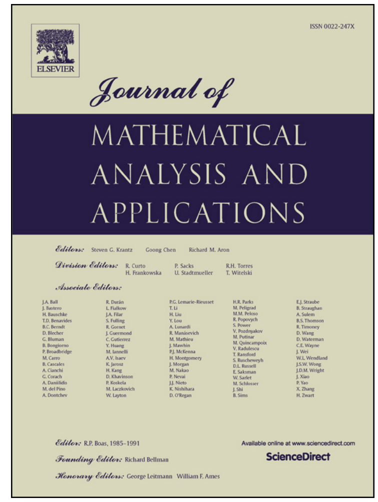

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On the first integrals of n-th order autonomous systems

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Abstract The first integrals of n-th order autonomous systems are considered. By getting integrating factors using $n-1$ first integrals of the systems, a necessary condition on the existence of global first integrals of n-th order autonomous systems is presented. It is also proved that $n-1$ functionally independent first integrals of an n-th order autonomous system can be obtained if the system possesses $n-1$ functionally independent integrating factors. Based on one-parameter Lie groups admitted by n-th order autonomous systems, several methods to obtain first integrals of the systems are presented. Simultaneously, several related examples are given to illustrate the feasibility and the effectiveness of the proposed method.

Key words First integral, Integrating factor, Lie group, n-th order autonomous system

MR(2010) Subject Classification 34A08

1 Introduction

We mainly concentrate on n-th order autonomous systems of the following form,

$$\frac{dx}{dt} = P(x), \tag{1}$$

where $x = (x_1, x_2, ..., x_n) \in \mathbf{D} \subset \mathbf{R}^n(or\ \mathbf{C}^n), P(x) = (P_1(x), ..., P_n(x)), P_i(x) : \mathbf{D} \to \mathbf{R}(or\ \mathbf{C})$. $P_i(x) \in C_\infty(\mathbf{D})$ and $t \in \mathbf{R}(or\ \mathbf{C})$. Associated to system (1), there is the vector field $X = \sum_{i=1}^{n} P_i(x)\frac{\partial}{\partial x_i}$. The divergence is $divX = \sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i}$. For n-th order non-autonomous systems

$$\frac{dx}{dt} = P(t, x), \tag{2}$$

it is often easy to be converted into the $(n+1)$-th order autonomous systems by letting $t = x_0$. In this paper, we are interested in first integrals in the sense that the open set $\mathbf{D} \subseteq \mathbf{R}^n$ where the first integral is well defined. A function $\Phi(x) \in C'(\mathbf{D}), \Phi(x) : \mathbf{D} \to \mathbf{R}$ is called a first integral of system (1) if it satisfies the following differential equation at every point in $\mathbf{D}$,

$$X\Phi(x) = 0.$$

First integrals are powerful tool in the study of ordinary differential equations and partial differential equations (see for instance Ref. [1]-[10] and the references therein). As we know, searching for first integrals of a differential equations system plays a very important role for integrating the system. The importance of the existence of a non-constant first integral $\Phi(x)$ of the vector field X

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lies in the fact that the trajectories of the vector field X leave invariant the level sets of the function $\Phi(x)$, and hence this is a strong constraint on the dynamical behavior of the vector field. So, one often wants to recognize if an autonomous system has first integrals, and how to get first integrals of the system. Many different methods have been used for studying the existence and searching for first integrals of ordinary differential systems. For example, the Lie groups ([1, 11]), the Darboux theory of integrability ([12]), the Painlevé analysis ([13]), the use of Lax pairs ([14]), etc. In [15], The necessary conditions for the existence of functionally independent generalized local rational first integrals of ordinary differential systems via resonances are studied. The main results in the paper extend some of the previous related ones. In [16], The relations between the existence of local analytic first integrals and resonance are studied, and a necessary condition for general nonlinear systems to have rational first integrals is given. By using the so-called Kowalevsky exponents, the author also presents a criterion for the nonexistence of rational first integrals for semiquasihomogeneous systems in [16].

As we know, there are two important tools, which can be used to search for first integrals of an autonomous system: One is inverse integrating factors and the other is Lie groups admitted by the autonomous system. The fact that the existence of Lie group gives rise to study first integrals is well known since Lie's work [1, 11]. Let's recall some basic concepts. We say that a function $\mu(x) \in C^{\infty}(\mathbf{D})$ is an inverse integrating factor of the vector field X if $X\mu(x) = \mu(x) divX$. Actually, if $\mu(x)$ is an inverse integrating factor of the vector field X, $\frac{1}{\mu(x)}(\mu(x) \neq 0)$ is an integrating factor, that is, $div(\frac{1}{\mu(x)}X) = 0$. A one-parameter Lie group G with the following generator

$$V = V_1 \frac{\partial}{\partial x_1} + ... + V_n \frac{\partial}{\partial x_n}$$

is admitted by system (1) if every transformations of G leave system (1) invariant, that is, $[X, V] = A(x)X$ for some function $A(x) \in C^{\infty}(\mathbf{D})$, where $[\cdot, \cdot]$ stands for the Lie bracket. Some recent results about first integrals can be consulted in [17]-[26] and references therein. In [17], a sufficient condition for the existence of explicit first integrals for vector fields which admit an integrating factor is proved, and the result of the literature extends existing results on the integrability of vector fields.

In this paper, we further consider the first integrals of n-th order systems. Firstly, a necessary condition on the existence of global first integrals of n-th order autonomous systems is presented and proved. A method to get integrating factors using $n-1$ first integrals of the systems is obtained. We also prove that an n-th order autonomous system has $n-1$ functionally independent first integrals under knowing $n-1$ functionally independent integrating factors of the system. Based on the Lie groups admitted by n-th order autonomous systems, several methods to obtain the first integrals of the systems are presented. Simultaneously, the corresponding results of n-th non-autonomous systems are given. Finally, several related examples are given to illustrate the feasibility and the effectiveness of the proposed methods.

2 The existence of global first integrals of n-th order autonomous systems

Definition 2.1 System (1) is called a conservative system or X is called a volume-preserving vector field if $divX = 0$.

Obviously, a conservative system has a divergence-free vector field X, and has a constant integrating factor. A conservative system has many excellent properties. Hamiltonian vector fields are a class of typical conservative vector fields.

Definition 2.2 If system (1) and other autonomous system have the same trajectories (except for individual points) in a subset of **D**, then we call that two system are equivalent in the subset.

Lemma 2.1 If system (1) has a non-constant integrating factor $\mu(x)$ in **D**, then system (1) is equivalent to a conservative system.

Proof. Multiplying system (1) by $\mu(x)$, and let $d\tau = \dfrac{dt}{\mu(x)}$, we have

$$\frac{dx}{d\tau} = \mu(x)P(x). \tag{3}$$

Obviously, system (1) has the same trajectories with system (3) except for individual points in **D**, that is, they are equivalent in **D** owing to Definition 2.2. Because $\mu(x)$ is an integrating factor of system (1), $div(\mu(x)X) = 0$. $\mu(x)X$ is a volume-preserving vector field. So, system (3) is a conservative system.

In physics or mechanics, conservative systems (or being equivalent to conservative systems) are a class of important systems in nature, and their vector fields maintain a constant volume of phase space. As we know, in the neighborhood of ordinary points of system (1), the system has $n-1$ functionally independent first integrals. In the neighborhood of equilibrium points of system (1), there is the complexity on the existence of first integrals of the system. Up to the present, we can see many literatures to study the existence of the first integral of autonomous systems in the neighborhood of the equilibrium points, see [15, 16] and the references therein. In this section, we consider the necessary on the existence of global first integrals of system (1) by getting integrating factors of system (1).

Theorem 2.1 If system (1) has $n-1$ functionally independent global first integrals $\Phi_i(x)(i = 1, 2, ..., n-1)$ in **D**, then system (1) has an integrating factor $\mu(x) = \dfrac{L_n}{P_n}$, where $L_n = |\dfrac{D(\Phi_1, ..., \Phi_{n-1})}{D(x_1, ..., x_{n-1})}|$ is the Jacobi determinant.

Proof. Because $\Phi_i(x)(i = 1, 2, ..., n-1)$ are $n-1$ functionally independent global first integrals of system (1), that is, $X\Phi_i(x) = 0, i = 1, 2, ..., n-1$, we have the following formulas in **D**,

$$\begin{cases} \dfrac{\partial \Phi_1}{\partial x_1}P_1 + ... + \dfrac{\partial \Phi_1}{\partial x_n}P_n = 0, \\ ...... \\ \dfrac{\partial \Phi_{n-1}}{\partial x_1}P_1 + ... + \dfrac{\partial \Phi_{n-1}}{\partial x_n}P_n = 0. \end{cases} \tag{4}$$

The above equations can be considered as a linear system in the unknown $P_1, P_2, ..., P_{n-1}$. Because $\Phi_i(i = 1, 2, ..., n-1)$ are functionally independent of each other in **D**, the Jacobi determinant $|\dfrac{D(\Phi_1, ..., \Phi_{n-1})}{D(x_1, ..., x_{n-1})}| \neq 0$. Denote the determinant by L_n :

$$L_n = |\frac{D(\Phi_1, ..., \Phi_{n-1})}{D(x_1, ..., x_{n-1})}|.$$

According to (4), $P_1, P_2, ..., P_{n-1}$ can be expressed by P_n as follows,

$$P_i = \frac{P_n}{L_n}(-1)^{n-i}L_i,\ i = 1, 2, ..., n-1,$$

where $L_i = |\dfrac{D(\Phi_1, ..., \Phi_{n-1})}{D(x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_n)}|,\ \ i = 1, 2, ..., n-1$. Let $\mu(x) = \dfrac{L_n}{P_n}$, one has

$$\frac{L_i}{P_i} = (-1)^{n-i}\mu(x),\ i = 1, 2, ..., n-1.$$

Therefore,

$$L_i = (-1)^{n-i}\mu(x)P_i, i = 1, 2, ..., n-1. \tag{5}$$

Morover, we have

$$\frac{\partial L_1}{\partial x_1} = |\frac{D(\Phi_1, ..., \Phi_{n-1})}{D(x_{2,1}, x_3, ..., x_n)}| + |\frac{D(\Phi_1, ..., \Phi_{n-1})}{D(x_2, x_{3,1}, ..., x_n)}| + ... + |\frac{D(\Phi_1, ..., \Phi_{n-1})}{D(x_2, ..., x_{n-1}, x_{n,1})}|,$$

$$\frac{\partial L_2}{\partial x_2} = |\frac{D(\Phi_1, ..., \Phi_{n-1})}{D(x_{1,2}, x_3, ..., x_n)}| + |\frac{D(\Phi_1, ..., \Phi_{n-1})}{D(x_1, x_{3,2}, ..., x_n)}| + ... + |\frac{D(\Phi_1, ..., \Phi_{n-1})}{D(x_1, x_3, ..., x_{n-1}, x_{n,2})}|,$$

$$......,$$

$$\frac{\partial L_n}{\partial x_n} = |\frac{D(\Phi_1, ..., \Phi_{n-1})}{D(x_{1,n}, x_2, ..., x_{n-1})}| + |\frac{D(\Phi_1, ..., \Phi_{n-1})}{D(x_1, x_{2,n}, ..., x_{n-1})}| + ... + |\frac{D(\Phi_1, ..., \Phi_{n-1})}{D(x_1, ..., x_{n-1,n})}|,$$

where

$$|\frac{D(\Phi_1, ..., \Phi_{n-1})}{D(x_1, x_2, ..., x_{li}, ..., x_{i-1}, x_{i+1}, ..., x_n)}| = \begin{vmatrix} \frac{\partial \Phi_1}{\partial x_1} & ... & \frac{\partial^2 \Phi_1}{\partial x_l \partial x_i} & ... & \frac{\partial \Phi_1}{\partial x_{i-1}} & \frac{\partial \Phi_1}{\partial x_{i+1}} & ... & \frac{\partial \Phi_1}{\partial x_n} \\ ... & ... & ... & ... & ... & ... & ... & ... \\ \frac{\partial \Phi_{n-1}}{\partial x_1} & ... & \frac{\partial^2 \Phi_{n-1}}{\partial x_l \partial x_i} & ... & \frac{\partial \Phi_{n-1}}{\partial x_{i-1}} & \frac{\partial \Phi_{n-1}}{\partial x_{i+1}} & ... & \frac{\partial \Phi_{n-1}}{\partial x_n} \end{vmatrix}.$$

Based on direct calculation, one can obtain

$$\frac{\partial L_1}{\partial x_1} + (-1)\frac{\partial L_2}{\partial x_2} + (-1)^2\frac{\partial L_3}{\partial x_3} + ... + (-1)^{n-1}\frac{\partial L_n}{\partial x_n} = 0.$$

That is,

$$\sum_{i=1}^{n}(-1)^{i-1}\frac{\partial L_i}{\partial x_i} = 0.$$

Substituting L_i of (5) into the above formulae, one has

$$\sum_{i=1}^{n}(-1)^{i-1}\frac{\partial((-1)^{n-i}\mu(x)P_i)}{\partial x_i} = \sum_{i=1}^{n}\frac{\partial((-1)^{n-1}\mu(x)P_i)}{\partial x_i} = 0,$$

that is,

$$\sum_{i=1}^{n}\frac{\partial(\mu(x)P_i)}{\partial x_i} = 0.$$

So, $\mu(x) = \frac{L_n}{P_n}$ is an integrating factor of system (1).

Based on Theorem 2.1, $\frac{L_i}{P_i}(-1)^{n-i}, i = 1, 2, ..., n-1$ are all equal to $\mu(x) = \frac{L_n}{P_n}$, and $\frac{L_i}{P_i}(-1)^{n-i}, i = 1, 2, ..., n-1$ are also integrating factors of system (1).

Theorem 2.2 If system (1) has $n-1$ functionally independent global first integrals $\Phi_i(x)(i = 1, 2, ..., n-1)$ in $\mathbf{D}$, then system (1) is a conservative system or is equivalent to a conservative system.

Proof. Because $\mu(x) = \frac{L_n}{P_n}$ is an integrating factor of system (1), system (1) is equivalent to a conservative system by using Lemma 2.1. When $\mu(x)$ is constant, system (1) is actually a conservative system.

Obviously, from the necessary condition on the existence of global first integrals of n-th order autonomous systems above, we have the following result.

Corollary 2.1 If system (1) is not a conservative system or is not equivalent to a conservative system, then it has not $n-1$ functionally independent global first integrals in $\mathbf{D}$.

For a conservative system (or a system being equivalent to a conservative system), under what conditions does it has $n-1$ functionally independent global first integrals? To begin this topic, we consider a system of two equations ($n = 2$). To look for a first integral for a given system (1) of $n = 2$

is an equivalent problem as to look for an integrating factor. As we know, If there exists a function $\mu(x_1, x_2) \in \mathbf{D}$ satisfying

$$\mu(x_1, x_2)(P_2 dx_1 - P_1 dx_2) = d\Omega(x_1, x_2),$$

then we call $\mu(x_1, x_2)$ an integrating factor, and $\Omega(x_1, x_2)$ a first integral of system (1). The first integral $\Omega(x_1, x_2)$ associated to the integrating factor can be computed through the integral

$$\Omega(x_1, x_2) = \int \mu(x_1, x_2)(P_2 dx_1 - P_1 dx_2).$$

Because a conservative system has a constant integrating factor, the system can be integrated directly.

For n-th order autonomous systems, We have the following result.

Theorem 2.3 If system (1) has $n-1$ functionally independent integrating factors in $\mathbf{D}$, then it has $n-1$ functionally independent first integrals in an open subset of $\mathbf{D}$.

Proof. Let $\mu_1(x_1, ..., x_n), ..., \mu_{n-1}(x_1, ..., x_n)$ be $n-1$ functionally independent integrating factors of system (1). Then we have

$$\frac{\partial(\mu_i(x_1, ..., x_n)P_1)}{x_1} + \frac{\partial(\mu_i(x_1, ..., x_n)P_2)}{x_2} + ... + \frac{\partial(\mu_i(x_1, ..., x_n)P_n)}{x_n} = 0, \ i = 1, 2, ..., n-1.$$

Furthermore, some easy calculations show that $\frac{\mu_i}{\mu_{n-1}}, \ i = 1, 2, ..., n-2$ is not a constant and

$$X(\frac{\mu_i}{\mu_{n-1}}) \equiv 0,$$

i.e. $\frac{\mu_i}{\mu_{n-1}}, \ i = 1, 2, ..., n-2$ are nontrivial first integrals of system (1). Because $\mu_1, ..., \mu_{n-1}$ are functionally independent,

$$\frac{\mu_1}{\mu_{n-1}}, ..., \frac{\mu_{n-2}}{\mu_{n-1}}$$

are functionally independent. Without loss of generality, we introduce the invertible transformation,

$$\begin{aligned} y_i &= \frac{\mu_i}{\mu_{n-1}}, \ i = 1, ..., n-2, \\ y_{n-1} &= x_{n-1}, \\ y_n &= x_n. \end{aligned} \tag{6}$$

Denote this transformation by $y = G(x)$. Then under it, system (1) is transformed to the following system,

$$\begin{cases} \dot{y}_i &= 0, \ i = 1, ..., n-2, \\ \dot{y}_{n-1} &= P_{n-1}(G^{-1}(y)), \\ \dot{y}_n &= P_n(G^{-1}(y)). \end{cases} \tag{7}$$

Apparently, system (7) has the first integrals $\Omega_i(y) = y_i, \ i = 1, ..., n-2$. In addition, we can know that system (7) has the integrating factor,

$$\nu(y) = \mu_{n-1}(G^{-1}(y))|D_y G^{-1}(y)|,$$

where $|D_y G^{-1}(y)|$ denotes the Jacobian determinant of $G^{-1}(y)$ with respect to y.

In fact, because $\mu_{n-1}(x)$ is an integrating factor of system (1), we have

$$\frac{\partial(\mu_{n-1}P_1)}{x_1} + \frac{\partial(\mu_{n-1}P_2)}{x_2} + ... + \frac{\partial(\mu_{n-1}P_n)}{x_n} = 0. \tag{8}$$

One can get the partial derivatives of $\mu_{n-1}P_1$ with respect to $y_1, y_2, ..., y_n$,

$$\begin{cases} \frac{\partial(\mu_{n-1}P_1)}{y_1} = \frac{\partial(\mu_{n-1}P_1)}{x_1} \cdot \frac{\partial x_1}{\partial y_1} + \frac{\partial(\mu_{n-1}P_1)}{x_2} \cdot \frac{\partial x_2}{\partial y_1} + ... + \frac{\partial(\mu_{n-1}P_1)}{x_n} \cdot \frac{\partial x_n}{\partial y_1}, \\ \frac{\partial(\mu_{n-1}P_1)}{y_2} = \frac{\partial(\mu_{n-1}P_1)}{x_1} \cdot \frac{\partial x_1}{\partial y_2} + \frac{\partial(\mu_{n-1}P_1)}{x_2} \cdot \frac{\partial x_2}{\partial y_2} + ... + \frac{\partial(\mu_{n-1}P_1)}{x_n} \cdot \frac{\partial x_n}{\partial y_2}, \\ ...... \\ \frac{\partial(\mu_{n-1}P_1)}{y_n} = \frac{\partial(\mu_{n-1}P_1)}{x_1} \cdot \frac{\partial x_1}{\partial y_n} + \frac{\partial(\mu_{n-1}P_1)}{x_2} \cdot \frac{\partial x_2}{\partial y_n} + ... + \frac{\partial(\mu_{n-1}P_1)}{x_n} \cdot \frac{\partial x_n}{\partial y_n}. \end{cases}$$

The above equations system can be considered as a linear system in

$$\frac{\partial(\mu_{n-1}P_1)}{x_1}, \frac{\partial(\mu_{n-1}P_1)}{x_2}, ..., \frac{\partial(\mu_{n-1}P_1)}{x_n}.$$

We can obtain the expression of $\frac{\partial(\mu_{n-1}P_1)}{x_1}$,

$$\frac{\partial(\mu_{n-1}P_1)}{x_1} = \frac{1}{|A|}|\frac{D(\mu_{n-1}P_1, x_2, ..., x_n)}{D(y_1, y_2, ..., y_n)}|,$$

where $A = \frac{D(x_1, x_2, ..., x_n)}{D(y_1, y_2, ..., y_n)}$. Obviously, by using (6), we have

$$|A| = |\frac{D(x_1, x_2, ..., x_{n-2})}{D(y_1, y_2, ..., y_{n-2})}|. \tag{9}$$

Similarly, one can calculate the partial derivatives of $\mu_{n-1}P_2$ with respect to $y_1, y_2, ..., y_n$. Then, from a linear system in $\frac{\partial(\mu_{n-1}P_2)}{x_1}, \frac{\partial(\mu_{n-1}P_2)}{x_2}, ..., \frac{\partial(\mu_{n-1}P_2)}{x_n}$, one can obtain the expression of $\frac{\partial(\mu_{n-1}P_2)}{x_2}$,

$$\frac{\partial(\mu_{n-1}P_2)}{x_2} = \frac{1}{|A|}|\frac{D(x_1, \mu_{n-1}P_2, x_3, ..., x_n)}{D(y_1, y_2, ..., y_n)}|.$$

Similarly, one can also obtain the expressions of $\frac{\partial(\mu_{n-1}P_3)}{x_3}, ..., \frac{\partial(\mu_{n-1}P_n)}{x_n}$ as follows,

$$\frac{\partial(\mu_{n-1}P_i)}{x_i} = \frac{1}{|A|}|\frac{D(x_1, x_2, ..., x_{i-1}, \mu_{n-1}P_i, x_{i+1}, ..., x_n)}{D(y_1, y_2, ..., y_n)}|, \; i = 3, 4, ..., n.$$

Substituting $\frac{\partial(\mu_{n-1}P_i)}{x_i}$, $i = 1, 2, ..., n$ into (8) and expanding each determinant

$$|\frac{D(x_1, x_2, ..., x_{i-1}, \mu_{n-1}P_i, x_{i+1}, ..., x_n)}{D(y_1, y_2, ..., y_n)}|$$

in the i-th column, by using (9), we have

$$\begin{aligned}
&|\frac{D(\mu_{n-1}P_1, x_2, ..., x_n)}{D(y_1, y_2, ..., y_n)}| + ... + |\frac{D(x_1, x_2, ..., x_{n-3}, \mu_{n-1}P_{n-2}, x_{n-1}, x_n)}{D(y_1, y_2, ..., y_n)}| \\
&+(-1)^{2n-2}\frac{\partial(\mu_{n-1}P_{n-1})}{\partial y_{n-1}}|A| + (-1)^{2n-3}\frac{\partial(\mu_{n-1}P_{n-1})}{\partial y_{n-2}}|\frac{D(x_1, x_2, ..., x_{n-2})}{D(y_1, y_2, ..., y_{n-3}, y_{n-1})}| + ... \\
&+(-1)^{n}\frac{\partial(\mu_{n-1}P_{n-1})}{\partial y_1}|\frac{D(x_1, x_2, ..., x_{n-2})}{D(y_2, ..., y_{n-1})}| \\
&+(-1)^{2n-2}\frac{\partial(\mu_{n-1}P_n)}{\partial y_n}|A| + (-1)^{2n-3}\frac{\partial(\mu_{n-1}P_n)}{\partial y_{n-2}}|\frac{D(x_1, x_2, ..., x_{n-2})}{D(y_1, y_2, ..., y_{n-3}, y_n)}| + ... \\
&+(-1)^{n}\frac{\partial(\mu_{n-1}P_n)}{\partial y_1}|\frac{D(x_1, x_2, ..., x_{n-2})}{D(y_2, ..., y_{n-2}, y_n)}| = 0.
\end{aligned} \tag{10}$$

Therefore, (10) can be rewritten as

$$\begin{aligned}
&\frac{\partial(\mu_{n-1}P_{n-1}|A|)}{\partial y_{n-1}} + \frac{\partial(\mu_{n-1}P_n|A|)}{\partial y_n} \\
&+[|\frac{D(\mu_{n-1}P_1, x_2, ..., x_n)}{D(y_1, y_2, ..., y_n)}| + ... + |\frac{D(x_1, x_2, ..., x_{n-3}, \mu_{n-1}P_{n-2}, x_{n-1}, x_n)}{D(y_1, y_2, ..., y_n)}| \\
&+(-1)^{2n-2}\frac{\partial(\mu_{n-1}P_{n-1})}{\partial y_{n-1}}|A| + (-1)^{2n-3}\frac{\partial(\mu_{n-1}P_{n-1})}{\partial y_{n-2}}|\frac{D(x_1, x_2, ..., x_{n-2})}{D(y_1, y_2, ..., y_{n-3}, y_{n-1})}| + ... \\
&+(-1)^{n}\frac{\partial(\mu_{n-1}P_{n-1})}{\partial y_1}|\frac{D(x_1, x_2, ..., x_{n-2})}{D(y_2, ..., y_{n-1})}| \\
&+(-1)^{2n-2}\frac{\partial(\mu_{n-1}P_n)}{\partial y_n}|A| + (-1)^{2n-3}\frac{\partial(\mu_{n-1}P_n)}{\partial y_{n-2}}|\frac{D(x_1, x_2, ..., x_{n-2})}{D(y_1, y_2, ..., y_{n-3}, y_n)}| + ... \\
&+(-1)^{n}\frac{\partial(\mu_{n-1}P_n)}{\partial y_1}|\frac{D(x_1, x_2, ..., x_{n-2})}{D(y_2, ..., y_{n-2}, y_n)}| \\
&-\mu_{n-1}P_{n-1}\frac{\partial|A|}{\partial y_{n-1}} - \mu_{n-1}P_n\frac{\partial|A|}{\partial y_n}] = 0.
\end{aligned} \tag{11}$$

For every equations of (6), one can get partial derivatives with respect to y_1 by the chain rule,

$$\begin{cases} 1 = \dfrac{\partial(\frac{\mu_1}{\mu_{n-1}})}{\partial x_1}\cdot\dfrac{\partial x_1}{\partial y_1} + \dfrac{\partial(\frac{\mu_1}{\mu_{n-1}})}{\partial x_2}\cdot\dfrac{\partial x_2}{\partial y_1} + ... + \dfrac{\partial(\frac{\mu_1}{\mu_{n-1}})}{\partial x_{n-2}}\cdot\dfrac{\partial x_{n-2}}{\partial y_1}, \\ 0 = \dfrac{\partial(\frac{\mu_2}{\mu_{n-1}})}{\partial x_1}\cdot\dfrac{\partial x_1}{\partial y_1} + \dfrac{\partial(\frac{\mu_2}{\mu_{n-1}})}{\partial x_2}\cdot\dfrac{\partial x_2}{\partial y_1} + ... + \dfrac{\partial(\frac{\mu_2}{\mu_{n-1}})}{\partial x_{n-2}}\cdot\dfrac{\partial x_{n-2}}{\partial y_1}, \\ ...... \\ 0 = \dfrac{\partial(\frac{\mu_{n-2}}{\mu_{n-1}})}{\partial x_1}\cdot\dfrac{\partial x_1}{\partial y_1} + \dfrac{\partial(\frac{\mu_{n-2}}{\mu_{n-1}})}{\partial x_2}\cdot\dfrac{\partial x_2}{\partial y_1} + ... + \dfrac{\partial(\frac{\mu_{n-2}}{\mu_{n-1}})}{\partial x_{n-2}}\cdot\dfrac{\partial x_{n-2}}{\partial y_1}. \end{cases}$$

The above equations can be considered as a linear system in $\dfrac{\partial x_i}{\partial y_1}$, $(i = 1, 2, ..., n-2)$. So, we get

$$\frac{\partial x_i}{\partial y_1} = \frac{1}{|\frac{D(\frac{\mu_1}{\mu_{n-1}}, ..., \frac{\mu_{n-2}}{\mu_{n-1}})}{D(x_1, ..., x_{n-2})}|}\cdot\begin{vmatrix} \frac{\partial(\frac{\mu_1}{\mu_{n-1}})}{\partial x_1} & ... & \frac{\partial(\frac{\mu_1}{\mu_{n-1}})}{\partial x_{i-1}} & 1 & \frac{\partial(\frac{\mu_1}{\mu_{n-1}})}{\partial x_{i+1}} & ... & \frac{\partial(\frac{\mu_1}{\mu_{n-1}})}{\partial x_{n-2}} \\ \frac{\partial(\frac{\mu_2}{\mu_{n-1}})}{\partial x_1} & ... & \frac{\partial(\frac{\mu_2}{\mu_{n-1}})}{\partial x_{i-1}} & 0 & \frac{\partial(\frac{\mu_2}{\mu_{n-1}})}{\partial x_{i+1}} & ... & \frac{\partial(\frac{\mu_2}{\mu_{n-1}})}{\partial x_{n-2}} \\ ... & & & & & & \\ \frac{\partial(\frac{\mu_{n-2}}{\mu_{n-1}})}{\partial x_1} & ... & \frac{\partial(\frac{\mu_{n-2}}{\mu_{n-1}})}{\partial x_{i-1}} & 0 & \frac{\partial(\frac{\mu_{n-2}}{\mu_{n-1}})}{\partial x_{i+1}} & ... & \frac{\partial(\frac{\mu_{n-2}}{\mu_{n-1}})}{\partial x_{n-2}} \end{vmatrix}, \ i = 1, 2, ..., n-2.$$

Similarly, we can get the expressions of $\dfrac{\partial x_i}{\partial y_j}$, $j = 2, ..., n-2$. Substituting $\dfrac{\partial x_i}{\partial y_j}$, $i, j = 1, 2, ..., n-2$, into the determinants of (11), by very tedious computation one can get the following formula,

$$\begin{aligned} & |\frac{D(\mu_{n-1}P_1, x_2, ..., x_n)}{D(y_1, y_2, ..., y_n)}| + ... + |\frac{D(x_1, x_2, ..., x_{n-3}, \mu_{n-1}P_{n-2}, x_{n-1}, x_n)}{D(y_1, y_2, ..., y_n)}| \\ & +(-1)^{2n-2}\frac{\partial(\mu_{n-1}P_{n-1})}{\partial y_{n-1}}|A| + (-1)^{2n-3}\frac{\partial(\mu_{n-1}P_{n-1})}{\partial y_{n-2}}|\frac{D(x_1, x_2, ..., x_{n-2})}{D(y_1, y_2, ..., y_{n-3}, y_{n-1})}| + ... \\ & +(-1)^{n}\frac{\partial(\mu_{n-1}P_{n-1})}{\partial y_1}|\frac{D(x_1, x_2, ..., x_{n-2})}{D(y_2, ..., y_{n-1})}| \\ & +(-1)^{2n-2}\frac{\partial(\mu_{n-1}P_n)}{\partial y_n}|A| + (-1)^{2n-3}\frac{\partial(\mu_{n-1}P_n)}{\partial y_{n-2}}|\frac{D(x_1, x_2, ..., x_{n-2})}{D(y_1, y_2, ..., y_{n-3}, y_n)}| + ... \\ & +(-1)^{n}\frac{\partial(\mu_{n-1}P_n)}{\partial y_1}|\frac{D(x_1, x_2, ..., x_{n-2})}{D(y_2, ..., y_{n-2}, y_n)}| \\ & -\mu_{n-1}P_{n-1}\frac{\partial|A|}{\partial y_{n-1}} - \mu_{n-1}P_n\frac{\partial|A|}{\partial y_n} = 0. \end{aligned}$$

From (11), one can get directly

$$\frac{\partial(\mu_{n-1}P_{n-1}|A|)}{\partial y_{n-1}} + \frac{\partial(\mu_{n-1}P_n|A|)}{\partial y_n} = 0,$$

where $|A|$ is actually $|D_yG^{-1}(y)|$. That means that $\mu_{n-1}(G^{-1}(y))|D_yG^{-1}(y)|$ is an integrating factor of system (7).

This shows that the two dimensional differential system

$$\begin{cases} \dot{y}_{n-1} & = P_{n-1}(G^{-1}(y)) \\ \dot{y}_n & = P_n(G^{-1}(y)) \end{cases} \tag{12}$$

has the integrating factor

$$\nu(y_{n-1}, y_n) = \mu_{n-1}(G^{-1}(y))D_yG^{-1}(\Omega_1(y), ..., \Omega_{n-2}(y), y_{n-1}, y_n),$$

where $\Omega_1(y), ..., \Omega_{n-2}(y)$ are constants. Hence, system (12) has the first integral

$$\Omega_{n-1}(y_{n-1}, y_n) = \int \nu P_n(G^{-1}(y))dy_{n-1} - \nu P_{n-1}(G^{-1}(y))dy_n.$$

Obviously, Ω_{n-1} is functionally independent of $\Omega_1, ..., \Omega_{n-2}$ because the latter are independent of y_{n-1} and y_n. Applying (6) to $\Omega_1, ..., \Omega_{n-1}$, we can get $n-1$ functionally independent first integrals

$$\Omega_1(x) = \Omega_1(G(x)), ..., \Omega_{n-1}(x) = \Omega_{n-1}(G(x))$$

of system (1). The proof is completed.

3 A necessary condition on the existence of global first integrals of n-th order non-autonomous systems

We consider n-th order non-autonomous systems (2), where $x = (x_1, x_2, ..., x_n) \in \mathbf{D} \subset \mathbf{R}^n(or\ \mathbf{C}^n)$, $P(t, x) = (P_1(t, x_1, x_2, ..., x_n), ..., P_n(t, x_1, x_2, ..., x_n))$, $P_i(t, x_1, x_2, ..., x_n) : \mathbf{R}\times\mathbf{D} \to \mathbf{R}(or\ \mathbf{C})$. $P_i(t, x_1, x_2, ..., x_n) \in C_\infty(\mathbf{R}\times\mathbf{D})$ and $t \in \mathbf{R}(or\ \mathbf{C})$. Let $t = x_0$, system (2) can be rewritten as an $(n+1)$-th order autonomous system,

$$\frac{dx}{dt} = P(x_0, x_1, x_2, ..., x_n). \tag{13}$$

Associated to system (13) there is the vector field $X = \sum_{i=0}^n P_i(x)\frac{\partial}{\partial x_i}$, where $P_0(x_0, x_1, ..., x_n) = 1$. The divergence is $divX = \sum_{i=1}^n \frac{\partial P_i}{\partial x_i}$. It is easy to have the following results.

Theorem 3.1 If system (2) has n functionally independent global first integrals $\Phi_i(x)(i = 0, 1, ..., n-1)$ in $\mathbf{R}\times\mathbf{D}$, then system (2) has an integrating factor $\mu(x) = \frac{L_n}{P_n}$, where $L_n = |\frac{D(\Phi_0, ..., \Phi_{n-1})}{D(x_0, ..., x_{n-1})}|$ is the Jacobi determinant.

Similarly, one can have that $\frac{L_i}{P_i}(-1)^{n-i}, i = 0, 1, ..., n-1$ are all also integrating factors of system (2), where $L_i = |\frac{D(\Phi_0, ..., \Phi_{n-1})}{D(x_0, x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)}|$, $i = 0, 1, ..., n-1$. Specially,

$$L_0 = |\frac{D(\Phi_0, ..., \Phi_{n-1})}{D(x_1, ..., x_n)}(-1)^n|$$

is an integrating factor of system (2).

Theorem 3.2 If system (2) has n functionally independent global first integrals $\Phi_i(x)(i = 0, 1, ..., n-1)$ in $\mathbf{R} \times \mathbf{D}$, then system (2) has a volume-preserving vector field (is a conservative system) or is equivalent to a system having volume-preserving vector field (a conservative system).

Corollary 3.1 If system (2) has not a volume-preserving vector field (is a conservative system) or is not equivalent to a system having volume-preserving vector field (a conservative system), then it has not n functionally independent global first integrals in $\mathbf{R} \times \mathbf{D}$.

Theorem 3.3 If system (2) has n functionally independent integrating factors in $\mathbf{D}$, then it has n functionally independent first integrals in subset of $\mathbf{D}$.

The following simple example are given to illustrate the feasibility and the effectiveness of the above proposed methods.

Example 3.1 Consider the following third order ordinary differential equation,

$$y''' = \frac{(3y' - 1)y''^2}{y'^2}.$$

Let us rewrite the equation as an autonomous system ,

$$\begin{cases} \dot{x} &=& 1 \\ \dot{y} &=& z \\ \dot{z} &=& w \\ \dot{w} &=& \dfrac{(3z-1)w^2}{z^2}. \end{cases} \tag{14}$$

Its corresponding vector field is $X = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{(3z-1)w^2}{z^2}\frac{\partial}{\partial w}$. The system has the following first integrals in $W = \{(x, y, z, w)|z > 0, w > 0\}$,

$$\Omega_1(x, y, z, w) = x - y - \frac{z^2}{w},$$

$$\Omega_2(x, y, z, w) = -y + \frac{z^3}{w}$$

and

$$\Omega_3(x, y, z, w) = 2\ln w - 6\ln z - \frac{2}{z}.$$

One can obtain

$$L_4 = |\frac{D(\Omega_1, \Omega_2, \Omega_3)}{D(x, y, z)}| = \frac{6}{z} - \frac{2}{z^2}.$$

Based on Theorem 3.1, one can get an integrating factor of system (14),

$$\mu(x, y, z, w) = \frac{L_4}{P_4} = \frac{2}{w^2}.$$

By directly calculating, we can obtain $\frac{L_2}{P_2} = \frac{L_4}{P_4}$ and $\frac{L_1}{P_1} = (-1)^{4-1}\frac{L_4}{P_4}, \frac{L_3}{P_3} = (-1)^{4-3}\frac{L_4}{P_4}$. So, system (14) is equivalent to a volume-preserving vector field (is equivalent to a conservative system) in W.

4 Methods to obtain first integrals of n-th order autonomous systems

In this section, we will discuss several flexible methods to obtain first integrals of n-th order autonomous system using one-parameter Lie groups admitted by the system. As we know, for a Lagrange system, Noether theorem ensures that one can get a first integral by knowing a one-parameter Lie group admitted by the system [27]. For an n-th order autonomous system, in order to obtain first integrals, multiple-parameter Lie groups admitted by the system are required in traditional theory of Lie groups [1, 11]. Generally, it is more complicated to calculate a multiple-parameter Lie group admitted by a given system. In [7], for avoiding the multiple-parameter Lie group, the spacial structure spanned by generators of a system of one-parameter Lie groups admitted by an n-th order autonomous system is studied, and the possibility and the flexibility of finding first integrals of the system by using several one-parameter Lie groups admitted by the system are revealed. Several methods to obtain first integrals of n-th order autonomous system by using one-parameter Lie groups are given and discussed in [7]. In order to make the paper be integrity, here first let us introduce simply the ideas of the methods and related contents.

Let

$$V_i = \sum_{j=1}^{n} \xi_j^i(x_1, x_2, ..., x_n)\frac{\partial}{\partial x_j} \ (i = 1, 2, ..., n-1) \tag{15}$$

be the generators of $n - 1$ independent one-parameter Lie groups admitted by system (1).

Lemma 4.1[22] If there exist functions $A_i(x) \in C_\infty$ satisfying $[X, V_i] = A_i(x)X(i = 1, 2, ..., n-1)$, then system (1) admits the Lie groups G_i $(i = 1, 2, ..., n-1)$ generated by $V_i(i = 1, 2, ..., n-1)$.

Lemma 4.2[22] If V is a generator of a one-parameter Lie group admitted by system (1) and $\Omega(x)$ is a first integral of system (1), then $V\Omega(x) = f(x)$, where $f(x)$ is either a first integral of system (1), or a constant.

In [22], the following results are also proved.

Theorem 4.1 If $V_i, i = 1, 2, ..., n-1$ are the generators of $n-1$ independent one-parameter Lie groups admitted by system (1), then under the operation of the Lie brackets,

$$[V_i, V_j] = \Sigma_{k=1}^{n-1} C_{i,j}^k(x)V_k + C_{i,j}^0(x)X,$$

where the coefficients $C_{i,j}^k(x)(i, j, k = 1, 2, ..., n-1)$ are either the first integrals of system (1) or constants, while $C_{i,j}^0(x)$ may be arbitrary functions.

Theorem 4.2 If $V_i, (i = 1, 2, ..., n-1)$ are the generators of $n-1$ independent one-parameter Lie groups admitted by system (1), then the one-parameter Lie group with a given generator V is admitted by system (1) if and only if the generator V can be expressed by

$$V = a(x)X + \Sigma_{i=1}^{n-1} a_i(x)V_i,$$

where $a_i(x), i = 1, 2, ..., n-1$ are either first integrals or constants, and $a(x)$ is an arbitrary function.

Theorem 4.1 shows that $C_{i,j}^k(x),\ i, j, k = 1, 2, ..., n-1$ may be first integrals of system (1) if $C_{i,j}^k(x),\ i, j, k = 1, 2, ..., n-1$ are functions. If $C_{i,j}^k(x), i, j, k = 1, 2, ..., n-1$ are all constants, a first integral of system (1) can still be found according to the method presented in [7, 23].

We can construct the following system of equations using the generators $V_i(i = 1, 2, ..., n-1)$ of Lie groups admitted by system (1),

$$\begin{bmatrix} P_1 & P_2 & \dots & P_n \\ \xi_1^1 & \xi_2^1 & \dots & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \dots & \xi_n^{n-1} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} 0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}, \tag{16}$$

where $a_i(i = 1, 2, ..., n-1)$ are constants or first integrals of (1). By the independence of $V_i(i = 1, 2, ..., n-1)$, we know that (16) has only a unique set of solutions $f_1, f_2, ..., f_n$.

Theorem 4.3[7]If there exists a set of constants or first integrals $a_1, a_2, ..., a_{n-1}$ of (1) such that the solutions $f_1, f_2, ..., f_n$ of equation (16) satisfies

$$\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}, i, j = 1, 2, ..., n$$

then the following first integral of (1) can be obtained,

$$\Omega = \int f_1 dx_1 + f_2 dx_2 + ...f_n dx_n.$$

In fact, we often only know $m(< n-1)$ independent one-parameter Lie groups admitted by (1). In this case, by constructing the similar equation for obtaining $f_1, f_2, ..., f_n$, one can still obtain a first integral of system (1) under selecting appropriate $a_1, a_2, ..., a_{n-1}$ according to the idea in Theorem 4.3. The selected $a_1, a_2, ..., a_{n-1}$ in the above method are important for obtaining a first integral. A set of suitable $a_1, a_2, ..., a_{n-1}$ can help us to get a first integral. In [23], the sufficient and necessary condition satisfied by a set of $a_1, a_2, ..., a_{n-1}$ is given for searching for a first integral of the system by the above method. Based on Theorem 4.3, the authors continue to discuss the method of obtaining first integrals of an n-th autonomous system when the system accepts a series of one-parameter Lie groups with the certain solvability in [24]. When an n-th autonomous system accepts a series of one-parameter Lie groups with the certain solvability, a first integral of the system can be obtained under the set of $a_1, a_2, ..., a_{n-1}$ equating concrete constants[24].

Next, a more flexible method to obtain first integrals using one-parameter Lie group admitted by n-th order autonomous systems is presented, which need not be considered to select a set of $a_1, a_2, ..., a_{n-1}$.

Let

$$\mu(x_1, x_2, ..., x_n) = \begin{vmatrix} P_1 & P_2 & ... & P_n \\ \xi_1^1 & \xi_2^1 & ... & \xi_n^1 \\ \vdots & \vdots & \vdots & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & ... & \xi_n^{n-1} \end{vmatrix}.$$

Since $V_i \ \ (i = 1, 2, ..., n-1)$ are independent, $\mu(x_1, x_2, ..., x_n) \neq 0$ holds.

Lemma 4.3[25] $\dfrac{1}{\mu(x_1, x_2, ..., x_n)}$ is an integrating factor of system (1).

Theorem 4.4 If $V_i (i = 1, 2, ..., n-1)$ are generators of $n-1$ independent one-parameter Lie groups admitted by system (1), then

$$\mu(x_1, x_2, ..., x_n)V_i(\frac{1}{\mu(x_1, x_2, ..., x_n)}) + div(V_i) - A_i(x_1, x_2, ..., x_n) \ (i = 1, 2, ..., n-1)$$

are all first integrals of system (1), where $A_i(x_1, x_2, ..., x_n)$ satisfies $[X, V_i] = A_i(x_1, x_2, ..., x_n)X$, and $[\cdot, \cdot]$ stands for the Lie bracket.

Proof. Let $\Omega = \mu(x_1, x_2, ..., x_n)V_i(\dfrac{1}{\mu(x_1, x_2, ..., x_n)}) + div(V_i) - A_i(x_1, x_2, ..., x_n)$. One has

$$\begin{aligned} X(\Omega) = \ & X(-\frac{1}{\mu}V_i(\mu)) + X(div(V_i)) - X(A_i) \\ = \ & \frac{1}{\mu^2}X(\mu)V_i(\mu) - \frac{1}{\mu}X(V_i(\mu)) + X(div(V_i)) - X(A_i) \\ = \ & \frac{1}{\mu^2}(\mu divX)V_i(\mu) - \frac{1}{\mu}(divX)V_i(\mu) - \frac{1}{\mu}\mu V_i(divX) - \frac{1}{\mu}A_i(\mu divX) + X(div(V_i)) - X(A_i) \\ = \ & -V_i(divX) - A_i divX + X(div(V_i)) - XA_i. \end{aligned} \tag{17}$$

The above last equality can be obtained by using the definition of inverse integrating factor and Lemma 4.1. By direct calculation, we have

$$div(V_iX) = V_i(divX) + \frac{\partial X_1}{\partial x_1}(\frac{\partial V_{i1}}{\partial x_1} + ... + \frac{\partial V_{i1}}{\partial x_n}) + ... + \frac{\partial X_1}{\partial x_n}(\frac{\partial V_{in}}{\partial x_1} + ... + \frac{\partial V_{in}}{\partial x_n}) \tag{18}$$

and

$$div(XV_i) = X(divV_i) + \frac{\partial V_{i1}}{\partial x_1}(\frac{\partial X_1}{\partial x_1} + ... + \frac{\partial X_1}{\partial x_n}) + ... + \frac{\partial V_{in}}{\partial x_1}(\frac{\partial X_n}{\partial x_1} + ... + \frac{\partial X_n}{\partial x_n}). \tag{19}$$

Obviously,

$$\begin{aligned} & \frac{\partial X_1}{\partial x_1}(\frac{\partial V_{i1}}{\partial x_1} + ... + \frac{\partial V_{i1}}{\partial x_n}) + ... + \frac{\partial X_1}{\partial x_n}(\frac{\partial V_{in}}{\partial x_1} + ... + \frac{\partial V_{in}}{\partial x_n}) \\ = & \frac{\partial V_{i1}}{\partial x_1}(\frac{\partial X_1}{\partial x_1} + ... + \frac{\partial X_1}{\partial x_n}) + ... + \frac{\partial V_{in}}{\partial x_1}(\frac{\partial X_n}{\partial x_1} + ... + \frac{\partial X_n}{\partial x_n}). \end{aligned}$$

Substituting (18) and (19) to the last equality of (17), we have

$$X(\Omega) = div(XV_i) - div(V_iX) - A_i divX - X(A_i). \tag{20}$$

Because $V_i \ (i = 1, 2, ..., n-1)$ are all the generators of Lie groups admitted by system (1), one has

$$XV_i - V_iX = A_iX$$

and

$$div(XV_i - V_iX) = div(A_iX) = A_i divX + X(A_i).$$

Substituting the above formulae to (20), we have $X(\Omega) = 0$. The proof is completed.

Remark 4.1 Theorem 4.4 presents a method to obtain first integrals of system (1) using Lie groups admitted by the system. If system (1) is a conservative system, that is, $divX = 0$, then system (1) has the constant integrating factor $\mu(x) \equiv 1$. One may get directly a first integral under knowing only a one-parameter Lie group admitted by system (1). If system (1) is not a conservative system, that is, $divX \neq 0$, one may get $n-1$ first integrals under knowing $n-1$ independent one-parameter Lie groups admitted by system (1).

Remark 4.2 By Theorem 4.4, we know

$$\Omega = \mu V_i(\frac{1}{\mu}) + div(V_i) - A_i (i = 1, 2, ..., n-1)$$

are all first integrals of system (1). The first integrals can also be written as

$$\Omega = \frac{V_i(\frac{1}{\mu}) + \frac{1}{\mu}div(V_i) - \frac{1}{\mu}A_i}{\frac{1}{\mu}}.$$

So, one can have

$$V_i(\frac{1}{\mu}) + \frac{1}{\mu}div(V_i) - \frac{1}{\mu}A_i \ \ (i = 1, 2, ..., n-1)$$

being also all integrating factors of system (1).

From Theorem 4.4, we can get the following result.

Corollary 4.1 If $V_i(i = 1, 2, ..., n-1)$ are generators of $n-1$ independent one-parameter Lie groups admitted by system (1), then $\Omega_k = V_i(V_i(...(\Omega))))$ $(i = 1, 2, ..., n-1)$ are also first integrals of system (1), where V_i is applied $k \geq 1$ times on Ω.

Proof. Because $[X, V_i] = XV_i - V_iX = A_iX, i = 1, 2, ..., n-1$,

$$\begin{aligned} X(V_i(\Omega)) &= V_i(X(\Omega)) + A_iX(\Omega) \\ &= 0. \end{aligned}$$

The proof is completed.

Example 4.1 Consider the following autonomous system,

$$\begin{cases} \dot{x}_1 &= y_1 \\ \dot{x}_2 &= y_2 \\ \dot{y}_1 &= -4x_1(x_1^2 + x_2^2) \\ \dot{y}_2 &= -4x_2(x_1^2 + x_2^2). \end{cases} \tag{21}$$

Its corresponding vector field is $X = y_1\frac{\partial}{\partial x_1} + y_2\frac{\partial}{\partial x_2} - 4x_1(x_1^2 + x_2^2)\frac{\partial}{\partial y_1} - 4x_2(x_1^2 + x_2^2)\frac{\partial}{\partial y_2}$, and $divX = 0$. It is easy to check that this system admits one Lie group with the following generator,

$$V = (x_1y_2 - y_1x_2)[\frac{1}{2}(x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}) + y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}],$$

and

$$[X, V] = -\frac{1}{2}(x_1y_2 - y_1x_2)X.$$

Based on Theorem 4.4, one can get a first integral of system (21),

$$\begin{aligned} \Omega = &\ div(V) - A(x) \\ = &\ \frac{7}{2}(x_1y_2 - y_1x_2) + \frac{1}{2}(x_1y_2 - y_1x_2) \\ = &\ 4(x_1y_2 - x_2y_1). \end{aligned}$$

Apparently, one has $V(\Omega) = 6(x_1y_2 - x_2y_1)^2$. It is a first integral of system (21). It is only not a new first integral because of the integral $\Omega = 4(x_1y_2 - x_2y_1)$.

The following simple system is integrable. Here it is only for illustrating the feasibility and the effectiveness of the proposed methods in Theorem 4.4.

Example 4.2 Consider the following high order ordinary differential equation,

$$y^{(iv)} = a^{\frac{1}{3}} y'''^{\frac{4}{3}},$$

where $a > 0$. Let us rewrite the equation as an autonomous system ,

$$\begin{cases} \dot{x} &= 1 \\ \dot{y} &= z \\ \dot{z} &= v \\ \dot{v} &= w \\ \dot{w} &= a^{\frac{1}{3}} w^{\frac{4}{3}}. \end{cases} \tag{22}$$

Its corresponding vector field is $X = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + v\frac{\partial}{\partial z} + w\frac{\partial}{\partial v} + a^{\frac{1}{3}} w^{\frac{4}{3}} \frac{\partial}{\partial w}$. It is easy to check that this system admits five Lie groups with the following generators,

$$\begin{aligned} V_1 &= \frac{\partial}{\partial x}, \\ V_2 &= \frac{\partial}{\partial y}, \\ V_3 &= x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \\ V_4 &= x^2\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial z} + 2\frac{\partial}{\partial v}, \\ V_5 &= x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z} - 2v\frac{\partial}{\partial v} - 3w\frac{\partial}{\partial w}, \end{aligned}$$

and

$$[X, V_i] = 0, \quad i = 1, 2, ..., 4; \quad [X, V_5] = X.$$

The following inverse integrating factor can be obtained,

$$\begin{aligned} \mu(x, y, z, v, w) &= \begin{vmatrix} 1 & z & v & w & a^{\frac{1}{3}} w^{\frac{4}{3}} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & x & 1 & 0 & 0 \\ 0 & x^2 & 2x & 2 & 0 \\ x & 0 & -z & -2v & -3w \end{vmatrix} \\ &= a^{\frac{1}{3}} w^{\frac{4}{3}} (2xz - 2x^2 v) + 3w(x^2 w + 2z - 2xv). \end{aligned}$$

Based on Theorem 4.4, by direct computing, one can get first integrals of system (22),

$$\begin{aligned} \Omega_1 &= -\frac{1}{\mu} V_1(\mu) + div(V_1) - A_1(x) \\ &= -\frac{2a^{\frac{1}{3}} w^{\frac{4}{3}} (z - 2xv) + 6w(xw - v)}{a^{\frac{1}{3}} w^{\frac{4}{3}} (2xz - 2x^2 v) + 3w(x^2 w + 2z - 2xv)}. \\ \Omega_3 &= -\frac{1}{\mu} V_3(\mu) + div(V_3) - A_3(x) \\ &= -\frac{2(a^{\frac{1}{3}} w^{\frac{4}{3}} x + 3w)}{a^{\frac{1}{3}} w^{\frac{4}{3}} (2xz - 2x^2 v) + 3w(x^2 w + 2z - 2xv)}. \end{aligned}$$

The above first integrals are independent of each other. One can also get new first integrals $V_i(\Omega_j), \quad i = 1, 2, ..., 5; \ j = 1, 3, 5$, For example,

$$\begin{aligned} \Omega_{11} &=: V_1(\Omega_1) \\ &= \frac{4(aw)^{\frac{2}{3}} (2x^2 v^2 - 2xvz + z^2) + 12(aw)^{\frac{1}{3}} x(zw - 2xvw + 2v^2) + 18(x^2 w^2 + 2v^2 - 2wz - 2xvw)}{[2a^{\frac{1}{3}} w^{\frac{1}{3}} (z - xv)x + 3(x^2 w + 2z - 2xv)]^2}. \end{aligned}$$

5 Conclusions

In this paper, we have studied the first integrals of n-th order autonomous systems. By considering the integrating factors generated by the global first integrals of n-th order autonomous systems, the necessary condition on the existence of $n-1$ functionally independent global first integrals is obtained, that is, if system (1) has $n-1$ functionally independent global first integrals $\Phi_i(x)(i = 1, 2, ..., n-1)$ in $\mathbf{D}$, then system (1) is a conservative system or is equivalent to a conservative system. In physics or mechanics, conservative systems (or being equivalent to conservative systems) are a class of important systems in nature, and their vector fields maintain a constant volume of phase space. Celestial mechanics is an important source of conservative systems. We also prove that an n-th order autonomous system has $n-1$ functionally independent first integrals under knowing $n-1$ functionally independent integrating factors of the system. From the above investigation, we see that n-th autonomous systems having $n-1$ functionally independent global first integrals in $\mathbf{D}$ are conservative systems or are equivalent to conservative systems. The $n-1$ independent global first integrals of the system can also help us get integrating factors of the system. We have also presented several flexible methods to obtain first integrals of the system under knowing one-parameter Lie groups admitted by the system. In particular, when system (1) is a conservative system or is equivalent to a conservative system, one can get directly a first integral under knowing only one one-parameter Lie group admitted by system (1). At last, several related examples are given to illustrate the feasibility and the effectiveness of the proposed method.

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