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On strongly Gauduchon metrics of almost complex manifolds ☆

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In this paper, we study strongly Gauduchon metrics on a special kind of compact almost complex manifolds which are called 2-integrable. In particular, we investigate the existence of strongly Gauduchon metrics on compact 2-integrable almost complex manifolds.

1. Introduction

Let M be an n-dimensional compact complex manifold with a hermitian metric F. We usually call a hermitian metric F a Gauduchon metric if $\partial\bar{\partial}F^{n-1} = 0$. Gauduchon (cf. [1]) proved that there always exists a smooth function f such that $\partial\bar{\partial}e^f F^{n-1} = 0$. That means Gauduchon metrics always exist on compact complex manifolds. In [9], D. Popovici first defined the strongly Gauduchon metric in the study of limits of projective manifolds under deformations. A strongly Gauduchon metric on a compact complex n-dimensional manifold is a hermitian metric F such that ∂F^{n-1} is $\bar{\partial}$-exact. A compact complex manifold is called a strongly Gauduchon manifold, if there exists a strongly Gauduchon metric on it. Note that $\bar{\partial}^2 = 0$ on a complex manifold, of course strongly Gauduchon metrics are Gauduchon metrics. However, a strongly Gauduchon metric does not always exist on a compact complex manifold. In [9], D. Popovici proved that a compact complex manifold M carries a strongly Gauduchon metric if and only if there is no non-zero $(1,1)$-current T such that $T \geq 0$ and T is d-exact on M. In this paper, we want to introduce the concept of strongly Gauduchon metric on compact almost complex manifold (M, J). But in general, $\bar{\partial}_J^2 \neq 0$ on an almost complex manifold, which implies that $\partial_J F^{n-1} = \bar{\partial}_J\beta$ need not imply $\partial_J\bar{\partial}_J F^{n-1} = 0$. Hence, we want to introduce a special kind of almost complex manifolds which are called q-integrable. We call J a q-integrable almost complex structure if

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$$\bar{\partial}_J^2 \mid_{\Omega^{2n-k}(M)} = 0, \ \ q \geq k \geq 2,$$

where $2n \geq q \geq 2$. An almost complex manifold (M, J) is called a q-integrable almost complex manifold if J is a q-integrable almost complex structure. Then the definition of strongly Gauduchon metric in classical complex analysis also makes sense on 2-integrable almost complex manifolds. In particular, we investigate the existence of strongly Gauduchon metrics on compact 2-integrable almost complex manifolds.

Main Theorem. *Let (M, J) be a compact 2-integrable almost complex manifold of dimension $2n$ such that $\dim H^{(n,n-1),(n-1,n)}_{\partial_J+\bar{\partial}_J}(M, \mathbb{R}) < \infty$. Then (M, J) carries a strongly Gauduchon metric F if and only if there are no nontrivial positive $(\partial_J + \bar{\partial}_J)$-exact $(1, 1)$-currents.*

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2. Definitions and preliminaries

Let M be a closed oriented smooth $2n$-manifold. An almost complex structure on M is a differentiable endomorphism on the tangent bundle

$$J : TM \to TM \ \text{ with } \ J^2 = -id.$$

A manifold M with a fixed almost complex structure J is called an almost complex manifold denoted by (M, J). Suppose that M is an almost complex manifold with almost complex structure J, then for any $x \in M$, $T_x(M) \otimes_{\mathbb{R}} \mathbb{C}$ which is the complexification of $T_x(M)$, has the following decomposition (cf. [3]):

$$T_x(M) \otimes_{\mathbb{R}} \mathbb{C} = T_x^{1,0} + T_x^{0,1},$$

where $T_x^{1,0}$ and $T_x^{0,1}$ are the eigenspaces of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. A complex tangent vector is of type $(1, 0)$ (resp. $(0, 1)$) if it belongs to $T_x^{1,0}$ (resp. $T_x^{0,1}$). Let $T(M) \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of the tangent bundle. Similarly, let $T^*M \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of the cotangent bundle T^*M. J can act on $T^*M \otimes_{\mathbb{R}} \mathbb{C}$ as follows:

$$\forall \alpha \in T^*M \otimes_{\mathbb{R}} \mathbb{C}, \ \ J\alpha(\cdot) = -\alpha(J\cdot).$$

Hence $T^*M \otimes_{\mathbb{R}} \mathbb{C}$ has the following decomposition according to the eigenvalues $\pm\sqrt{-1}$:

$$T^*M \otimes_{\mathbb{R}} \mathbb{C} = \Lambda_J^{1,0} \oplus \Lambda_J^{0,1}.$$

We can form exterior bundle $\Lambda_J^{p,q} = \Lambda^p \Lambda_J^{1,0} \otimes \Lambda^q \Lambda_J^{0,1}$. Let $\Omega_J^{p,q}(M)$ denote the space of C^∞ sections of the bundle $\Lambda_J^{p,q}$. Then we have a direct sum decomposition $\Omega^k(M) = \bigoplus_{p+q=k} \Omega_J^{p,q}(M)$. We denote the projections $\Omega^k(M) \to \Omega_J^{p,q}(M)$ by $\Pi^{p,q}$. Note the fact that the total differential form space $\Omega(M) = \bigoplus_{p,q=0}^n \Omega_J^{p,q}(M)$ is locally generated by $\Omega_J^{0,0}(M)$, $\Omega_J^{1,0}(M)$ and $\Omega_J^{0,1}(M)$ and from the following inclusions:

$$d\Omega_J^{0,0}(M) \subset \Omega_J^{1,0}(M) \oplus \Omega_J^{0,1}(M), \ \ \Omega_J^{1,0}(M) \subset \Omega_J^{2,0}(M) \oplus \Omega_J^{1,1}(M) \oplus \Omega_J^{0,2}(M),$$
$$d\Omega_J^{0,1}(M) \subset \Omega_J^{2,0}(M) \oplus \Omega_J^{1,1}(M) \oplus \Omega_J^{0,2}(M).$$

The exterior differential operator acts on $\Omega_J^{p,q}(M)$ as follows ([3]):

$$d\Omega_J^{p,q}(M) \subset \Omega_J^{p-1,q+2}(M) + \Omega_J^{p+1,q}(M) + \Omega_J^{p,q+1}(M) + \Omega_J^{p+2,q-1}(M). \tag{2.1}$$

Hence, d has the following decomposition:

$$d = A_J \oplus \partial_J \oplus \bar{\partial}_J \oplus \bar{A}_J, \tag{2.2}$$

where $A_J \triangleq \Pi^{p-1,q+2} \circ d$, $\partial_J \triangleq \Pi^{p+1,q} \circ d$, $\bar{\partial}_J \triangleq \Pi^{p,q+1} \circ d$ and $\bar{A}_J \triangleq \Pi^{p+2,q-1} \circ d$. Let α be a (p, q)-form. We have following formulas (cf. [8]):

$$\begin{aligned}
&\partial_J \alpha(\xi_1, \cdots, \xi_{p+1}, \bar{\eta}_1, \cdots, \bar{\eta}_q)\\
&= \sum_{k=1}^{p+1} (-1)^{k+1} \xi_k \alpha(\xi_1, \cdots, \hat{\xi}_k, \cdots, \bar{\eta}_q)\\
&\quad + \sum_{1 \le k < l \le p+1} (-1)^{k+1} \alpha([\xi_k, \xi_l], \xi_1, \cdots, \hat{\xi}_k, \cdots, \hat{\xi}_l, \cdots, \bar{\eta}_q)\\
&\quad + \sum_{1 \le k \le p+1, 1 \le l \le q} (-1)^{k+l+p+1} \alpha([\xi_k, \bar{\eta}_l], \xi_1, \cdots, \hat{\xi}_k, \cdots, \hat{\bar{\eta}}_l, \cdots, \bar{\eta}_q),\\
&\bar{\partial}_J \alpha(\xi_1, \cdots, \xi_p, \bar{\eta}_1, \cdots, \bar{\eta}_{q+1})\\
&= \sum_{k=1}^{q+1} (-1)^{k+p+1} \bar{\eta}_k \alpha(\xi_1, \cdots, \hat{\bar{\eta}}_k, \cdots, \bar{\eta}_{q+1})\\
&\quad + \sum_{1 \le k < l \le q+1} (-1)^{k+1} \alpha([\bar{\eta}_k, \bar{\eta}_l], \xi_1, \cdots, \hat{\bar{\eta}}_k, \cdots, \hat{\bar{\eta}}_l, \cdots, \bar{\eta}_{q+1})\\
&\quad + \sum_{1 \le k \le p, 1 \le l \le q+1} (-1)^{k+l+p} \alpha([\xi_k, \bar{\eta}_l], \xi_1, \cdots, \hat{\xi}_k, \cdots, \hat{\bar{\eta}}_l, \cdots, \bar{\eta}_{q+1}),\\
&A_J \alpha(\xi_1, \cdots, \xi_{p-1}, \bar{\eta}_1, \cdots, \bar{\eta}_{q+2})\\
&= \sum_{1 \le k < l \le q+2} (-1)^{k+l} \alpha([\bar{\eta}_k, \bar{\eta}_l], \xi_1, \cdots, \hat{\bar{\eta}}_k, \cdots, \hat{\bar{\eta}}_l, \cdots, \bar{\eta}_{q+2})
\end{aligned}$$

and

$$\begin{aligned}
&\bar{A}_J \alpha(\xi_1, \cdots, \xi_{p+2}, \bar{\eta}_1, \cdots, \bar{\eta}_{q-1})\\
&= \sum_{1 \le k < l \le p+2} (-1)^{k+l} \alpha([\xi_k, \xi_l], \xi_1, \cdots, \hat{\xi}_k, \cdots, \hat{\xi}_l, \cdots, \bar{\eta}_{q-1}),
\end{aligned}$$

where $\xi_1, \cdots, \xi_{p+2}, \eta_1, \cdots, \eta_{q+2}$ are vector fields of type $(1, 0)$. It is easy to see that A_J and $\bar{A}_J$ are $\mathbb{R}$-linear operators of order 0. Moreover, the operator $S = A_J, \partial_J, \bar{\partial}_J, \bar{A}_J$ satisfies the Leibnitz rule

$$S(\alpha \wedge \beta) = S\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge S\beta. \tag{2.3}$$

After a simple calculation, we can get the following properties:

$$d = \partial_J + \bar{\partial}_J : \Omega^0(M) \longrightarrow \Omega^1(M); \tag{2.4}$$

$$A_J \circ \partial_J + \bar{\partial}_J^2 + \bar{A}_J \circ \bar{\partial}_J + \partial_J^2 = 0 : \Omega^0(M) \longrightarrow (\Omega_J^{2,0}(M) + \Omega_J^{0,2}(M)); \tag{2.5}$$

$$\partial_J \circ \bar{\partial}_J + \bar{\partial}_J \circ \partial_J = 0 : \Omega^0(M) \longrightarrow \Omega_J^{1,1}(M); \tag{2.6}$$

$$\partial_J \circ \bar{\partial}_J + \bar{\partial}_J \circ \partial_J = 0 : \Omega_J^{n-1,n-1}(M) \longrightarrow \Omega^{2n}(M). \tag{2.7}$$

Here, we just explain the formula (2.7). For any $\alpha \in \Omega_J^{n-1,n-1}(M)$,

$$d\alpha = (A_J + \partial_J + \bar{\partial}_J + \bar{A}_J)\alpha = (\partial_J + \bar{\partial}_J)\alpha$$

and

$$0 = d^2\alpha = d(\partial_J + \bar{\partial}_J)\alpha = (\partial_J + \bar{\partial}_J)(\partial_J\alpha + \bar{\partial}_J\alpha) = \bar{\partial}_J\partial_J\alpha + \partial_J\bar{\partial}_J\alpha.$$

Recall that on an almost complex manifold (M, J), there exists Nijenhuis tensor $\mathcal{N}_J$ as follows:

$$4\mathcal{N}_J = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y], \tag{2.8}$$

where $X, Y \in TM$. J is said to be integrable if $\mathcal{N}_J = 0$ (cf. [6]), then J is a complex structure and (M, J) is a complex manifold. Moreover, we have the following equivalence on a compact almost complex manifold (for details, see [3,7]):

$$J \text{ is } \ \text{integrable} \Longleftrightarrow \bar{\partial}_J^2 = 0.$$

Generally speaking, the integrability is too strong for some almost complex structures. There are many known almost complex manifolds which are not complex manifolds. Hence, we want to introduce a special kind of almost complex manifolds which are called q-integrable.

Definition 2.1. Let (M, J) be an almost complex manifold of dimension $2n$. We call J a q-integrable almost complex structure if

$$\bar{\partial}_J^2 \mid_{\Omega^{2n-k}(M)} = 0, \ \ q \geq k \geq 2,$$

where $2n \geq q \geq 2$. An almost complex manifold (M, J) is called a q-integrable almost complex manifold if J is a q-integrable almost complex structure.

It is easy to see that a $2n$-integrable almost complex structure is just a complex structure.

Suppose (M, J) is a closed almost complex $2n$-manifold. One can construct a J-invariant Riemannian metric g on M. g could be constructed as $g(\cdot, \cdot) = \frac{1}{2}\{h(\cdot, \cdot) + h(J\cdot, J\cdot)\}$ for some Riemannian metric h. This then in turn gives a J-compatible non-degenerate 2-form F by $F(X, Y) = g(JX, Y)$, called the fundamental 2-form. Such a quadruple (M, g, J, F) is called a closed almost Hermitian manifold. For convenience, we call F the almost Hermitian metric.

Definition 2.2. (cf. [9]) A strongly Gauduchon metric on a 2-integrable almost complex $2n$-dimensional manifold (M, J) is an almost Hermitian metric F such that $\partial_J F^{n-1}$ is $\bar{\partial}_J$-exact. A compact 2-integrable almost complex manifold is called a strongly Gauduchon manifold, if there exists a strongly Gauduchon metric on it.

3. Proof of main result

A form $\alpha \in \Omega^k(M)$ is said to be real if it satisfies $\alpha = \bar{\alpha}$. Denote the space of real k-forms on (M, J) by $\Omega^k(M)_{\mathbb{R}}$. It is easy to see that all positive forms are real. The following proposition gives some relations between strongly Gauduchon metrics and some special real forms.

Proposition 3.1. *(cf. [5,9]) Let (M, J) be a compact 2-integrable almost complex manifold of dimension $2n$. Then the following properties are equivalent.*

(1) *(M, J) is a strongly Gauduchon manifold.*
(2) *There exists a strictly positive $(n-1, n-1)$-form Ω, such that $\partial_J\Omega$ is $\bar{\partial}_J$-exact.*

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(3) *There exists a real* $(\partial_J + \bar{\partial}_J)$*-closed* $(2n-2)$*-form* Ω *whose* $(n-1, n-1)$*-component* $\Omega^{n-1,n-1}$ *is strictly positive.*

Proof. "(1) ⇒ (2)" Suppose that F is a strongly Gauduchon metric on (M, J). We choose Ω to be F^{n-1}. By the definition of strongly Gauduchon metric, it is easy to see that $\Omega = F^{n-1}$ is a strictly positive $(n-1, n-1)$-form and $\partial_J \Omega$ is $\bar{\partial}_J$-exact.

"(2) ⇒ (1)" Suppose that Ω is a strictly positive $(n-1, n-1)$-form and $\partial_J \Omega$ is $\bar{\partial}_J$-exact. Then there exists a unique strictly positive $(1, 1)$-form F such that $F^{n-1} = \Omega$ (see [5, page 280], the proof is independent of the integrability of J). It is easy to verify that F is a strongly Gauduchon metric.

"(1) ⇒ (3)" Suppose that F is a strongly Gauduchon metric on (M, J). Thus, there exists a $(n, n-2)$-form β such that $\partial_J F^{n-1} = \bar{\partial}_J \beta$. Let $\Omega = F^{n-1} - \beta - \bar{\beta}$. Obviously, the $(n-1, n-1)$-component of Ω is F^{n-1} which is strictly positive. In addition, we have $(\partial_J + \bar{\partial}_J)\Omega = \partial_J F^{n-1} - \partial_J \bar{\beta} + \bar{\partial}_J F^{n-1} - \bar{\partial}_J \beta = 0$.

"(3) ⇒ (1)" Let $\Omega = \Omega^{n-1,n-1} + \Omega^{n,n-2} + \Omega^{n-2,n}$ be a real $(\partial_J + \bar{\partial}_J)$-closed $(2n-2)$-form and $\Omega^{n-1,n-1}$ is strictly positive. Since $(\partial_J + \bar{\partial}_J)\Omega = \partial_J \Omega^{n-1,n-1} + \partial_J \Omega^{n-2,n} + \bar{\partial}_J \Omega^{n-1,n-1} + \bar{\partial}_J \Omega^{n,n-2} = 0$, we can obtain

$$\partial_J \Omega^{n-1,n-1} = -\bar{\partial}_J \Omega^{n,n-2}, \;\; \partial_J \Omega^{n-2,n} = -\bar{\partial}_J \Omega^{n-1,n-1}.$$

On the other hand, there exists a strictly positive $(1, 1)$-form F such that $F^{n-1} = \Omega^{n-1,n-1}$ (see [5, page 280]). Hence, $\partial_J F^{n-1} = \bar{\partial}_J(-\Omega^{n,n-2})$. Therefore, F is a strongly Gauduchon metric. □

By using Proposition 3.1, we can obtain the following interesting proposition.

Proposition 3.2. *(cf. [4,5]) Let* (M, J_M) *and* (N, J_N) *be compact 2-integrable almost complex manifolds of dimension* $2m$ *and* $2n$*, respectively.*

(1) *If* $f : (M, J_M) \to (N, J_N)$ *is a* (J_M, J_N)*-holomorphic submersion and* (M, J_M) *is a strongly Gauduchon manifold, then* (N, J_N) *is a strongly Gauduchon manifold.*
(2) $(M \times N, J = J_M \times J_N)$ *is a strongly Gauduchon manifold if and only of* (M, J_M) *and* (N, J_N) *are both strongly Gauduchon manifolds.*

Proof. (1) By Proposition 3.1, there exists a strictly positive $(m-1, m-1)$-form Ω_M, such that $\partial_{J_M} \Omega_M = \bar{\partial}_{J_M} \beta$, where β is a $(2m-2)$-form on M. Define $\Omega_N = f_* \Omega_M$. Ω_N is simply the $(2n-2)$-form obtained by integration over the fibers of f (the proof of Proposition 1.9 in [5]). It is well known that a (J_M, J_N)-holomorphic map $f : (M, J_M) \to (N, J_N)$ between two almost complex manifolds is a smooth map whose differential f_* satisfies $f_* J_M = J_N f_*$ at every point. Thus, we have

$$\partial_{J_N} \Omega_N = \partial_{J_N} f_* \Omega_M = f_* \partial_{J_M} \Omega_M = f_* \bar{\partial}_{J_M} \beta = \bar{\partial}_{J_N} f_* \beta.$$

By the proof of Proposition 1.9 in [5] (the proof is independent of the integrability of J_N), we know Ω_N is a strictly positive $(n-1, n-1)$-form. So (N, J_N) is a strongly Gauduchon manifold.

(2) Let (M, J_M) and (N, J_N) be both strongly Gauduchon manifolds. Suppose F_M and F_N are strongly Gauduchon metrics on (M, J_M) and (N, J_N) respectively, such that $\partial_{J_M} F_M^{m-1} = \bar{\partial}_{J_M} \beta$ and $\partial_{J_N} F_N^{n-1} = \bar{\partial}_{J_N} \gamma$. Here β and γ are $(2m-2)$ and $(2n-2)$-form on M and N respectively. Denote by π_1 and π_2 the projections $\pi_1 : (M \times N, J) \to (M, J_M)$ and $\pi_2 : (M \times N, J) \to (N, J_N)$ respectively. It is easy to verify that both π_1 and π_2 are (J, J_M)-holomorphic and (J, J_N)-holomorphic maps. We define a form on $M \times N$ by $F \triangleq \pi_1^* F_M + \pi_2^* F_N$. Then $F^{m+n-1} = C_1 \pi_1^* F_M^{m-1} \wedge \pi_2^* F_N^n + C_2 \pi_1^* F_M^m \wedge \pi_2^* F_N^{n-1}$, where C_1, C_2 are constants. Hence, $\partial_J F^{m+n-1} = C_1 \pi_1^*(\partial_{J_M} F_M^{m-1}) \wedge \pi_2^* F_N^n + C_2 \pi_1^* F_M^m \wedge \pi_2^*(\partial_{J_N} F_N^{n-1})$. For simplicity, we will omit pullbacks π_1^* and π_2^* in the following proof. By direct calculation, we can obtain

$$\begin{aligned}\partial_J F^{m+n-1} &= C_1 \partial_{J_M} F_M^{m-1} \wedge F_N^n + C_2 F_M^m \wedge \partial_{J_N} F_N^{n-1} \\ &= C_1 \bar{\partial}_{J_M} \beta \wedge F_N^n + C_2 F_M^m \wedge \bar{\partial}_{J_N} \gamma \\ &= \bar{\partial}_J (C_1 \beta \wedge F_N^n) + \bar{\partial}_J (C_2 F_M^m \wedge \gamma) \\ &= \bar{\partial}_J (C_1 \beta \wedge F_N^n + C_2 F_M^m \wedge \gamma).\end{aligned}$$

Hence, $(M \times N, J = J_M \times J_N)$ is a strongly Gauduchon manifold.

The converse is an obvious result following (1). □

Let (M, J) be a compact 2-integrable almost complex manifold of dimension $2n$. Suppose $\alpha \in \Omega^{2n-2}(M)_{\mathbb{R}}$, then $(\partial_J + \bar{\partial}_J)\alpha \in [\Omega_J^{n,n-1}(M) \oplus \Omega_J^{n-1,n}(M)]_{\mathbb{R}}$. Moreover, by (2.7), we have $(\partial_J + \bar{\partial}_J)^2\alpha = \partial_J^2\alpha + \bar{\partial}_J^2\alpha + \partial_J\bar{\partial}_J\alpha + \bar{\partial}_J\partial_J\alpha = 0$. This gives a differential complex

$$\Omega^{2n-2}(M)_{\mathbb{R}} \xrightarrow{\partial_J + \bar{\partial}_J} [\Omega_J^{n,n-1}(M) \oplus \Omega_J^{n-1,n}(M)]_{\mathbb{R}} \xrightarrow{\partial_J + \bar{\partial}_J} \Omega^{2n}(M)_{\mathbb{R}}. \tag{3.9}$$

Considering the cohomology associated with this complex leads us to introduce

$$H_{\partial_J + \bar{\partial}_J}^{(n,n-1),(n-1,n)}(M, \mathbb{R}) = \frac{\ker(\partial_J + \bar{\partial}_J) \cap [\Omega_J^{n,n-1}(M) \oplus \Omega_J^{n-1,n}(M)]_{\mathbb{R}}}{\operatorname{im}(\partial_J + \bar{\partial}_J) \cap [\Omega_J^{n,n-1}(M) \oplus \Omega_J^{n-1,n}(M)]_{\mathbb{R}}}. \tag{3.10}$$

If J is integrable, $H_{\partial_J + \bar{\partial}_J}^{(n,n-1),(n-1,n)}(M, \mathbb{R})$ is finite dimensional since it is just the de Rham cohomology $H_{dR}^{2n-1}(M, \mathbb{R})$. There's a natural question: is the dimension of $H_{\partial_J + \bar{\partial}_J}^{(n,n-1),(n-1,n)}(M, \mathbb{R})$ finite on a compact 2-integrable almost complex manifold or under what condition the dimension of $H_{\partial_J + \bar{\partial}_J}^{(n,n-1),(n-1,n)}(M, \mathbb{R})$ is finite? In order to prove the following results, we need the technical condition $\dim H_{\partial_J + \bar{\partial}_J}^{(n,n-1),(n-1,n)}(M, \mathbb{R}) < \infty$.

Lemma 3.3. *Suppose that (M, J) is a compact 2-integrable almost complex manifold of dimension $2n$. If $\dim H_{\partial_J + \bar{\partial}_J}^{(n,n-1),(n-1,n)}(M, \mathbb{R}) < \infty$, then the operator*

$$(\partial_J + \bar{\partial}_J) : \Omega^{2n-2}(M)_{\mathbb{R}} \to [\Omega_J^{n,n-1}(M) \oplus \Omega_J^{n-1,n}(M)]_{\mathbb{R}}$$

has closed range.

Proof. Define $Z_J^{(n,n-1),(n-1,n)}(M)_{\mathbb{R}} \triangleq \{\alpha \in [\Omega_J^{n,n-1}(M) \oplus \Omega_J^{n-1,n}(M)]_{\mathbb{R}} : (\partial_J + \bar{\partial}_J)\alpha = 0\}$. With the condition $\dim H_{\partial_J + \bar{\partial}_J}^{(n,n-1),(n-1,n)}(M, \mathbb{R}) < \infty$ we can easily get that the image of

$$(\partial_J + \bar{\partial}_J) : \Omega^{2n-2}(M)_{\mathbb{R}} \to Z_J^{(n,n-1),(n-1,n)}(M)_{\mathbb{R}}$$

has finite codimension. Therefore, by the classical theory of functional analysis the image of $(\partial_J + \bar{\partial}_J)$ is closed in $Z_J^{(n,n-1),(n-1,n)}(M)_{\mathbb{R}}$ which, in turn, is closed in $[\Omega_J^{n,n-1}(M) \oplus \Omega_J^{n-1,n}(M)]_{\mathbb{R}}$, hence, the image is closed in the last space. □

Let (M, J) be a compact 2-integrable almost complex manifold of dimension $2n$. Let $\mathcal{E}_k(M)$ denote the dual space of the Fréchet space $\Omega^k(M)$. An element $T \in \mathcal{E}_k(M)$ is called a current of dimension k (or a $(2n-k)$-current). Similarly, let $\mathcal{E}_{p,q}(M)$ denote the dual space of $\Omega_J^{p,q}(M)$. An element $T \in \mathcal{E}_{p,q}(M)$ is called a current of bidimension (p, q) (or a $(n-p, n-q)$-current). The closed range theorem (cf. [10, Sect. 7 of Chap. IV]) says that the adjoint of a map with closed range has closed range. Hence, we have the following corollary.

Corollary 3.4. $(\partial_J + \bar{\partial}_J) : [\mathcal{E}_{n,n-1}(M) \oplus \mathcal{E}_{n-1,n}(M)]_{\mathbb{R}} \to \mathcal{E}_{2n-2}(M)_{\mathbb{R}}$ *has closed range too.*

In the following, we will give a characterisation of strongly Gauduchon manifolds in terms of non-existence of certain currents. Our approach is along the lines used in [2,9]. Let (M, J) be a compact 2-integrable almost complex manifold of dimension $2n$. Suppose that there exists a real $(\partial_J + \bar{\partial}_J)$-closed $(2n-2)$-form Ω whose $(n-1, n-1)$-component $\Omega^{n-1,n-1}$ is strictly positive on (M, J). Then for any real $(1,1)$-current $T = (\partial_J + \bar{\partial}_J)S$, we have

$$(\Omega, T) = (\Omega, (\partial_J + \bar{\partial}_J)S) = ((\partial_J + \bar{\partial}_J)\Omega, S) = 0.$$

On the other hand, for each nontrivial positive $(1,1)$-current T, we obtain

$$(\Omega, T) = (\Omega^{n-1,n-1}, T) > 0.$$

This means that if (M, J) admits a strongly Gauduchon metric, then there are no nontrivial positive $(\partial_J + \bar{\partial}_J)$-exact $(1,1)$-currents. Moreover, we have the following theorem.

Theorem 3.5. *Let (M, J) be a compact 2-integrable almost complex manifold of dimension $2n$ such that $\dim H^{(n,n-1),(n-1,n)}_{\partial_J+\bar{\partial}_J}(M, \mathbb{R}) < \infty$. Then (M, J) carries a strongly Gauduchon metric F if and only if there are no nontrivial positive $(\partial_J + \bar{\partial}_J)$-exact $(1,1)$-currents.*

Proof. Let Ω be any real smooth $(2n-2)$-form on compact almost complex (M, J). We claim that the condition $(\partial_J + \bar{\partial}_J)\Omega = 0$ is equivalent to the property

$$\int_M \Omega \wedge (\partial_J + \bar{\partial}_J)T = 0 \tag{3.11}$$

for every real 1-current T on (M, J). Indeed, by (2.3), we have $\partial_J(\Omega \wedge T) = \partial_J\Omega \wedge T + \Omega \wedge \partial_J T$ and $\bar{\partial}_J(\Omega \wedge T) = \bar{\partial}_J\Omega \wedge T + \Omega \wedge \bar{\partial}_J T$. Then

$$\begin{aligned}
\int_M \Omega \wedge (\partial_J + \bar{\partial}_J)T &= \int_M (\partial_J + \bar{\partial}_J)(\Omega \wedge T) - \int_M (\partial_J + \bar{\partial}_J)\Omega \wedge T \\
&= \int_M (\partial_J + \bar{\partial}_J)(\Omega^{n-1,n-1} \wedge T^{1,0} + \Omega^{n-1,n-1} \wedge T^{0,1} \\
&\qquad + \Omega^{n-2,n} \wedge T^{1,0} + \Omega^{n,n-2} \wedge T^{0,1}) - \int_M (\partial_J + \bar{\partial}_J)\Omega \wedge T \\
&= \int_M (\partial_J + \bar{\partial}_J + A_J + \bar{A}_J)(\Omega^{n-1,n-1} \wedge T^{1,0} \\
&\qquad + \Omega^{n-1,n-1} \wedge T^{0,1} + \Omega^{n-2,n} \wedge T^{1,0} + \Omega^{n,n-2} \wedge T^{0,1}) \\
&\qquad - \int_M (\partial_J + \bar{\partial}_J)\Omega \wedge T \\
&= \int_M d(\Omega \wedge T) - \int_M (\partial_J + \bar{\partial}_J)\Omega \wedge T \\
&= -\int_M (\partial_J + \bar{\partial}_J)\Omega \wedge T,
\end{aligned} \tag{3.12}$$

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where we write $\Omega = \Omega^{n-1,n-1} + \Omega^{n,n-2} + \Omega^{n-2,n}$ and $T = T^{1,0} + T^{0,1}$. On the other hand, the duality between strictly positive smooth $(n-1, n-1)$-forms and non-zero positive $(1,1)$-currents on (M, J) shows that the condition $\Omega^{n-1,n-1} > 0$ is equivalent to the property

$$\int_M \Omega \wedge T = \int_M \Omega^{n-1,n-1} \wedge T > 0 \tag{3.13}$$

for every non-zero positive $(1,1)$-current T on (M, J). Denote the space of real 2-currents on (M, J) by $\mathcal{E}^2(M)_{\mathbb{R}} = \mathcal{E}_{2n-2}(M)_{\mathbb{R}}$ which is a locally convex space. Define $\mathcal{A} = \{\text{real } (\partial_J + \bar{\partial}_J)\text{-exact 2-currents}\}$. By Corollary 3.4, we can get that $\mathcal{A}$ is a closed vector subspace of $\mathcal{E}^2(M)_{\mathbb{R}}$. If we fix a smooth, strictly positive $(n-1, n-1)$-form F on (M, J), positive non-zero $(1,1)$-currents T on (M, J) can be normalised such that $\int_M T \wedge F = 1$ and it suffices to guarantee property (3.13) for normalised currents. Clearly, these normalised positive $(1,1)$-currents form a compact convex subset $\mathcal{B}$ of $\mathcal{E}^2(M)_{\mathbb{R}}$. Then the Hahn–Banach separation Theorem for locally convex space (cf. [2]) implies that there exists a linear functional vanishing identically on a given closed subset and assuming only positive values on a given compact subset if the two subsets are convex and do not intersect. Thus, in our case, there exists a real $(\partial_J + \bar{\partial}_J)$-closed $(2n-2)$-form Ω whose $(n-1, n-1)$-component $\Omega^{n-1,n-1}$ is strictly positive (that is, there exists a real $(2n-2)$-form satisfying conditions (3.11) and (3.13)) if and only if $\mathcal{A} \cap \mathcal{B} = \emptyset$. This amounts to there being no nontrivial positive $(\partial_J + \bar{\partial}_J)$-exact $(1,1)$-currents on (M, J). □

By the above description, we know that $\mathcal{A}$ is a closed vector subspace on a compact 2-integrable almost complex manifold if $\dim H^{(n,n-1),(n-1,n)}_{\partial_J+\bar{\partial}_J}(M, \mathbb{R}) < \infty$. Is $\mathcal{A}$ also closed if we drop the assumption $\dim H^{(n,n-1),(n-1,n)}_{\partial_J+\bar{\partial}_J}(M, \mathbb{R}) < \infty$? In follows we will consider the case without $\dim H^{(n,n-1),(n-1,n)}_{\partial_J+\bar{\partial}_J}(M, \mathbb{R}) < \infty$. Suppose that there exists a real $(\partial_J + \bar{\partial}_J)$-closed $(2n-2)$-form Ω whose $(n-1, n-1)$-component $\Omega^{n-1,n-1}$ is strictly positive and $T \in \overline{\mathcal{A}} \cap \mathcal{B} \neq \emptyset$. Moreover, suppose $T = \lim_{k\to+\infty}(\partial_J + \bar{\partial}_J)T_k$ for $\{T_k\} \subseteq [\mathcal{E}_{n,n-1}(M) \oplus \mathcal{E}_{n-1,n}(M)]_{\mathbb{R}}$, then we will get

$$\begin{aligned} 0 &< (\Omega^{n-1,n-1}, T) \\ &= (\Omega, T) \\ &= (\Omega, \lim_{k\to+\infty}(\partial_J + \bar{\partial}_J)T_k) \\ &= \lim_{k\to+\infty}(\Omega, (\partial_J + \bar{\partial}_J)T_k) \\ &= \lim_{k\to+\infty}((\partial_J + \bar{\partial}_J)\Omega, T_k) \\ &= 0. \end{aligned}$$

It is a contradiction.

Corollary 3.6. *Let (M, J) be a compact 2-integrable almost complex manifold of dimension $2n$. Then (M, J) carries a strongly Gauduchon metric F if and only if $\overline{\mathcal{A}} \cap \mathcal{B} = \emptyset$.*

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