



LIPSCHITZ (p, r, s) -INTEGRAL OPERATORS AND LIPSCHITZ (p, r, s) -NUCLEAR OPERATORS

AMAR BELACEL AND DONGYANG CHEN*

ABSTRACT. We introduce the notions of strongly Lipschitz (p, r, s) -nuclear operators and strongly Lipschitz (p, r, s) -integral operators. We develop a theory of strongly Lipschitz (p, r, s) -nuclear operators and strongly Lipschitz (p, r, s) -integral operators which closely parallels the theory for linear operators. Their close relationships with other Lipschitz operator ideals are studied.

1. INTRODUCTION

In 2009, J. Farmer and W. B. Johnson started in [10] the study of a natural nonlinear version of linear p -summing operators, which they call *Lipschitz p -summing operators*. This is a true generalization of the concept of linear p -summing operators, since it is shown in [10, Theorem 2] that the Lipschitz p -summing norm of a linear operator is the same as its p -summing norm. The paper [10] has motivated the study of Lipschitz versions of different classes of bounded linear operators such as, Lipschitz (p, r, s) -summing operators in [3] and Lipschitz (q, p) -mixing operators in [4], Lipschitz p -integral operators and Lipschitz p -nuclear operators in [7], Lipschitz Grothendieck-integral operators in [6], (p, σ) -absolutely Lipschitz operators in [1], strongly Lipschitz p -summing operators in [18]. A. Jiménez-Vargas, J. M. Sepulcre and M. Villegas-Vallecillos [13] introduced the notions of Lipschitz compact, Lipschitz weakly compact, Lipschitz finite-rank and Lipschitz approximable operators from a pointed metric space into a Banach space. D. Achour, P. Rueda, E. A. Sánchez-Pérez and R. Yahi [2] introduce the concept of Lipschitz operator ideals and develop a basic theory of the Lipschitz operator ideals that unifies all the recently introduced new classes of Lipschitz operators.

Our purpose at the present paper is to introduce and investigate Lipschitz versions of (p, r, s) -nuclear operators and (p, r, s) -integral operators. We study their relationships with other classes of Lipschitz operators as Lipschitz (p, r, s) -summing operators, Lipschitz finite-rank operators, Lipschitz approximable operators, Lipschitz compact operators, Lipschitz weakly compact operators.

We now describe the contents of this paper. In Section 2, we introduce the notion of Lipschitz w^* - p -summable sequences and use it to introduce strongly Lipschitz (p, r, s) -nuclear operators, which is a nonlinear analogue of (p, r, s) -nuclear operators ([15, Definition 18.1.1]). It is proved in [13, Proposition 2.7] that every strongly Lipschitz p -nuclear operator is Lipschitz compact. In this section, we generalize and strengthen this result and prove that every strongly Lipschitz (p, r, s) -nuclear operator is Lipschitz approximable. It is easy to see that strongly Lipschitz $(p, p, 1)$ -nuclear operators are precisely strongly Lipschitz p -nuclear operators introduced in [7]. We

2010 *Mathematics Subject Classification.* 46B28, 47B10, 47L20.

Key words and phrases. Lipschitz map, strongly Lipschitz (p, r, s) -nuclear operators, strongly Lipschitz (p, r, s) -integral operators, Lipschitz r -compact operators.

*Corresponding author

Dongyang Chen's project was supported by the Natural Science Foundation of Fujian Province of China(No.2015J01026).

also prove that every strongly Lipschitz p -nuclear operator is Lipschitz p -summing. Moreover, a canonical correspondence between (p, r, s) -nuclear operators and strongly Lipschitz (p, r, s) -nuclear operators is established. The concept of the generic Lipschitz operator Banach ideal was introduced in [5]. In Section 2, we prove that the Lipschitz ideal of strongly Lipschitz $(1, 1, 1)$ -nuclear operators is the smallest generic Lipschitz operator Banach ideal. Furthermore, we identify the linear dual of space of strongly Lipschitz (p, r, s) -nuclear operators with the space of (p^*, s^*, r^*) -summing operators. The Lipschitz dual of an operator ideal is introduced by K. Saadi [16]. In this section, we introduce the Lipschitz regular hull of a generic Lipschitz operator β -Banach ideal and prove that the Lipschitz dual of the space of (p, r, s) -nuclear operators is precisely the Lipschitz regular hull of the space of strongly Lipschitz (p, s, r) -nuclear operators.

Section 3 focuses on an important specific example of strongly Lipschitz (p, r, s) -nuclear operators, which is called *Lipschitz r -compact operators*. We prove that a Lipschitz map from a pointed metric space to l_r is Lipschitz r -compact if and only if it is Lipschitz compact. In this section, we introduce the notion of Lipschitz w^* -unconditionally- p -summable sequences and use it to characterize Lipschitz r -compact operators.

The final section is concerned with strongly Lipschitz (p, r, s) -integral operators that is a nonlinear analogue of (p, r, s) -integral operators. D. Chen and B. Zheng [7] introduced a class of Lipschitz operators called *strongly Lipschitz p -integral operators*. A. Jiménez-Vargas, J. M. Sepulcre and M. Villegas-Vallecillos [13] introduced the notion of strongly Lipschitz p -integral operators which differ from the one defined by D. Chen and B. Zheng [7]. In this section, we introduce the concept of strongly Lipschitz (p, r, s) -integral operators that extend the notion of strongly Lipschitz p -integral operators in [13]. As expected, we show that every strongly Lipschitz (p, r, s) -nuclear is strongly Lipschitz (p, r, s) -integral. We also prove that every strongly Lipschitz (p, r, s) -integral operator is Lipschitz weakly compact. This result generalizes [13, Proposition 2.8]. Finally, we prove that the Lipschitz dual of space of (p, r, s) -integral operators is the space of strongly Lipschitz (p, s, r) -integral operators.

Notation and Preliminary. Our notation and terminology are standard as may be found in [15] and [17]. Throughout the paper, X, Y, Z will always denote metric spaces, whereas E, F, G will denote real Banach spaces. We use the convention of having pointed metric spaces, i.e., with a base point denoted by 0. In the case that X is a normed space, the base point of X will be the origin. As customary, B_E denotes the closed unit ball of E , E^* its linear dual, I_E the identity map on E , and $\mathcal{L}(E, F)$ is the space of bounded linear operators from E to F . We denote by $\mathcal{K}(E, F)$ the space of all linear compact operators from E to F . The letters p, r, s will designate elements of $[1, +\infty]$, and p^* denotes the exponent conjugate to p (i.e., $\frac{1}{p} + \frac{1}{p^*} = 1$). $\|\cdot\|_p$ denotes the norm on ℓ_p of a sequence of real numbers. We denote by $Lip_0(X, E)$ the Banach space of Lipschitz maps $T : X \rightarrow E$ such that $T(0) = 0$ with pointwise addition and under the Lipschitz norm given by

$$Lip(T) := \sup\left\{\frac{\|Tx - Tx'\|}{d(x, x')} : x, x' \in X, x \neq x'\right\}.$$

We use the shorthand $X^\# := Lip_0(X, \mathbb{R})$. The unit ball $B_{X^\#}$ of $X^\#$ is a compact Hausdorff space in the topology of pointwise convergence on X . For $x \in X$, we denote by $\delta_x \in (X^\#)^*$ the evaluation functional, i.e., $\langle \delta_x, f \rangle = f(x)$ for every $f \in X^\#$. The closed linear span of $\{\delta_x : x \in X\}$ in $(X^\#)^*$ is denoted by $\mathcal{F}(X)$ and called the *Lipschitz-free space over X* .

The Dirac map $\delta_X : x \rightarrow \delta_x$ is an isometric embedding of X into $(X^\#)^*$. It follows from the compactness of $B_{X^\#}$ with respect to the topology of pointwise convergence,

that $\mathcal{F}(X)$ can be seen as the canonical predual of $X^\#$. Then the *weak**-topology induced by $\mathcal{F}(X)$ on $X^\#$ coincides with the topology of pointwise convergence on the bounded subsets of $X^\#$. The space $\mathcal{F}(X)$ has the following universal property:

Let E be a Banach space and $T \in \text{Lip}_0(X, E)$. Then there is a unique linear operator $\widehat{T} : \mathcal{F}(X) \rightarrow E$ such that $T = \widehat{T}\delta_X$. Moreover, $\|\widehat{T}\| = \text{Lip}(T)$.

For each $T \in \text{Lip}_0(X, E)$, the linear operator $T^t : E^* \rightarrow X^\#$, given by $T^t u^* = u^* \circ T$ for all $u^* \in E^*$, is called the *Lipschitz transpose map* of T . It is easy to see that $\|T^t\| = \text{Lip}(T)$. For $f \in X^\#$ and $u \in E$, let us define a Lipschitz mapping $f \otimes u : X \rightarrow E$ by $(f \otimes u)(x) = f(x)u$.

Let $1 \leq p \leq \infty$ and let $l_p^w(E)$ be the set of all weakly p -summable sequences in E . Then $l_p^w(E)$ becomes a Banach space with the norm

$$\omega_p((u_n)_n) := \sup_{u^* \in B_{E^*}} \|(\langle u^*, u_n \rangle)_n\|_p, \quad (u_n)_n \in l_p^w(E).$$

It is a well-known result of A. Grothendieck ([12], [9, Proposition 2.2]) that the canonical correspondence $T \mapsto (Te_n)_n$ provides an isometric isomorphism of $\mathcal{L}(l_{p^*}, E)$ onto $l_p^w(E)$ when $1 < p < \infty$; For $p = 1$, the isometric isomorphism is from $\mathcal{L}(c_0, E)$ onto $l_1^w(E)$; For $p = \infty$, the isometric isomorphism is from $\mathcal{L}(l_1, E)$ onto $l_\infty(E)$. Recall that a sequence $(u_n^*)_n$ in E^* is called *w^* - p -summable* if $(\langle u_n^*, u \rangle)_n$ is in $l_p(c_0)$ for $p = \infty$ for every $u \in E$. The space of all *w^* - p -summable* sequences in E^* is denoted by $l_p^{w^*}(E^*)$. It is also proved in [12] that the canonical correspondence $T \mapsto (\langle e_j^*, T(\cdot) \rangle)_j$ provides an isometric isomorphism of $\mathcal{L}(E, l_p)(\mathcal{L}(E, c_0)$ for $p = \infty$) onto $l_p^{w^*}(E^*)$.

The reader is referred to [9], [15] and [17] for any unexplained notation or terminology.

2. STRONGLY LIPSCHITZ (p, r, s) -NUCLEAR OPERATORS

Suppose that $1 + \frac{1}{p} \geq \frac{1}{r} + \frac{1}{s}$. Recall that a linear operator $S : E \rightarrow F$ is called *(p, r, s) -nuclear* ([15, Definition 18.1.1]) if

$$S = \sum_{j=1}^{\infty} \lambda_j u_j^* \otimes v_j,$$

with $(\lambda_j)_{j=1}^\infty \in l_p$, $(u_j^*)_{j=1}^\infty \in l_{s^*}^{w^*}(E^*)$ and $(v_j)_{j=1}^\infty \in l_r^w(F)$. In the case $p = \infty$ we suppose that $(\lambda_j)_{j=1}^\infty \in c_0$.

We put

$$N_{(p,r,s)}(S) := \inf \|(\lambda_j)_{j=1}^\infty\|_p \cdot \omega_{s^*}((u_j^*)_{j=1}^\infty) \cdot \omega_r((v_j)_{j=1}^\infty),$$

where the infimum is taken over all so-called *(p, r, s) -nuclear representations* described above.

Let $\lambda = (\lambda_j)_j \in l_p$ if $1 \leq p < \infty$ and $\lambda = (\lambda_j)_j \in c_0$ if $p = \infty$. Throughout this paper, we denote the diagonal operator of the form $(\xi_j)_j \mapsto (\lambda_j \xi_j)_j$ from l_{s^*} to l_r by D_λ .

Theorem 18.1.3 in [15] states that a linear operator $S : E \rightarrow F$ is *(p, r, s) -nuclear* if and only if there exists a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{S} & F \\ B \downarrow & & \uparrow A \\ l_{s^*} & \xrightarrow{D_\lambda} & l_r \end{array}$$

In this case,

$$N_{(p,r,s)}(S) = \inf \|A\| \cdot \|(\lambda_j)_{j=1}^\infty\|_p \cdot \|B\|,$$

where the infimum is taken over all possible factorizations. The class of all (p, r, s) -nuclear operators from E to F is denoted by $\mathcal{N}_{(p,r,s)}(E, F)$.

Definition 2.1. Suppose that $1 + \frac{1}{p} \geq \frac{1}{r} + \frac{1}{s}$ and $T \in Lip_0(X, E)$. We say that T is *Lipschitz (p, r, s) -nuclear* if there are two Lipschitz maps $A \in Lip_0(l_r, E)$, $B \in Lip_0(X, l_{s^*})$ and $\lambda = (\lambda_j)_j \in l_p$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{T} & E \\ B \downarrow & & \uparrow A \\ l_{s^*} & \xrightarrow{D\lambda} & l_r \end{array}$$

We put

$$N_{(p,r,s)}^L(T) := \inf Lip(A) \cdot \|(\lambda_j)_{j=1}^\infty\|_p \cdot Lip(B),$$

the infimum being taken over all factorizations as above.

A proof similar to [7, Theorem 2.1] shows that the Lipschitz (p, r, s) -nuclear norm is equal to the (p, r, s) -nuclear norm for a linear operator from a separable space to a dual space.

Proposition 2.2. Suppose that $1 + \frac{1}{p} \geq \frac{1}{r} + \frac{1}{s}$ and $1 \leq r < \infty$. Let T be a bounded linear operator from a separable Banach space E into a dual space F . Then

$$N_{(p,r,s)}^L(T) = N_{(p,r,s)}(T).$$

To introduce the concept of strongly Lipschitz (p, r, s) -nuclear operators, we introduce the notion of Lipschitz- w^* - p -summable sequences as follows.

Definition 2.3. Let $(f_j)_j$ be a sequence in $X^\#$. We say that $(f_j)_j$ is *Lipschitz- w^* - p -summable* if there is a constant C such that for all $n \in \mathbb{N}$ and for all $x, x' \in X$ we have

$$\|(f_j(x) - f_j(x'))_{j=1}^n\|_p \leq C \cdot d(x, x').$$

The smallest such constant C will be denoted by $\omega_p^{L,w^*}((f_j)_j)$ and $l_p^{L,w^*}(X^\#)$ will denote the set of all Lipschitz- w^* - p -summable sequences in $X^\#$. Clearly,

$$\omega_p^{L,w^*}((f_j)_j) = \sup_{\substack{x \neq x' \\ x, x' \in X}} \frac{\|(f_j(x) - f_j(x'))_j\|_p}{d(x, x')}.$$

The following lemma is immediate from Definition 2.3.

Lemma 2.4. The canonical correspondence

$$T \mapsto (\langle e_j^*, T(\cdot) \rangle)_j$$

provides an isometric isomorphism of $Lip_0(X, l_p)$ onto $l_p^{L,w^*}(X^\#)$.

Proposition 2.5. $[l_p^{L,w^*}(X^\#), \omega_p^{L,w^*}(\cdot)] = [l_p^w(X^\#), \omega_p(\cdot)]$.

Proof. Let $(f_j)_j \in l_p^{L,w^*}(X^\#)$. Define an operator $T : X \rightarrow l_p$ by $Tx = (f_j(x))_j$. Then T is Lipschitz and $Lip(T) = \omega_p^{L,w^*}((f_j)_j)$. It is easy to see that $T^t e_j^* = f_j$ for each j . Hence $(f_j)_j \in l_p^w(X^\#)$ and $\omega_p((f_j)_j) \leq \|T\| = \omega_p^{L,w^*}((f_j)_j)$.

On the other hand, take any $(f_j)_j \in l_p^w(X^\#)$. Define a linear operator $T : l_{p^*} \rightarrow X^\#$ ($c_0 \rightarrow X^\#$ for $p = 1$) by $T e_j^* = f_j$ for each j . Then T is bounded, $\|T\| = \omega_p((f_j)_j)$ and $\langle e_j^*, T^* \delta_X(x) \rangle = f_j(x)$ for all $x \in X$. This yields

$$\omega_p^{L,w^*}((f_j)_j) = \sup_{\substack{x \neq x' \\ x, x' \in X}} \frac{\|T^* \delta_X(x) - T^* \delta_X(x')\|_p}{d(x, x')} \leq \|T\| = \omega_p((f_j)_j).$$

□

Definition 2.6. Suppose that $1 + \frac{1}{p} \geq \frac{1}{r} + \frac{1}{s}$ and $T \in Lip_0(X, E)$. We say that $T : X \rightarrow E$ is *strongly Lipschitz (p, r, s) -nuclear* if T can be written in the form

$$T = \sum_j \lambda_j f_j \otimes v_j,$$

where $(\lambda_j)_j \in \ell_p$ if $1 \leq p < \infty$ or $(\lambda_j)_j \in c_0$ if $p = \infty$, $(f_j)_j \in l_{s^*}^{L, w^*}(X^\#)$ and $(v_j)_j \in l_r^w(E)$.

The set of all strongly Lipschitz (p, r, s) -nuclear operators will be denoted by $SN_{(p, r, s)}^L(X, E)$ and we set

$$SN_{(p, r, s)}^L(T) := \inf \|(\lambda_j)_j\|_p \cdot \omega_{s^*}^{L, w^*}((f_j)_j) \cdot \omega_{r^*}((v_j)_j),$$

where the infimum is taken over all the strongly Lipschitz (p, r, s) -nuclear representations of T .

If X is a pointed metric space and E is a Banach space, by the Lipschitz image of a map $T : X \rightarrow E$ we mean the set $\{\frac{Tx - Tx'}{d(x, x')} : x, x' \in X, x \neq x'\}$. A Lipschitz map $T : X \rightarrow E$ is said to have *Lipschitz finite dimensional rank* [13] if the linear hull of its Lipschitz image is finite-dimensional. We denote by $Lip_{0F}(X, E)$ the set of all Lipschitz finite-rank operators from X into E . It is said in [13] that a Lipschitz map $T : X \rightarrow E$ is *Lipschitz approximable* if it is the limit in the Lipschitz norm of a sequence of Lipschitz finite-rank operators from X to E .

Remark 2.7. It is easy to see that $\sum_j \lambda_j f_j \otimes v_j$ converges in the Lipschitz norm. Hence every strongly Lipschitz (p, r, s) -nuclear operator is Lipschitz approximable.

Theorem 2.8. Suppose that $1 + \frac{1}{p} \geq \frac{1}{r} + \frac{1}{s}$ and $T \in Lip_0(X, E)$. Then $T : X \rightarrow E$ is strongly Lipschitz (p, r, s) -nuclear if and only if T has a factorization $T = AD_\lambda B$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{T} & E \\ B \downarrow & & \uparrow A \\ l_{s^*} & \xrightarrow{D_\lambda} & l_r \end{array}$$

where $B \in Lip_0(X, l_{s^*})$, $A \in \mathcal{L}(l_r, E)$ and $\lambda = (\lambda_j)_j \in l_p, 1 \leq p < \infty (\lambda = (\lambda_j)_j \in c_0, p = \infty)$. Moreover,

$$SN_{(p, r, s)}^L(T) = \inf \|A\| \cdot \|(\lambda_j)_j\|_p \cdot Lip(B),$$

where the infimum is taken over all the above factorizations.

Proof. First note that $D_\lambda = \sum_j \lambda_j e_j \otimes e_j$, where $\sum_j \lambda_j e_j \otimes e_j$ converges in operator norm.

Assume that T is strongly Lipschitz (p, r, s) -nuclear and let $T = \sum_j \lambda_j f_j \otimes v_j$ be a strongly Lipschitz (p, r, s) -nuclear representation of T . Let us define $B : X \rightarrow l_{s^*}$ by $Bx = (f_j(x))_j (x \in X)$. By Lemma 2.4, we get that $B \in Lip_0(X, l_{s^*})$ and $Lip(B) = \omega_{s^*}^{L, w^*}((f_j)_j)$. We define a linear operator $A : l_r \rightarrow E$ by $Ae_j = v_j$ for each j . It follows from [9, Proposition 2.2] that A is bounded and $\|A\| = \omega_{r^*}((v_j)_j)$. Moreover, for each $x \in X$, we have

$$AD_\lambda Bx = A\left(\sum_j \lambda_j \langle e_j, Bx \rangle e_j\right) = \sum_j \lambda_j f_j(x) v_j = Tx.$$

Thus, $T = AD_\lambda B$ and $\inf \|A\| \cdot \|(\lambda_j)_j\|_p \cdot Lip(B) \leq SN_{(p,r,s)}^L(T)$.

Conversely, suppose that T has a factorization $T = AD_\lambda B$ as stated in this theorem. For each j , let $f_j = \langle e_j, B(\cdot) \rangle$. Again by Lemma 2.4, we get that $(f_j)_j$ is Lipschitz- w^* - p -summable and $\omega_{s^*}^{L,w^*}((f_j)_j) = Lip(B)$. Set $v_j = Ae_j$ for each j . Then $\|A\| = \omega_{r^*}((v_j)_j)$. It is easy to see that $Tx = \sum_j \lambda_j f_j(x) v_j$ for all $x \in X$. Hence, we get that $SN_{(p,r,s)}^L(T) \leq \|A\| \cdot \|(\lambda_j)_j\|_p \cdot Lip(B)$. By taking the infimum, we have $SN_{(p,r,s)}^L(T) \leq \inf \|A\| \cdot \|(\lambda_j)_j\|_p \cdot Lip(B)$. \square

An immediate consequence of Theorem 2.8 is that strongly Lipschitz (p, r, s) -nuclear operators are Lipschitz (p, r, s) -nuclear.

Furthermore, if we denote by \mathcal{SN}_p^L the class of strongly Lipschitz p -nuclear operators introduced in [7], then a combination of Theorem 2.8 and [7, Theorem 2.2] yields the following corollary.

Corollary 2.9. $[\mathcal{SN}_{(p,p,1)}^L, SN_{(p,p,1)}^L(\cdot)] = [\mathcal{SN}_p^L, s\nu_p^L(\cdot)]$.

Recall that a Lipschitz map $T : X \rightarrow Y$ is called *Lipschitz p -summing* [10] if there exists a constant $C \geq 0$ such that regardless of $n \in \mathbb{N}$ and regardless of the choice of points $x_1, \dots, x_n, x'_1, \dots, x'_n$ in X and the choice of positive reals a_1, \dots, a_n , one has

$$\left[\sum_{j=1}^n a_j d(Tx_j, Tx'_j)^p \right]^{\frac{1}{p}} \leq C \cdot \sup_{f \in B_{X^\#}} \left[\sum_{j=1}^n a_j |f(x_j) - f(x'_j)|^p \right]^{\frac{1}{p}}.$$

The infimum of such constants is denoted by $\pi_p^L(T)$ and the class of Lipschitz p -summing operators from X to Y is denoted by $\Pi_p^L(X, Y)$. It is noted in [10] that the above definition is the same if one restricts to $a_j = 1$ for each j .

Proposition 2.10. $[\mathcal{SN}_p^L, s\nu_p^L(\cdot)] \subseteq [\Pi_p^L, \pi_p^L(\cdot)]$.

Proof. Suppose that $T : X \rightarrow E$ is strongly Lipschitz p -nuclear and let $\epsilon > 0$. By Corollary 2.9, the operator T has a strongly Lipschitz $(p, p, 1)$ -nuclear representation $T = \sum_j \lambda_j f_j \otimes v_j$, where $(\lambda_j)_j \in \ell_p$, $(f_j)_j \in l_\infty^{L,w^*}(X^\#)$ and $(v_j)_j \in l_{p^*}^w(E)$, such that

$$\|(\lambda_j)_j\|_p \cdot \omega_\infty^{L,w^*}((f_j)_j) \cdot \omega_{p^*}((v_j)_j) \leq (1 + \epsilon) s\nu_p^L(T).$$

We may assume that $\|(\lambda_j)_j\|_p = 1$, $\omega_\infty^{L,w^*}((f_j)_j) = 1$ and then

$$\omega_{p^*}((v_j)_j) \leq (1 + \epsilon) s\nu_p^L(T).$$

Hence, each f_j is in $B_{X^\#}$.

For $x, x' \in X$, we get

$$\begin{aligned} \|Tx - Tx'\| &= \left\| \sum_j \lambda_j (f_j(x) - f_j(x')) v_j \right\| \\ &= \sup_{v^* \in B_{F^*}} \left| \sum_j \lambda_j (f_j(x) - f_j(x')) \langle v^*, v_j \rangle \right| \\ &\leq \sup_{v^* \in B_{F^*}} \left(\sum_j |\lambda_j|^p |f_j(x) - f_j(x')|^p \right)^{\frac{1}{p}} \left(\sum_j |\langle v^*, v_j \rangle|^{p^*} \right)^{\frac{1}{p^*}} \\ &\leq (1 + \epsilon) s\nu_p^L(T) \left(\sum_j |\lambda_j|^p |f_j(x) - f_j(x')|^p \right)^{\frac{1}{p}} \\ &= (1 + \epsilon) s\nu_p^L(T) \left(\int_{B_{X^\#}} |f(x) - f(x')|^p d\mu(f) \right)^{\frac{1}{p}}, \end{aligned}$$

where $\mu = \sum_j |\lambda_j|^p \delta_{f_j}$ is a probability on $B_{X^\#}$.

It follows from [10, Theorem 1] that T is Lipschitz p -summing and $\pi_p^L(T) \leq (1 + \epsilon) s \nu_p^L(T)$. Letting $\epsilon \rightarrow 0$, we get $\pi_p^L(T) \leq s \nu_p^L(T)$. \square

By Theorem 2.8 and a similar argument as [7, Theorem 2.1], we can show that the strongly Lipschitz (p, r, s) -nuclear norm is equal to the (p, r, s) -nuclear norm for a linear operator whose domain is separable.

Proposition 2.11. *Suppose that $1 + \frac{1}{p} \geq \frac{1}{r} + \frac{1}{s}$. Let T be a bounded linear operator from a separable Banach space E into a Banach space F . Then*

$$SN_{(p,r,s)}^L(T) = N_{(p,r,s)}(T).$$

Proposition 2.12. *Suppose that $1 + \frac{1}{p} \geq \frac{1}{r} + \frac{1}{s}$. The canonical mapping*

$$T \mapsto \hat{T}$$

provides an isometric isomorphism from $SN_{(p,r,s)}^L(X, E)$ onto $N_{(p,r,s)}(\mathcal{F}(X), E)$.

Proof. Suppose that $T : X \rightarrow E$ is strongly Lipschitz (p, r, s) -nuclear. It follows from Theorem 2.8 that $T = AD_\lambda B$, where $\lambda = (\lambda_j)_j \in l_p$, $B \in Lip_0(X, l_{s^*})$ and $A \in \mathcal{L}(l_r, E)$. Then

$$\hat{T}\delta = T = AD_\lambda B = AD_\lambda \hat{B}\delta,$$

where $Lip(B) = \|\hat{B}\|$.

Since A is linear, it follows from uniqueness of linearization that $\hat{T} = AD_\lambda \hat{B}$. Moreover, we have

$$N_{(p,r,s)}(\hat{T}) \leq \|A\| \|(\lambda_j)_j\|_p \|\hat{B}\| = \|A\| \|(\lambda_j)_j\|_p Lip(B).$$

This yields

$$N_{(p,r,s)}(\hat{T}) \leq SN_{(p,r,s)}(T).$$

On the other hand, suppose that \hat{T} is (p, r, s) -nuclear and let $\epsilon > 0$. By [15, Theorem 18.1.3], we get that $\hat{T} = AD_\lambda B$, $\lambda = (\lambda_j)_j \in l_p$, $B \in \mathcal{L}(X, l_{s^*})$ and $A \in \mathcal{L}(l_r, E)$, such that

$$\|A\| \|(\lambda_j)_j\|_p \|B\| \leq (1 + \epsilon) N_{(p,r,s)}(\hat{T}).$$

Then $T = \hat{T}\delta = AD_\lambda B\delta$ and hence

$$\begin{aligned} SN_{(p,r,s)}^L(T) &\leq \|A\| \|(\lambda_j)_j\|_p Lip(B\delta) \\ &\leq \|A\| \|(\lambda_j)_j\|_p \|B\| \\ &\leq (1 + \epsilon) N_{(p,r,s)}(\hat{T}). \end{aligned}$$

Letting $\epsilon \rightarrow 0$, we get

$$SN_{(p,r,s)}^L(T) \leq N_{(p,r,s)}(\hat{T}).$$

\square

The following inclusion result is an immediate consequence of [15, Proposition 18.1.5] and Proposition 2.12.

Corollary 2.13. *Suppose that $p_1 \leq p_2, s_1 \leq s_2, r_1 \leq r_2$ and*

$$\left(\frac{1}{s_2^*} - \frac{1}{s_1^*}\right) + \left(\frac{1}{r_2^*} - \frac{1}{r_1^*}\right) \leq \frac{1}{p_1} - \frac{1}{p_2}.$$

Then

$$[SN_{(p_1, r_1, s_1)}^L, SN_{(p_1, r_1, s_1)}^L(\cdot)] \subseteq [SN_{(p_2, r_2, s_2)}^L, SN_{(p_2, r_2, s_2)}^L(\cdot)].$$

Corollary 2.14. If $p_1 \leq p_2$, then $[\mathcal{SN}_{p_1}^L, s\nu_{p_1}^L(\cdot)] \subseteq [\mathcal{SN}_{p_2}^L, s\nu_{p_2}^L(\cdot)]$.

Suppose that $1 + \frac{1}{p} \geq \frac{1}{r} + \frac{1}{s}$. For $T \in Lip_{0F}(X, E)$, we set

$$SN_{(p,r,s)}^{L_0}(T) := \inf \|(\lambda_j)_{j=1}^n\|_p \cdot \omega_{s^*}^{L,w^*}((f_j)_{j=1}^n) \cdot \omega_{r^*}((u_j)_{j=1}^n),$$

where the infimum is extended over all representations $T = \sum_{j=1}^n \lambda_j f_j \otimes u_j$, $f_j \in X^\#$, $u_j \in E$ ($j = 1, 2, \dots, n$).

For $S \in \mathcal{F}(E, F)$, we put

$$N_{(p,r,s)}^0(S) := \inf \|(\lambda_j)_{j=1}^n\|_p \cdot \omega_{s^*}((v_j^*)_{j=1}^n) \cdot \omega_{r^*}((u_j)_{j=1}^n),$$

where the infimum is taken over all finite representations $S = \sum_{j=1}^n \lambda_j v_j^* \otimes u_j$. See [15, p.249] for this definition.

Theorem 2.15. Let $T \in Lip_{0F}(X, E)$. Then

$$SN_{(p,r,s)}^{L_0}(T) = N_{(p,r,s)}^0(\hat{T}).$$

Proof. Let $\epsilon > 0$. Choose a finite representation $T = \sum_{j=1}^n \lambda_j f_j \otimes u_j$ such that

$$\|(\lambda_j)_{j=1}^n\|_p \omega_{s^*}^{L,w^*}((f_j)_{j=1}^n) \omega_{r^*}((u_j)_{j=1}^n) \leq (1 + \epsilon) SN_{(p,r,s)}^{L_0}(T).$$

Define linear operators $A : l_r^n \rightarrow E$ by $Ae_j = u_j$ for each j and $B : X \rightarrow l_{s^*}^n$ by $Bx = (f_j(x))_{j=1}^n$ for all $x \in X$. Then $\|A\| = \omega_{r^*}((u_j)_{j=1}^n)$, $Lip(B) = \omega_{s^*}^{L,w^*}((f_j)_{j=1}^n)$ and $T = AD_\lambda B$, where $\lambda = (\lambda_j)_{j=1}^n$. Hence, $T = AD_\lambda B = AD_\lambda \hat{B} \delta_X$. By uniqueness of linearization, we get $\hat{T} = AD_\lambda \hat{B}$. For each j , we set $g_j = (\hat{B})^* e_j \in (\mathcal{F}(X))^*$ and $u_j = Ae_j$. Then $\omega_{s^*}((g_j)_{j=1}^n) = \|\hat{B}\|$, $\omega_{r^*}((u_j)_{j=1}^n) = \|A\|$. It is easy to see that $\hat{T} = \sum_{j=1}^n \lambda_j g_j \otimes u_j$ and

$$\begin{aligned} N_{(p,r,s)}^0(\hat{T}) &\leq \|(\lambda_j)_{j=1}^n\|_p \omega_{s^*}((g_j)_{j=1}^n) \omega_{r^*}((u_j)_{j=1}^n) \\ &= \|(\lambda_j)_{j=1}^n\|_p \|\hat{B}\| \|A\| \\ &= \|(\lambda_j)_{j=1}^n\|_p Lip(B) \|A\| \\ &= \|(\lambda_j)_{j=1}^n\|_p \omega_{s^*}^{L,w^*}((f_j)_{j=1}^n) \omega_{r^*}((u_j)_{j=1}^n) \\ &\leq (1 + \epsilon) SN_{(p,r,s)}^{L_0}(T). \end{aligned}$$

Letting $\epsilon \rightarrow 0$, we get $N_{(p,r,s)}^0(\hat{T}) \leq SN_{(p,r,s)}^{L_0}(T)$.

Conversely, let $\epsilon > 0$. We choose a finite representation $\hat{T} = \sum_{j=1}^n \lambda_j v_j^* \otimes u_j$, $v_j^* \in (\mathcal{F}(X))^*$, $u_j \in E$ ($j = 1, 2, \dots$) such that

$$\|(\lambda_j)_{j=1}^n\|_p \omega_{s^*}((v_j^*)_{j=1}^n) \omega_{r^*}((u_j)_{j=1}^n) \leq (1 + \epsilon) N_{(p,r,s)}^0(\hat{T}).$$

Similarly, we define two linear operators $R : l_r^n \rightarrow E$ by $Re_j = u_j$ for each j and $S : \mathcal{F}(X) \rightarrow l_{s^*}^n$ by $Sv = (v_j^*, v)_{j=1}^n$ for all $v \in \mathcal{F}(X)$. Then $\|R\| = \omega_{r^*}((u_j)_{j=1}^n)$, $\|S\| = \omega_{s^*}((v_j^*)_{j=1}^n)$ and $\hat{T} = RD_\lambda S$. Thus, we have $T = \hat{T} \delta_X = RD_\lambda S \delta_X$. For each j , define

$f_j(x) = \langle S\delta_X(x), e_j \rangle (x \in X)$. Clearly, each f_j is in $X^\#$ and $\omega_{s^*}^{L,w^*}((f_j)_{j=1}^n) = \text{Lip}(S\delta_X)$.

It is easy to see that $T = \sum_{j=1}^n \lambda_j f_j \otimes u_j$ and

$$\begin{aligned} SN_{(p,r,s)}^{L_0}(T) &\leq \|(\lambda_j)_{j=1}^n\|_p \omega_{s^*}^{L,w^*}((f_j)_{j=1}^n) \omega_{r^*}((u_j)_{j=1}^n) \\ &= \|(\lambda_j)_{j=1}^n\|_p \text{Lip}(S\delta_X) \|R\| \\ &\leq \|(\lambda_j)_{j=1}^n\|_p \|S\| \|R\| \\ &\leq \|(\lambda_j)_{j=1}^n\|_p \omega_{s^*}((v_j^*)_{j=1}^n) \omega_{r^*}((u_j)_{j=1}^n) \\ &\leq (1 + \epsilon) N_{(p,r,s)}^0(\widehat{T}). \end{aligned}$$

Again letting $\epsilon \rightarrow 0$, we get $SN_{(p,r,s)}^{L_0}(T) \leq N_{(p,r,s)}^0(\widehat{T})$. \square

Corollary 2.16. Suppose that $X^\#$ or E has the metric approximation property. Then

$$SN_{(p,r,s)}^{L_0}(T) = SN_{(p,r,s)}^L(T), \text{ for all } T \in \text{Lip}_{0F}(X, E).$$

Proof. Let $T \in \text{Lip}_{0F}(X, E)$. Combining Theorem 2.15, [13, Proposition 2.4], Proposition 2.12 and [15, Proposition 18.1.15], we get

$$SN_{(p,r,s)}^{L_0}(T) = N_{(p,r,s)}^0(\widehat{T}) = N_{(p,r,s)}(\widehat{T}) = SN_{(p,r,s)}^L(T).$$

The first equality follows from Theorem 2.15, the second from [13, Proposition 2.4] and [15, Proposition 18.1.15], the third from Proposition 2.12. \square

Recall that for $0 < \beta \leq 1$, a non-negative absolutely homogeneous functional m defined on a vector space V is called a β -norm if m vanishes only at 0 and $m(x+y)^\beta \leq m(x)^\beta + m(y)^\beta$ for all $x, y \in V$. The following notion of β -Banach ideal of Lipschitz maps is a slight generalization of Banach ideal of Lipschitz maps introduced in [5].

Definition 2.17. [5] Let $0 < \beta \leq 1$. By a *generic Lipschitz operator β -Banach ideal* \mathcal{A} , we mean an assignment, for each pointed metric space X and each Banach space E , of a linear subspace $\mathcal{A}(X, E)$ of $\text{Lip}_0(X, E)$ together with a non-negative function defined on $\mathcal{A}(X, E)$ and denoted by $\|\cdot\|_{\mathcal{A}}$, satisfying the following properties:

- (i) $\text{Lip}(T) \leq \|T\|_{\mathcal{A}}$ for all $T \in \mathcal{A}(X, E)$.
- (ii) $\|\cdot\|_{\mathcal{A}}$ is a complete, β -norm on $\mathcal{A}(X, E)$.
- (iii) If $f \in X^\#$ and $u \in E$, then $f \otimes u$ is in $\mathcal{A}(X, E)$ with $\|f \otimes u\|_{\mathcal{A}} = \text{Lip}(f)\|u\|$.
- (iv) If Y is a pointed metric space, F is a Banach space, $R \in \text{Lip}_0(Y, X)$, $T \in \mathcal{A}(X, E)$, and $S \in \mathcal{L}(E, F)$, then $STR \in \mathcal{A}(Y, F)$ and $\|STR\|_{\mathcal{A}} \leq \|S\| \|T\|_{\mathcal{A}} \text{Lip}(R)$.

Clearly, The generic Lipschitz operator 1-Banach ideal is precisely the generic Lipschitz operator Banach ideal introduced in [5].

Theorem 2.18. Let $\frac{1}{\beta} := \frac{1}{p} + \frac{1}{s^*} + \frac{1}{r^*} \geq 1$. Then $[\mathcal{SN}_{(p,r,s)}^L, SN_{(p,r,s)}^L(\cdot)]$ is a generic Lipschitz operator β -Banach ideal.

Proof. Let us fix a pointed metric space X and a Banach space E .

- (i) Let $T \in \mathcal{SN}_{(p,r,s)}^L(X, E)$. It follows from Theorem 2.8 that $\text{Lip}(T) \leq SN_{(p,r,s)}^L(T)$.
- (ii) Combining [15, Theorem 18.1.2] with Proposition 2.12, we get that $SN_{(p,r,s)}^L(\cdot)$ is a complete, β -norm on $\mathcal{SN}_{(p,r,s)}^L(X, E)$.
- (iii) is trivial.
- (iv) follows from Theorem 2.8. \square

Corollary 2.19. $[\mathcal{SN}_{(1,1,1)}^L, SN_{(1,1,1)}^L(\cdot)]$ is the smallest generic Lipschitz operator Banach ideal.

Proof. Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be any generic Lipschitz operator Banach ideal. Let $T \in \mathcal{SN}_{(1,1,1)}^L(X, E)$ and $\epsilon > 0$. It follows from Corollary 2.9 that

$$T = \sum_j f_j \otimes u_j, \quad (f_j)_j \subseteq X^\#, (u_j)_j \subseteq E, \left(\sum_j \text{Lip}(f_j)\right) \left(\sup_j \|u_j\|\right) < \infty$$

such that

$$\left(\sum_j \text{Lip}(f_j)\right) \left(\sup_j \|u_j\|\right) \leq (1 + \epsilon) \mathcal{SN}_{(1,1,1)}^L(T).$$

Let $T_n = \sum_{j=1}^n f_j \otimes u_j$ for each n . Then, for $m < n$, we have

$$\begin{aligned} \|T_n - T_m\|_{\mathcal{A}} &= \left\| \sum_{j=m+1}^n f_j \otimes u_j \right\|_{\mathcal{A}} \\ &\leq \sum_{j=m+1}^n \|f_j \otimes u_j\|_{\mathcal{A}} \\ &= \sum_{j=m+1}^n \text{Lip}(f_j) \|u_j\| \rightarrow 0 \quad (m \rightarrow \infty). \end{aligned}$$

Hence, $(T_n)_n$ is a $\|\cdot\|_{\mathcal{A}}$ -Cauchy sequence in $\mathcal{A}(X, E)$. It follows from the completeness of $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ that $T_n \xrightarrow{\|\cdot\|_{\mathcal{A}}} S$ for some $S \in \mathcal{A}(X, E)$. In particular, $\text{Lip}(T_n - S) \rightarrow 0$. It follows from uniqueness of the limit that $T = S \in \mathcal{A}(X, E)$. Moreover,

$$\begin{aligned} \|T\|_{\mathcal{A}} &= \lim_{n \rightarrow \infty} \|T_n\|_{\mathcal{A}} \\ &\leq \lim_{n \rightarrow \infty} \sum_{j=1}^n \text{Lip}(f_j) \|u_j\| \\ &= \left(\sum_j \text{Lip}(f_j)\right) \left(\sup_j \|u_j\|\right) \\ &\leq (1 + \epsilon) \mathcal{SN}_{(1,1,1)}^L(T). \end{aligned}$$

Letting $\epsilon \rightarrow 0$, we get

$$\|T\|_{\mathcal{A}} \leq \mathcal{SN}_{(1,1,1)}^L(T).$$

□

Let $\frac{1}{p} \leq \frac{1}{r} + \frac{1}{s}$. Recall that an operator $S \in \mathcal{L}(E, F)$ is called *absolutely* (p, r, s) -*summing* ([15, p.228]) if there is a constant $C \geq 0$ such that

$$\|(\langle v_j^*, Su_j \rangle)_{j=1}^n\|_p \leq C \omega_r((u_j)_{j=1}^n) \cdot \omega_s((v_j^*)_{j=1}^n),$$

for all finite sequences $(u_j)_{j=1}^n$ in E and $(v_j^*)_{j=1}^n$ in F^* .

We put

$$\pi_{(p,r,s)}(S) := \inf C.$$

The class of all absolutely (p, r, s) -summing operators from E to F is denoted by $\Pi_{(p,r,s)}(E, F)$.

Theorem 2.20. $[\mathcal{SN}_{(p,r,s)}^L(X, E)]^*$ is isometrically isomorphic to $\Pi_{(p^*, s^*, r^*)}(X^\#, E^*)$.

Proof. Let us define two linear operators

$$V_1 : \Pi_{(p^*, s^*, r^*)}(X^\#, E^*) \rightarrow [\mathcal{SN}_{(p,r,s)}^L(X, E)]^*$$

by

$$\langle V_1 S, T \rangle = \sum_j \lambda_j \langle S f_j, u_j \rangle, \quad S \in \Pi_{(p^*, s^*, r^*)}(X^\#, E^*), \quad T = \sum_j \lambda_j f_j \otimes u_j.$$

and

$$V_2 : [\mathcal{SN}_{(p, r, s)}^L(X, E)]^* \rightarrow \Pi_{(p^*, s^*, r^*)}(X^\#, E^*)$$

by

$$\langle (V_2 \phi)(f), u \rangle = \phi(f \otimes u), \quad \phi \in (\mathcal{SN}_{(p, r, s)}^L(X, E))^*, \quad f \in X^\#, \quad u \in E.$$

It can be checked that $\|V_1\| \leq 1, \|V_2\| \leq 1$ and

$$V_2 V_1 = I_{\Pi_{(p^*, s^*, r^*)}(X^\#, E^*)}, \quad V_1 V_2 = I_{[\mathcal{SN}_{(p, r, s)}^L(X, E)]^{**}}.$$

Hence, V_1, V_2 are linear, surjective isometries. \square

Let \mathcal{U} be an operator ideal. The dual of \mathcal{U} is defined by

$$\mathcal{U}^{dual}(E, F) := \{S \in \mathcal{L}(E, F) : S^* \in \mathcal{U}(F^*, E^*)\},$$

for each pair of Banach spaces E, F .

The Lipschitz dual of an operator ideal \mathcal{U} is defined by K. Saadi [16] as follows.

$$\mathcal{U}^{Lip-dual}(X, E) := \{T \in Lip_0(X, E) : T^t \in \mathcal{U}(E^*, X^\#)\},$$

for a pointed metric space X and a Banach space E .

If $[\mathcal{U}, \|\cdot\|_{\mathcal{U}}]$ is a normed (Banach) operator ideal, we define

$$\|T\|_{\mathcal{U}^{Lip-dual}} = \|T^t\|_{\mathcal{U}}.$$

Then $[\mathcal{U}^{Lip-dual}, \|\cdot\|_{\mathcal{U}^{Lip-dual}}]$ becomes a normed (Banach) Lipschitz ideal.

Recall that the regular hull \mathcal{U}^{reg} of an operator ideal \mathcal{U} is defined by

$$\mathcal{U}^{reg}(E, F) := \{S \in \mathcal{L}(E, F) : J_F S \in \mathcal{U}(E, F^{**})\},$$

for each pair of Banach spaces E, F .

We introduce the Lipschitz analogue of the regular hull.

Definition 2.21. Let $0 < \beta \leq 1$ and $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be a generic Lipschitz operator β -Banach ideal. We set

$$\mathcal{A}^{Lip-reg}(X, E) := \{T \in Lip_0(X, E) : J_E T \in \mathcal{A}(X, E^{**})\},$$

for a pointed metric space X and a Banach space E .

Moreover, we set

$$\|T\|_{\mathcal{A}^{Lip-reg}} := \|J_E T\|_{\mathcal{A}}, \quad \text{for all } T \in \mathcal{A}^{Lip-reg}(X, E).$$

It is straightforward that $[\mathcal{A}^{Lip-reg}, \|\cdot\|_{\mathcal{A}^{Lip-reg}}]$ is a generic Lipschitz operator β -Banach ideal. Clearly, $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}] \subseteq [\mathcal{A}^{Lip-reg}, \|\cdot\|_{\mathcal{A}^{Lip-reg}}]$.

Definition 2.22. Let $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ be a generic Lipschitz operator β -Banach ideal. We say that $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is *Lipschitz regular* if

$$[\mathcal{A}, \|\cdot\|_{\mathcal{A}}] = [\mathcal{A}^{Lip-reg}, \|\cdot\|_{\mathcal{A}^{Lip-reg}}].$$

It is easy to see that the Lipschitz operator Banach ideal $[\Pi_{(p, r, s)}^L, \pi_{(p, r, s)}^L(\cdot)]$ is Lipschitz regular. See [3] for the definition of $[\Pi_{(p, r, s)}^L, \pi_{(p, r, s)}^L(\cdot)]$.

Proposition 2.23. Let $[\mathcal{U}, \|\cdot\|_{\mathcal{U}}]$ be a Banach operator ideal. Then $[\mathcal{U}^{Lip-dual}, \|\cdot\|_{\mathcal{U}^{Lip-dual}}]$ is Lipschitz regular.

Proof. Let $T \in (\mathcal{U}^{Lip-dual})^{Lip-reg}(X, E)$. Then $(J_ET)^t \in \mathcal{U}(E^{***}, X^\#)$ and hence we get

$$T^t = T^t J_E^* J_{E^*} = (J_ET)^t J_{E^*} \in \mathcal{U}(E^*, X^\#).$$

Hence, $T \in \mathcal{U}^{Lip-dual}(X, E)$. Moreover, we have

$$\begin{aligned} \|T\|_{\mathcal{U}^{Lip-dual}} &= \|T^t\|_{\mathcal{U}} \\ &= \|(J_ET)^t J_{E^*}\|_{\mathcal{U}} \\ &\leq \|(J_ET)^t\|_{\mathcal{U}} \\ &= \|J_ET\|_{\mathcal{U}^{Lip-dual}} \\ &= \|T\|_{(\mathcal{U}^{Lip-dual})^{Lip-reg}}. \end{aligned}$$

This completes the proof. \square

Theorem 2.24. *Let $1 \leq r < \infty$ and $1 < s \leq \infty$. Then*

$$\mathcal{N}_{(p,r,s)}^{Lip-dual} = [\mathcal{SN}_{(p,s,r)}^L]^{Lip-reg}.$$

Proof. Let us fix a pointed metric space X and a Banach space E .

Let $T \in \mathcal{N}_{(p,r,s)}^{Lip-dual}(X, E)$ and $\epsilon > 0$. Then T^t is (p, r, s) -nuclear. By [15, Theorem 18.1.3], there exists a commutative diagram

$$\begin{array}{ccc} E^* & \xrightarrow{T^t} & X^\# \\ B \downarrow & & \uparrow A \\ l_{s^*} & \xrightarrow{D_\lambda} & l_r \end{array}$$

where $A \in \mathcal{L}(l_r, X^\#)$, $B \in \mathcal{L}(E^*, l_{s^*})$ and $\lambda = (\lambda_j)_j \in l_p$ such that

$$\|A\| \|\lambda\|_p \|B\| \leq (1 + \epsilon) N_{(p,r,s)}(T^t).$$

By dualizing, we get

$$\begin{array}{ccc} (X^\#)^* & \xrightarrow{(T^t)^*} & E^{**} \\ A^* \downarrow & & \uparrow B^* \\ l_{r^*} & \xrightarrow{\tilde{D}_\lambda} & l_s \end{array}$$

where $\tilde{D}_\lambda = (D_\lambda)^*$ is the diagonal operator from l_{r^*} to l_s defined by $\lambda = (\lambda_j)_j$.

Hence, we have

$$(T^t)^* \delta_X = (A D_\lambda B)^* \delta_X = B^* \tilde{D}_\lambda A^* \delta_X.$$

It is easy to see that $(T^t)^* \delta_X = J_ET$. By Theorem 2.8, we get that J_ET is strongly Lipschitz (p, s, r) -nuclear and

$$\begin{aligned} \|T\|_{(\mathcal{SN}_{(p,s,r)}^L)^{Lip-reg}} &= \mathcal{SN}_{(p,s,r)}^L(J_ET) \\ &\leq Lip(A^* \delta_X) \|\lambda\|_p \|B^*\| \\ &\leq \|A\| \|\lambda\|_p \|B\| \\ &\leq (1 + \epsilon) N_{(p,r,s)}(T^t) \\ &= (1 + \epsilon) \|T\|_{\mathcal{N}_{(p,r,s)}^{Lip-dual}}. \end{aligned}$$

Letting $\epsilon \rightarrow 0$, we get

$$\|T\|_{(\mathcal{SN}_{(p,s,r)}^L)^{Lip-reg}} \leq \|T\|_{\mathcal{N}_{(p,r,s)}^{Lip-dual}}.$$

□

3. LIPSCHITZ r -COMPACT OPERATORS

Let us recall that a bounded linear operator $S : E \rightarrow F$ is called r -compact [15, Definition 18.3.1] if it belongs to

$$[\mathcal{K}_r, k_r] := [\mathcal{N}_{(\infty, r, r^*)}^L, N_{(\infty, r, r^*)}^L(\cdot)].$$

Definition 3.1. Let $1 \leq r \leq \infty$ and $T \in Lip_0(X, E)$. We say that $T : X \rightarrow E$ is *Lipschitz r -compact* if it belongs to

$$[\mathcal{K}_r^L, k_r^L] := [\mathcal{SN}_{(\infty, r, r^*)}^L, SN_{(\infty, r, r^*)}^L(\cdot)].$$

Recall that a Lipschitz map $T : X \rightarrow E$ is called *Lipschitz compact* [13] if its Lipschitz image $\{\frac{Tx - Tx'}{d(x, x')} : x, x' \in X, x \neq x'\}$ is relatively norm compact in E . The set of all Lipschitz compact operators from X to E is denoted by $Lip_{0K}(X, E)$.

Theorem 3.2. Let $1 \leq r \leq \infty$ and $T \in Lip_0(X, E)$. The following statements are equivalent:

- (i) T is Lipschitz r -compact.
- (ii) T has a factorization $T = RS$, where $S \in Lip_{0K}(X, l_r)$ and $R \in \mathcal{K}(l_r, E)$.
- (iii) T admits a factorization $T = RS$, where $S \in Lip_{0K}(X, l_r)$ and $R \in \mathcal{L}(l_r, E)$.
- (iv) For $1 < r \leq \infty$, T has a factorization $T = RS$, where $S \in Lip_0(X, l_r)$ and $R \in \mathcal{K}(l_r, E)$.

Moreover,

$$k_r^L(T) = \inf Lip(S)\|R\|,$$

where the infimum extends over all factorizations as in (ii), (iii) or (iv).

Proof. (i) \Rightarrow (ii). Suppose that $T : X \rightarrow E$ is Lipschitz r -compact and let $\epsilon > 0$. It follows from Theorem 2.8 that $T = AD_\lambda B$, $\lambda = (\lambda_j)_j \in c_0$, $B \in Lip_0(X, l_r)$ and $A \in \mathcal{L}(l_r, E)$ such that

$$Lip(B) \sup_j |\lambda_j| \|A\| \leq (1 + \epsilon) k_r^L(T).$$

Choose a sequence $(\xi_j)_j$ such that $1 \leq \xi_j \rightarrow \infty$, $(\xi_j \lambda_j)_j \in c_0$ and

$$\sup_j |\xi_j \lambda_j| \leq (1 + \epsilon) \sup_j |\lambda_j|.$$

Put $\tau = (\xi_j \lambda_j)_j$ and $\alpha_j = \frac{1}{\xi_j}$ for each j .

Define an operator $C : l_r \rightarrow l_r$ by $(t_j)_j \mapsto (\sqrt{\alpha_j} t_j)_j$. It is easy to see that the operator C is compact, $\|C\| \leq 1$ and $D_\lambda = CD_\tau C$. Since C is linear and compact, CB is Lipschitz compact. Let $R = ACD_\tau$ and $S = CB$.

Thus, we get

$$T = AD_\lambda B = ACD_\tau CB = RS$$

and

$$\begin{aligned} \|R\| Lip(S) &\leq \|A\| \sup_j |\xi_j \lambda_j| Lip(B) \\ &\leq \|A\| (1 + \epsilon) \sup_j |\lambda_j| Lip(B) \\ &\leq (1 + \epsilon)^2 k_r^L(T). \end{aligned}$$

Letting $\epsilon \rightarrow 0$, we get $\inf Lip(S)\|R\| \leq k_r^L(T)$.

The implication (ii) \Rightarrow (iii) is trivial.

(iii) \Rightarrow (i). Assume that T has a factorization $T = RS$, $S : X \rightarrow l_r$ is Lipschitz compact and $R : l_r \rightarrow E$ is linear, bounded. It follows from [13, Proposition 2.1]

that \widehat{S} is compact and $\|\widehat{S}\| = \text{Lip}(S)$. Hence $T = RS = R\widehat{S}\delta_X$. By uniqueness of linearization, we get $\widehat{T} = R\widehat{S} : \mathcal{F}(X) \xrightarrow{\widehat{S}} l_r \xrightarrow{R} E$. By [15, Theorem 18.3.2], the operator \widehat{T} is r -compact and $k_r(\widehat{T}) \leq \|R\|\|\widehat{S}\|$. It follows from Proposition 2.12 that T is Lipschitz r -compact and

$$k_r^L(T) = k_r(\widehat{T}) \leq \|R\|\|\widehat{S}\| = \|R\|\text{Lip}(S).$$

Hence, $k_r^L(T) \leq \inf \|R\|\text{Lip}(S)$.

The implication (i) \Rightarrow (iv) is obvious.

(iv) \Rightarrow (i). Take any factorization $T = RS$, where $S \in \text{Lip}_0(X, l_r)$ and $R \in \mathcal{K}(l_r, E)$. Let $\epsilon > 0$. By [11, Proposition 2.2 (b)], we get $\lambda = (\lambda_j)_j \in c_0, \|\lambda\| = 1$ and an operator $\widetilde{R} \in \mathcal{K}(l_r, E)$ such that $R = \widetilde{R}D_\lambda$ and $\|\widetilde{R}\| \leq (1 + \epsilon)\|R\|$. Hence $T = \widetilde{R}D_\lambda S$ is Lipschitz r -compact and

$$k_r^L(T) \leq \|\widetilde{R}\|\|\lambda\|\text{Lip}(S) \leq (1 + \epsilon)\|R\|\text{Lip}(S).$$

Letting $\epsilon \rightarrow 0$, we are done. \square

Corollary 3.3. Let $1 \leq r \leq \infty$ and $T \in \text{Lip}_0(X, l_r)$. Then T is Lipschitz r -compact if and only if T is Lipschitz compact. Moreover, $k_r^L(T) = \text{Lip}(T)$.

Recall that a sequence $(u_n)_n \in l_p^w(E)$ is *unconditionally p -summable* if

$$\sup_{x^* \in B_{E^*}} \left(\sum_{j=m}^{\infty} |\langle x^*, x_j \rangle|^p \right)^{\frac{1}{p}} \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

The space of unconditionally p -summable sequences in E is denoted by $l_p^u(E)$. For $p = \infty$, the space $l_\infty^u(X)$ is identical to $c_0(X)$, the space of all norm null sequences in X . J. H. Fourie and J. Swart ([11, Theorem 1.4(b)]) proved that the correspondence $T \mapsto (Te_j)_j$ provides an isometric isomorphism of $\mathcal{K}(l_p^*, E)$ onto $l_p^u(E)$ when $1 < p < \infty$; For $p = 1$, the isometric isomorphism is from $\mathcal{K}(c_0, E)$ onto $l_1^u(E)$; Nevertheless, the correspondence $T \mapsto (Te_j)_j$ does not provide an isometric isomorphism of $\mathcal{K}(l_1, E)$ onto $c_0(E)$. Indeed, it is easy to see that a bounded linear operator T from l_1 to a Banach space E is compact if and only if the sequence $(Te_j)_j$ is relatively norm compact (see [8, p.114]). A sequence $(u_n^*)_n$ in E^* is called *w^* -unconditionally p -summable* if $(u_n^*)_n$ is in $l_p^{w^*}(E^*)$ and

$$\sup_{u \in B_E} \left(\sum_{j=n}^{\infty} |\langle u_j^*, u \rangle|^p \right)^{\frac{1}{p}} \rightarrow 0 \quad (n \rightarrow \infty).$$

The space of all w^* -unconditionally p -summable sequences in E^* is denoted by $l_p^{w^*,u}(E^*)$. It is proved in [11] that $l_p^{w^*,u}(E^*)$ is isometrically isomorphic to $\mathcal{K}(E, l_p)(\mathcal{K}(E, c_0)$ for $p = \infty$).

Definition 3.4. let $1 \leq p \leq \infty$. We say that a sequence $(f_j)_j$ in $X^\#$ is *Lipschitz w^* -unconditionally p -summable* if $(f_j)_j$ is in $l_p^{L,w^*}(X^\#)$ and

$$\sup_{\substack{x \neq x' \\ x, x' \in X}} \frac{(\sum_{j=n}^{\infty} |f_j(x) - f_j(x')|^p)^{\frac{1}{p}}}{d(x, x')} \rightarrow 0 \quad (n \rightarrow \infty).$$

The set of all Lipschitz w^* -unconditionally p -summable sequences in $X^\#$ is denoted by $l_p^{L,w^*,u}(X^\#)$.

The following lemma follows immediately from [8, Exercise 6, p.6].

Lemma 3.5. *The canonical correspondence*

$$T \mapsto (\langle e_j^*, T(\cdot) \rangle)_j$$

provides an isometric isomorphism of $Lip_{0K}(X, l_p)(Lip_{0K}(X, c_0)$ for $p = \infty$) onto $l_p^{L, w^*, u}(X^\#)$.

Proposition 3.6. $[l_p^{L, w^*, u}(X^\#), \omega_p^{L, w^*}(\cdot)] = [l_p^u(X^\#), \omega_p(\cdot)]$.

Proof. For $p = \infty$, it is obvious that $l_p^{L, w^*, u}(X^\#) = c_0(X^\#) = l_\infty^u(X^\#)$. Hence we only consider the case $1 \leq p < \infty$.

Take any $(f_j)_j$ in $l_p^{L, w^*, u}(X^\#)$. Let us define a Lipschitz map $T : X \rightarrow l_p$ by $Tx = (f_j(x))_j (x \in X)$. It follows from Lemma 3.5 that T is Lipschitz compact and $Lip(T) = \omega_p^{L, w^*}((f_j)_j)$. By [13, Proposition 2.1], T^t is compact and $\|T^t\| = Lip(T)$. It is easy to see that $T^t e_j^* = f_j$ for each j . For $p = 1$, the restriction of T^t to c_0 is also compact. By [11, Theorem 1.4(b)], we get $(f_j)_j \in l_p^u(X^\#)$ and $\omega_p((f_j)_j) \leq \|T^t\| = \omega_p^{L, w^*}((f_j)_j)$.

Conversely, choose any $(f_j)_j$ in $l_p^u(X^\#)$. Define a linear operator $T : l_{p^*} \rightarrow X^\# (c_0 \rightarrow X^\#$ for $p = 1)$ by $T e_j^* = f_j$ for each j . Again by [11, Theorem 1.4(b)], we get that T is compact and $\|T\| = \omega_p((f_j)_j)$. Since T^* is also compact, the set $\{\frac{T^* \delta_X(x) - T^* \delta_X(x')}{d(x, x')} : x \neq x', x, x' \in X\}$ is relatively norm compact in l_p . Furthermore, $\langle e_j^*, T^* \delta_X(x) \rangle = f_j(x)$ for all $x \in X$. By the well-known characterization of relatively norm compact sets in l_p (see [8, Exercise 6, p.6]), we get

$$\sup_{\substack{x \neq x' \\ x, x' \in X}} \frac{(\sum_{j=n}^{\infty} |f_j(x) - f_j(x')|^p)^{\frac{1}{p}}}{d(x, x')} = \sup_{\substack{x \neq x' \\ x, x' \in X}} (\sum_{j=n}^{\infty} |\langle e_j^*, \frac{T^* \delta_X(x) - T^* \delta_X(x')}{d(x, x')} \rangle|^p)^{\frac{1}{p}} \rightarrow 0 \quad (n \rightarrow \infty).$$

Thus $(f_j)_j$ is in $l_p^{L, w^*, u}(X^\#)$ and

$$\omega_p^{L, w^*}((f_j)_j) = \sup_{\substack{x \neq x' \\ x, x' \in X}} \frac{\|T^* \delta_X(x) - T^* \delta_X(x')\|}{d(x, x')} \leq \|T\| = \omega_p((f_j)_j).$$

This completes the proof. \square

Combining Theorem 3.2 and Lemma 3.5, we get the following corollary.

Corollary 3.7. Let $1 \leq r \leq \infty$ and $T \in Lip_0(X, E)$. The following statements are equivalent:

- (i) T is Lipschitz r -compact.
- (ii) $T = \sum_j f_j \otimes u_j$, where $(f_j)_j \in l_r^{L, w^*, u}(X^\#)$, $(u_j)_j \in l_{r^*}^u(E)$.
- (iii) $T = \sum_j f_j \otimes u_j$, where $(f_j)_j \in l_r^{L, w^*, u}(X^\#)$, $(u_j)_j \in l_{r^*}^w(E)$.
- (vi) For $r > 1$, $T = \sum_j f_j \otimes u_j$, where $(f_j)_j \in l_r^{L, w^*}(X^\#)$, $(u_j)_j \in l_{r^*}^u(E)$.

Furthermore,

$$k_r^L(T) = \inf \omega_r^{L, w^*}((f_j)_j) \cdot \omega_r((u_j)_j),$$

with the infimum taken over all representations of T as in (ii), (iii) and (vi).

Proof. Only the implication (i) \Rightarrow (ii) for $r = 1$ needs to be showed.

Let $\epsilon > 0$. Choose a representation $T = \sum_j \lambda_j f_j \otimes u_j$, where $(\lambda_j)_j \in c_0$, $(f_j)_j \in l_1^{L, w^*}(X^\#)$ and $(u_j)_j \in l_\infty^w(E)$ such that

$$\omega_1^{L, w^*}((f_j)_j) \cdot \sup_j |\lambda_j| \cdot \sup_j \|u_j\| \leq (1 + \epsilon) k_1^L(T).$$

Let $B : X \rightarrow l_1$ be the Lipschitz mapping corresponding to $(f_j)_j$ as in Lemma 2.4. Choose a sequence $(\xi_j)_j$ such that $1 \leq \xi_j \rightarrow \infty$, $(\xi_j \lambda_j)_j \in c_0$ and

$$\sup_j |\xi_j \lambda_j| \leq (1 + \epsilon) \sup_j |\lambda_j|.$$

Set $\alpha_j = \frac{1}{\xi_j}$ for each j and $\tau = (\xi_j \lambda_j)_j$. Let us define a linear operator $b : l_1 \rightarrow l_1$ by $b((t_j)_j) = (\sqrt{\alpha_j} t_j)_j$, $(t_j)_j \in l_1$. Then the operator b is compact, $\|b\| \leq 1$ and $D_\lambda = bD_\tau b$. For each j , define $g_j \in X^\#$ by $g_j(x) = \langle e_j^*, bBx \rangle$, $x \in X$. It follows from Lemma 3.5 that $(g_j)_j$ is in $l_1^{L,w*,u}(X^\#)$ and $\omega_1^{L,w*}((g_j)_j) = \text{Lip}(bB)$. It is easy to check that $T = \sum_j g_j \otimes (\sqrt{\xi_j} \lambda_j u_j)$, $(\sqrt{\xi_j} \lambda_j u_j)_j \in c_0(E)$. Moreover, we have

$$\begin{aligned} \omega_1^{L,w*}((g_j)_j) \sup_j \|\sqrt{\xi_j} \lambda_j u_j\| &= \text{Lip}(bB) \sup_j \|\sqrt{\xi_j} \lambda_j u_j\| \\ &\leq \text{Lip}(B) \sup_j \|\sqrt{\xi_j} \lambda_j u_j\| \\ &= \omega_1^{L,w*}((f_j)_j) \sup_j \|\sqrt{\xi_j} \lambda_j u_j\| \\ &\leq \omega_1^{L,w*}((f_j)_j) \sup_j \|\xi_j \lambda_j u_j\| \\ &\leq \omega_1^{L,w*}((f_j)_j) (1 + \epsilon) \sup_j |\lambda_j| \sup_j \|u_j\| \\ &\leq (1 + \epsilon)^2 k_1^L(T). \end{aligned}$$

Letting $\epsilon \rightarrow 0$, we are done. □

4. STRONGLY LIPSCHITZ (p, r, s) -INTEGRAL OPERATORS

Let $\frac{1}{p} + \frac{1}{r} + \frac{1}{s} = 1$. Recall in [14, Definiton 2.1, p.69] that a bounded linear operator $S : E \rightarrow F$ is said to be (r, s) -factorisable if it admits a factorization $J_F S : E \xrightarrow{B} L_r(\mu) \xrightarrow{M_g} L_{s^*}(\mu) \xrightarrow{A} F^{**}$, where M_g is the multiplication operator induced by $g \in L_p(\mu)$, $A \in \mathcal{L}(L_{s^*}(\mu), F^{**})$, $B \in \mathcal{L}(E, L_r(\mu))$ and μ is a measure. Here, the term “factorisable” comes from the French paper [14]. Note that $\frac{1}{p} + \frac{1}{r} + \frac{1}{s} = 1$ is equivalent to $1 + \frac{1}{p} = \frac{1}{r^*} + \frac{1}{s^*}$. For the sake of notation consistency throughout the paper, we replace the r, s^* in the above definition by s^*, r and prefer to adopting the following equivalent format of (r, s) -factorisable operators at the present paper:

For $1 + \frac{1}{p} = \frac{1}{r} + \frac{1}{s}$, a bounded linear operator $S : E \rightarrow F$ is said to be (r, s) -factorisable if it admits a factorization $J_F S : E \xrightarrow{B} L_{s^*}(\mu) \xrightarrow{M_g} L_r(\mu) \xrightarrow{A} F^{**}$, where M_g is the multiplication operator induced by $g \in L_p(\mu)$, $A \in \mathcal{L}(L_r(\mu), F^{**})$, $B \in \mathcal{L}(E, L_{s^*}(\mu))$ and μ is a measure. Thus, according to our definition, the (r, s) -factorisable operators in [14] are precisely the (s^*, r^*) -factorisable operators.

Similarly, the (p, r, s) -nuclear operators in [14] is defined in the case of $\frac{1}{p} + \frac{1}{r} + \frac{1}{s} \geq 1$, which is equivalent to $1 + \frac{1}{p} \geq \frac{1}{r^*} + \frac{1}{s^*}$. According to [14, Theorem 2.5, p.55], the (p, r, s) -nuclear operators in [14] are precisely the (p, s^*, r^*) -nuclear operators in the definition of our paper.

According to [15, Definition 19.1.1], a bounded linear operator $S : E \rightarrow F$ is called (p, r, s) -integral if S belongs to the maximal hull of the space of (p, r, s) -nuclear operators. It follows from [14, Theorem 1.3, p.67], [14, Corollary 3.6, p.77] and [15, Theorem 9.3.1] that in the case $1 + \frac{1}{p} = \frac{1}{r} + \frac{1}{s}$, a bounded linear operator $S : E \rightarrow F$ is (p, r, s) -integral if and only if it admits a factorization $J_F S : E \xrightarrow{B} L_{s^*}(\mu) \xrightarrow{M_g} L_r(\mu) \xrightarrow{A}$

F^{**} , where M_g is the multiplication operator induced by $g \in L_p(\mu)$, $A \in \mathcal{L}(L_r(\mu), F^{**})$, $B \in \mathcal{L}(E, L_{s^*}(\mu))$ and μ is a measure. Hence, (r, s) -factorisable operators and (p, r, s) -integral operators coincide. At the present paper, the class of all (r, s) -factorisable operators from E to F is denoted by $\Gamma_{(r,s)}(E, F)$ and the (r, s) -factorisable norm of S is defined by $\gamma_{(r,s)}(S) := \inf \|A\| \cdot \|g\|_p \cdot \|B\|$. In this section, we deal with the (p, r, s) -integral operators in more general case $1 + \frac{1}{p} \leq \frac{1}{r} + \frac{1}{s}$.

Definition 4.1. Let $1 + \frac{1}{p} \leq \frac{1}{r} + \frac{1}{s}$. We say that a linear operator $S : E \rightarrow F$ is (p, r, s) -integral if it admits a factorization $J_F S : E \xrightarrow{B} L_{s^*}(\mu) \xrightarrow{M_g} L_r(\mu) \xrightarrow{A} F^{**}$, where μ is a probability measure, M_g is the multiplication operator induced by $g \in L_p(\mu)$, $A \in \mathcal{L}(L_r(\mu), F^{**})$ and $B \in \mathcal{L}(E, L_{s^*}(\mu))$.

We set

$$I_{(p,r,s)}(S) := \inf \|A\| \cdot \|g\|_p \cdot \|B\|.$$

The collection of all (p, r, s) -integral operators from E to F is denoted by $\mathcal{I}_{(p,r,s)}(E, F)$.

Using the same technique and argument as in [14, Theorem 2.4, p.70] or [9, Theorem 5.2], we can show that $[\mathcal{I}_{(p,r,s)}, I_{(p,r,s)}(\cdot)]$ is a quasi-normed operator ideal in the sense of [15, Definition 6.1.1].

Definition 4.2. Suppose that $1 + \frac{1}{p} \leq \frac{1}{r} + \frac{1}{s}$. We say that a Lipschitz map $T \in \text{Lip}_0(X, E)$ is *strongly Lipschitz (p, r, s) -integral* if there exist a probability measure μ , a Lipschitz map $B \in \text{Lip}_0(X, L_{s^*}(\mu))$, a p -integral function $g \in L_p(\mu)$ and a linear operator $A \in \mathcal{L}(L_r(\mu), E^{**})$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{J_E T} & E^{**} \\ B \downarrow & & \uparrow A \\ L_{s^*}(\mu) & \xrightarrow{M_g} & L_r(\mu), \end{array}$$

where $M_g : L_{s^*}(\mu) \rightarrow L_r(\mu)$ is the multiplication operator induced by g . The collection of all strongly Lipschitz (p, r, s) -integral operators from X to E is denoted by $\mathcal{ST}_{(p,r,s)}^L(X, E)$. With each $T \in \mathcal{ST}_{(p,r,s)}^L(X, E)$, we associate its *strongly Lipschitz (p, r, s) -integral norm*

$$SI_{(p,r,s)}^L(T) := \inf \text{Lip}(B) \cdot \|g\|_p \cdot \|A\|,$$

where the infimum is extended over all probability measures μ , linear operators A , Lipschitz maps B and p -integral functions g as above.

Remark 4.3.

- (i) It is easy to see that strongly Lipschitz $(\infty, r, 1)$ -integral operators are precisely the strongly Lipschitz r -integral operators that is introduced in [13]. Indeed, for $g \in L_\infty(\mu)$, we have

$$M_g = \widetilde{M}_g I_{\infty, r} : L_\infty(\mu) \xrightarrow{I_{\infty, r}} L_r(\mu) \xrightarrow{\widetilde{M}_g} L_r(\mu),$$

where $I_{\infty, r} : L_\infty(\mu) \rightarrow L_r(\mu)$ is the formal identity and $\widetilde{M}_g : L_r(\mu) \rightarrow L_r(\mu)$ is the multiplication operator induced by g .

- (ii) $[\mathcal{ST}_{(p,r,s)}^L, SI_{(p,r,s)}^L(\cdot)]$ is Lipschitz regular.
- (iii) In the case $1 + \frac{1}{p} = \frac{1}{r} + \frac{1}{s}$, the probability measures μ in Definition 4.2 can be extended to arbitrary measures without changing the strongly Lipschitz (p, r, s) -integral norm. Indeed, given any measure space (Ω, Σ, μ) , $g \in L_p(\Omega, \Sigma, \mu)$ and A, B as in Definition 4.2, we set $\mu_1(S) = \int_S |g|^p d\mu$ for $S \in \Sigma$. Then μ_1 is a finite measure. Let us

define two linearly surjective isometries

$$D : L_{s^*}(\mu) \rightarrow L_{s^*}(\mu_1), f \mapsto |g|^{1-\frac{p}{r}} f$$

and

$$C : L_r(\mu_1) \rightarrow L_r(\mu), f \mapsto |g|^{\frac{p}{r}} f.$$

Let $\tilde{g} = \text{sgn}(g)$ and $M_{\tilde{g}} : L_{s^*}(\mu_1) \rightarrow L_r(\mu)$ be the multiplication operator induced by \tilde{g} . It can be checked that $M_g = CM_{\tilde{g}}D$ and $\|\tilde{g}\|_{L_p(\mu_1)} = \|g\|_{L_p(\mu)}$.

We set $\nu = \frac{\mu_1}{\|\mu_1\|}$. Then ν is a probability measure. We let

$$V_1 : L_{s^*}(\mu_1) \rightarrow L_{s^*}(\nu), f \mapsto \|\mu_1\|^{\frac{1}{s^*}} f$$

and

$$V_2 : L_r(\nu) \mapsto L_r(\mu_1), f \mapsto \frac{1}{\|\mu_1\|^{\frac{1}{r}}} f.$$

Then V_1, V_2 are linearly surjective isometries. Let $g' = \frac{\|\mu_1\|^{\frac{1}{r}}}{\|\mu_1\|^{\frac{1}{s^*}}} \tilde{g}$. It is easy to see that $M_{\tilde{g}} = V_2 M_{g'} V_1$ and $\|\tilde{g}\|_{L_p(\mu_1)} = \|g'\|_{L_p(\nu)}$. We set $A' = ACV_2$ and $B' = V_1 DB$. Then $AM_g B = A' M_{g'} B'$ and $\text{Lip}(B') = \text{Lip}(B)$, $\|A'\| = \|A\|$ and $\|g'\|_{L_p(\nu)} = \|g\|_{L_p(\mu)}$.

The following corollary follows immediately from Remark 4.3(iii).

Corollary 4.4. Let $1 + \frac{1}{p} = \frac{1}{r} + \frac{1}{s}$. Then

$$[\mathcal{SN}_{(p,r,s)}^L, \mathcal{SN}_{(p,r,s)}^L(\cdot)] \subseteq [\mathcal{SI}_{(p,r,s)}^L, \mathcal{SN}_{(p,r,s)}^L(\cdot)].$$

Using an argument similar to Proposition 2.12 and Remark 4.3(iii), we get the following result.

Proposition 4.5.

(i) Suppose that $1 + \frac{1}{p} \leq \frac{1}{r} + \frac{1}{s}$. The canonical mapping

$$T \mapsto \hat{T}$$

provides an isometric isomorphism from $\mathcal{SI}_{(p,r,s)}^L(X, E)$ onto $\mathcal{I}_{(p,r,s)}(\mathcal{F}(X), E)$.

(ii) Suppose that $1 + \frac{1}{p} = \frac{1}{r} + \frac{1}{s}$. The canonical mapping

$$T \mapsto \hat{T}$$

provides an isometric isomorphism from $\mathcal{SI}_{(p,r,s)}^L(X, E)$ onto $\Gamma_{(r,s)}(\mathcal{F}(X), E)$.

Corollary 4.6. Let $1 + \frac{1}{p} = \frac{1}{r} + \frac{1}{s}$. Suppose that $X^\#$ or E has the metric approximation property. Then

$$\mathcal{SN}_{(p,r,s)}^{L_0}(T) = \mathcal{SI}_{(p,r,s)}^L(T), \text{ for all } T \in \text{Lip}_0(X, E).$$

Proof. Let $T \in \text{Lip}_0(X, E)$. By Corollary 2.16, [13, Proposition 2.4], Proposition 2.12, [14, Lemma 2.6, p.71] and Proposition 4.5, we get

$$\mathcal{SN}_{(p,r,s)}^{L_0}(T) = \mathcal{SN}_{(p,r,s)}^L(T) = N_{(p,r,s)}(\hat{T}) = \gamma_{(r,s)}(\hat{T}) = \mathcal{SI}_{(p,r,s)}^L(T).$$

The first equality follows from Corollary 2.16, the second from Proposition 2.12, the third from [13, Proposition 2.4] and [14, Lemma 2.6, p.71], the last from Proposition 4.5. \square

Definition 4.7. Let Lip_0 denote the class of all Lipschitz mappings that vanish at 0. A generic Lipschitz operator quasi-Banach ideal $[\mathcal{A}, \|\cdot\|_{\mathcal{A}}]$ is a subclass of Lip_0 such that the linear subspaces

$$\mathcal{A}(X, E) := \mathcal{A} \cap \text{Lip}_0(X, E)$$

satisfy the following conditions:

- (i) $Lip(T) \leq \|T\|_{\mathcal{A}}$ for all $T \in \mathcal{A}(X, E)$.
- (ii) $\|\cdot\|_{\mathcal{A}}$ is a complete, quasi-norm on $\mathcal{A}(X, E)$.
- (iii) If $f \in X^{\#}$ and $u \in E$, then $f \otimes u$ is in $\mathcal{A}(X, E)$ with $\|f \otimes u\|_{\mathcal{A}} = Lip(f)\|u\|$.
- (iv) If Y is a pointed metric space, F is a Banach space, $R \in Lip_0(Y, X)$, $T \in \mathcal{A}(X, E)$, and $S \in \mathcal{L}(E, F)$, then $STR \in \mathcal{A}(Y, F)$ and $\|STR\|_{\mathcal{A}} \leq \|S\|\|T\|_{\mathcal{A}}Lip(R)$.

An immediate consequence of Definition 4.2 and Proposition 4.5 is the following:

Proposition 4.8. $[\mathcal{ST}_{(p,r,s)}^L, SI_{(p,r,s)}^L(\cdot)]$ is a generic Lipschitz operator quasi-Banach ideal.

Let us recall that a Lipschitz map $T : X \rightarrow E$ is called *Lipschitz weakly compact* [13] if its Lipschitz image $\{\frac{Tx-Tx'}{d(x,x')} : x, x' \in X, x \neq x'\}$ is relatively weakly compact in E .

Proposition 4.9. Let $1 \leq p, r, s < \infty$. Then every strongly Lipschitz (p, r, s) -integral operator is Lipschitz weakly compact.

Proof. Suppose that $T : X \rightarrow E$ is strongly Lipschitz (p, r, s) -integral. We choose a Lipschitz (p, r, s) -integral factorization

$$\begin{array}{ccc} X & \xrightarrow{J_E T} & E^{**} \\ B \downarrow & & \uparrow A \\ L_{s^*}(\mu) & \xrightarrow{M_g} & L_r(\mu). \end{array}$$

We only consider the case $s^* = \infty$ and $r = 1$. The other case is trivial.

If $p > 1$, it is noted that

$$M_g = I_{p,1} \widetilde{M}_g : L_{\infty}(\mu) \xrightarrow{\widetilde{M}_g} L_p(\mu) \xrightarrow{I_{p,1}} L_1(\mu),$$

where $\widetilde{M}_g : L_{\infty}(\mu) \rightarrow L_p(\mu)$ is the multiplication operator induced by g and $I_{p,1} : L_p(\mu) \rightarrow L_1(\mu)$ is the formal identity. Clearly, \widetilde{M}_g is weakly compact and so is M_g . Thus T is Lipschitz weakly compact.

If $p = 1$, it follows from

$$\sup_{f \in B_{L_{\infty}(\mu)}} \int_D |fg| d\mu \leq \int_D |g| d\mu \rightarrow 0, \quad \text{as } \mu(D) \rightarrow 0,$$

that $M_g : L_{\infty}(\mu) \rightarrow L_1(\mu)$ is weakly compact. Hence T is Lipschitz weakly compact. \square

Theorem 4.10. Let $1 < r, s < \infty$. Then

$$\mathcal{I}_{(p,r,s)}^{Lip-dual} = \mathcal{ST}_{(p,s,r)}^L.$$

Proof. Let $T \in \mathcal{I}_{(p,r,s)}^{Lip-dual}(X, E)$. Then T^t is (p, r, s) -integral and consider any (p, r, s) -integral factorization

$$J_{X^{\#}} T^t : E^* \xrightarrow{B} L_{s^*}(\mu) \xrightarrow{M_g} L_r(\mu) \xrightarrow{A} (X^{\#})^{**}.$$

Taking an adjoint, we get

$$(T^t)^* J_{X^{\#}}^* = B^* M_g^* A^* = B^* \widetilde{M}_g A^* : (X^{\#})^{***} \xrightarrow{A^*} L_{r^*}(\mu) \xrightarrow{\widetilde{M}_g} L_s(\mu) \xrightarrow{B^*} E^{**},$$

where $\widetilde{M}_g : L_{r^*}(\mu) \rightarrow L_s(\mu)$ is the multiplication operator induced by g .

Define an isometric embedding $R : X \rightarrow (X^{\#})^{***}$ by

$$\langle Rx, \phi \rangle = \langle \phi, \delta_X(x) \rangle, \quad \phi \in (X^{\#})^{**}.$$

It is easy to check that $J_ET = (T^t)^* J_{X^\#}^* R$. Hence, we get $J_ET = B^* \widetilde{M}_g A^* R$. This implies that T is strongly Lipschitz (p, s, r) -integral and

$$SI_{(p,s,r)}^L(T) \leq Lip(A^* R) \|g\|_p \|B^*\| \leq \|A\| \|g\|_p \|B\|.$$

Thus, we get

$$SI_{(p,s,r)}^L(T) \leq I_{(p,r,s)}(T^t) = \|T\|_{\mathcal{I}_{(p,r,s)}^{Lip-dual}}.$$

Conversely, assume that $T \in \mathcal{SI}_{(p,s,r)}^L(X, E)$. Take any Lipschitz (p, s, r) -integral factorization

$$J_ET : X \xrightarrow{B} L_{r^*}(\mu) \xrightarrow{M_g} L_s(\mu) \xrightarrow{A} E^{**}.$$

Taking the transpose map of J_ET , we get

$$T^t J_E^* = B^t M_g^* A^* = B^t \widetilde{M}_g A^* : E^{***} \xrightarrow{A^*} L_{s^*}(\mu) \xrightarrow{\widetilde{M}_g} L_r(\mu) \xrightarrow{B^t} X^\#,$$

where $\widetilde{M}_g : L_{s^*}(\mu) \rightarrow L_r(\mu)$ is the multiplication operators induced by g .

It follows from $J_E^* J_{E^*} = I_{E^*}$ that

$$J_{X^\#} T^t = J_{X^\#} T^t J_E^* J_{E^*} = J_{X^\#} B^t \widetilde{M}_g A^* J_{E^*}.$$

This yields that T^t is (p, r, s) -integral and

$$\begin{aligned} \|T\|_{\mathcal{I}_{(p,r,s)}^{Lip-dual}} &= I_{(p,r,s)}(T^t) \\ &\leq \|J_{X^\#} B^t\| \|g\|_p \|A^* J_{E^*}\| \\ &\leq Lip(B) \|g\|_p \|A\| \end{aligned}$$

Hence, we get

$$\|T\|_{\mathcal{I}_{(p,r,s)}^{Lip-dual}} \leq SI_{(p,s,r)}^L(T).$$

This completes the proof. \square

Acknowledgements. The authors thank the anonymous referees for valuable comments and suggestions which improve our paper substantially.

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LABORATOIRE DE MATHÉMATIQUES PURES ET APPLIQUÉES (LMPA), UNIVERSITY OF LAGHOUAT,
LAGHOUAT, ALGERIA

E-mail address: a.belacel@lagh-univ.dz

SCHOOL OF MATHEMATICAL SCIENCES, XIAMEN UNIVERSITY, XIAMEN, 361005, CHINA

E-mail address: cdy@xmu.edu.cn