J. Math. Anal. Appl. 482 (2020) 123521

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On an isoperimetric problem with power-law potentials and external attraction

Guoqing Zhang *, Xiaoqian Geng

College of Sciences, University of Shanghai for Science and Technology, Shanghai, 200093, PR China

ARTICLE INFO

Article history:
Received 27 April 2019
Available online 23 September 2019
Submitted by V. Radulescu

Keywords:
Isoperimetric problem
Power-law potentials
Minimizers

ABSTRACT

In this paper, an isoperimetric problem with power-law potentials and external attraction is considered. Using concentration-compactness lemma for sets of finite perimeter which was introduced by Frank and Lieb, we prove the existence of minimizers for the isoperimetric problem. In the special case of quadratic attraction, we also obtain that the ball is the unique minimizer of the isoperimetric problem.

1. Introduction

This paper is concerned with a study of the following nonlocal isoperimetric problem: minimize the energy functional

$$E(u) := \int_{\mathbb{R}^N} |\nabla u| + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K(x-y)u(x)u(y)dxdy - \int_{\mathbb{R}^N} V(x)u(x)dx, \tag{1.1}$$

subject to the constraint

$$\mathcal{A} := \{u : BV(\mathbb{R}^N; \{0,1\}), \int_{\mathbb{R}^N} udx = m\}. \tag{1.2}$$

Here $N > 2$, the power-law potentials $K(x) = \frac{1}{p}|x|^{-p} - \frac{1}{q}|x|^{-q}$ where $q < p < N$, the external potential $V(x)$ satisfies some appropriate conditions, and the first term in $E(u)$ computes the total variation of the function u, i.e.,

* Corresponding author.
E-mail addresses: shzhangguoqing@126.com (G. Zhang), 1013073420@qq.com (X. Geng).

https://doi.org/10.1016/j.jmaa.2019.123521
0022-247X/

$$\int_{\mathbb{R}^N} |\nabla u| = \sup\{\int_{\mathbb{R}^N} u \ \mathrm{div}\phi \ dx : \phi \in C_0^1(\mathbb{R}^N;\mathbb{R}^N), \ |\phi| \le 1\}. \tag{1.3}$$

In other world, the problem can be seen as a minimization problem over sets of finite perimeter:

$$\min\{P(E) + \int_E \int_E K(x-y)dxdy - \int_E V(x)dx : \ |E| = m\},$$

where $|E|$ denotes the Lebesgue measure of E and $P(E)$ is its perimeter.

Recently, a rather detailed study of such variational problem described above was performed. For the nonlocal isoperimetric problem with the repulsive Coulombic interaction

$$E_{nlip}(u) = \int_{\mathbb{R}^3} |\nabla u| + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u(x)u(y)}{|x-y|}dxdy, \tag{1.4}$$

over $\int_{\mathbb{R}^N} udx = m$, Choksi and Peletier ([5]) conjectured that the minimizer of the problem (1.4) is a ball whenever it exists. The answer to this question is not obvious at all, since the two terms in the problem (1.4) are in direct competition with each other. Lu and Otto ([15]) proved the non-existence of the problem (3) for sufficiently large m. Julin ([8]), Knüepfer and Muratov ([9]) obtained the existence of a radially symmetric minimizer (i.e., a ball) for m sufficiently small. On the other hand, the problem (1.4) corresponds to the classical Gamow's liquid drop model of an atomic nucleus ([7,10–13,18]). In 2015, Frank and Lieb ([6]) obtained the existence of a nucleus with minimal binding energy per particle. For the external potential $V(x)$, Alama, Bronsard, Choksi and Topaloglu ([1]) considered the following a variant of liquid drop problem

$$E_V(u) = \int_{\mathbb{R}^3} |\nabla u| + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u(x)u(y)}{|x-y|}dxdy - \int_{\mathbb{R}^3} V(x)u(x)dx, \tag{1.5}$$

where the potential $V(x)$, with long range decay, in the sense that $V(x) \gg |x|^{-1}$ for large $|x|$, and obtained the existence of ground states.

Potentials in power-law form have been frequently considered ([2,4,20,18,21]). In these works, the delicate balance between attraction and repulsion often leads to complex equilibrium configurations. In 2015, Choksi, Fetecau and Topaloglu ([4]) established the existence of global minimizers for a class of energy functionals consisting of power-law potentials.

In this paper, firstly, as $0 < p < q < N$, the external potential $V(x)$ satisfies

$(L1)$ $V \ge 0$, and $V \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$,
$(L2)$ $\lim_{|x|\to\infty} V(x) = 0$,

we have

Theorem 1.1. *Suppose that $0 < q < p < N$, $V(x)$ satisfies (L1) and (L2). Then for any $m > 0$, there exists a minimizer of the problem (1.1) in $\mathcal{A}$.*

Secondly, as $q = -2$, $0 < p < N$, and $V(x) = 0$, we have

Theorem 1.2. *Suppose that $q = -2$, $0 < p < N$, and $V(x) = 0$. Then for sufficiently large $m > 0$, the problem (1.1) in $\mathcal{A}$ has a unique solution up to translation, the ball of volume m.*

Remark 1.3. When $0 < q < p < N$, we utilize a technical concentration-compactness result for sets of finite perimeter by Frank and Lieb ([6]), and prove a lower bound on the energy in case minimizing sequence $\{u_n\}$ lose compactness via splitting, of the form

$$\lim_{n\to\infty} E(u_n) \geq e(m_0) + e_0(m_1) + e_0(m - m_0 - m_1), \tag{1.6}$$

where

$$e(m) = \inf\{E(u) : u \in BV(\mathbb{R}^N; \{0,1\}), \int_{\mathbb{R}^N} u dx = m\},$$

$0 < m_i < m (i = 0, 1)$ with $m_0 + m_1 \leq m$ and $e_0(m)$ is defined in section 3. However, by the external attraction, we will show that $e(m)$ actually lies strictly below the value given in (1.6).

Remark 1.4. For the liquid drop model, the nonlocal energy is purely repulsive. However, Alama ([1]) considers the liquid model with an attractive background nucleus. It is natural that we consider more general potentials, the power-law potentials with attractive and repulsive parts.

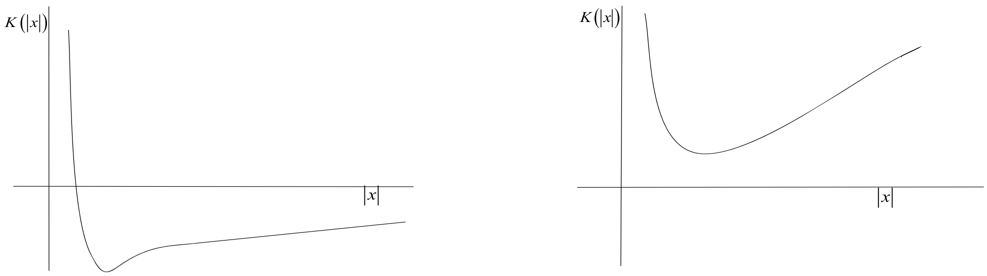
In the application of Frank and Lieb ([6]), the nonlocal energy is purely repulsive (the interaction kernel is monotonically decreasing) and there exists a minimizer only for small mass. However, here we obtain that there exists a minimizer for any mass in Theorem 1.1. The reason for the existence of a minimizer for any mass is chiefly due to the property that interaction potential is attractive for large distances. Alama utilizes a compactness result by Frank and Lieb to prove the liquid drop model with an attractive background in ([1]), and Choksi uses the concentration-compactness lemma to establish the existence of global minimizers of interaction functionals with competing attractive and repulsive potentials in ([4]). Motivated by ([1]) and ([4]), we prove that the isoperimetric problem with power-law potentials and external attraction has a minimizer for any mass by using the concentration-compactness lemma.

Remark 1.5. When $q = -2$ and $0 < p < N$, our approach is via a relaxation of the problem (1.1). Then we show that the ball satisfies the first-order variational inequality corresponding to the relaxed problem when the mass is sufficiently large, and obtain that the ball is the solution of (1.1) in $\mathcal{A}$.

Remark 1.6. Let us now comment on the mathematical motivations of the power-law potentials. It is worthy to notice that these functionals appear in biological swarms, granular media, self-assembly of nanoparticles and molecular dynamics simulations of matter ([4,20,21]). In the context of biological swarms, K incorporates social interactions (attraction and repulsion) between ground individuals. These sums of attractive and repulsive power-law potentials have collective effect which is repulsive at short ranges but attractive at long ranges. Hence, the equilibrium solution is complex and properties of the potential $K(x)$ are important for us to analyze the behavior of solutions. As we can see from the Fig. 1, when $0 < q < p < N$, the potential K becomes negative, approaches 0 as $|x| \to \infty$ and is not convex, whereas when $0 < p < N$ and $q = -2$, K is positive, convex and $K \to \infty$ as $|x| \to \infty$. In addition, recently there are many works about 1-Laplace operator ([3,19]) and TFDW models ([11,14,12,15]).

2. Preliminaries

In the case $0 < p < q < N$, our proof relies on a technical concentration-compactness lemma for sets of finite perimeter. As noted in the introduction, our goal is to obtain a splitting property. Hence, we will use the following splitting form.



$(a) 0 < q < p < N$, $(b) 0 < p < N$, and $q = -2$

Fig. 1. Generic examples of K for various values of p and q.

Lemma 2.1. *Let $\{F_n\}$ and $\{E_n\}$ be sequences of measurable sets in $\mathbb{R}^N$ with uniformly bounded measure, and $|F_n \cap E_n| = 0$ for all n. Assume that*

$$F_n \to F \ (globally) \quad and \quad E_n \to 0 \ (locally),$$

for some set F. Then we have

$$I(F_n \cup E_n) = I(F_n) + I(E_n) + o(1), \tag{2.1}$$

and

$$I(F_n) = I(F) + o(1), \tag{2.2}$$

where

$$I(A) = \int\limits_A \int\limits_A D(x-y)dxdy,$$

where $D(x) \in L^1_{loc}(\mathbb{R}^N)$, $\lim_{|x|\to\infty} D(x) = 0$ and A is a measurable set in $\mathbb{R}^N$.

Proof. First note that

$$|I(F_n, E_n) - I(F, E_n)| \leq |I(F_n \Delta F, E_n)| = |\int\limits_{F_n \Delta F} \int\limits_{E_n} D(x-y)dxdy| \to 0, \tag{2.3}$$

as $n \to \infty$, where $F_n \Delta F := (F_n \setminus F) \cup (F \setminus F_n)$. Now we claim that

$$I(F_n, E_n) = \int\limits_{F_n} \int\limits_{E_n} D(x-y)dxdy \to 0, \text{ as } n \to \infty.$$

Indeed, for given $\varepsilon > 0$, we choose $R > 0$ such that

$$|\int\limits_F D(x-y)dy| \leq \varepsilon,$$

for any $|x| \geq R$. Then we have

$$
\begin{aligned}
|I(F, E_n)| &= |I(F, E_n \cap B_R) + I(F, E_n \setminus B_R)| \\
&\leq \int_F \int_{E_n \cap B_R} D(x-y)dxdy + \varepsilon |E_n \setminus B_R|,
\end{aligned}
$$

where B_R denotes the open ball of radius R centered at $0 \in \mathbb{R}^N$. Since $E_n \to 0$ (locally) as $n \to \infty$, we have $|E_n \cap B_R| \to 0$ as $n \to \infty$, and therefore $I(F, E_n) \to 0$ as $n \to \infty$. By (2.3), we have

$$
I(F_n, E_n) = \int_{F_n} \int_{E_n} D(x-y)dxdy \to 0, \text{ as } n \to \infty.
$$

That is,

$$
I(F_n \cup E_n) = I(F_n) + I(E_n) + o(1).
$$

The proof of (2.2) is similar. □

Lemma 2.2. *Let*

$$
E_0(u) := \int_{\mathbb{R}^N} |\nabla u| + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x-y)u(x)u(y)dxdy
$$

where $0 < q < p < N$*, then, for* $e(m)$ *given as the infimum of the problem (1.1), we have*

$$
e(m) \leq e(m_0) + e_0(m_1) + e_0(m - m_0 - m_1), \tag{2.4}
$$

where $0 \leq m_0 + m_1 \leq m$ *and*

$$
e_0 = \inf\{E_0(u) : u \in BV(\mathbb{R}^N; \{0,1\}), \int_{\mathbb{R}^N} udx = m\}.
$$

Proof. First, we prove the following inequality

$$
e(m) \leq e(m_0) + e_0(m_1), \text{ where } m = m_0 + m_1. \tag{2.5}
$$

Inequality (2.5) can be obtained as follows: for given $\varepsilon > 0$, there exist bounded sets $\Omega_i (i = 0, 1)$, with $E(\chi_{\Omega_i}) \leq e_0(m_i) + \varepsilon$. Hence, we can choose $R > 0$ such that $\Omega_i \subset B_R(0)$. Define $\tilde{\Omega} = \Omega_0 + (d + \Omega_1)$ with a shift vector d with $|d| > 2R$. Note that for $x \in \Omega_0$ and $y \in d + \Omega$, we have

$$
0 < |d| - 2R \leq |x - y| \leq |d| + 2R.
$$

In particular, we have $\int_{\mathbb{R}^N} |\nabla \chi_{\tilde{\Omega}}| = \int_{\mathbb{R}^N} |\nabla \chi_{\Omega_0}| + \int_{\mathbb{R}^N} |\nabla \chi_{\Omega_1}|$ and $|\tilde{\Omega}| = |\Omega_0| + |\Omega_1| = m_0 + m_1$, so that we may use $\tilde{\Omega}$ in the definition of $e(m)$, and

$$
\begin{aligned}
e(m) \leq E(\chi_{\tilde{\Omega}}) &\leq E(\chi_{\Omega_0}) + E_0(\chi_{\Omega_1}) + \frac{2}{p} m_0 m_1 (d - 2R)^{-p} + \frac{2}{q} m_0 m_1 (d + 2R)^{-q} + \int_{\mathbb{R}^N} V(x) \chi_{d + \Omega_1} dx \\
&\leq e(m_0) + e_0(m_1) + 2\varepsilon + \frac{2}{p} m_0 m_1 (d - 2R)^{-p} + \frac{2}{q} m_0 m_1 (d + 2R)^{-q} + \int_{\mathbb{R}^N} V(x) \chi_{d + \Omega_1} dx.
\end{aligned}
$$

Let first d tend to infinity, and then ε to 0 yields (2.5).

Next, we show

$$e_0(m) \le e_0(m_0) + e_0(m_1), \text{ where } m = m_0 + m_1. \tag{2.6}$$

In the same argument of the above proof of (2.5), assume Ω_0 and Ω_1 are bounded sets that approximately minimize $e_0(m_0)$ and $e_0(m_1)$ respectively, by considering $\tilde{\Omega} = \Omega_0 + (d + \Omega_1)$ with a shift vector d, we have

$$\begin{aligned} e_0(m) &\le E_0(\chi_{\tilde{\Omega}}) \le E_0(\chi_{\Omega_0}) + E_0(\chi_{\Omega_1}) + \frac{2}{p} m_0 m_1 (d - 2R)^{-p} + \frac{2}{q} m_0 m_1 (d + 2R)^{-q} \\ &\le e_0(m_0) + e_0(m_1) + 2\varepsilon + \frac{2}{p} m_0 m_1 (d - 2R)^{-p} + \frac{2}{q} m_0 m_1 (d + 2R)^{-q}. \end{aligned}$$

As d tend to infinity, we have ε to 0 yields (2.6).

In conclusion, using (2.5) and (2.6), we have,

$$e(m) \le e(m_0) + e_0(m - m_0) \le e(m_0) + e_0(m_1) + e_0(m - m_0 - m_1). \quad \square$$

Definition 2.3. If $u(x) \in L^1(\mathbb{R}^N)$, the Fourier transform of the function $u(x)$ is defined as

$$\hat{u}(\xi) = \int\limits_{\mathbb{R}^N} e^{-2\pi i \xi x} u(x) dx.$$

Lemma 2.4. *For $q = -2$, $0 < p < N$ and $V(x) = 0$, the minimizer of the problem*

$$E(u) = \int\limits_{\mathbb{R}^N} |\nabla u| + \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} (\frac{1}{2}|x - y|^2 + \frac{1}{p}|x - y|^{-p}) u(x) u(y) dx dy$$

over

$$\mathcal{A}_{m,1} = \{u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap BV(\mathbb{R}^N) : \|u\|_{L^1(\mathbb{R}^N)} = m,\ 0 \le u(x) \le 1, \text{and} \int\limits_{\mathbb{R}^N} |\nabla u| < \infty\},$$

is unique up to translation.

Proof. Now, we split the energy into two parts $E = E_1 + E_2$, where

$$E_1 = \int\limits_{\mathbb{R}^N} |\nabla u|, \text{ and } E_2 = \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} (\frac{1}{2}|x - y|^2 + \frac{1}{p}|x - y|^{-p}) u(x) u(y) dx dy.$$

Since the energy E is invariant translation, we assume, without loss of generality, that the center of mass of admissible densities is zero, that is, $\int_{\mathbb{R}^N} x u(x) dx = 0$.

We firstly show $E_2 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (\frac{1}{2}|x - y|^2 + \frac{1}{p}|x - y|^{-p}) u(x) u(y) dx dy$ is strictly convex over $\mathcal{A}_{m,1}$. Since $\int_{\mathbb{R}^N} x u(x) dx = 0$, this implies that

$$\begin{aligned} \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} |x - y|^2 u(x) u(y) dx dy &= \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} (|x|^2 - 2xy + |y|^2) u(x) u(y) dx dy \\ &= \int\limits_{\mathbb{R}^N} \int\limits_{\mathbb{R}^N} (|x|^2 + |y|^2) u(x) u(y) dx dy \\ &= 2m \int\limits_{\mathbb{R}^N} |x|^2 u(x) dx. \end{aligned}$$

Hence, we get that $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|x-y|^2u(x)u(y)dxdy$ is linear in u. On the other hand, when $0<p<N$, by Theorem 5.10 in ([13]), we have

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{1}{p}|x-y|^{-p}u(x)u(y)dxdy=\int\limits_{\mathbb{R}^N}\frac{C(N-p)}{p}|\xi|^{-(N-p)}|\hat{u}(\xi)|^2d\xi.$$

Since $\int_{\mathbb{R}^N}\frac{C(N-p)}{p}|\xi|^{-(N-p)}|\hat{u}(\xi)|^2d\xi$ is strictly convex, we get that

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{1}{p}|x-y|^{-p}u(x)u(y)dxdy$$

is strictly convex. Thus, E_2 is strictly convex among all functions in $\mathcal{A}_{m,1}$, that is,

$$E_2(tu_1+(1-t)u_2)\le tE_2(u_1)+(1-t)E_2(u_2), \tag{2.7}$$

for any u_1 and $u_2\in\mathcal{A}_{m,1}$.

Clearly, $E_1=\int_{\mathbb{R}^N}|\nabla u|$ is convex, that is,

$$E_1(tu_1+(1-t)u_2)\le tE_1(u_1)+(1-t)E_1(u_2), \tag{2.8}$$

for any u_1 and $u_2\in\mathcal{A}_{m,1}$.

Thus, using (2.7) and (2.8), we have that

$$E(tu_1+(1-t)u_2)\le tE(u_1)+(1-t)E(u_2),$$

for any u_1 and $u_2\in\mathcal{A}_{m,1}$, that is, E is strictly convex among all functions in $\mathcal{A}_{m,1}$ when $q=-2$, $0<p<N$ and $V(x)=0$. Hence, the solution of the problem (1.1) is unique up to translation. □

3. Proof of main theorems

Our approach to Theorem 1.1 is via a concentration-compactness lemma for sets of finite perimeter. We define “minimization problem at infinity” e_0 by

$$e_0(m):=\inf\{E_0(u):u\in BV(\mathbb{R}^N;\{0,1\}),\int\limits_{\mathbb{R}^N}udx=m\},$$

where

$$E_0(u):=\int\limits_{\mathbb{R}^N}|\nabla u|+\int\limits_{\mathbb{R}^N}K(x-y)u(x)u(y)dxdy.$$

Proof of Theorem 1.1. When $0<q<p<N$, the interaction potential satisfies

$$K(x)\ge\frac{1}{p}-\frac{1}{q}.$$

By $(L1)$ and $(L2)$, we choose $R>0$ such that

$$V(x)\le 1,$$

for $|x| \geq R$. Hence, we have

$$E(u) = \int\limits_{\mathbb{R}^N} |\nabla u| + \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} K(x-y)u(x)u(y)dxdy - \int\limits_{\mathbb{R}^N} V(x)u(x)dx$$
$$\geq (\frac{1}{p}-\frac{1}{q})\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} u(x)u(y)dxdy - \int\limits_{|x|<R} V(x)u(x)dx - \int\limits_{|x|\geq R} u(x)dx$$
$$\geq (\frac{1}{p}-\frac{1}{q})m^2 - \|V\|_{L^1(B_R)} - m.$$

This implies that

$$e(m) = \inf\{E(u) : u \in BV(\mathbb{R}^N;\{0,1\}), \int\limits_{\mathbb{R}^N} udx = m\} > -\infty.$$

Now let $\{u_n\} \subset BV(\mathbb{R}^N;\{0,1\})$ with $\int_{\mathbb{R}^N} u_n dx = m$ be a minimizing sequence of the problem (1.1) in $\mathcal{A}$, we have that the minimizing sequence has uniformly bounded perimeter. To simplify the proof, we define the sets of finite perimeter $\{\Omega_n\} \subset \mathbb{R}^N$ satisfying $\chi_{\Omega_n} = u_n$ and $|\Omega_n| = m$ for $n \in \mathbb{N}$. Applying Corollary 12.27 in ([17]), there exist a subsequence $\{u_n\}$ and a set of finite perimeter $\Omega^0 \subset \mathbb{R}^N$ satisfying

$$u_n \to \chi_{\Omega^0}, \text{ in } L^1_{\text{loc}}(\mathbb{R}^N).$$

At this point, we admit the possibility that $\chi_{\Omega^0} \equiv 0$, i.e., $|\Omega^0| = 0$. However, in step 3 we show that $\chi_{\Omega^0} \neq 0$. Next, by contradiction argument, indeed we show $|\Omega^0| = m$.

Step 1. We will show that the energy splits. Indeed, we assume that $0 < |\Omega^0| < m$. By Lemma 2.2 in ([6]), there exists a sequence $\{r_n\}$ such that the sets

$$F_n^0 = \Omega_n \cap B_{r_n} \text{ and } E_n^0 = \Omega_n \cap (\mathbb{R}^N \setminus \bar{B}_{r_n}),$$

satisfy

$$\lim_{n\to\infty}(\text{Per}(\Omega_n) - \text{Per}(F_n^0) - \text{Per}(E_n^0)) = 0,$$

and

$$F_n^0 \to \Omega^0 \text{ (globally) and } E_n^0 \to \emptyset \text{ (locally)}.$$

In particular,

$$\lim_{n\to\infty}|F_n^0| = |\Omega^0| = m_0 \text{ and } \liminf_{n\to\infty} \text{Per}(F_n^0) \geq \text{Per}(\Omega^0).$$

We denote $f_n^0 = \chi_{F_n^0},\ f^0 = \chi_{\Omega^0},\ \Omega_n^0 = E_n^0$ and $u_n^0 = \chi_{\Omega_n^0}$ such that

$$u_n = f_n^0 + u_n^0 = f^0 + u_n^0 + o(1) \text{ in } L^1(\mathbb{R}^N), \text{ and } u_n^0 \to 0 \text{ in } L^1_{\text{loc}}(\mathbb{R}^N).$$

Using Lemma 4 in ([1]), we obtain

$$\int\limits_{\mathbb{R}^N} Vu_n dx = \int\limits_{\mathbb{R}^N} V(f_n^0 + u_n^0)dx = \int\limits_{\mathbb{R}^N} Vf^0 dx + o(1).$$

By Lemma 2.1, we have

$$E(u_n) = \int\limits_{\mathbb{R}^N} |\nabla u_n| + \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} K(x-y)u_n(x)u_n(y)dxdy - \int\limits_{\mathbb{R}^N} Vu_n dx$$
$$\geq E(f^0) + E_0(u_n^0) + o(1).$$

Step 2. Now we locate a concentration set for the remainder. We repeat the procedure in step 1, that is to say, we replace $\{u_n\}$ by $\{u_n^0\}$ and $\{\Omega_n\}$ by $\{\Omega_n^0\}$. It is known that

$$u_n^0 = \chi_{\Omega_n^0} \to 0 \text{ in } L^1_{\text{loc}}(\mathbb{R}^N) \text{ and } |\Omega_n^0| = m - m_0 + o(1).$$

By Proposition 2.1 in ([6]), we obtain there exist a set Ω^1 with $0 < |\Omega^1| \leq m - m_0$ and a sequence $\{x_n\} \subset \mathbb{R}^N$ such that

$$\chi_{\Omega_n^0 - x_n} \to \chi_{\Omega^1} \text{ (locally)}.$$

Note that $\chi_{\Omega_n^0} \to 0$ in $L^1_{\text{loc}}(\mathbb{R}^N)$, we have $x_n \to \infty$ as $n \to \infty$. Moreover, by a similar argument, we also obtain that the sets $\{\Omega_n^0 - x_n\} = \{F_n^1\} \cup \{E_n^1\}$ satisfy

$$\chi_{F_n^1} \to \chi_{\Omega^1} \text{ in } L^1(\mathbb{R}^N) \text{ and } \chi_{E_n^1} \to 0 \text{ in } L^1_{\text{loc}}(\mathbb{R}^N).$$

Again, we have that

$$E(u_n^0) = E_0(u_n^0) + o(1) \geq E_0(f^1) + E_0(u_n^1) + o(1),$$

where $f^1 = \chi_{\Omega^1}$, $u_n^1 = \chi_{E_n^1 + x_n}$ and $|E_n^1| = |E_n^0| - m_1 + o(1)$. We denote the re-centered remainder set $\Omega_n^1 = E_n^1 + x_n$, so that $u_n^1(x) = \chi_{E_n^1 + x_n}(x)$. In conclusion, we have

$$E(u_n) \geq E(f^0) + E_0(u_n^0) + o(1) \geq E(f^0) + E_0(f^1) + E_0(u_n^1) + o(1),$$

and $m = m_0 + m_1 + |\Omega_n^1| + o(1)$. By Lemma 4.8 in ([10]), we obtain

$$e(m) \geq e(m_0) + e_0(m_1) + e_0(m - m_0 - m_1). \tag{3.1}$$

Step 3. Now, we show $|\Omega^0| \neq 0$. Indeed, if $\chi_{\Omega^0} \equiv 0$, we have $\{u_n^1\} = \{u_n\}$. Using the translation sequence obtained above, we define a sequence $\{f_n\} = \{u_n(x + x_n)\}$, and obtain

$$\chi_{F_n^1} \to \chi_{\Omega^1} \text{ in } L^1(\mathbb{R}^N), \text{ and } \chi_{E_n^1} \to 0 \text{ in } L^1_{\text{loc}}(\mathbb{R}^N).$$

Hence, using Lemma 4 in ([1]) and the translation invariance of the first two terms of $E(u)$, we get that

$$E(f_n) - E(u_n) = -\int\limits_{R^N} V\chi_{\Omega^1} dx < 0.$$

Since $E(f_n) - E(u_n) > 0$, it is a contradiction, so we have $|\Omega^0| \neq 0$.

Step 4. Now we prove that $e(m_0) = E(f^0)$ and $e_0(m_1) = E_0(f^1)$. By Lemma 2.2, we have

$$e(m) \leq e(m_0) + e_0(m_1) + e_0(m - m_0 - m_1),$$

and by (3.1), we obtain

$$\begin{aligned} e(m_0) + e_0(m_1) + e_0(m - m_0 - m_1) &\geq e(m) \\ &\geq E(f^0) + E_0(f^1) + \liminf_{n\to\infty} E_0(u_n^1) \\ &\geq e(m_0) + e_0(m_1) + e_0(m - m_0 - m_1). \end{aligned}$$

Therefore, that implies

$$E(f^0) - e(m_0) + E_0(f^1) - e_0(m_1) + \liminf_{n\to\infty} E_0(u_n^1) - e_0(m - m_0 - m_1) = 0.$$

Thus, we have

$$E(f^0) = e(m_0),\ E_0(f^1) = e_0(m_1) \text{ and } \liminf_{n\to\infty} E_0(u_n^1) = e_0(m - m_0 - m_1).$$

Step 5. By the regularity of minimizers in ([16]), there exists $R > 0$ such that $\Omega^0,\ \Omega^1 \in B_R(0)$. Let $a \in S^{N-1}$ be any unit vector, where S^{N-1} denotes the unit sphere in $\mathbb{R}^N$. For t large enough, we have $\Omega^0 \bigcap(\Omega^1 + ta) = \phi$. We define

$$h_1(t) = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} (\frac{1}{p}|x-y|^{-p} - \frac{1}{q}|x-y|^{-q}) f^0(x) f^1(y - ta)dxdy,$$

and

$$h_2(t) = \int\limits_{\mathbb{R}^N} V(x) f^1(x - ta)dx.$$

Since $K(x) < 0$ for $|x|$ large enough, we have

$$h_1(t) = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} (\frac{1}{p}|x-y|^{-p} - \frac{1}{q}|x-y|^{-q}) f^0(x) f^1(y - ta)dxdy < 0,$$

and

$$h_2(t) = \int\limits_{\mathbb{R}^N} V(x) f^1(x - ta)dx > 0,$$

for all t large enough. Hence, for given $\varepsilon > 0$, we can choose $t_0 > 0$ such that

$$h_1(t_0) - h_2(t_0) < -\varepsilon < 0,$$

and there exists a compact set $G = G(\varepsilon)$ with $|G| = m - m_0 - m_1$ such that

$$E_0(\chi_G) < e_0(m - m_0 - m_1) + \frac{\varepsilon}{3}.$$

Next, we can choose $\tau > 0$ large enough, and let $G_\tau = G - \tau a$, and

$$-\frac{\varepsilon}{3} < \int\limits_{\Omega^0}\int\limits_{G_\tau} K(x-y)dxdy < 0 \text{ and } -\frac{\varepsilon}{3} < \int\limits_{\Omega^1 + ta}\int\limits_{G_\tau} K(x-y)dxdy < 0.$$

By $f(x) = f^0(x) + f^1(x - t_0 a) + \chi_{G_\tau}$ as a test function which is admissible for $e(m)$, we have

$$\begin{aligned} e(m) \le E(f) &= E(f^0) + E_0(f^1) + E_0(\chi_{G_\tau}) + 2h_1(t_0) - h_2(t_0) \\ &\quad + 2\int\limits_{\Omega^0}\int\limits_{G_\tau} K(x-y)dxdy + 2\int\limits_{\Omega^1+ta}\int\limits_{G_\tau} K(x-y)dxdy - \int\limits_{G_\tau} V(x)dx \\ &\le e(m_0) + e_0(m_1) + e_0(m - m_0 - m_1) - \frac{2\varepsilon}{3}. \end{aligned}$$

Hence, it is a contradiction, that is $|\Omega^0| = m$.

Step 6. Since $\{u_n\}_{n\in N}$ is locally convergent in $L^1(\mathbb{R}^N)$, there exists a subsequence converges almost everywhere in $\mathbb{R}^N$. In addition, since $\|u_n\|_{L^1(\mathbb{R}^N)} = \|\chi_{\Omega^0}\|_{L^1(\mathbb{R}^N)} = m$, we have

$$u_n \to \chi_{\Omega^0} \quad \text{in } L^1(\mathbb{R}^N),$$

by Brezis-Lieb Lemma in ([13]). The weak lower semicontinuity follows directly Proposition 4.29 in ([17]) and Lemma 2.1. Thus, we obtain that the minimization problem (1.1) in $\mathcal{A}$ has a solution. □

Now, we prove Theorem 1.2, and consider

$$\epsilon(u) := \inf\{E(u) : u \in \mathcal{A}_{m,1}\}, \tag{3.2}$$

where

$$E(u) = \int\limits_{\mathbb{R}^N} |\nabla u| + \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} (\frac{1}{2}|x-y|^2 + \frac{1}{p}|x-y|^{-p})u(x)u(y)dxdy,$$

and

$$\mathcal{A}_{m,1} = \{u \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \cap BV(\mathbb{R}^N) : \|u\|_{L^1(\mathbb{R}^N)} = m,\ 0 \le u(x) \le 1, \text{and} \int\limits_{\mathbb{R}^N} |\nabla u| < \infty\}.$$

Clearly, $\mathcal{A} \subset \mathcal{A}_{m,1}$, and

$$\inf_{u\in\mathcal{A}} E(u) \ge \inf_{u\in\mathcal{A}_{m,1}} E(u).$$

If any the function $u \in \mathcal{A}$, then $u \in \mathcal{A}_{m,1}$. Hence, if $u \in BV(\mathbb{R}^N; \{0,1\})$ is a global minimizer for E in $\mathcal{A}_{m,1}$, then clearly it is a global minimizer for E in $\mathcal{A}$. So we will show the ball of volume m satisfies the first-order variational inequality corresponding to the problem (3.2) when the mass is sufficiently large. That is, the ball of volume m is the solution of the problem (1.1) in $\mathcal{A}$.

Lemma 3.1. *If $T(u)$ is strictly convex over any admissible class $\mathcal{A}_{m,1}$, and if $u \in \mathcal{A}_{m,1}$ satisfies*

$$(K * u)(x) \begin{cases} = \lambda, & 0 < u(x) < 1, \\ \ge \lambda, & u(x) = 0, \\ \le \lambda, & u(x) = 1, \end{cases} \tag{3.3}$$

where $K(x) = \frac{1}{2}|x|^2 + \frac{1}{p}|x|^{-p}$ and

$$T(u) = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} (\frac{1}{2}|x-y|^2 + \frac{1}{p}|x-y|^{-p})u(x)u(y)dxdy.$$

Then u is the unique minimizer of $T(u)$ over $\mathcal{A}_{m,1}$.

Proof. Suppose u_1 and u_2 satisfy (3.3), we will obtain a contradiction.

Case 1. If $T(u_1) \neq T(u_2)$, without loss of generality, we can assume that $T(u_1) < T(u_2)$. Since u_1 and u_2 satisfies (3.3), we have

$$\begin{aligned}
\lim_{t\to 0^+} \frac{T(u_2 + t(u_1 - u_2)) - T(u_2)}{t} &= 2\int\limits_{\mathbb{R}^N} ((\frac{1}{2}|x|^2 + \frac{1}{p}|x|^{-p}) * u_2(x))(u_1(x) - u_2(x))dx \\
&\geq 2 \int\limits_{\{u_2(x)\neq 1\}} ((K * u_2)(x))u_1(x)dx - 2 \int\limits_{\{u_2(x)\neq 0\}} ((K * u_2)(x))u_2(x)dx \\
&\geq 2 \int\limits_{\{u_1(x)\neq 0\}} \lambda u_1(x)dx - 2 \int\limits_{\{u_2(x)\neq 0\}} \lambda u_2(x)dx \\
&\geq 0. \qquad (3.4)
\end{aligned}$$

Moreover, since $T(u)$ is strictly convex, we have

$$T(tu_1 + (1-t)u_2) \leq tT(u_1) + (1-t)T(u_2). \qquad (3.5)$$

Combining (3.4) with (3.5), we obtain

$$0 \leq \lim_{t\to 0^+} \frac{T(tu_1 + (1-t)u_2) - T(u_2)}{t} \leq T(u_1) - T(u_2).$$

Hence, it is a contradiction.

Case 2. Suppose $T(u_1) = T(u_2)$. Since $T(u)$ is strictly convex, there exists a function $\psi \in \mathcal{A}_{m,1}$ such that $T(\varphi) < T(u_1)$. Again, by the strict convexity of $T(u)$ and u_1 satisfies (3.3), we have

$$0 \leq \lim_{t\to 0^+} \frac{T((1-t)u_1 + t\psi) - T(u_1)}{t} \leq T(\varphi) - T(u_1).$$

It is a contradiction, since $T(\varphi) < T(u_1)$. So we have $u_1 = u_2$. In conclusion, the proof is completed. □

Proof of Theorem 1.2. We divided into the following three steps.

Step 1. Firstly, we will show that the characteristic function of a ball satisfies (3.3) for large mass m. By inspecting the interaction function $K(x)$, we split it into two parts:

$$K_2 = \frac{1}{2}|x|^2 \text{ and } K_p = \frac{1}{|p|}|x|^{-p}.$$

Now, denote by R the radius of a ball with mass m. Note that K_2 and K_p are radially symmetric, we have $K_2 * \chi_{B_R}$ and $K_p * \chi_{B_R}$ are radially symmetric. We have $K_2 * \chi_{B_R}$ is radial increasing following from K_2. Similarly, $K_p * \chi_{B_R}$ is radial decreasing. For $|x| \geq \frac{R}{2}$, we have

$$(\nabla(K_2 * \chi_{B_R})(x) \cdot \frac{x}{|x|}) = \int\limits_{\{|y| \leq R\}} (x-y) \cdot \frac{x}{|x|} dy|_{|x|=\frac{R}{2}} \geq C_2 R^{N+1}, \tag{3.6}$$

where

$$C_2 = 2^{-N} \inf_{t \geq \frac{1}{2}} \int\limits_{\{|y| \leq 1\}} (t - y_1) dy.$$

Similarly, for $|x| \geq \frac{R}{2}$, we also get

$$(\nabla(K_p * \chi_{B_R})(x) \cdot \frac{x}{|x|}) \leq C_p R^{N-p-1}, \tag{3.7}$$

for some constant C_p. Since $-p < 2$, there exists R such that $C_2 R^2 > C_p R^{-p}$. Combining (3.6) with (3.7), we have

$$\nabla(K_2 + K_p) * \chi_{B_R}(x) \cdot \frac{x}{|x|} \geq 0,$$

for $|x| \geq \frac{R}{2}$. Hence,

$$K * \chi_{B_R}(x) = (K_2 + K_p) * \chi_{B_R}(x) \geq K * \chi_{B_R}(x)|_{|x|=R} = \lambda_R,$$

for $|x| \geq R$. Moreover, $K * \chi_{B_R}(x) \leq \lambda_R$ for $\frac{R}{2} \leq |x| < R$.

Secondly, we show that $K * \chi_{B_R}(x) \leq \lambda_R$ for $|x| < \frac{R}{2}$. Indeed, since $K_p * \chi_{B_R}(x)$ and $K_2 * \chi_{B_R}(x)$ are radially symmetric, we have

$$\begin{aligned}
\lambda_R &= \int\limits_{\{|y| \leq R\}} (\frac{|Re_1 - y|^2}{2} + \frac{|Re_1 - y|^{-p}}{p}) dy \\
&= R^{N+2} \int\limits_{\{|y| \leq 1\}} \frac{|e_1 - y|^2}{2} dy + R^{N-p} \int\limits_{\{|y| \leq 1\}} \frac{|e_1 - y|^{-p}}{p} dy \\
&= \bar{C_2} R^{N+2} + \bar{C_p} R^{N-p},
\end{aligned} \tag{3.8}$$

where $\bar{C_2} = K_2 * \chi_{B_1}(x)|_{|x|=1} > 0$, $\bar{C_p} = K_p * \chi_{B_1}(x)|_{|x|=1} > 0$ and e_1 denotes a unit vector in $\mathbb{R}^N$. Since $(K_2 * \chi_{B_R})$ is increasing in $|x|$ and $(K_p * \chi_{B_R})$ is decreasing in $|x|$, we have

$$(K * \chi_{B_R})(x) \leq (K_2 * \chi_{B_R})(x)|_{x=\frac{R}{2}} + (k_q * \chi_{B_R})(0) = \bar{\bar{C}}_2 R^{N+2} + \bar{\bar{C}}_p R^{N-p},$$

where

$$\bar{\bar{C}}_2 = K_2 * \chi_{B_1}(x)|_{|x|=\frac{1}{2}}.$$

Since $K_2 * \chi_{B_R}$ is radially increasing, we get $\bar{\bar{C}}_2 < \bar{C}_2$ for R large enough. Combining with (3.8), we obtain

$$(K * \chi_{B_R})(x) \leq \lambda_R, \tag{3.9}$$

for $|x| \leq R/2$, if R is sufficiently large.

Step 2. We will show the characteristic function of a ball for large enough mass m satisfies the first-order variational inequality corresponding to $E(u)$ in $\mathcal{A}_{m,1}$. That is, the characteristic function is the solution of the problem (3.2).

We construct perturbations that are nonnegative on $S_0 := \{x : |x| > R\}$, nonpositive on $S_1 = \{x : |x| \leq R\}$, and preserve mass. Let ϕ and $\varphi \in BV(\mathbb{R}^N)$ be compactly supported, bounded, nonnegative functions with $\phi = 0$, a.e. in S_1, $\varphi = 0$, a.e. in S_0, and

$$\int_{\mathbb{R}^N} \phi(x)dx = \int_{\mathbb{R}^N} \varphi(x)dx = 1.$$

By construction, $\chi_{B_R} + t(\phi - \varphi)$ lies in $\mathcal{A}_{m,1}$ and the perturbation is small for sufficiently small values of $t > 0$. We claim that

$$\frac{d}{dt}|_{t=0^+} E(\chi_{B_R} + t(\phi - \varphi)) = \int_{\mathbb{R}^N} |\nabla(\phi - \varphi)| + 2\int_{\mathbb{R}^N} K * \chi_{B_R}(\phi - \varphi)(x)dx \geq 0. \tag{3.10}$$

For the second term in the right hand of (3.10), combining (3.3) with (3.9), we get

$$\int_{\mathbb{R}^N} K * \chi_{B_R}(\phi - \varphi)(x)dx = \int_{\{\phi(x)\neq 0\}} K * \chi_{B_R}(x)\phi(x)dx - \int_{\{\varphi(x)\neq 0\}} K * \chi_{B_R}(x)\varphi(x)dx \geq 0. \tag{3.11}$$

Clearly, we have

$$\int_{\mathbb{R}^N} |\nabla(\phi - \varphi)| \geq 0. \tag{3.12}$$

Hence, using (3.11) and (3.12), we have

$$\frac{d}{dt}|_{t=0^+} E(\chi_{B_R} + t(\phi - \varphi)) \geq 0.$$

That implies the function $\chi_{B(0,R)}$ with $R = (\frac{m}{w_N})^{\frac{1}{N}}$ is the solution of the problem (3.2) for large enough m. By Lemma 2.4, $E(u)$ is strictly convex. Hence, the function $\chi_{B(0,R)}$ is the unique solution of the problem (3.2) up to translation.

Step 3. Since

$$\inf_{u\in\mathcal{A}} E(u) \geq \inf_{u\in\mathcal{A}_{m,1}} E(u),$$

and $\chi_{B(0,R)} \in \mathcal{A}$, we obtain that the ball of volume m is the unique solution of the problem (1.1) in $\mathcal{A}$ for sufficiently large m. $\square$

4. Conclusions

In this paper, for the nonlocal isoperimetric with power-law potentials and external attraction, we consider various minimization problem, depending on the signs of the repulsive and attraction power exponents of potential and external attraction. As $0 < q < p < N$, we prove the existence of minimizer. As $q = -2$, $0 < p < N$ and $V(x) = 0$, the proofs rely on a relation between the problem (1.1) in $\mathcal{A}$ and the relaxed problem (3.2).

Note that the other cases $q < 0 < p < N$ and $q < p < 0$ have been not considered in this paper. For $q < 0 < p < N$, we conjecture that the existence of minimizers would be true for sufficiently small m as the energy is dominated by the external attraction and the perimeter which is minimized by the balls of measure m and Theorem 1.2 could extend to the general case $q < -1$ since the energy is dominated by the attractive term which is minimized by the balls of measure m. For $q < p < 0$, the existence of minimizers would be true for any $m > 0$ as the repulsion is bounded and the energy is dominated by the perimeter and the attractive term. In the forthcoming work, we will continue to discuss the existence of solution for the problem (1.1) as $q < 0 < p < N$ and $q < p < 0$ respectively.

Acknowledgments

All authors would like to express their gratitude to the anonymous referee for their invaluable comments and suggestions. This work is supported by NNSF of China (No. 11771291).

References

[1] S. Alama, L. Bronsard, R. Choksi, I. Topaloglu, Ground-states for the liquid drop and TFDW models with long-range attraction, J. Math. Phys. 58 (2017) 1–11.
[2] A. Burchard, R. Choksi, I. Topaloglu, Nonlocal shape optimization via interactions of attractive and repulsive potentials, Indiana Univ. Math. J. 67 (2018) 375–395.
[3] K.C. Chang, The spectrum of the 1-Laplace operator, Commun. Contemp. Math. 11 (2009) 865–894.
[4] R. Choksi, R.C. Fetecau, I. Topaloglu, On minimizers of interaction functionals with competing attractive and repulsive potentials, Ann. Inst. H. Poincaré Anal. Non Linéaire 32 (2015) 1283–1305.
[5] R. Choksi, M.A. Peletier, Small volume fraction limit of the diblock copolymer problem: II. Diffuse interface functional, SIAM J. Math. Anal. 43 (2011) 739–763.
[6] R.L. Frank, E.H. Lieb, A compactness lemma and its application to the existence of minimizers for the liquid drop mode, SIAM J. Math. Anal. 47 (2015) 4436–4450.
[7] N. Fusco, F. Maggi, A. Pratelli, The sharp quantitative isoperimetric inequality, Ann. of Math. 168 (2008) 941–980.
[8] V. Julin, Isoperimetric problem with a Coulombic repulsive term, Indiana Univ. Math. J. 63 (2014) 77–89.
[9] H. Knüepfer, C. Muratov, On an isoperimetric problem with a competing non-local term. II. The general case, Comm. Pure Appl. Math. 66 (2014) 1129–1162.
[10] H. Knüpfer, C.B. Muratov, M. Novaga, Low density phases in a uniformly charged liquid, Comm. Math. Phys. 345 (2016) 141–183.
[11] C. Le Bris, Some results on the Thomas-Fermi-Dirac-von Weizsäcker model, Differential Integral Equations 6 (1993) 337–353.
[12] C. Le Bris, P.L. Lions, From atoms to crystals: a mathematical journey, Bull. Amer. Math. Soc. 42 (2005) 291–363.
[13] E.H. Lieb, M. Loss, Analysis, 2nd edition, AMS, Providence, 2001.
[14] P.L. Lions, Solutions of Hartree-Fock equations for Coulomb systems, Comm. Math. Phys. 109 (1987) 33–97.
[15] J. Lu, F. Otto, Nonexistence of a minimizer for Thomas-Fermi-Dirac-von Weizsäcker model, Comm. Pure Appl. Math. 67 (2014) 1605–1617.
[16] J. Lu, F. Otto, An isoperimetric problem with Coulomb repulsion and attraction to a background nucleus, arXiv:1508.07172.
[17] F. Maggi, Sets of Finite Perimeter and Geometric Variational Problems, Cambridge University Press, 2012.
[18] C.B. Maratov, A. Zaleski, On an isoperimetric problem with a competing non-local term: quantitative results, Ann. Global Anal. Geom. 47 (2015) 63–80.
[19] Z. Milbers, F. Schuricht, Some special aspects related to the 1-Laplace operator, Adv. Calc. Var. 4 (2010) 101–126.
[20] A. Mogilner, L. Edelstein-Keshet, A non-local model for a swarm, J. Math. Biol. 38 (1999) 534–570.
[21] G. Toscani, One-dimensional kinetic models of granular flows, Modél. Math. Anal. Numér. 34 (2000) 1277–1291.