\binom{m}{i}T^{*i}T^i = I - \binom{m}{1}T^*T + \binom{m}{2}T^{*2}T^2 - \ldots$ denote the m-th difference for an operator T. We first define an analog of the Bram matrix in (0.1): for $k \ge 1$ and $m \ge 0$,

$$B(k\times k,m) := \begin{pmatrix} D_m & T^*D_m & T^{*2}D_m & \ldots & T^{*k}D_m\\ D_mT & T^*D_mT & T^{*2}D_mT & \ldots & T^{*k}D_mT\\ D_mT^2 & T^*D_mT^2 & T^{*2}D_mT^2 & \ldots & T^{*k}D_mT^2\\ \vdots & & \vdots & & \vdots\\ D_mT^k & T^*D_mT^k & T^{*2}D_mT^k & \ldots & T^{*k}D_mT^k\end{pmatrix}. \tag{2.1}$$

We may also define the analog of the Embry matrix in (0.2); given $k \ge 1$, and $m \ge 0$, define $E = E(k\times k,m)$ by

$$E(k\times k,m) := \begin{pmatrix} D_m & T^*D_mT & T^{*2}D_mT^2 & \ldots & T^{*k-1}D_mT^{k-1}\\ T^*D_mT & T^{*2}D_mT^2 & T^{*3}D_mT^3 & \ldots & T^{*k}D_mT^k\\ T^{*2}D_mT^2 & & & \ldots & T^{*k+1}D_mT^{k+1}\\ \vdots & & \ddots & & \vdots\\ T^{*k-1}D_mT^{k-1} & T^{*k}D_mT^k & T^{*k+1}D_mT^{k+1} & \ldots & T^{*2k-2}D_mT^{2k-2}\end{pmatrix}. \tag{2.2}$$

Obviously $B((k+1)\times(k+1),0)$ (respectively, $E((k+1)\times(k+1),0)$) is just the standard operator matrix whose positivity is the requirement for Bram (respectively Embry) k-hyponormality.

In [21] are established the following relationships between positivity of the $(k+1)$ by $(k+1)$ Bram matrix $B((k+1)\times(k+1),0)$ as in (0.1) and the $(k+1)$ by $(k+1)$ Embry matrix $E((k+1)\times(k+1),0)$ as in (0.2): first, Bram positivity implies Embry positivity. Second, the reverse does not hold, but does if T is invertible. Third, for unilateral weighted shifts Bram and Embry positivity are equivalent, and fourth (originally proved in [7][Theorem 4]), that for unilateral weighted shifts positivity of either $(k+1)$ by $(k+1)$ matrix is equivalent to positivity of the $(k+1)$ by $(k+1)$ scalar moment matrices (0.4) for $n = 0,1,2,\ldots$. It turns out that each of the analogous results holds in our generalized case, and the proofs are the same, *mutatis mutandis*, so we omit them.

Theorem 2.1. *Let T be an operator, with $B(k\times k,m)$ and $E(k\times k,m)$ its generalized Bram and Embry matrices. For each $k \ge 2$ and $m \ge 0$, $B(k\times k,m)$ positive implies $E(k\times k,m)$ positive. The reverse implication need not hold, but does if T is invertible.*

Theorem 2.2. *Let W_α be a weighted shift, with $B(k\times k,m)$ and $E(k\times k,m)$ its generalized Bram and Embry matrices. For each $k \ge 1$ and $m \ge 0$, $B(k\times k,m)$ is positive if and only if $E(k\times k,m)$ is positive. Either of these is in turn equivalent to positivity of $\Gamma(k\times k,m,j)$ for $j = 0,1,2,\ldots$.*

These results encourage one to believe that the definitions in (2.1) and (2.2) are the "right" ones. Given some T, we denote by $PE(k \times k, m)$ that $E(k \times k, m)$ is non-negative, with similar notation for the Bram matrices. The following result is also encouraging, since it generalizes the result that Embry (and hence Bram) k-hyponormality implies $2k$-contractivity ([16][Theorem 1.2]).

Theorem 2.3. *Let T be any operator and $k \geq 1$ and $m \geq 0$ be arbitrary. Then $PE(k \times k, m)$ implies T is $(2(k-1)+m)$-contractive.*

Proof (sketch). The proof requires some induction using the recurrence relationships amongst the differences (it helps to use a certain operation "$\diamond$" from [15]) but a sample computation is as follows: for any vector v, we have

$$(v, -v)\begin{pmatrix} I - 2T^*T + T^{*2}T^2 & T^*(I - 2T^*T + T^{*2}T^2)T \\ T^*(I - 2T^*T + T^{*2}T^2)T & T^{*2}(I - 2T^*T + T^{*2}T^2)T^2 \end{pmatrix}\begin{pmatrix} v \\ -v \end{pmatrix}$$
$$= \langle (I - 3T^*T + 3{T^*}^2T^2 - {T^*}^3T^3)v, v\rangle,$$

and so the operator matrix positivity, and the right choice of vector, yields the positivity of the appropriate difference operator. In general, given a vector v at which one wishes to show $(2(k-1)+m)$-contractivity, one uses $E(k \times k, m)$ to yield a quadratic form applied to the vector

$$(v, -\binom{k}{1}v, \binom{k}{2}v, \ldots, \pm\binom{k}{k}v)^T,$$

as in the proof of [16][Theorem 1.2], and with the aid of [15][Lemma 2.1] to cope with differences of differences (for this last, see, for example, equation (2.3) below). □

Remarks and Questions. In the weighted shift case, matrices of differences of moments appear in [18] in the context of expansivity, with, as one might expect, reversed inequality constraints. (Recall that complete hyperexpansivity is defined by negativity, as opposed to positivity, of all m-th order operator differences: $I - T^*T \leq 0, I - 2T^*T + {T^*}^2T^2 \leq 0$, and so on.)

There is an intriguing condition which arose in passing in *A characterization of k-hyponormality via weak subnormality* [11]. The authors show that if T is 2-hyponormal then $T^*[T^*, T]T \leq \|T\|^2[T^*, T]$. If we suppose for a moment that $\|T\| = 1$, this becomes

$$(T^*T - TT^*) - T^*(T^*T - TT^*)T \geq 0,$$

which "looks like" the 1-hyponormality condition inserted into the first difference $I - T^*T \geq 0$.

In [15] the "$\diamond$" operation previously mentioned was defined: ${T^*}^nT^n \diamond {T^*}^mT^m := {T^*}^n{T^*}^mT^mT^n$ and $I \diamond {T^*}^nT^n := {T^*}^nT^n$ amounts to some sort of "insertion in the middle." In terms of this operation one obtains, for example,

$$D_{m+1} = (I - T^*T) \diamond D_m, \tag{2.3}$$

where D_j is the j-th difference of T.

The relationship from [11]

$$(T^*T - TT^*) - T^*(T^*T - TT^*)T \geq 0,$$

is the 1-hyponormality condition inserted into the first difference $I - T^*T \geq 0$. Is there a version for k-hyponormality, $k \geq 2$?

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