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Doubly nonlocal Fisher–KPP equation: Speeds and uniqueness of traveling waves

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ABSTRACT

We study traveling waves for a reaction-diffusion equation with nonlocal anisotropic diffusion and a linear combination of local and nonlocal monostable-type reactions. We describe relations between speeds and asymptotic of profiles of traveling waves, and prove the uniqueness of the profiles up to shifts.

1. Introduction

We will study traveling wave solutions to the equation

$$\begin{aligned}\frac{\partial u}{\partial t}(x,t) &= \varkappa^+ \int_{\mathbb{R}^d} \boldsymbol{a}^+(x-y)u(y,t)dy - mu(x,t) - u(x,t)\, G\big(u(x,t)\big),\\ G\big(u(x,t)\big) &:= \varkappa_\ell u(x,t) + \varkappa_{n\ell} \int_{\mathbb{R}^d} \boldsymbol{a}^-(x-y)u(y,t)dy.\end{aligned} \tag{1.1}$$

Here $d \in \mathbb{N}$; $\varkappa^+, m > 0$ and $\varkappa_\ell, \varkappa_{n\ell} \geq 0$ are constants, such that

$$\varkappa^- := \varkappa_\ell + \varkappa_{n\ell} > 0; \tag{1.2}$$

the kernels $0 \leq \boldsymbol{a}^\pm \in L^1(\mathbb{R}^d)$ are probability densities, i.e. $\int_{\mathbb{R}^d} \boldsymbol{a}^\pm(y)dy = 1$.

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The present paper is a continuation of [15]; they both are based on thesis [29] and, in the particular case $\varkappa_\ell = 0$, on our unpublished preprint [13]. For the case of the local nonlinearity in (1.1), when $\varkappa_{n\ell} = 0$, the equation (1.1) was considered, in particular, in [1,5,6,20,23,25–27,31,33]. For a nonlocal nonlinearity and, especially, for the case $\varkappa_\ell = 0$ in (1.1), see e.g. [8,10–13,16,19,24,32]. For details, see the introduction to [15] and also the comments below.

The solution $u = u(x,t)$ describes the local density of a species at the point $x \in \mathbb{R}^d$ at the moment of time $t \geq 0$. The individuals of the species spread over the space $\mathbb{R}^d$ according to the dispersion kernel $\boldsymbol{a}^+$ and the fecundity rate $\varkappa^+$. The individuals may die according to both constant mortality rate m and density dependent competition, described by the rate $\varkappa^-$. The competition may be *local*, when the density $u(x,t)$ at a point x is influenced by itself only, with the rate $\varkappa_\ell$, or *nonlocal*, when the density $u(x,t)$ is influenced by all values $u(y,t)$, $y \in \mathbb{R}^d$, averaged over $\mathbb{R}^d$ according to the competition kernel $\boldsymbol{a}^-$ with the rate $\varkappa_{n\ell}$. See also [2,5,8,10,11,19,24–26].

Under assumption

$$\varkappa^+ > m, \tag{A1}$$

the equation (1.1) has two constant stationary solutions: $u \equiv 0$ and $u \equiv \theta$, where

$$\theta := \frac{\varkappa^+ - m}{\varkappa^-} > 0. \tag{1.3}$$

Moreover, one can then also rewrite the equation in a reaction-diffusion form

$$\frac{\partial u}{\partial t}(x,t) = \varkappa^+ \int_{\mathbb{R}^d} \boldsymbol{a}^+(x-y)\big(u(y,t) - u(x,t)\big)dy + u(x,t)\Big(\beta - G\big(u(x,t)\big)\Big),$$

where $\beta = \varkappa^+ - m > 0$. We treat then (1.1) as a doubly nonlocal Fisher–KPP equation, see the introduction to [15] for details.

By a (monotone) traveling wave solution to (1.1) in a direction $\xi \in S^{d-1}$ (the unit sphere in $\mathbb{R}^d$), we will understand a solution of the form

$$\begin{aligned} u(x,t) &= \psi(x\cdot\xi - ct), \quad t \geq 0, \text{ a.a. } x \in \mathbb{R}^d, \\ \psi(-\infty) &= \theta, \qquad \psi(+\infty) = 0, \end{aligned} \tag{1.4}$$

where $c \in \mathbb{R}$ is called the speed of the wave and the function $\psi \in \mathcal{M}_\theta(\mathbb{R})$ is called the profile of the wave. Here $\mathcal{M}_\theta(\mathbb{R})$ denotes the set of all decreasing and right-continuous functions $f : \mathbb{R} \to [0,\theta]$, and $x\cdot\xi$ denotes the scalar product in $\mathbb{R}^d$.

In [15, Propositions 3.7], we have shown (cf. also [5]) that the study of a traveling wave solution (1.4) with a fixed $\xi \in S^{d-1}$ can be reduced to the study of the one-dimensional version of (1.1) with the kernels

$$a^\pm(s) := \int_{\{\xi\}^\perp} \boldsymbol{a}^\pm(s\xi + \eta)\, d\eta, \quad s \in \mathbb{R}, \tag{1.5}$$

where $\{\xi\}^\perp := \{x \in \mathbb{R}^d \mid x\cdot\xi = 0\}$. For $d = 1$ and $\xi \in S^0 = \{-1,1\}$, (1.5) reads as follows: $a^\pm(s) = \boldsymbol{a}^\pm(s\xi)$, $s \in \mathbb{R}$. Clearly, $\int_{\mathbb{R}} a^\pm(s)\, ds = 1$.

For simplicity, we omit ξ from the notations for functions $a^\pm$, assuming that the direction $\xi \in S^{d-1}$ is fixed for the sequel. We denote also

$$J_\theta(s) := \varkappa^+ a^+(s) - \theta\varkappa_{n\ell} a^-(s), \quad s \in \mathbb{R}. \tag{1.6}$$

Under (A1), we assume that

$$J_\theta \geq 0, \quad \text{a.a. } s \in \mathbb{R}, \tag{A2}$$

and that there exists $\mu = \mu(\xi) > 0$, such that

$$\int_{\mathbb{R}^d} \boldsymbol{a}^+(x) e^{\mu x\cdot\xi}\, dx = \int_{\mathbb{R}} a^+(s) e^{\mu s}\, ds < \infty. \tag{A3}$$

Suppose also, that $\boldsymbol{a}^+$ is not degenerated in the direction $\xi \in S^{d-1}$, i.e.

$$\begin{gathered}\text{there exist } r = r(\xi) \geq 0,\ \rho = \rho(\xi) > 0,\ \delta = \delta(\xi) > 0, \text{ such that}\\ a^+(s) \geq \rho, \text{ for a.a. } s \in [r-\delta, r+\delta].\end{gathered} \tag{A4}$$

A sufficient condition for (A4) is that $\boldsymbol{a}^+(x) \geq \rho'$ for a.a. $x \in \mathbb{R}^d$ such that $|x - r\xi| \leq \delta'$, for some $\rho', \delta' > 0$.

By [17], (A2) implies the comparison principle for the mentioned one-dimensional version of (1.1), that was a background for results we obtained in [15], see Theorem 1.1 below. Stress that the assumption (A2) is redundant for the case of the local nonlinearity in (1.1), when $\varkappa_{n\ell} = 0$. For $\varkappa_{n\ell} > 0$, note that (A2) always holds, in particular, for equal kernels, $\boldsymbol{a}^- = \boldsymbol{a}^+$ (or just $a^- = a^+$), because of (1.2) and (1.3). On the other hand, if $\varkappa_{n\ell} > 0$ and (A2) fails, the bifurcation of the constant solution $u \equiv \theta$ is possible, developing an infinite family of spatially periodic stationary solutions (see [22] for more details). For example, consider, for some $h \in \mathbb{R}$,

$$a^+(s) = \frac{1}{\sqrt{4\pi}} e^{-\frac{s^2}{4}}, \quad a^-(s) = \frac{1}{2\sqrt{4\pi}}\big(e^{-\frac{(s-h)^2}{4}} + e^{-\frac{(s+h)^2}{4}}\big), \quad s \in \mathbb{R}.$$

Then, under (A1), there exists h_0 such that, for all $h \leq |h_0|$, the assumption (A2) holds true, and, for all $h > |h_0|$, it does not hold.

The assumption (A3) is necessary for existence of traveling waves (see [14, Proposition 1.4]). In fact, if (A3) fails, then the solution propagates with a superlinear rate which depends on the asymptotic of a^+. See e.g. [3,16,20] for more details.

Theorem 1.1 *([15, Theorem 1.1, Propositions 3.7, 3.14, 3.15]). Let $\xi \in S^{d-1}$ be fixed and suppose that (A1), (A2), (A3) hold. Then there exists $c_* = c_*(\xi) \in \mathbb{R}$, such that, for any $c < c_*$, a traveling wave solution to (1.1) of the form (1.4) with $\psi \in \mathcal{M}_\theta(\mathbb{R})$ does not exist; whereas for any $c \geq c_*$,*

1) there exists a traveling wave solution to (1.1) with the speed c and a profile $\psi \in \mathcal{M}_\theta(\mathbb{R})$ such that (1.4) holds;
2) if $c \neq 0$, then the profile $\psi \in C_b^\infty(\mathbb{R})$ (the class of infinitely many times differentiable functions on $\mathbb{R}$ with bounded derivatives); if $c = 0$ (in the case $c_* \leq 0$), then $\psi \in C(\mathbb{R})$;
3) there exists $\mu = \mu(c, a^+, \varkappa^-, \theta) > 0$ such that

$$\int_{\mathbb{R}} \psi(s) e^{\mu s}\, ds < \infty; \tag{1.7}$$

4) if, additionally, (A4) holds, then, for any $c \neq 0$, there exists $\nu > 0$, such that $\psi(t)e^{\nu t}$ is a strictly increasing function;
5) if, additionally, (A4) holds with $r = 0$, then the profile ψ is a strictly decreasing function on $\mathbb{R}$.

The smoothness of the profile ψ implies, see [15, Proposition 3.11] for details, that ψ satisfies the equation

$$c\psi'(s) + \varkappa^+(a^+ * \psi)(s) - m\psi(s) - \varkappa_{n\ell}\psi(s)(a^- * \psi)(s) - \varkappa_\ell\psi^2(s) = 0 \tag{1.8}$$

for all $s \in \mathbb{R}$. Here $*$ denotes the classical convolution of functions on $\mathbb{R}$, i.e.

$$(a^\pm * \psi)(s) := \int\limits_{\mathbb{R}} a^\pm(s-\tau)\psi(\tau)\, d\tau, \quad s \in \mathbb{R}.$$

To study (1.8), we will use a bilateral-type Laplace transform

$$(\mathfrak{L}f)(z) = \int\limits_{\mathbb{R}} f(s)e^{zs}\, ds, \quad \operatorname{Re} z > 0,\ f \in L^\infty(\mathbb{R}). \tag{1.9}$$

For each $f \in L^\infty(\mathbb{R})$, there exists $\sigma(f) \in [0,\infty]$, called the abscissa of f, such that the integral in (1.9) is convergent for $0 < \operatorname{Re} z < \sigma(f)$ and divergent for $\operatorname{Re} z > \sigma(f)$, see Lemma 2.1 below for details.

We assume that

$$a^+ \in L^\infty(\mathbb{R}), \tag{A5}$$

that is evidently fulfilled if e.g. $\boldsymbol{a}^+ \in L^\infty(\mathbb{R}^d)$. Then, under (A3) and (A5), there exists $\sigma(a^+) \in (0,\infty]$. Similarly, because of (1.7), for any profile ψ of a traveling wave solution to (1.1), there exists $\sigma(\psi) \in (0,\infty]$.

Finally, for the fixed $\xi \in S^{d-1}$, we assume that

$$\int\limits_{\mathbb{R}^d} |x\cdot\xi|\, \boldsymbol{a}^+(x)\, dx = \int\limits_{\mathbb{R}} |s| a^+(s)\, ds < \infty. \tag{A6}$$

Under assumption (A6), we define

$$\mathfrak{m}_\xi := \int\limits_{\mathbb{R}^d} x\cdot\xi\ \boldsymbol{a}^+(x)\, dx = \int\limits_{\mathbb{R}} s\, a^+(s)\, ds. \tag{1.10}$$

We formulate now the first main result of the present paper.

Theorem 1.2. *Let, for the fixed $\xi \in S^{d-1}$, the conditions (A1)–(A6) hold. Let $c_* = c_*(\xi) \in \mathbb{R}$ be the minimal traveling wave speed according to Theorem 1.1, and let, for any $c \geq c_*$, the function $\psi = \psi_c \in \mathcal{M}_\theta(\mathbb{R})$ be a traveling wave profile corresponding to the speed c. Then*

1. *There exists a unique $\lambda_* \in \mathbb{R}$, such that*

$$\begin{aligned} c_* &= \min_{\lambda>0} \frac{1}{\lambda}\left(\varkappa^+ \int\limits_{\mathbb{R}} a^+(s)e^{\lambda s}\, ds - m\right) \\ &= \frac{1}{\lambda_*}\left(\varkappa^+ \int\limits_{\mathbb{R}} a^+(s)e^{\lambda_* s}\, ds - m\right) > \varkappa^+\mathfrak{m}_\xi. \end{aligned} \tag{1.11}$$

2. *For any $c \geq c_*$ the abscissa of the corresponding profile ψ_c is finite:*

$$\sigma(\psi_c) \in (0, \lambda_*], \tag{1.12}$$

and the mapping $(0, \lambda_*] \ni \sigma(\psi_c) \mapsto c \in [c_*, \infty)$ *is a (strictly) decreasing bijection, given by*

$$c = \frac{1}{\sigma(\psi_c)} \Bigl(\varkappa^+ \int_{\mathbb{R}} a^+(s) e^{\sigma(\psi_c)\, s}\, ds - m \Bigr). \tag{1.13}$$

In particular,

$$\sigma(\psi_{c_*}) = \lambda_*. \tag{1.14}$$

3. *For any* $c \geq c_*$,

$$(\mathfrak{L}\psi_c)\bigl(\sigma(\psi_c)\bigr) = \infty. \tag{1.15}$$

Note that, in light of (1.11), the kernel $\boldsymbol{a}^+$ may be so *slanted* to the direction opposite to ξ, that $c_*(\xi) < 0$. A sufficient condition to exclude this, by the inequality in (1.11), is that $\mathfrak{m}_\xi = 0$; in particular, this evidently holds if a^+ is symmetric.

We will show also that $\sigma(\psi_{c_*}) = \lambda_* \leq \sigma(a^+)$. We will distinguish two cases: the non-critical case when $\sigma(\psi_{c_*}) < \sigma(a^+)$, and the critical case when $\sigma(\psi_{c_*}) = \sigma(a^+)$. Note that a kernel a^+ which is compactly supported or decreases faster than any exponential function corresponds to the non-critical case, as then $\lambda_* < \infty = \sigma(a^+)$.

The critical case is characterized by the following conditions (cf. Proposition 2.5 and Definition 2.6 below)

$$\widehat{\sigma} := \sigma(a^+) < \infty, \qquad \int_{\mathbb{R}} (1+|s|) a^+(s) e^{\widehat{\sigma} s}\, ds < \infty, \tag{1.16}$$

$$m \leq \varkappa^+ \int_{\mathbb{R}} (1 - \widehat{\sigma} s) a^+(s) e^{\widehat{\sigma} s}\, ds. \tag{1.17}$$

Note that, informally, (1.17) implies upper bounds for both m and $\widehat{\sigma}$; cf. also the example (1.21) below.

Our second main result is about the asymptotic and the uniqueness (up to a shift) of the profile for a traveling wave with a given speed $c \geq c_*(\xi)$, $c \neq 0$.

Theorem 1.3. *Let* $\xi \in S^{d-1}$ *be fixed, and let conditions* (A1)–(A6) *hold. Let* $c_* = c_*(\xi) \in \mathbb{R}$ *be the minimal traveling wave speed given by* (1.11), *and let, for any speed* $c \geq c_*$, $\psi_c \in \mathcal{M}_\theta(\mathbb{R})$ *be the corresponding profile with the abscissa* $\sigma(\psi_c)$. *If* (1.16) *holds and if, cf.* (1.17), *for* $\widehat{\sigma} = \sigma(a^+)$,

$$m = \varkappa^+ \int_{\mathbb{R}} (1 - \widehat{\sigma} s) a^+(s) e^{\widehat{\sigma} s}\, ds, \tag{1.18}$$

we assume, additionally, that

$$\int_{\mathbb{R}} s^2 a^+(s) e^{\widehat{\sigma} s}\, ds < \infty. \tag{1.19}$$

Let $c \geq c_*$ *and* $c \neq 0$; *then the following holds.*

1) *There exists* $D > 0$, *such that*

$$\psi_c(s) \sim D\, s^{j-1}\, e^{-\sigma(\psi_c) s}, \quad s \to \infty. \tag{1.20}$$

Here $j = 1$ in two cases: 1) $c > c_*$; 2) $c = c_*$ and (1.16) holds as well as the strict inequality in (1.17). Otherwise, $j = 2$, i.e. when $c = c_*$ and either (1.16) fails or both (1.16) and (1.18) hold. Moreover, $D = D_j$ may be chosen equal to 1 by a shift of ψ_c.

2) If, additionally, there exist $\rho, \delta > 0$, such that

$$J_\theta(s) \geq \rho, \text{ for a.a. } |s| \leq \delta, \tag{A7}$$

then the traveling wave profile ψ_c is unique up to a shift.

Clearly, (A7) implies that (A4) holds with $r = 0$. If, additionally, (A4) holds with $r = 0$, e.g. if $\boldsymbol{a}^+$ is separated from 0 in a neighborhood of the origin, then (A7) holds as well.

Therefore, in the non-critical case, the profile of a traveling wave with a non-minimal speed decays exponentially at infinity with the rate equal to the abscissa of the profile, whereas for the minimal speed it decays slower: with an additional linear factor. However, in the critical case, the profile of the traveling wave with the minimal speed will not have that additional factor, unless both (1.16) and (1.18) hold (and we can prove the latter under the additional assumption (1.19) only).

To demonstrate the critical case, consider the kernel

$$a^+(s) := \frac{\alpha e^{-\mu|s|}}{1+|s|^q}, \qquad s \in \mathbb{R},\ q \geq 0,\ \mu > 0, \tag{1.21}$$

where $\alpha > 0$ is a normalizing constant. Then (A3)–(A6) evidently hold and $\widehat{\sigma} = \sigma(a^+) = \mu$. In Example 2.8 below, we will show that, for $q > 2$, there exist $\mu_* > 0$ and $m_* \in (0, \varkappa^+)$, such that $\sigma(\psi_{c_*(\xi)}) = \widehat{\sigma}$, if only $\mu \in (0, \mu_*]$ and $m \in (0, m_*]$. The condition (1.19) does not take place only for $q \in (2, 3]$, $\mu \in (0, \mu_*]$ and $m = m_*$.

Another specific of the critical case is visible from the behavior on the positive half-line of the so-called characteristic function $\mathfrak{h}_{\xi,c}$, corresponding to the traveling wave with a speed $c \geq c_*$, see (3.1) and Proposition 3.1 below:

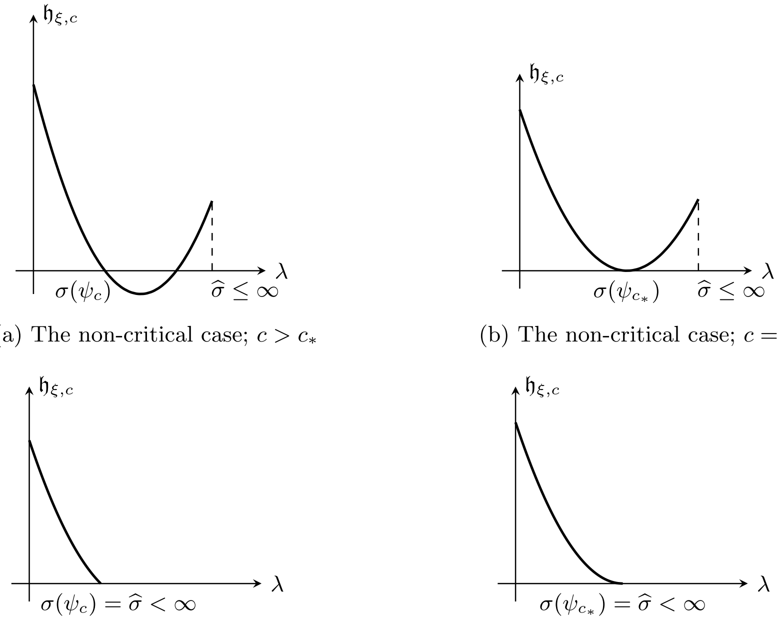
$$\mathfrak{h}_{\xi,c}(\lambda) := \varkappa^+(\mathfrak{L}a^+)(\lambda) - m - c\lambda,$$

cf. e.g. [25]. (This function is equal to infinity for $\lambda > \widehat{\sigma}$.) Then the minimal positive root of $\mathfrak{h}_{\xi,c}$ is $\sigma(\psi_c)$. The sketches on Fig. 1 reflect the difference between the critical and non-critical cases for the function $\mathfrak{h}_{\xi,c}$.

In the case of the local nonlinearity in (1.1), when $\varkappa_{n\ell} = 0$, the results of Theorems 1.2–1.3 were mainly known in the literature under additional assumptions. For example, in [26], the kernel $\boldsymbol{a}^+$ was symmetric and compactly supported; in [5], the kernel $\boldsymbol{a}^+$ was anisotropic, but a^+ was supposed to be compactly supported; whereas the conditions in [33] corresponded to a symmetric a^+, such that the inequality in (A3) holds *for all* $\mu > 0$. In these both cases, $\widehat{\sigma} = \sigma(a^+) = \infty$; and hence, recall, $\sigma(\psi_{c_*(\xi)}) < \widehat{\sigma}$. In [1], an anisotropic kernel which satisfies (A3) was considered (that allows $\widehat{\sigma} < \infty$ as well), however, it was assumed that $\sigma(\psi_{c_*(\xi)}) < \widehat{\sigma}$. The critical case $\sigma(\psi_{c_*(\xi)}) = \widehat{\sigma}$, therefore, remained an open problem.

For a nonlocal nonlinearity in (1.1), i.e. when $\varkappa_{n\ell} \neq 0$, the only known results [32] also concerned the more simple case $\sigma(a^+) = \infty$.

The paper is organized as follows: in Section 2 we prove Theorem 1.2 for both critical and non-critical cases, and in Section 3 we discuss properties of the function $\mathfrak{h}_{\xi,c}$ and prove Theorem 1.3.



(a) The non-critical case; $c > c_*$

(b) The non-critical case; $c = c_*$

(c) The critical case; (1.18) fails; $c \geq c_*$

(d) The critical case; (1.18) holds; $c = c_*$; the graph touches λ-axis

Fig. 1. Sketches of the characteristic function $\mathfrak{h}_{\xi,c}$ for the critical (where (1.16)–(1.17) hold) and the non-critical cases.

2. Speed and profile of a traveling wave

2.1. Properties of the bilateral-type Laplace transform

For an $f \in L^\infty(\mathbb{R})$, let $\mathfrak{L}f$ be the bilateral-type Laplace transform of f given by (1.9), cf. [30, Chapter VI]. We collect several results about $\mathfrak{L}$ in the following lemma.

Lemma 2.1. *Let $f \in L^\infty(\mathbb{R})$.*

(L1) *There exists $\sigma(f) \in [0,\infty]$ such that the integral (1.9) converges in the strip $\{0 < \operatorname{Re} z < \sigma(f)\}$ (provided that $\sigma(f) > 0$) and diverges in the half plane $\{\operatorname{Re} z > \sigma(f)\}$ (provided that $\sigma(f) < \infty$).*

(L2) *Let $\sigma(f) > 0$. Then $(\mathfrak{L}f)(z)$ is analytic in $\{0 < \operatorname{Re} z < \sigma(f)\}$, and, for any $n \in \mathbb{N}$,*

$$\frac{d^n}{dz^n}(\mathfrak{L}f)(z) = \int_{\mathbb{R}} e^{zs} s^n f(s)\, ds, \quad 0 < \operatorname{Re} z < \sigma(f).$$

(L3) *Let $f \geq 0$ a.e. and $0 < \sigma(f) < \infty$. Then $(\mathfrak{L}f)(z)$ has a singularity at $z = \sigma(f)$. In particular, $\mathfrak{L}f$ has not an analytic extension to a strip $0 < \operatorname{Re} z < \nu$, with $\nu > \sigma(f)$.*

(L4) *Let $f' := \frac{d}{ds} f \in L^\infty(\mathbb{R})$, $f(\infty) = 0$, and $\sigma(f') > 0$. Then $\sigma(f) \geq \sigma(f')$ and, for any $0 < \operatorname{Re} z < \sigma(f')$,*

$$(\mathfrak{L}f')(z) = -z(\mathfrak{L}f)(z). \tag{2.1}$$

(L5) *Let $g \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\sigma(f) > 0$, $\sigma(g) > 0$. Then $\sigma(f * g) \geq \min\{\sigma(f), \sigma(g)\}$ and, for any $0 < \operatorname{Re} z < \min\{\sigma(f), \sigma(g)\}$,*

$$\big(\mathfrak{L}(f * g)\big)(z) = (\mathfrak{L}f)(z)(\mathfrak{L}g)(z). \tag{2.2}$$

(L6) *Let* $0 \le f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *and* $\sigma(f) > 0$. *Then*

$$\lim_{\lambda\to 0+} (\mathfrak{L}f)(\lambda) = \int\limits_{\mathbb{R}} f(s)\,ds.$$

(L7) *Let* $f \ge 0$, $\sigma(f) \in (0,\infty)$ *and* $A := \int_{\mathbb{R}} f(s)e^{\sigma(f)s}\,ds < \infty$. *Then*

$$\lim_{\lambda\to \sigma(f)-} (\mathfrak{L}f)(\lambda) = A.$$

(L8) *Let* $f \ge 0$ *be decreasing on* $\mathbb{R}$, *and let* $\sigma(f) > 0$. *Then, for any* $0 < \lambda < \sigma(f)$,

$$f(s) \le \frac{\lambda e^\lambda}{e^\lambda - 1}(\mathfrak{L}f)(\lambda)e^{-\lambda s}, \quad s \in \mathbb{R}. \tag{2.3}$$

Moreover,

$$\sigma\bigl(f^2\bigr) \ge 2\sigma(f), \tag{2.4}$$

and for any $0 \le g \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, $\sigma(g) > 0$,

$$\sigma\bigl(f(g*f)\bigr) \ge \sigma(f) + \min\bigl\{\sigma(g), \sigma(f)\bigr\}. \tag{2.5}$$

Proof. We can rewrite $\mathfrak{L} = \mathfrak{L}^+ + \mathfrak{L}^-$, where

$$(\mathfrak{L}^\pm f)(z) = \int\limits_{\mathbb{R}_\pm} f(s)e^{zs}\,ds, \quad \operatorname{Re} z > 0,$$

$\mathbb{R}_+ = [0,\infty)$, $\mathbb{R}_- = (\infty, 0]$. Let $\mathcal{L}$ denote the classical (unilateral) Laplace transform:

$$(\mathcal{L}f)(z) = \int\limits_{\mathbb{R}_+} f(s)e^{-zs}\,ds,$$

and $\mathfrak{s}(f)$ be its abscissa of convergence (see details, e.g. in [30, Chapter II]). Then, clearly, $(\mathfrak{L}^+ f)(z) = (\mathcal{L}f)(-z)$, $(\mathfrak{L}^- f)(z) = (\mathcal{L}f^-)(z)$, where $f^-(s) = f(-s)$, $s \in \mathbb{R}$. As a result, $\sigma(f) = -\mathfrak{s}(f)$.

It is easily seen that, for $f \in L^\infty(\mathbb{R})$, $\mathfrak{s}(f^-) \le 0$, in particular, the function $(\mathfrak{L}^- f)(z)$ is analytic on $\operatorname{Re} z > 0$.

Therefore, the Properties (L1)–(L3) are direct consequences of [30, Theorems II.1, II.5a, II.5b], respectively. The Property (L4) may be easily derived from [30, Theorem II.2.3a, II.2.3b], taking into account that $f(\infty) = 0$. The Property (L5) one gets by a straightforward computation, cf. [30, Theorem VI.16a]; note that $f * g \in L^\infty(\mathbb{R})$.

Next, $\sigma(f) > 0$ implies $\mathfrak{s}(f) < 0$, therefore, $\mathfrak{L}^+ f$ can be analytically continued to 0. If $\mathfrak{s}(f^-) < 0$, then $\mathfrak{L}^- f$ can be analytically continued to 0 as well, and (L6) will be evident. Otherwise, if $\mathfrak{s}(f^-) = 0$ then (L6) follows from [30, Theorem V.1]. Similar arguments prove (L7).

To prove (L8) for a decreasing nonnegative f, note that, for any $0 < \lambda < \sigma(f)$,

$$f(s)\int\limits_{s-1}^{s} e^{\lambda\tau}\,d\tau \le \int\limits_{s-1}^{s} f(\tau)e^{\lambda\tau}\,d\tau \le (\mathfrak{L}f)(\lambda), \quad s \in \mathbb{R},$$

that implies (2.3). Next, by (L5), $\sigma(g*f)>0$, and conditions on g yield that $g*f\geq 0$ is decreasing as well. Therefore, by (2.3), for any $0<\lambda<\sigma(g*f)$,

$$\begin{aligned}\bigl|\bigl(\mathfrak{L}(f(g*f))\bigr)(z)\bigr| &\leq \int\limits_{\mathbb{R}} f(s)(g*f)(s)e^{s\operatorname{Re} z}\,ds\\ &\leq \frac{\lambda e^{\lambda}}{e^{\lambda}-1}\bigl(\mathfrak{L}(g*f)\bigr)(\lambda)\int\limits_{\mathbb{R}} f(s)e^{-s\lambda}e^{s\operatorname{Re} z}\,ds<\infty,\end{aligned}$$

provided that $\operatorname{Re} z<\sigma(f)+\lambda<\sigma(f)+\sigma(g*f)$. As a result, $\sigma\bigl(f(g*f)\bigr)\geq\sigma(f)+\sigma(g*f)$ that, by (L5), implies (2.5). Similarly one can prove (2.4). □

2.2. Proof of Theorem 1.2

Through the rest of the paper we will always assume that (A1) holds. Note also, that (A2) and (A5) imply $a^-\in L^\infty(\mathbb{R})$.

Remark 2.2. By [15, Remark 3.6], if $\psi\in\mathcal{M}_\theta(\mathbb{R})$, $c\in\mathbb{R}$ gets (1.4) then, for any $s\in\mathbb{R}$, $\psi(\cdot+s)$ is a traveling wave to (1.1) with the same c.

Fix any $\xi\in S^{d-1}$. For $\mu>0$, we denote, cf. (1.5),

$$\mathfrak{a}_\xi(\mu):=\int\limits_{\mathbb{R}^d}\boldsymbol{a}^+(x)e^{\mu x\cdot\xi}\,dx=\int\limits_{\mathbb{R}} a^+(s)e^{\mu s}\,ds\in(0,\infty]. \tag{2.6}$$

Under (A2), (A3) and (A5), $\sigma(a^\pm)>0$ and

$$\mathfrak{a}_\xi(\mu)=(\mathfrak{L}a^+)(\mu)<\infty,\quad 0<\mu<\sigma(a^+).$$

Consider, the following complex-valued function, cf. (A3),

$$G_\xi(z):=\frac{\varkappa^+(\mathfrak{L}a^+)(z)-m}{z},\quad \operatorname{Re} z>0, \tag{2.7}$$

which is well-defined on $0<\operatorname{Re} z<\sigma(a^+)$. We have proved in [15, formula (3.18)] that

$$c_*(\xi)\leq\inf_{\lambda>0}G_\xi(\lambda), \tag{2.8}$$

where $c_*(\xi)$ is the minimal speed of traveling waves, cf. Theorem 1.1. We will show below that in fact there exists equality in (2.8).

We start with the following notations to simplify the further statements.

Definition 2.3. Let $m>0$, $\varkappa^+>0$, $\varkappa_\ell,\varkappa_{n\ell}\geq 0$, $0\leq a^-\in L^1(\mathbb{R})$ be fixed, and (A1) and (1.2) hold. For an arbitrary $\xi\in S^{d-1}$, denote by $\mathcal{U}_\xi$ the subset of functions $0\leq a^+\in L^1(\mathbb{R})$ such that (A2)–(A6) hold.

For $a^+\in\mathcal{U}_\xi$, denote also the interval $I_\xi\subset(0,\infty)$ by

$$I_\xi:=\begin{cases}(0,\infty), & \text{if } \sigma(a^+)=\infty,\\ \bigl(0,\sigma(a^+)\bigr), & \text{if } \sigma(a^+)<\infty \text{ and } (\mathfrak{L}a^+)\bigl(\sigma(a^+)\bigr)=\infty,\\ \bigl(0,\sigma(a^+)\bigr], & \text{if } \sigma(a^+)<\infty \text{ and } (\mathfrak{L}a^+)\bigl(\sigma(a^+)\bigr)<\infty.\end{cases} \tag{2.9}$$

Proposition 2.4. *Let $\xi \in S^{d-1}$ be fixed and $a^+ \in \mathcal{U}_\xi$. Then there exists a unique $\lambda_* = \lambda_*(\xi) \in I_\xi$ such that*

$$\inf_{\lambda>0} G_\xi(\lambda) = \min_{\lambda\in I_\xi} G_\xi(\lambda) = G_\xi(\lambda_*) > \varkappa^+\mathfrak{m}_\xi. \tag{2.10}$$

Moreover, G_ξ is strictly decreasing on $(0, \lambda_*]$ and G_ξ is strictly increasing on $I_\xi \setminus (0, \lambda_*]$ (the latter interval may be empty).

Proof. We continue to use the notation $\widehat{\sigma} := \sigma(a^+) \in (0, \infty]$. Denote also

$$F_\xi(\lambda) := \varkappa^+\mathfrak{a}_\xi(\lambda) - m = \lambda G_\xi(\lambda), \qquad \lambda \in I_\xi. \tag{2.11}$$

By (L2), for any $\lambda \in (0, \widehat{\sigma})$,

$$\mathfrak{a}''_\xi(\lambda) = \int_{\mathbb{R}} s^2 a^+(s)e^{\lambda s}\,ds > 0, \tag{2.12}$$

therefore, $\mathfrak{a}'_\xi(\lambda)$ is increasing on $(0, \widehat{\sigma})$; in particular, by (A6), we have, for any $\lambda \in (0, \widehat{\sigma})$,

$$\int_{\mathbb{R}} sa^+(s)e^{\lambda s}\,ds = \mathfrak{a}'_\xi(\lambda) > \mathfrak{a}'_\xi(0) = \int_{\mathbb{R}} sa^+(s)\,ds = \mathfrak{m}_\xi. \tag{2.13}$$

Next, by (L6), $F_\xi(0+) = \varkappa^+ - m > 0$, hence,

$$G_\xi(0+) = \infty. \tag{2.14}$$

Finally, for $\lambda \in (0, \widehat{\sigma})$, we have

$$G'_\xi(\lambda) = \lambda^{-2}\bigl(\lambda F'_\xi(\lambda) - F_\xi(\lambda)\bigr) = \lambda^{-1}\bigl(F'_\xi(\lambda) - G_\xi(\lambda)\bigr), \tag{2.15}$$

$$G''_\xi(\lambda) = \lambda^{-1}(F''_\xi(\lambda) - 2G'_\xi(\lambda)). \tag{2.16}$$

We will distinguish two cases.

Case 1. There exists $\mu \in (0, \widehat{\sigma})$ with $G'_\xi(\mu) = 0$. Then, by (2.16), (2.12),

$$G''_\xi(\mu) = \mu^{-1}F''_\xi(\mu) = \mu^{-1}\varkappa^+\mathfrak{a}''_\xi(\mu) > 0.$$

Hence any stationary point of G_ξ is with necessity a point of local minimum, therefore, G_ξ has at most one such a point, thus it will be a global minimum. Moreover, by (2.15), (2.13), $G'(\mu) = 0$ implies

$$G_\xi(\mu) = F'_\xi(\mu) = \varkappa^+\mathfrak{a}'_\xi(\mu) > \varkappa^+\mathfrak{m}_\xi. \tag{2.17}$$

Therefore, in the Case 1, one can choose $\lambda_* = \mu$ (which is unique then) to fulfill the statement.

List the conditions under which the Case 1 is possible.

1. Let $\widehat{\sigma} = \infty$. Then, by (A4),

$$\frac{1}{\lambda}\mathfrak{a}_\xi(\lambda) \geq \frac{1}{\lambda}\int_r^{r+\delta} a^+(s)e^{\lambda s}\,ds \geq \rho\frac{1}{\lambda^2}\bigl(e^{\lambda(r+\delta)} - e^{\lambda r}\bigr) \to \infty, \tag{2.18}$$

as $\lambda \to \infty$. Then, in such a case, $G_\xi(\infty) = \infty$. Therefore, by (2.14), there exists a zero of G'_ξ.

2. Let $\widehat{\sigma}<\infty$ and $\mathfrak{a}_\xi(\widehat{\sigma})=\infty$. Then, again, (2.14) implies the existence of a zero of G'_ξ on $(0,\widehat{\sigma})$.
3. Let $\widehat{\sigma}<\infty$ and $\mathfrak{a}_\xi(\widehat{\sigma})<\infty$. By (2.11), (2.15),

$$\lim_{\lambda\to 0+}\lambda^2 G'_\xi(\lambda)=-F_\xi(0+)=-(\varkappa^+-m)<0.$$

   Therefore, the function G'_ξ has a zero on $(0,\widehat{\sigma})$ if and only if takes a positive value at some point from $(0,\widehat{\sigma})$.

Now, one can formulate and consider the opposite to the Case 1.
Case 2. Let $\widehat{\sigma}<\infty$, $\mathfrak{a}_\xi(\widehat{\sigma})<\infty$, and

$$G'_\xi(\lambda)<0,\qquad \lambda\in(0,\widehat{\sigma}). \tag{2.19}$$

Therefore,

$$\inf_{\lambda>0}G_\xi(\lambda)=\inf_{\lambda\in(0,\widehat{\sigma}]}G_\xi(\lambda)=\lim_{\lambda\to\widehat{\sigma}-}G_\xi(\lambda)=G_\xi(\widehat{\sigma}), \tag{2.20}$$

by (L7). Hence we have the first equality in (2.10), by setting $\lambda_*:=\widehat{\sigma}$. To prove the second inequality in (2.10), note that, by (2.15), the inequality (2.19) is equivalent to $F'_\xi(\lambda)<G_\xi(\lambda)$, $\lambda\in(0,\widehat{\sigma})$. Therefore, by (2.20), (2.11), (2.13),

$$G_\xi(\widehat{\sigma})=\inf_{\lambda\in(\frac{\widehat{\sigma}}{2},\widehat{\sigma})}G_\xi(\lambda)\ge\inf_{\lambda\in(\frac{\widehat{\sigma}}{2},\widehat{\sigma})}F'_\xi(\lambda)\ge\varkappa^+\mathfrak{a}'_\xi\Bigl(\frac{\widehat{\sigma}}{2}\Bigr)>\varkappa^+\mathfrak{m}_\xi,$$

where we used again that, by (2.12), $\mathfrak{a}'_\xi$ and hence F'_ξ are increasing on $(0,\widehat{\sigma})$. The statement is fully proved now. $\square$

The second case in the proof of Proposition 2.4 requires additional analysis. Let $\xi\in S^{d-1}$ be fixed and $a^+\in\mathcal{U}_\xi$, $\widehat{\sigma}:=\sigma(a^+)$. By (L2), one can define the following function

$$\mathfrak{t}_\xi(\lambda):=\varkappa^+\int_{\mathbb{R}}(1-\lambda s)a^+(s)e^{\lambda s}\,ds\in\mathbb{R},\qquad \lambda\in[0,\widehat{\sigma}). \tag{2.21}$$

Note that

$$\int_{\mathbb{R}_-}|s|a^+(s)e^{\widehat{\sigma}s}\,ds<\infty, \tag{2.22}$$

and $\int_{\mathbb{R}_+}sa^+(s)e^{\widehat{\sigma}s}\,ds\in(0,\infty]$ is well-defined. Then, in the case $\widehat{\sigma}<\infty$ and $\mathfrak{a}_\xi(\widehat{\sigma})<\infty$, one can continue $\mathfrak{t}_\xi$ at $\widehat{\sigma}$, namely,

$$\mathfrak{t}_\xi(\widehat{\sigma}):=\varkappa^+\int_{\mathbb{R}}(1-\widehat{\sigma}s)a^+(s)e^{\widehat{\sigma}s}\,ds\in[-\infty,\varkappa^+). \tag{2.23}$$

To prove the latter inclusion, i.e. the strict inequality $\mathfrak{t}_\xi(\widehat{\sigma})<\varkappa^+$, consider the function $f_0(s):=(1-\widehat{\sigma}s)e^{\widehat{\sigma}s}$, $s\in\mathbb{R}$. Then, $f'_0(s)=-\widehat{\sigma}^2se^{\widehat{\sigma}s}$, and thus $f_0(s)<f_0(0)=1$, $s\neq 0$. Moreover, the function $g_0(s)=f_0(-s)-f_0(s)$, $s\ge 0$ is such that $g'_0(s)=\widehat{\sigma}^2s(e^{\widehat{\sigma}s}-e^{-\widehat{\sigma}s})>0$, $s>0$. As a result, for any $\delta>0$, $f_0(-\delta)>f_0(\delta)$, and

$$\int_{\mathbb{R}}f_0(s)a^+(s)\,ds\le f_0(-\delta)\int_{\mathbb{R}\setminus[-\delta,\delta]}a^+(s)\,ds+\int_{[-\delta,\delta]}a^+(s)\,ds<\int_{\mathbb{R}}a^+(s)\,ds=1.$$

Proposition 2.5. *Let $\xi \in S^{d-1}$ be fixed and $a^+ \in \mathcal{U}_\xi$. Suppose also that $\widehat{\sigma} := \sigma(a^+) < \infty$ and $\mathfrak{a}_\xi(\widehat{\sigma}) < \infty$. Then (2.19) holds iff*

$$\mathfrak{t}_\xi(\widehat{\sigma}) \in (0, \varkappa^+), \tag{2.24}$$

$$m \le \mathfrak{t}_\xi(\widehat{\sigma}). \tag{2.25}$$

Proof. Define the function, cf. (2.11),

$$H_\xi(\lambda) := \lambda F'_\xi(\lambda) - F_\xi(\lambda), \quad \lambda \in (0, \widehat{\sigma}). \tag{2.26}$$

By (2.15), the condition (2.19) holds iff H_ξ is negative on $(0, \widehat{\sigma})$. By (2.26), (2.12), one has $H'_\xi(\lambda) = \lambda F''_\xi(\lambda) > 0$, $\lambda \in (0, \widehat{\sigma})$ and, therefore, H_ξ is (strictly) increasing on $(0, \widehat{\sigma})$. By Proposition 2.4, G'_ξ, and hence H_ξ, are negative on a right-neighborhood of 0. As a result, $H_\xi(\lambda) < 0$ on $(0, \widehat{\sigma})$ iff

$$\lim_{\lambda \to \widehat{\sigma}-} H_\xi(\lambda) \le 0. \tag{2.27}$$

On the other hand, by (2.11), (2.21), one can rewrite $H_\xi(\lambda)$ as follows:

$$H_\xi(\lambda) = -\mathfrak{t}_\xi(\lambda) + m, \quad \lambda \in (0, \widehat{\sigma}). \tag{2.28}$$

By the monotone convergence theorem,

$$\lim_{\lambda \to \widehat{\sigma}-} \int\limits_{\mathbb{R}_+} (\lambda s - 1) a^+(s) e^{\lambda s}\, ds = \int\limits_{\mathbb{R}_+} (\widehat{\sigma} s - 1) a^+(s) e^{\widehat{\sigma} s}\, ds \in (-1, \infty].$$

Therefore, by (2.22), (2.28), $\mathfrak{t}_\xi(\widehat{\sigma}) \in \mathbb{R}$ iff $H_\xi(\widehat{\sigma}) = \lim\limits_{\lambda \to \widehat{\sigma}-} H_\xi(\lambda) \in \mathbb{R}$. Next, clearly, $H_\xi(\widehat{\sigma}) \in (m - \varkappa^+, 0]$ holds true iff both (2.25) and (2.24) hold.

As a result, (2.19) is equivalent to (2.27) and the latter, by (2.22), implies that $\mathfrak{t}_\xi(\widehat{\sigma}) \in \mathbb{R}$ and hence $H_\xi(\widehat{\sigma}) \in (m - \varkappa^+, 0]$. Vice versa, (2.24) yields $\mathfrak{t}_\xi(\widehat{\sigma}) \in \mathbb{R}$ that together with (2.25) give that $H_\xi(\widehat{\sigma}) \le 0$, i.e. that (2.19) holds. □

According to the above, it is natural to consider two subclasses of functions from $\mathcal{U}_\xi$, cf. Definition 2.3.

Definition 2.6. Let $\xi \in S^{d-1}$ be fixed. We denote by $\mathcal{V}_\xi$ the class of all kernels $a^+ \in \mathcal{U}_\xi$ such that one of the following assumptions does hold:

1. $\widehat{\sigma} := \sigma(a^+) = \infty$;
2. $\widehat{\sigma} < \infty$ and $\mathfrak{a}_\xi(\widehat{\sigma}) = \infty$;
3. $\widehat{\sigma} < \infty$, $\mathfrak{a}_\xi(\widehat{\sigma}) < \infty$ and $\mathfrak{t}_\xi(\widehat{\sigma}) \in [-\infty, m)$, where $\mathfrak{t}_\xi(\widehat{\sigma})$ is given by (2.23).

Correspondingly, we denote by $\mathcal{W}_\xi$ the class of all kernels $a^+ \in \mathcal{U}_\xi$ such that $\widehat{\sigma} < \infty$, $\mathfrak{a}_\xi(\widehat{\sigma}) < \infty$, and $\mathfrak{t}_\xi(\widehat{\sigma}) \in [m, \varkappa^+)$. Clearly, $\mathcal{U}_\xi = \mathcal{V}_\xi \cup \mathcal{W}_\xi$.

As a result, combining the proofs and statements of Propositions 2.4 and 2.5, one immediately gets the following corollary.

Corollary 2.7. *Let $\xi \in S^{d-1}$ be fixed, $a^+ \in \mathcal{U}_\xi$, and λ_* be the same as in Proposition 2.4. Then $\lambda_* < \widehat{\sigma} := \sigma(a^+)$ iff $a^+ \in \mathcal{V}_\xi$; moreover, then $G'(\lambda_*) = 0$. Correspondingly, $\lambda_* = \widehat{\sigma}$ iff $a^+ \in \mathcal{W}_\xi$; in this case,*

$$\lim_{\lambda \to \widehat{\sigma}-} G'_\xi(\lambda) = \frac{m - \mathfrak{t}_\xi(\widehat{\sigma})}{(\widehat{\sigma})^2} \le 0. \tag{2.29}$$

Example 2.8. To demonstrate the cases of Definition 2.6 on an example, consider the following family of functions, cf. (1.21),

$$a^+(s) := \frac{\alpha e^{-\mu|s|^p}}{1+|s|^q}, \quad s \in \mathbb{R}, p \geq 0, q \geq 0, \mu > 0, \tag{2.30}$$

where $\alpha > 0$ is a normalizing constant to get $\int_{\mathbb{R}} a^+(s)\,ds = 1$. Clearly, the case $p \in [0,1)$ implies $\sigma(a^+) = 0$, that is impossible under assumption (A3). Next, $p > 1$ leads to $\sigma(a^+) = \infty$, in particular, the corresponding $a^+ \in \mathcal{V}_\xi$. Let now $p = 1$, then $\sigma(a^+) = \mu$. The case $q \in [0,1]$ gives $\mathfrak{a}_\xi(\widehat{\sigma}) = \infty$, i.e. $a^+ \in \mathcal{V}_\xi$ as well. In the case $q \in (1,2]$, we will have that $\mathfrak{a}_\xi(\widehat{\sigma}) < \infty$, however, $\int_{\mathbb{R}} sa^+(s)e^{\mu s}\,ds = \infty$, i.e. $\mathfrak{t}_\xi(\mu) = -\infty$, and again $a^+ \in \mathcal{V}_\xi$. Let $q > 2$; then, by (2.21),

$$\begin{aligned}
\mathfrak{t}_\xi(\mu) &= \varkappa^+\alpha \int\limits_{\mathbb{R}_-} \frac{1-\mu s}{1+|s|^q} e^{2\mu s}\,ds + \varkappa^+\alpha \int\limits_{\mathbb{R}_+} \frac{1-\mu s}{1+s^q}\,ds \\
&\geq \varkappa^+\alpha \int\limits_{\mathbb{R}_+} \frac{1-\mu s}{1+s^q}\,ds = \frac{\pi\varkappa^+\alpha}{q}\left(\frac{1}{\sin\frac{\pi}{q}} - \frac{\mu}{\sin\frac{2\pi}{q}}\right) \geq m,
\end{aligned}$$

if only $\mu \leq 2\cos\frac{\pi}{q} - \frac{mq}{\varkappa^+\alpha\pi}\sin\frac{2\pi}{q}$ (note that $q > 2$ implies $\sin\frac{2\pi}{q} > 0$); then we have $a^+ \in \mathcal{W}_\xi$. On the other hand, using the inequality $te^{-t} \leq e^{-1}$, $t \geq 0$, one gets

$$\begin{aligned}
\mathfrak{t}_\xi(\mu) &= \varkappa^+\alpha \int\limits_{\mathbb{R}_+} \frac{(1+\mu s)e^{-2\mu s} + 1 - \mu s}{1+s^q}\,ds \\
&\leq \varkappa^+\alpha \int\limits_{\mathbb{R}_+} \frac{1 + \frac{1}{2e} + 1 - \mu s}{1+s^q}\,ds = \frac{\pi\varkappa^+\alpha}{q}\left(\frac{1+4e}{2e\sin\frac{\pi}{q}} - \frac{\mu}{\sin\frac{2\pi}{q}}\right) < m,
\end{aligned} \tag{2.31}$$

if only $\mu > \frac{1+4e}{e}\cos\frac{\pi}{q} - \frac{mq}{\varkappa^+\alpha\pi}\sin\frac{2\pi}{q}$; then we have $a^+ \in \mathcal{V}_\xi$. Since

$$\frac{d}{d\mu}\big((1+\mu s)e^{-2\mu s} + 1 - \mu s\big) = -se^{-2\mu s}(1+2s\mu) - s < 0, \quad s > 0, \mu > 0,$$

we have from (2.31), that $\mathfrak{t}_\xi(\mu)$ is strictly decreasing and continuous in μ, therefore, there exists a critical value

$$\mu_* \in \left(2\cos\frac{\pi}{q} - \frac{mq}{\varkappa^+\alpha\pi}\sin\frac{2\pi}{q}, (4+e^{-1})\cos\frac{\pi}{q} - \frac{mq}{\varkappa^+\alpha\pi}\sin\frac{2\pi}{q}\right),$$

such that, for all $\mu > \mu_*$, $a^+ \in \mathcal{V}_\xi$, whereas, for $\mu \in (0,\mu_*]$, $a^+ \in \mathcal{W}_\xi$.

Now we are ready to prove the main statement of this subsection.

Theorem 2.9. *Let $\xi \in S^{d-1}$ be fixed and $a^+ \in \mathcal{U}_\xi$. Let $c_*(\xi)$ be the minimal traveling wave speed according to Theorem 1.1, and let, for any $c \geq c_*(\xi)$, the function $\psi = \psi_c \in \mathcal{M}_\theta(\mathbb{R})$ be a traveling wave profile corresponding to the speed c. Let $\lambda_* \in I_\xi$ be the same as in Proposition 2.4. Denote, as usual, $\widehat{\sigma} := \sigma(a^+)$. Then*

1. *Theorem 1.2 holds.*
2. *For $a^+ \in \mathcal{V}_\xi$, one has $\lambda_* < \widehat{\sigma}$ and there exists another representation for the minimal speed than (1.13), namely,*

$$
\begin{aligned}
c_*(\xi) &= \varkappa^+ \int\limits_{\mathbb{R}^d} x\cdot\xi\, \boldsymbol{a}^+(x) e^{\lambda_* x\cdot\xi}\, dx \\
&= \varkappa^+ \int\limits_{\mathbb{R}} s a^+(s) e^{\lambda_* s}\, ds > \varkappa^+ \mathfrak{m}_\xi.
\end{aligned} \tag{2.32}
$$

   Moreover, for all $\lambda \in (0, \lambda_*]$,

$$
\mathfrak{t}_\xi(\lambda) \geq m, \tag{2.33}
$$

   and the equality holds for $\lambda = \lambda_*$ only.
3. *For $a^+ \in \mathcal{W}_\xi$, one has $\lambda_* = \widehat{\sigma}$. Moreover, the inequality (2.33) also holds as well as, for all $\lambda \in (0, \lambda_*]$,*

$$
c \geq \varkappa^+ \int\limits_{\mathbb{R}} s a^+(s) e^{\lambda s}\, ds, \tag{2.34}
$$

   whereas the equalities in (2.33) and (2.34) hold true now for $m = \mathfrak{t}_\xi(\widehat{\sigma})$, $\lambda = \lambda_*$, $c = c_*(\xi)$ only.

Proof. By Theorem 1.1, for any $c \geq c_*(\xi)$, there exists a profile $\psi \in \mathcal{M}_\theta(\mathbb{R})$, cf. Remark 2.2, which define a traveling wave solution (1.4) to (1.1) in the direction ξ. Then, by (1.8), we get

$$
\begin{aligned}
-c\psi'(s) &= \varkappa^+ (a^+ * \psi)(s) - m\psi(s) \\
&\quad - \varkappa_\ell \psi^2(s) - \varkappa_{n\ell}\psi(s)(a^- * \psi)(s), \ \ s \in \mathbb{R}.
\end{aligned} \tag{2.35}
$$

Step 1. By (1.7), we have that $\sigma(\psi) > 0$. Rewrite (A2) as follows

$$
\varkappa^+ a^+(s) \geq \varkappa_{n\ell}\theta a^-(s), \quad \text{a.a. } s \in \mathbb{R}, \tag{2.36}
$$

therefore, $\sigma(a^-) \geq \sigma(a^+) > 0$, if $\varkappa_{n\ell} > 0$. Take any $z \in \mathbb{C}$ with

$$
\begin{aligned}
0 < \operatorname{Re} z &< \min\bigl\{\sigma(a^+), \sigma(\psi)\bigr\} \leq \sigma(\psi) \\
&< \min\{\sigma(\psi^2), \sigma\bigl(\psi(a^- * \psi)\bigr)\},
\end{aligned} \tag{2.37}
$$

where the later inequality holds by (2.4) and (2.5). As a result, by (L5), (L8), being multiplied on e^{zs} the l.h.s. of (2.35) will be integrable (in s) over $\mathbb{R}$. Hence, for any z which satisfies (2.37), $(\mathfrak{L}\psi')(z)$ converges. By (L4), it yields $\sigma(\psi) \geq \sigma(\psi') \geq \min\bigl\{\sigma(a^+), \sigma(\psi)\bigr\}$.

Therefore, by (2.1), (2.2), we get from (2.35)

$$
\begin{aligned}
cz(\mathfrak{L}\psi)(z) &= \varkappa^+ (\mathfrak{L}a^+)(z)(\mathfrak{L}\psi)(z) - m(\mathfrak{L}\psi)(z) \\
&\quad - \varkappa_\ell\bigl(\mathfrak{L}(\psi^2)\bigr)(z) - \varkappa_{n\ell}\bigl(\mathfrak{L}(\psi(a^- * \psi))\bigr)(z),
\end{aligned} \tag{2.38}
$$

if only

$$
0 < \operatorname{Re} z < \min\bigl\{\sigma(a^+), \sigma(\psi)\bigr\}. \tag{2.39}
$$

Since $\psi \not\equiv 0$, we have that $(\mathfrak{L}\psi)(z) \neq 0$, therefore, one can rewrite (2.38) as follows

$$G_\xi(z) - c = \frac{\varkappa_\ell\big(\mathfrak{L}(\psi^2)\big)(z) + \varkappa_{n\ell}\big(\mathfrak{L}(\psi(a^- * \psi))\big)(z)}{z(\mathfrak{L}\psi)(z)}, \tag{2.40}$$

if (2.39) holds. By (2.37), both nominator and denominator in the r.h.s. of (2.40) are analytic on $0 < \operatorname{Re} z < \sigma(\psi)$, therefore. Suppose that $\sigma(\psi) > \sigma(a^+)$, then (2.40) holds on $0 < \operatorname{Re} z < \sigma(a^+)$, however, the r.h.s. of (2.40) would be analytic at $z = \sigma(a^+)$, whereas, by (L3), the l.h.s. of (2.40) has a singularity at this point. As a result,

$$\sigma(a^+) \geq \sigma(\psi), \tag{2.41}$$

for any traveling wave profile $\psi \in \mathcal{M}_\theta(\mathbb{R})$. Thus one gets that (2.40) holds true on $0 < \operatorname{Re} z < \sigma(\psi)$.

Prove that

$$\sigma(\psi) < \infty. \tag{2.42}$$

Since $0 \leq \psi \leq \theta$ yields $0 \leq \boldsymbol{a}^- * \psi \leq \theta$, one gets from (2.40) that, for any $0 < \lambda < \sigma(\psi)$,

$$c \geq G_\xi(\lambda) - \varkappa^- \frac{\theta}{\lambda} = \frac{\varkappa^+(\mathfrak{L}a^+)(\lambda) - \varkappa^+}{\lambda}. \tag{2.43}$$

If $\sigma(a^+) < \infty$ then (2.42) holds by (2.41). Suppose that $\sigma(a^+) = \infty$. By (2.18), the r.h.s. of (2.43) tends to ∞ as $\lambda \to \infty$, thus the latter inequality cannot hold for all $\lambda > 0$; and, as a result, (2.42) does hold.

Step 2. Recall that (2.8) holds. Suppose that $c \geq c_*(\xi)$ is such that, cf. (2.10),

$$c \geq G_\xi(\lambda_*) = \inf_{\hat{\sigma} \in (0,\lambda_*]} G_\xi(\lambda) = \inf_{\hat{\sigma} \in I_\xi} G_\xi(\lambda). \tag{2.44}$$

Then, by Proposition 2.4, the equation $G_\xi(\lambda) = c$, $\lambda \in I_\xi$, has one or two solutions. Let λ_c be the unique solution in the first case or the smaller of the solutions in the second one. Since G_ξ is decreasing on $(0, \lambda_*]$, we have $\lambda_c \leq \lambda_*$. Since the nominator in the r.h.s. of (2.40) is positive, we immediately get from (2.40) that

$$(\mathfrak{L}\psi)(\lambda_c) = \infty, \tag{2.45}$$

therefore, $\lambda_c \geq \sigma(\psi)$. On the other hand, one can rewrite (2.40) as follows

$$(\mathfrak{L}\psi)(z) = \frac{\varkappa_\ell\big(\mathfrak{L}(\psi^2)\big)(z) + \varkappa_{n\ell}\big(\mathfrak{L}(\psi(a^- * \psi))\big)(z)}{z(G_\xi(z) - c)}. \tag{2.46}$$

By (2.40), $G_\xi(z) \neq c$, for all $0 < \operatorname{Re} z < \sigma(\psi) \leq \lambda_c \leq \lambda_* \leq \sigma(a^+)$. As a result, by (2.37), (L1), and (L3), $\lambda_c = \sigma(\psi)$, that together with (2.45) proves (1.12) and (1.15), for waves whose speeds satisfy (2.44). By (A3), (2.6), we immediately get, for such speeds, (1.13) as well. Moreover, (1.13) defines a strictly monotone function $(0, \lambda_*] \ni \sigma(\psi) \mapsto c \in [G_\xi(\lambda_*), \infty)$.

Next, by (2.21), (L2), (2.11), (2.15), we have that, for any $\lambda \in I_\xi$,

$$\mathfrak{t}_\xi(\lambda) = \varkappa^+ \mathfrak{a}_\xi(\lambda) - \varkappa^+ \lambda \mathfrak{a}_\xi'(\lambda) = m + F_\xi(\lambda) - \lambda F_\xi'(\lambda) = m - \lambda^2 G_\xi'(\lambda). \tag{2.47}$$

Recall that, by Proposition 2.4, the function G_ξ is strictly decreasing on $(0, \lambda_*)$. Then (2.47) implies that $\mathfrak{t}_\xi(\lambda) > m$, $\lambda \in (0, \lambda_*)$. On the other hand, by the second equality in (2.15), the inequality $G_\xi'(\lambda) < 0$, $\lambda \in (0, \lambda_*)$, yields $G_\xi(\lambda) > F_\xi'(\lambda)$, for such a λ. Let $c > G_\xi(\lambda_*)$. By (1.13), (2.11), we have then $c > \varkappa^+ \mathfrak{a}_\xi'(\lambda)$, for all $\lambda \in [\sigma(\psi), \lambda_*)$. By (2.12), F'_ξ is increasing, hence, by (L2), the strict inequality in (2.34) does hold, for $\lambda \in (0, \lambda_*)$.

Let again $c \geq G_\xi(\lambda_*)$, and let $a^+ \in \mathcal{V}_\xi$. Then, by Corollary 2.7, $\lambda_* < \sigma(a^+)$ and $G'(\lambda_*) = 0$. By (2.15), the latter equality and (2.47) give $\mathfrak{t}_\xi(\lambda_*) = m$, that fulfills the proof of (2.33), for such a^+ and m. Moreover, by (2.17),

$$G_\xi(\lambda_*) = \varkappa^+ \mathfrak{a}'_\xi(\lambda_*) = \varkappa^+ \int\limits_{\mathbb{R}} s a^+(s) e^{\lambda_* s}\, ds. \tag{2.48}$$

Let $a^+ \in \mathcal{W}_\xi$, then $\lambda_* = \sigma(a^+)$. It means that $\mathfrak{t}_\xi(\lambda_*) = m$ if $m = \mathfrak{t}_\xi(\widehat{\sigma})$ only, otherwise, $\mathfrak{t}_\xi(\lambda_*) > m$. Next, we get from (2.44), (2.15) (2.29),

$$c \geq G_\xi(\lambda_*) \geq \lim_{\lambda \to \lambda_*-} F'_\xi(\lambda) = \varkappa^+ \int\limits_{\mathbb{R}} s a^+(s) e^{\lambda_* s}\, ds, \tag{2.49}$$

where the latter equality may be easily verified if we rewrite, for $\lambda \in (0, \lambda_*)$,

$$F'_\xi(\lambda) = \varkappa^+ \int\limits_{\mathbb{R}_-} s a^+(s) e^{\lambda s}\, ds + \varkappa^+ \int\limits_{\mathbb{R}_+} s a^+(s) e^{\lambda s}\, ds, \tag{2.50}$$

and apply the dominated convergence theorem to the first integral and the monotone convergence theorem for the second one. On the other hand, (2.29) implies that the second inequality in (2.49) will be strict iff $m < \mathfrak{t}_\xi(\widehat{\sigma})$, whereas, for $c = G_\xi(\lambda_*) = \inf\limits_{\lambda>0} G_\xi(\lambda)$ and $m = \mathfrak{t}_\xi(\widehat{\sigma})$, we will get all equalities in (2.49).

Step 3. Let now $c \geq c_*(\xi)$ and suppose that $\sigma(a^+) > \sigma(\psi)$. Prove that (2.44) does hold. On the contrary, suppose that the c is such that

$$c_*(\xi) \leq c < \inf_{\lambda \in (0,\lambda_*]} G_\xi(\lambda) = \inf_{\lambda>0} G_\xi(\lambda). \tag{2.51}$$

Again, by (2.40), $G_\xi(z) \neq c$, for all $0 < \operatorname{Re} z < \sigma(\psi)$, and (2.46) holds, for such a z. Since we supposed that $\sigma(a^+) > \sigma(\psi)$, one gets from (2.37), that both nominator and denominator of the r.h.s. of (2.46) are analytic on

$$\{0 < \operatorname{Re} z < \nu\} \supsetneq \{0 < \operatorname{Re} z < \sigma(\psi)\},$$

where $\nu = \min\bigl\{\sigma(a^+), \sigma\bigl(\psi(a^- * \psi)\bigr), \sigma(\psi^2)\bigr\}$. On the other hand, (L3) implies that $\mathfrak{L}\psi$ has a singularity at $z = \sigma(\psi)$. Since

$$\min\{\bigl(\mathfrak{L}(\psi^2)\bigr)(\sigma(\psi)), \bigl(\mathfrak{L}(\psi(a^- * \psi))\bigr)(\sigma(\psi))\} > 0,$$

the equality (2.46) would be possible if only $G_\xi(\sigma(\psi)) = c$, that contradicts (2.51).

Step 4. By (2.41), it remains to prove that, for $c \geq c_*(\xi)$, (2.44) does holds, provided that we have $\sigma(a^+) = \sigma(\psi)$. Again on the contrary, suppose that (2.51) holds. For $0 < \operatorname{Re} z < \sigma(\psi)$, we can rewrite (2.38) as follows

$$z(\mathfrak{L}\psi)(z)(G_\xi(z) - c) = \varkappa_\ell\bigl(\mathfrak{L}(\psi^2)\bigr)(z) + \varkappa_{n\ell}\bigl(\mathfrak{L}(\psi(a^- * \psi))\bigr)(z). \tag{2.52}$$

In the notations of the proof of Lemma 2.1, the functions $\mathfrak{L}^-\psi$ and $\mathfrak{L}^-a^+$ are analytic on $\operatorname{Re} z > 0$. Moreover, $(\mathfrak{L}^+\psi)(\lambda)$ and $(\mathfrak{L}^+a^+)(\lambda)$ are increasing on $0 < \lambda < \sigma(a^+) = \sigma(\psi)$. Then, cf. (2.50), by the monotone convergence theorem, we will get from (2.52) and (2.37), that

$$\int_{\mathbb{R}} \psi(s)e^{\sigma(\psi)s}\,ds < \infty, \qquad \int_{\mathbb{R}} a^+(s)e^{\sigma(a^+)s}\,ds < \infty. \tag{2.53}$$

We are going to apply now [15, Proposition 2.10], in the case $d = 1$, to the equation

$$\begin{cases} \dfrac{\partial \phi}{\partial t}(s,t) = \varkappa^+(a^+ * \phi)(s,t) - m\phi(s,t) - \varkappa_\ell \phi^2(s,t) \\ \qquad\qquad - \varkappa_{n\ell}\phi(s,t)(a^- * \phi)(s,t), \qquad t > 0, \text{ a.a. } s \in \mathbb{R}, \\ \phi(s,0) = \psi(s), \qquad \text{a.a. } s \in \mathbb{R}, \end{cases} \tag{2.54}$$

where the initial condition ψ is a wave profile with the speed c which satisfies (2.51). Namely, we set $\Delta_R := (-\infty, R) \nearrow \mathbb{R}$, $R \to \infty$ and

$$a_R^\pm(s) := 1\!\!1_{\Delta_R}(s)a^\pm(s), \quad s \in \mathbb{R}, \tag{2.55}$$

$$A_R^\pm := \int_{\Delta_R} a^\pm(x)\,dx \nearrow 1, \quad R \to \infty. \tag{2.56}$$

Consider a strictly monotone sequence $\{R_n \mid n \in \mathbb{N}\}$, such that $0 < R_n \to \infty$, $n \to \infty$ and

$$A_{R_n}^+ > \frac{m}{\varkappa^+} \in (0,1). \tag{2.57}$$

Let $\theta_n := \theta_{R_n}$ be given by

$$\theta_{R_n} = \frac{\varkappa^+ A_{R_n}^+ - m}{\varkappa_{n\ell} A_{R_n}^- + \varkappa_\ell} \to \theta, \quad R_n \to \infty. \tag{2.58}$$

Then, by [15, formula (2.17)], $\theta_n \le \theta$, $n \in \mathbb{N}$.

Fix an arbitrary $n \in \mathbb{N}$. Consider the 'truncated' equation (2.54) with $a^\pm$ replaced by $a_{R_n}^\pm$, and the initial condition $w_0(s) := \min\{\psi(s), \theta_n\} \in C_{ub}(\mathbb{R})$. By [15, Proposition 2.10], there exists the unique solution $w^{(n)}(s,t)$ of the latter equation. Moreover, if we denote the corresponding nonlinear mapping by $\tilde{Q}_t^{(n)}$, we will have from [15, formulas (2.15)–(2.16)], that

$$(\tilde{Q}_t^{(n)} w_0)(s) \le \theta_n, \quad s \in \mathbb{R}, t \ge 0, \tag{2.59}$$

and

$$(\tilde{Q}_t^{(n)} w_0)(s) \le \phi(s,t), \tag{2.60}$$

where ϕ solves (2.54).

By [15, Remark 3.4], we get from (2.60) that $(\tilde{Q}_1^{(n)} w_0)(s+c) \le \psi(s)$, $s \in \mathbb{R}$. The latter inequality together with (2.59) imply

$$(\tilde{Q}_1^{(n)} w_0)(s+c) \le w_0(s). \tag{2.61}$$

Then, by the same arguments as in the proof of [15, Theorem 1.1], we obtain from [31, Theorem 5] that there exists a traveling wave ψ_n for the equation (2.54) with $a^\pm$ replaced by $a_{R_n}^\pm$, whose speed will be exactly c (and c satisfies (2.51)).

Now we are going to get a contradiction, by proving that

$$\inf_{\lambda>0} G_\xi(\lambda) = \lim_{n\to\infty} \inf_{\lambda>0} G_\xi^{(n)}(\lambda), \tag{2.62}$$

where $G_\xi^{(n)}$ is given by (2.7) with $a^\pm$ replaced by $a_n^\pm := a_{R_n}^\pm$. The sequence of functions $G_\xi^{(n)}$ is point-wise monotone in n and it converges to G_ξ point-wise, for $0 < \lambda \le \sigma(a^+)$; note we may include $\sigma(a^+)$ here, according to (2.53). Moreover, $G_\xi^{(n)}(\lambda) \le G_\xi(\lambda)$, $0 < \lambda \le \sigma(a^+)$. As a result, for any $n \in \mathbb{N}$,

$$G_\xi^{(n)}(\lambda_*^{(n)}) = \inf_{\lambda>0} G_\xi^{(n)}(\lambda) \le \inf_{\lambda>0} G_\xi(\lambda) = G_\xi(\lambda_*). \tag{2.63}$$

Hence if we suppose that (2.62) does not hold, then

$$\inf_{\lambda>0} G_\xi(\lambda) - \lim_{n\to\infty} \inf_{\lambda>0} G_\xi^{(n)}(\lambda) > 0.$$

Therefore, there exist $\delta > 0$ and $N \in \mathbb{N}$, such that

$$G_\xi^{(n)}(\lambda_*^{(n)}) = \inf_{\lambda>0} G_\xi^{(n)}(\lambda) \le \inf_{\lambda>0} G_\xi(\lambda) - \delta = G_\xi(\lambda_*) - \delta, \quad n \ge N. \tag{2.64}$$

Clearly, (2.55) with $\Delta_{R_n} = (-\infty, R_n)$ implies that $\sigma(a_n^+) = \infty$, hence $G_\xi^{(n)}$ is analytic on $\operatorname{Re} z > 0$. One can repeat all considerations of the first three steps of this proof for the equation (2.54). Let $c_*^{(n)}(\xi)$ be the corresponding minimal traveling wave speed, according to Theorem 1.1. Then the corresponding inequality (2.42) will show that the abscissa of an arbitrary traveling wave to (2.54) (with $a^\pm$ replaced by $a_{R_n}^\pm$) is less than $\sigma(a_n^+) = \infty$. As a result, the inequality $c_*^{(n)}(\xi) < \inf\limits_{\lambda>0} G_\xi^{(n)}(\lambda)$, cf. (2.51), is impossible, and hence, by the Step 3,

$$c \ge c_*^{(n)}(\xi) = \inf_{\lambda>0} G_\xi^{(n)}(\lambda) = G_\xi^{(n)}(\lambda_*^{(n)}), \tag{2.65}$$

where $\lambda_*^{(n)}$ is the unique zero of the function $\frac{d}{d\lambda} G_\xi^{(n)}(\lambda)$. Let $\mathfrak{t}_\xi^{(n)}$ be given on $(0,\infty)$ by (2.21) with a^+ replaced by a_n^+. Then

$$\frac{d}{d\lambda}\mathfrak{t}_\xi^{(n)}(\lambda) = -\lambda\varkappa^+ \int_{-\infty}^{R_n} a^+(s) s^2 e^{\lambda s}\, ds < 0, \quad \lambda > 0. \tag{2.66}$$

By (2.33), the unique point of intersection of the strictly decreasing function $y = \mathfrak{t}_\xi^{(n)}(\lambda)$ and the horizontal line $y = m$ is exactly the point $(\lambda_*^{(n)}, 0)$.

Prove that there exist $\lambda_1 > 0$, such that $\lambda_*^{(n)} > \lambda_1$, $n \ge N$, and there exists $N_1 \ge N$, such that $\mathfrak{t}_\xi^{(n)}(\lambda) \le \mathfrak{t}_\xi^{(m)}(\lambda)$, $n > m \ge N_1$, $\lambda \ge \lambda_1$. Recall that (2.57) holds; we have

$$\begin{aligned}
\lambda G_\xi^{(n)}(\lambda) &= \varkappa^+ \int_{\mathbb{R}} a_n^+(s)(e^{\lambda s} - 1)\, ds + \varkappa^+ A_{R_n}^+ - m \\
&\ge \varkappa^+ \int_{-\infty}^{0} a_n^+(s)(e^{\lambda s} - 1)\, ds + \varkappa^+ A_{R_1}^+ - m,
\end{aligned}$$

and the inequality $1 - e^{-s} \le s$, $s \ge 0$ implies that

$$\biggl| \int_{-\infty}^{0} a_n^+(s)(e^{\lambda s}-1)\,ds \biggr| \le \lambda \int_{-\infty}^{0} a_n^+(s)|s|\,ds \le \lambda \int_{\mathbb{R}} a^+(s)|s|\,ds < \infty,$$

by (A6). As a result, if we set

$$\lambda_1 := (\varkappa^+ A^+_{R_1} - m)\biggl(\varkappa^+ \int_{\mathbb{R}} a^+(s)|s|\,ds + |G_\xi(\lambda_*)| \biggr)^{-1} > 0,$$

then, for any $\lambda \in (0,\lambda_1)$, we have

$$\lambda G_\xi^{(n)}(\lambda) \ge \varkappa^+ A^+_{R_1} - m - \lambda_1 \varkappa^+ \int_{\mathbb{R}} a_n^+(s)|s|\,ds = \lambda_1 |G_\xi(\lambda_*)| \ge \lambda G_\xi(\lambda_*),$$

i.e. $G_\xi^{(n)}(\lambda) \ge G_\xi(\lambda_*) = \inf\limits_{\lambda>0} G_\xi(\lambda)$. Then, for any $n \ge N$, (2.64) implies that $\lambda_*^{(n)}$, being the minimum point for $G_\xi^{(n)}$, does not belong to the interval $(0,\lambda_1)$. Next, let $N_1 \ge N$ be such that $R_n \ge \frac{1}{\lambda_1}$, for all $n \ge N_1$. Then, for any $\lambda \ge \lambda_1$, and for any $n > m \ge N_1$, we have $R_n > R_m$ and

$$\begin{aligned} \mathfrak{t}_\xi^{(n)}(\lambda) - \mathfrak{t}_\xi^{(m)}(\lambda) &= \varkappa^+ \int_{R_m}^{R_n} (1-\lambda s) a^+(s) e^{\lambda s}\,ds \\ &\le \varkappa^+ \int_{R_m}^{R_n} (1-\lambda_1 s) a^+(s) e^{\lambda s}\,ds \le 0. \end{aligned}$$

As a result, the sequence $\{\lambda_*^{(n)} \mid n \ge N_1\} \subset [\lambda_1,\infty)$ is monotonically decreasing (cf. (2.66)). We set

$$\vartheta := \lim_{n\to\infty} \lambda_*^{(n)} \ge \lambda_1. \tag{2.67}$$

Next, for any $n, m \in \mathbb{N}$, $n > m \ge N_1$,

$$G_\xi^{(n)}(\lambda_*^{(n)}) \ge G_\xi^{(m)}(\lambda_*^{(n)}) \ge G_\xi^{(m)}(\lambda_*^{(m)}), \tag{2.68}$$

where we used that $G_\xi^{(n)}$ is increasing in n and $\lambda_*^{(m)}$ is the minimum point of $G_\xi^{(m)}$. Therefore, the sequence $\{G_\xi^{(n)}(\lambda_*^{(n)})\}$ is increasing and, by (2.64), is bounded. Then, there exists

$$\lim_{n\to\infty} G_\xi^{(n)}(\lambda_*^{(n)}) =: g \le G_\xi(\lambda_*) - \delta. \tag{2.69}$$

Fix $m \ge N_1$ in (2.68) and pass n to infinity; then, by the continuity of $G_\xi^{(n)}$,

$$g \ge \lim_{\lambda\to\vartheta+} G_\xi^{(m)}(\lambda) = G_\xi^{(m)}(\vartheta) \ge G_\xi^{(m)}(\lambda^{(m)}), \tag{2.70}$$

in particular, $\vartheta > 0$, as $G_\xi^{(m)}(0+) = \infty$. Next, if we pass m to ∞ in (2.70), we will get from (2.69)

$$\lim_{m\to\infty} G_\xi^{(m)}(\vartheta) = g \le G_\xi(\lambda_*) - \delta < G_\xi(\lambda_*). \tag{2.71}$$

If $0 < \vartheta \le \sigma(a^+)$ then

$$\lim_{m\to\infty} G_\xi^{(m)}(\vartheta) = G_\xi(\vartheta) \geq G_\xi(\lambda_*),$$

that contradicts (2.71). If $\vartheta > \sigma(a^+)$, then $\lim\limits_{m\to\infty} G_\xi^{(m)}(\vartheta) = \infty$ (recall again that $\mathfrak{L}^-(a^+)(\lambda)$ is analytic and $\mathfrak{L}^-(a^+)(\lambda)$ is monotone in λ), that contradicts (2.71) as well.

The contradiction we obtained shows that (2.62) does hold. Then, for the chosen $c \geq c_*(\xi)$ which satisfies (2.51), one can find n big enough to ensure that, cf. (2.65),

$$c < \inf_{\lambda>0} G_\xi^{(n)}(\lambda) = c_*^{(n)}(\xi).$$

However, as it was shown above, for this n there exists a profile ψ_n of a traveling wave to the 'truncated' equation (2.54) with $a^\pm$ replaced by $a^\pm_{R_n}$. The latter contradicts the statement of Theorem 1.1 applied to this equation, as $c_*^{(n)}(\xi)$ has to be a minimal possible speed for such waves.

Therefore, the strict inequality in (2.51) is impossible, hence, we have equality in (2.8). As a result, (A3) and (2.6) imply (1.11), and (2.48) may be read as (2.32). The rest of the statement is evident now. □

Remark 2.10. Clearly, the assumption $\boldsymbol{a}^+(-x) = \boldsymbol{a}^+(x)$, $x \in \mathbb{R}^d$, implies $\mathfrak{m}_\xi = 0$, for any $\xi \in S^{d-1}$. As a result, all speeds of traveling waves in any directions are positive, by (1.11).

3. Asymptotic and uniqueness

In this subsection we will prove the uniqueness (up to shifts) of a profile ψ for a traveling wave with given speed $c \geq c_*(\xi)$, $c \neq 0$. We will use the almost traditional now approach, namely, we find an *a priori* asymptotic for $\psi(t)$, $t \to \infty$, cf. e.g. [1,4] and the references therein.

We start with the so-called characteristic function of the equation (1.1). Namely, for a given $\xi \in S^{d-1}$ and for any $c \in [c_*(\xi), \infty)$, we set

$$\mathfrak{h}_{\xi,c}(z) := \varkappa^+(\mathfrak{L}a^+)(z) - m - zc = zG_\xi(z) - zc, \qquad \operatorname{Re} z \in I_\xi. \tag{3.1}$$

Proposition 3.1. *Let $\xi \in S^{d-1}$ be fixed, $a^+ \in \mathcal{U}_\xi$, $\widehat{\sigma} := \sigma(a^+)$, $c_*(\xi)$ be the minimal traveling wave speed in the direction ξ. Let, for any $c \geq c_*(\xi)$, the function $\psi \in \mathcal{M}_\theta(\mathbb{R})$ be a traveling wave profile corresponding to the speed c. For the case $a^+ \in \mathcal{W}_\xi$ with $m = \mathfrak{t}_\xi(\widehat{\sigma})$, we will assume, additionally, that*

$$\int_{\mathbb{R}} s^2 a^+(s) e^{\widehat{\sigma} s}\, ds < \infty. \tag{3.2}$$

Then the function $\mathfrak{h}_{\xi,c}$ is analytic on $\{0 < \operatorname{Re} z < \sigma(\psi)\}$. Moreover, for any $\beta \in (0, \sigma(\psi))$, the function $\mathfrak{h}_{\xi,c}$ is continuous and does not equal to 0 on the closed strip $\{\beta \leq \operatorname{Re} z \leq \sigma(\psi)\}$, except the root at $z = \sigma(\psi)$, whose multiplicity j may be 1 or 2 only.

Proof. By (2.40) and the arguments around, $\mathfrak{h}_{\xi,c}(z) = z(G_\xi(z) - c)$ is analytic on $\{0 < \operatorname{Re} z < \sigma(\psi)\} \subset I_\xi$ and does not equal to 0 there. Then, by (1.13) and Proposition 2.4, the smallest positive root of the function $\mathfrak{h}_{\xi,c}(\lambda)$ on $\mathbb{R}$ is exactly $\sigma(\psi)$. Prove that if $z_0 := \sigma(\psi) + i\beta$ is a root of $\mathfrak{h}_{\xi,c}$, then $\beta = 0$. Indeed, $\mathfrak{h}_{\xi,c}(z_0) = 0$ yields

$$\varkappa^+ \int_{\mathbb{R}} a^+(s) e^{\sigma(\psi)s} \cos\beta s\, ds = m + c\sigma(\psi),$$

that together with (1.13) leads to

$$\varkappa^+ \int_{\mathbb{R}} a^+(s) e^{\sigma(\psi)s} (\cos \beta s - 1)\, ds = 0,$$

and thus $\beta = 0$.

Regarding multiplicity of the root $z = \sigma(\psi)$, we note that, by Proposition 2.4 and Corollary 2.7, there exist two possibilities. If $a^+ \in \mathcal{V}_\xi$, then $\sigma(\psi) \le \lambda_* < \sigma(a^+)$ and, therefore, G_ξ is analytic at $z = \sigma(\psi)$. By the second equality in (3.1), the multiplicity j of this root for $\mathfrak{h}_{\xi,c}$ is the same as for the function $G_\xi(z) - c$. By Proposition 2.4, G_ξ is strictly decreasing on $(0, \lambda_*)$ and, therefore, $j = 1$ for $c > c_*(\xi)$. By Corollary 2.7, for $c = c_*(\xi)$, we have $G'_\xi(\sigma(\psi)) = G'_\xi(\lambda_*) = 0$ and, since $\mathfrak{h}''_{\xi,c}(\widehat{\sigma}) > 0$, one gets $j = 2$.

Let now $a^+ \in \mathcal{W}_\xi$. Then, we recall, $\lambda_* = \widehat{\sigma} := \sigma(a^+) < \infty$, $G_\xi(\widehat{\sigma}) < \infty$ and (2.29) hold. For $c > c_*(\xi)$, the arguments are the same as before, and they yield $j = 1$. Let $c = c_*(\xi)$. Then $\mathfrak{h}_{\xi,c}(\widehat{\sigma}) = 0$, and, for all $z \in \mathbb{C}$, $\operatorname{Re} z \in (0, \widehat{\sigma})$, one has

$$\begin{aligned} \mathfrak{h}_{\xi,c}(\widehat{\sigma} - z) = \mathfrak{h}_{\xi,c}(\widehat{\sigma} - z) - \mathfrak{h}_{\xi,c}(\widehat{\sigma}) &= \varkappa^+ \int_{\mathbb{R}} a^+(\tau)(e^{(\widehat{\sigma}-z)\tau} - e^{\widehat{\sigma}\tau}) d\tau + cz \\ &= z\Bigl(-\varkappa^+ \int_{\mathbb{R}} a^+(\tau) e^{\widehat{\sigma}\tau} \int_0^\tau e^{-zs}\, ds d\tau + c \Bigr). \end{aligned} \tag{3.3}$$

Let $z = \alpha + \beta i$, $\alpha \in (0, \widehat{\sigma})$. Then $\bigl|e^{\widehat{\sigma}\tau} e^{-zs}\bigr| = e^{\widehat{\sigma}\tau - \alpha s}$. Next, for $\tau \ge 0$, $s \in [0, \tau]$, we have $e^{\widehat{\sigma}\tau - \alpha s} \le e^{\widehat{\sigma}\tau}$; whereas, for $\tau < 0$, $s \in [\tau, 0]$, one has $e^{\widehat{\sigma}\tau - \alpha s} = e^{\sigma(\tau - s)} e^{(\widehat{\sigma} - \alpha)s} \le 1$. As a result, $\bigl|e^{\widehat{\sigma}\tau} e^{-zs}\bigr| \le e^{\widehat{\sigma} \max\{\tau, 0\}}$. Then, using that $a^+ \in \mathcal{W}_\xi$ implies $\int_{\mathbb{R}} a^+(\tau) e^{\widehat{\sigma} \max\{\tau,0\}}\, ds < \infty$, one can apply the dominated convergence theorem to the double integral in (3.3); we get then

$$\lim_{\substack{\operatorname{Re} z \to 0+ \\ \operatorname{Im} z \to 0}} \frac{\mathfrak{h}_{\xi,c}(\widehat{\sigma} - z)}{z} = -\varkappa^+ \int_{\mathbb{R}} a^+(\tau) e^{\widehat{\sigma}\tau} \tau d\tau + c. \tag{3.4}$$

According to the statement 3 of Theorem 2.9, for $m < \mathfrak{t}_\xi(\widehat{\sigma})$, the r.h.s. of (3.4) is positive, i.e. $j = 1$ in such a case. Let now $m = \mathfrak{t}_\xi(\widehat{\sigma})$, then the r.h.s. of (3.4) is equal to 0. It is easily seen that one can rewrite then (3.3) as follows

$$\begin{aligned} \frac{\mathfrak{h}_{\xi,c}(\widehat{\sigma} - z)}{z} &= \varkappa^+ \int_{\mathbb{R}} a^+(\tau) e^{\widehat{\sigma}\tau} \int_0^\tau (1 - e^{-zs})\, ds d\tau \\ &= z\varkappa^+ \int_{\mathbb{R}} a^+(\tau) e^{\widehat{\sigma}\tau} \int_0^\tau \int_0^s e^{-zt}\, dt\, ds\, d\tau. \end{aligned} \tag{3.5}$$

Similarly to the above, for $\operatorname{Re} z \in (0, \widehat{\sigma})$, one has that $|e^{\widehat{\sigma}\tau - zt}| \le e^{\widehat{\sigma} \max\{\tau, 0\}}$. Then, by (3.2) and the dominated convergence theorem, we get from (3.5) that

$$\lim_{\substack{\operatorname{Re} z \to 0+ \\ \operatorname{Im} z \to 0}} \frac{\mathfrak{h}_{\xi,c}(\widehat{\sigma} - z)}{z^2} = \frac{\varkappa^+}{2} \int_{\mathbb{R}} a^+(\tau) e^{\widehat{\sigma}\tau} \tau^2 d\tau \in (0, \infty).$$

Thus $j = 2$ in such a case. The statement is fully proved now. □

Remark 3.2. Combining results of Theorem 2.9 and Proposition 3.1, we immediately get that, for the case $j = 2$, the minimal traveling wave speed $c_*(\xi)$ always satisfies (2.32).

Remark 3.3. If a^+ is given by (2.30), then, cf. Example 2.8, the case $a^+ \in \mathcal{W}_\xi$, $m = \mathfrak{t}_\xi(\widehat{\sigma})$ together with (3.2) requires $p = 1$, $\mu < \mu_*$, $q > 3$.

In order to include the critical case $\sigma(a^+) = \sigma(\psi_{c_*})$, we consider the following analogue of the Ikehara complex Tauberian theorem, cf. [7,21,28]. Let, for any $D \subset \mathbb{C}$, $\mathcal{H}(D)$ be the class of all holomorphic functions on D.

Proposition 3.4 *([18, Theorem 2]). Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+ := [0, \infty)$ be a non-increasing function such that, for some $\mu > 0$, $\nu > 0$,*

$$\text{the function } e^{\nu t}\varphi(t) \text{ is non-decreasing,} \tag{3.6}$$

and

$$\int_0^\infty e^{zt} d\varphi(t) < \infty, \quad 0 < \operatorname{Re} z < \mu. \tag{3.7}$$

Let also the following assumptions hold.

1. *There exist a constant $j > 0$ and complex-valued functions*

$$H \in \mathcal{H}(0 < \operatorname{Re} z \le \mu), \qquad F \in \mathcal{H}(0 < \operatorname{Re} z < \mu) \cap C(0 < \operatorname{Re} z \le \mu),$$

such that the following representation holds

$$\int_0^\infty e^{zt}\varphi(t)dt = \frac{F(z)}{(\mu - z)^j} + H(z), \quad 0 < \operatorname{Re} z < \mu. \tag{3.8}$$

2. *For any $T > 0$,*

$$\lim_{\sigma \to 0+} q_j(\sigma) \sup_{|\tau| \le T} \big| F(\mu - 2\sigma - i\tau) - F(\mu - \sigma - i\tau) \big| = 0, \tag{3.9}$$

where, for $\sigma > 0$,

$$q_j(\sigma) := \begin{cases} \sigma^{j-1}, & 0 < j < 1, \\ \log \sigma, & j = 1, \\ 1, & j > 1. \end{cases} \tag{3.10}$$

Then φ has the following asymptotic

$$\varphi(t) \sim \frac{F(\mu)}{\Gamma(j)} t^{j-1} e^{-\mu t}, \quad t \to \infty. \tag{3.11}$$

Now, we can apply Proposition 3.4 to find the asymptotic of the profile of a traveling wave.

Proposition 3.5. *In conditions and notations of Proposition 3.1, for $c \ne 0$, there exists $D = D_j > 0$, such that*

$$\psi(t) \sim D e^{-\sigma(\psi)t} t^{j-1}, \quad t \to \infty. \tag{3.12}$$

Proof. We set $\mu := \sigma(\psi)$ and

$$\begin{aligned} f(z) &:= \varkappa_\ell\bigl(\mathfrak{L}(\psi^2)\bigr)(z) + \varkappa_{n\ell}\bigl(\mathfrak{L}(\psi(a^- * \psi))\bigr)(z), & g_j(z) &:= \frac{\mathfrak{h}_{\xi,c}(z)}{(z-\mu)^j}, \\ H(z) &:= -\int_{-\infty}^{0} \psi(t)e^{zt}dt, & F(z) &:= \frac{f(z)}{g_j(z)}. \end{aligned} \tag{3.13}$$

For any $\mu > \beta > 0$, $T > 0$, we set

$$K_{\beta,\mu,T} := \bigl\{z \in \mathbb{C} \bigm| \beta \le \operatorname{Re} z \le \mu,\ |\operatorname{Im} z| \le T\bigr\}.$$

By (2.37) and Lemma 2.1, we have that $f, H \in \mathcal{H}(0 < \operatorname{Re} z \le \mu)$; in particular, for any $T > 0$, $\beta > 0$,

$$\bar{f} := \sup_{z \in K_{\beta,\mu,T}} |f(z)| < \infty. \tag{3.14}$$

By Proposition 3.1, the function g_j is continuous and does not equal to 0 on the strip $\{0 < \operatorname{Re} z \le \mu\}$, in particular, for any $T > 0$, $\beta > 0$,

$$\bar{g}_j := \inf_{z \in K_{\beta,\mu,T}} |g(z)| > 0. \tag{3.15}$$

Therefore, $F \in \mathcal{H}(0 < \operatorname{Re} z < \mu) \cap C(0 < \operatorname{Re} z \le \mu)$. As a result, one can rewrite (2.46) in the form (3.8), with $\varphi = \psi$ and with F, H as in (3.13).

Taking into account the fourth statement of Theorem 1.1, to apply Proposition 3.4 it is enough to prove that (3.9) holds. Assume that $0 < 2\sigma < \mu$.

Let $j = 2$. Clearly, $F \in C(0 < \operatorname{Re} z \le \mu)$ implies that F is uniformly continuous on $K_{\beta,\mu,T}$. Then, for any $\varepsilon > 0$ there exists $\delta > 0$ such that, for any $\tau \in [-T, T]$, the inequality

$$|\sigma| = |(\mu - 2\sigma - i\tau) - (\mu - \sigma - i\tau)| < \delta,$$

implies

$$|F(\mu - 2\sigma - i\tau) - F(\mu - \sigma - i\tau)| < \varepsilon,$$

and hence (3.9) holds (with $j = 2$).

Let now $j = 1$. If $F \in \mathcal{H}(K_{\beta,\mu,T})$, we have, evidently, that F' is bounded on $K_{\beta,\mu,T}$, and one can apply a mean-value-type theorem for complex-valued functions, see e.g. [9], to get that F is a Lipschitz function on $K_{\beta,\mu,T}$. Therefore, for some $K > 0$,

$$|F(\mu - 2\sigma - i\tau) - F(\mu - \sigma - i\tau)| < K|\sigma|,$$

for all $\tau \in [-T, T]$, that yields (3.9) (with $j = 1$). By Proposition 2.4 and Corollary 2.7, the inclusion $F \in \mathcal{H}(K_{\beta,\mu,T})$ always holds for $c > c_*$; whereas, for $c = c_*$ it does hold iff $a^+ \in \mathcal{V}_\xi$. Moreover, the case $a^+ \in \mathcal{W}_\xi$ with $m = \mathfrak{t}_\xi(\widehat{\sigma})$ and $c = c_*$ implies, by Proposition 3.1, $j = 2$ and hence it was considered above.

Therefore, it remains to prove (3.9) for the case $a^+ \in \mathcal{W}_\xi$ with $m < \mathfrak{t}_\xi(\widehat{\sigma})$, $c = c_*$ (then $j = 1$). Denote, for simplicity,

$$z_1 := \mu - \sigma - i\tau, \qquad z_2 := \mu - 2\sigma - i\tau. \tag{3.16}$$

Then, by (3.13), (3.14), (3.15), one has

$$\begin{aligned}\big|F(z_2)-F(z_1)\big| &\le \Big|\frac{f(z_2)}{g_1(z_2)}-\frac{f(z_1)}{g_1(z_2)}\Big|+\Big|\frac{f(z_1)}{g_1(z_2)}-\frac{f(z_1)}{g_1(z_1)}\Big| \\ &\le \frac{1}{\bar{g}_1}\big|f(z_2)-f(z_1)\big|+\frac{\bar{f}}{\bar{g}_1^2}|g_1(z_1)-g_1(z_2)|. \end{aligned}\tag{3.17}$$

Note that, if $0<\phi\in L^\infty(\mathbb{R})\cap L^1(\mathbb{R})$ be such that $\sigma(\phi)>\mu$ then

$$\begin{aligned}\big|(\mathfrak{L}\phi)(z_2)-(\mathfrak{L}\phi)(z_1)\big| &\le \int_{\mathbb{R}}\phi(s)e^{\mu s}|e^{-2\sigma s}-e^{-\sigma s}|ds \\ &\le \sigma\int_0^\infty \phi(s)e^{(\mu-\sigma)s}s\,ds+\sigma\int_{-\infty}^0\phi(s)e^{(\mu-2\sigma)s}|s|\,ds=O(\sigma), \end{aligned}\tag{3.18}$$

as $\sigma\to 0+$, where we used that $\sup_{s<0}e^{(\mu-2\sigma)s}|s|<\infty$, $0<2\sigma<\mu$, and that (L2) holds. Applying (3.18) to $\phi=\psi(a^-*\psi)\le\theta^2a^-\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$, one gets

$$\sup_{\tau\in[-T,T]}\big|f(z_2)-f(z_1)\big|=O(\sigma),\quad \sigma\to 0+\,.$$

Therefore, by (3.17), it remains to show that

$$\lim_{\sigma\to0+}\log\sigma\sup_{\tau\in[-T,T]}|g_1(z_1)-g_1(z_2)|=0.\tag{3.19}$$

Recall that, in the considered case $c=c_*$, one has $\mathfrak{h}_{\xi,c}(\mu)=0$. Therefore, by (3.1), (3.13), (3.16), we have

$$\begin{aligned}|g_1(z_1)-g_1(z_2)| &= \bigg|\frac{\mathfrak{h}_{\xi,c}(z_1)-\mathfrak{h}_{\xi,c}(\mu)}{z_1-\mu}-\frac{\mathfrak{h}_{\xi,c}(z_2)-\mathfrak{h}_{\xi,c}(\mu)}{z_2-\mu}\bigg| \\ &= \bigg|\frac{\varkappa^+(\mathfrak{L}a^+)(z_1)-\varkappa^+(\mathfrak{L}a^+)(\mu)}{z_1-\mu}-\frac{\varkappa^+(\mathfrak{L}a^+)(z_2)-\varkappa^+(\mathfrak{L}a^+)(\mu)}{z_2-\mu}\bigg| \\ &\le \varkappa^+\int_{\mathbb{R}}a^+(s)e^{\mu s}\bigg|\frac{1-e^{(-\sigma-i\tau)s}}{\sigma+i\tau}-\frac{1-e^{(-2\sigma-i\tau)s}}{2\sigma+i\tau}\bigg|ds \\ &= \varkappa^+\int_{\mathbb{R}}a^+(s)e^{\mu s}\bigg|\int_0^s\Big(e^{(-\sigma-i\tau)t}-e^{(-2\sigma-i\tau)t}\Big)\,dt\bigg|ds \\ &\le \varkappa^+\int_0^\infty a^+(s)e^{\mu s}\int_0^s\big|e^{-\sigma t}-e^{-2\sigma t}\big|\,dt\,ds \\ &\quad+\varkappa^+\int_{-\infty}^0 a^+(s)e^{\mu s}\int_s^0\big|e^{-\sigma t}-e^{-2\sigma t}\big|\,dt\,ds \end{aligned}\tag{3.20}$$

and since, for $t\ge0$, $\big|e^{-\sigma t}-e^{-2\sigma t}\big|\le\sigma t$; and, for $s\le t\le 0$,

$$\big|e^{-\sigma t}-e^{-2\sigma t}\big|=e^{-2\sigma t}\big|e^{\sigma t}-1\big|\le e^{-2\sigma s}\sigma|t|,$$

one can continue (3.20)

$$\le \frac{1}{2}\sigma\varkappa^+ \int\limits_0^\infty a^+(s)e^{\mu s}s^2\,ds + \frac{1}{2}\sigma\varkappa^+ \int\limits_{-\infty}^0 a^+(s)e^{(\mu-2\sigma)s}s^2\,ds.$$

Since $\mu > 2\sigma$, one has $\sup\limits_{s\le 0} e^{(\mu-2\sigma)s}s^2 < \infty$, therefore, by (3.2), one gets

$$\sup_{\tau\in[-T,T]} |g_1(z_1) - g_1(z_2)| \le \mathrm{const}\cdot\sigma,$$

that proves (3.19). The statement is fully proved now. □

Remark 3.6. By (3.11) and (3.13), one has that the constant $D = D_j$ in (3.12) is given by

$$D = D(\psi) = \bigl(\varkappa_\ell\bigl(\mathfrak{L}(\psi^2)\bigr)(\mu) + \varkappa_{n\ell}\bigl(\mathfrak{L}(\psi(a^- * \psi))\bigr)(\mu)\bigr) \lim_{z\to\mu} \frac{(z-\mu)^j}{\mathfrak{h}_{\xi,c}(z)},$$

where $\mu = \sigma(\psi)$. Note that, by Proposition 3.1, the limit above is finite and does not depend on ψ. Next, by Remark 2.2, for any $q\in\mathbb{R}$, $\psi_q(s) := \psi(s+q)$, $s\in\mathbb{R}$ is a traveling wave with the same speed, and hence, by Theorem 2.9, $\sigma(\psi_q) = \sigma(\psi)$. Moreover,

$$\begin{aligned}
\bigl(\mathfrak{L}(\psi_q(a^- * \psi_q))\bigr)(\mu) &= \int\limits_{\mathbb{R}} \psi(s+q) \int\limits_{\mathbb{R}} a^-(t)\psi(s-t+q)\,dt\,e^{\mu s}\,ds \\
&= e^{-\mu q}\bigl(\mathfrak{L}(\psi(a^- * \psi))\bigr)(\mu), \\
\bigl(\mathfrak{L}(\psi_q^2)\bigr)(\mu) &= \int\limits_{\mathbb{R}} \psi^2(s+q)e^{\mu s}ds = e^{-\mu q}\bigl(\mathfrak{L}(\psi^2)\bigr)(\mu).
\end{aligned}$$

Thus, for a traveling wave profile ψ one can always choose a $q\in\mathbb{R}$ such that, for the shifted profile ψ_q, the corresponding $D = D(\psi_q)$ will be equal to 1.

Finally, we are ready to prove the uniqueness result.

Theorem 3.7. *Let $\xi \in S^{d-1}$ be fixed and $a^+ \in \mathcal{U}_\xi$. Suppose, additionally, that (A7) holds. Let $c_*(\xi)$ be the minimal traveling wave speed according to Theorem 1.1. For the case $a^+ \in \mathcal{W}_\xi$ with $m = \mathfrak{t}_\xi(\widehat{\sigma})$, we will assume, additionally, that (3.2) holds. Then, for any $c \ge c_*$, such that $c \ne 0$, there exists a unique, up to a shift, traveling wave profile ψ for (1.1).*

Proof. We will follow the sliding technique from [5]. Let $\psi_1, \psi_2 \in C^1(\mathbb{R})\cap\mathcal{M}_\theta(\mathbb{R})$ are traveling wave profiles with a speed $c \ge c_*$, $c\ne 0$, cf. Theorem 1.1. By Proposition 3.5 and Remark 3.6, we may assume, without lost of generality, that (3.12) holds for both ψ_1 and ψ_2 with $D = 1$. By the proof of Proposition 3.1, the corresponding $j\in\{1,2\}$ depends on $a^\pm$, $\varkappa^\pm$, m only, and does not depend on the choice of ψ_1, ψ_2. By Theorem 2.9, $\sigma(\psi_1) = \sigma(\psi_2) =: \lambda_c \in (0,\infty)$.

Step 1. Prove that, for any $\tau > 0$, there exists $T = T(\tau) > 0$, such that

$$\psi_1^\tau(s) := \psi_1(s-\tau) > \psi_2(s), \quad s \ge T. \tag{3.21}$$

Indeed, take an arbitrary $\tau > 0$. Then (3.12) with $D = 1$ yields

$$\lim_{s\to\infty} \frac{\psi_1^\tau(s)}{(s-\tau)^{j-1}e^{-\lambda_c(s-\tau)}} = 1 = \lim_{s\to\infty} \frac{\psi_2(s)}{s^{j-1}e^{-\lambda_c s}}.$$

Then, for any $\varepsilon > 0$, there exists $T_1 = T_1(\varepsilon) > \tau$, such that, for any $s > T_1$,

$$\frac{\psi_1^\tau(s)}{(s-\tau)^{j-1}e^{-\lambda_c(s-\tau)}} - 1 > -\varepsilon, \qquad \frac{\psi_2(s)}{s^{j-1}e^{-\lambda_c s}} - 1 < \varepsilon.$$

As a result, for $s > T_1 > \tau$,

$$\begin{aligned}
&\psi_1^\tau(s) - \psi_2(s) > (1-\varepsilon)(s-\tau)^{j-1}e^{-\lambda_c(s-\tau)} - (1+\varepsilon)s^{j-1}e^{-\lambda_c s}\\
&= s^{j-1}e^{-\lambda_c s}\biggl(\Bigl(1-\frac{\tau}{s}\Bigr)^{j-1}e^{\lambda_c\tau} - 1 - \varepsilon\Bigl(\Bigl(1-\frac{\tau}{s}\Bigr)^{j-1}e^{\lambda_c\tau}+1\Bigr)\biggr)\\
&\geq s^{j-1}e^{-\lambda_c s}\biggl(\Bigl(1-\frac{\tau}{T_1}\Bigr)^{j-1}e^{\lambda_c\tau} - 1 - \varepsilon\bigl(e^{\lambda_c\tau}+1\bigr)\biggr) > 0,
\end{aligned} \tag{3.22}$$

if only

$$0 < \varepsilon < \frac{\Bigl(1-\dfrac{\tau}{T_1}\Bigr)^{j-1}e^{\lambda_c\tau} - 1}{e^{\lambda_c\tau}+1} =: g(\tau, T_1). \tag{3.23}$$

For $j = 1$, the nominator in the r.h.s. of (3.23) is positive. For $j = 2$, consider $f(t) := \bigl(1 - \frac{t}{T_1}\bigr)e^{\lambda_c t} - 1$, $t \geq 0$. Then $f'(t) = \frac{1}{T_1}e^{\lambda_c t}(\lambda_c T_1 - \lambda_c t - 1) > 0$, if only $T_1 > t + \frac{1}{\lambda_c}$, that implies $f(t) > f(0) = 0$, $t \in \bigl(0, T_1 - \frac{1}{\lambda_c}\bigr)$.

As a result, choose $\varepsilon = \varepsilon(\tau) > 0$ with $\varepsilon < g\bigl(\tau, \tau + \frac{1}{\lambda_c}\bigr)$, then, without loss of generality, suppose that $T_1 = T_1(\varepsilon) = T_1(\tau) > \tau + \frac{1}{\lambda_c} > \tau$. Therefore, $0 < \varepsilon < g\bigl(\tau, \tau + \frac{1}{\lambda_c}\bigr) \leq g(\tau, T_1)$, that fulfills (3.23), and hence (3.22) yields (3.21), with any $T > T_1$.

Step 2. Prove that there exists $\nu > 0$, such that, cf. (3.21),

$$\psi_1^\nu(s) \geq \psi_2(s), \quad s \in \mathbb{R}. \tag{3.24}$$

Let $\tau > 0$ be arbitrary and $T = T(\tau)$ be as above. Choose any $\delta \in \bigl(0, \frac{\theta}{4}\bigr)$. By (1.4), and the dominated convergence theorem,

$$\lim_{s\to-\infty}(a^- * \psi_2)(s) = \lim_{s\to-\infty}\int_{\mathbb{R}} a^-(\tau)\psi_2(s-\tau)\,d\tau = \theta > \delta. \tag{3.25}$$

Then, one can choose $T_2 = T_2(\delta) > T$, such that, for all $s < -T_2$,

$$\psi_1^\tau(s) > \theta - \delta, \tag{3.26}$$

$$\varkappa_\ell\psi_2(s) + \varkappa_{n\ell}(a^- * \psi_2)(s) > \delta. \tag{3.27}$$

Note also that (3.21) holds, for all $s \geq T_2 > T$, as well. Clearly, for any $\nu \geq \tau$,

$$\psi_1^\nu(s) = \psi_1(s-\nu) \geq \psi_1(s-\tau) > \psi_2(s), \quad s > T_2.$$

Next, $\lim\limits_{\nu\to\infty}\psi_1^\nu(T_2) = \theta > \psi_2(-T_2)$ implies that there exists $\nu_1 = \nu_1(T_2) = \nu_1(\delta) > \tau$, such that, for all $\nu > \nu_1$,

$$\psi_1^\nu(s) \geq \psi_1^\nu(T_2) > \psi_2(-T_2) \geq \psi_2(s), \quad s \in [-T_2, T_2].$$

Let such a $\nu > \nu_1$ be chosen and fixed. As a result,

$$\psi_1^\nu(s) \geq \psi_2(s), \quad s \geq -T_2, \tag{3.28}$$

and, by (3.26),

$$\psi_1^\nu(s) + \delta > \theta > \psi_2(s), \quad s < -T_2. \tag{3.29}$$

For the $\nu > \nu_1$ chosen above, define

$$\varphi_\nu(s) := \psi_1^\nu(s) - \psi_2(s), \quad s \in \mathbb{R}. \tag{3.30}$$

To prove (3.24), it is enough to show that $\varphi_\nu(s) \geq 0$, $s \in \mathbb{R}$.

On the contrary, suppose that φ_ν takes negative values. By (3.28), (3.29),

$$\varphi_\nu(s) \geq -\delta, \quad s < -T_2; \qquad \varphi_\nu(s) \geq 0, \quad s \geq -T_2. \tag{3.31}$$

Since $\lim\limits_{s\to-\infty} \varphi_\nu(s) = 0$ and $\varphi_\nu \in C^1(\mathbb{R})$, our assumption implies that there exists $s_0 < -T_2$, such that

$$\varphi_\nu(s_0) = \min_{s\in\mathbb{R}} \varphi_\nu(s) \in [-\delta, 0). \tag{3.32}$$

We set also

$$\delta_* := -\varphi_\nu(s_0) = \psi_2(s_0) - \psi_1^\nu(s_0) \in (0, \delta]. \tag{3.33}$$

Next, both ψ_1^ν and ψ_2 solve (1.8). By (1.6), $\int_{\mathbb{R}} J_\theta(s)\,ds = \varkappa^+ - \varkappa_{n\ell}\theta$. Denote $L_\theta\varphi := J_\theta * \varphi - (\varkappa^+ - \varkappa_{n\ell}\theta)\varphi$. Then one can rewrite (1.8)

$$c\psi'(s) + (L_\theta\psi)(s) + (\theta - \psi(s))\bigl(\varkappa_\ell\psi(s) + \varkappa_{n\ell}(a^- * \psi)(s)\bigr) = 0.$$

Writing the latter equation for ψ_1^ν and ψ_2 and subtracting the results, one gets

$$\begin{aligned} &c\varphi_\nu'(s) + (L_\theta\varphi_\nu)(s) + A(s) = 0,\\ &A(s) := (\theta - \psi_1^\nu(s))\bigl(\varkappa_\ell\psi_1^\nu(s) + \varkappa_{n\ell}(a^- * \psi_1^\nu)(s)\bigr)\\ &\qquad -(\theta - \psi_2(s))\bigl(\varkappa_\ell\psi_2(s) + \varkappa_{n\ell}(a^- * \psi_2)(s)\bigr). \end{aligned} \tag{3.34}$$

Consider (3.34) at the point s_0. By (3.32),

$$\varphi_\nu'(s_0) = 0, \qquad (L_\theta\varphi_\nu)(s_0) \geq 0. \tag{3.35}$$

Next, (3.33) yields

$$\begin{aligned} A(s_0) =&(\theta - \psi_1^\nu(s_0))\bigl(\varkappa_\ell\psi_1^\nu(s_0) + \varkappa_{n\ell}(a^- * \psi_1^\nu)(s_0)\bigr)\\ &+ (\delta_* - (\theta - \psi_1^\nu(s_0))\bigl(\varkappa_\ell\psi_2(s_0) + \varkappa_{n\ell}(a^- * \psi_2)(s_0)\bigr)\\ =&(\theta - \psi_1^\nu(s_0))\bigl(\varkappa_\ell\varphi_\nu(s_0) + \varkappa_{n\ell}(a^- * \varphi_\nu)(s_0)\bigr)\\ &+ \delta_*\bigl(\varkappa_\ell\psi_2(s_0) + \varkappa_{n\ell}(a^- * \psi_2)(s_0)\bigr)\\ =&(\theta - \psi_1^\nu(s_0))\bigl(\varkappa_\ell\varphi_\nu(s_0) + \varkappa_{n\ell}(a^- * (\varphi_\nu + \delta_*))(s_0)\bigr)\\ &+ \delta_*\bigl(\varkappa_\ell\psi_2(s_0) + \varkappa_{n\ell}(a^- * \psi_2)(s_0) - (\theta - \psi_1^\nu(s_0))\bigr)\\ >&0, \end{aligned} \tag{3.36}$$

because of (3.32), (3.26), and (3.27). The strict inequality in (3.36) together with (3.35) contradict to (3.34). Therefore, (3.24) holds, for any $\nu > \nu_1$.

Step 3. Prove that, cf. (3.24),

$$\vartheta_* := \inf\{\vartheta > 0 \mid \psi_1^\vartheta(s) \geq \psi_2(s), s \in \mathbb{R}\} = 0. \tag{3.37}$$

On the contrary, suppose that $\vartheta_* > 0$. Let $\varphi_* := \varphi_{\vartheta_*}$ be given by (3.30). By the continuity of the profiles, $\varphi_* \geq 0$.

First, assume that $\varphi_*(s_0) = 0$, for some $s_0 \in \mathbb{R}$, i.e. φ_* attains its minimum at s_0. Then (3.35) holds with ϑ replaced by ϑ_*, and, moreover, cf. (3.34),

$$A(s_0) = \varkappa_{n\ell}(\theta - \psi_1^\vartheta(s_0))(a^- * \varphi_*)(s_0) \geq 0.$$

Therefore, (3.34) implies

$$(L_\theta \varphi_*)(s_0) = 0. \tag{3.38}$$

By the same arguments as in the proof of Proposition 2.4, one can show that (A7) implies that the function J_θ also satisfies (A7), for $d = 1$, with some another constants. Then, arguing in the same way as in the proof of [17, Proposition 5.2] (with $d = 1$ and a^+ replaced by J_θ), one gets that (3.38) implies that φ_* is a constant, and thus $\varphi_* \equiv 0$, i.e. $\psi_1^{\vartheta_*} \equiv \psi_2$. The latter contradicts (3.21).

Therefore, $\varphi_*(s) > 0$, i.e. $\psi_1^{\vartheta_*}(s) > \psi_2(s)$, $s \in \mathbb{R}$. By (3.21) and (3.25), there exists $T_3 = T_3(\vartheta_*) > 0$, such that $\psi_1^{\frac{\vartheta_*}{2}}(s) > \psi_2(s)$, $s > T_3$, and also, for any $s < -T_3$, (3.27) holds and (3.29) holds with ϑ replaced by $\frac{\vartheta_*}{2}$ (for some fixed $\delta \in \left(0, \frac{\theta}{4}\right)$). For any $\varepsilon \in \left(0, \frac{\vartheta_*}{2}\right)$, $\psi_1^{\vartheta_* - \varepsilon} \geq \psi_1^{\frac{\vartheta_*}{2}}$, therefore,

$$\psi_1^{\vartheta_* - \varepsilon}(s) > \psi_2(s), \quad s > T_3,$$

and also (3.29) holds with ϑ replaced by $\vartheta_* - \varepsilon$, for $s < -T_3$. We set

$$\alpha := \inf_{t \in [-T_3, T_3]} (\psi_1^{\vartheta_*}(s) - \psi_2(s)) > 0.$$

Since the family $\left\{\psi_1^{\vartheta_* - \varepsilon} \mid \varepsilon \in \left(0, \frac{\vartheta_*}{2}\right)\right\}$ is monotone in ε, and $\lim\limits_{\varepsilon \to 0} \psi_1^{\vartheta_* - \varepsilon}(t) = \psi_1^{\vartheta_*}(t)$, $t \in \mathbb{R}$, we have, by Dini's theorem, that the latter convergence is uniform on $[-T_3, T_3]$. As a result, there exists $\varepsilon = \varepsilon(\alpha) \in \left(0, \frac{\vartheta_*}{2}\right)$, such that

$$\psi_1^{\vartheta_*}(s) \geq \psi_1^{\vartheta_* - \varepsilon}(s) \geq \psi_2(s), \quad s \in [-T_3, T_3].$$

Then, the same arguments as in the Step 2 prove that $\psi_1^{\vartheta_* - \varepsilon}(s) \geq \psi_2(s)$, for all $s \in \mathbb{R}$, that contradicts the definition (3.37) of ϑ_*.

As a result, $\vartheta_* = 0$, and by the continuity of profiles, $\psi_1 \geq \psi_2$. By the same arguments, $\psi_2 \geq \psi_1$, that fulfills the statement. □

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