J. Math. Anal. Appl. 485 (2020) 123837

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Discontinuous Galerkin method for the fully dynamic Biot's model ☆

Jing Wen, Yinnian He, Hongbin Chen *

Department of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, PR China

ARTICLE INFO

Article history:
Received 26 September 2019
Available online 7 January 2020
Submitted by G. Chen

Keywords:
Fully dynamic Biot's model
Discontinuous Galerkin method
Fully discrete scheme
Priori error estimates

ABSTRACT

In this paper, a fully discrete scheme of the fully dynamic Biot's model problem is proposed, which is constructed by using interior penalty discontinuous Galerkin method for the spatial approximation and a tailor difference scheme to approximate the first and second order temporal derivative terms. First of all, we prove the existence and uniqueness of solutions of proposed fully discrete scheme in proper norms. Then, based on the error equations a priori error estimates shall be derived for both primal variables displacement and pore pressure. Finally, a series of numerical examples are given to examine the convergence results by using the proposed numerical scheme to solve the fully dynamic Biot's model problem.

1. Introduction

The fully dynamic Biot's model (known as poroelastic model) describes the time-dependent fluid flow through a porous media material with elastic deformation and it can trace back to the work of Terzaghi and Biot. Terzaghi [25] built the corresponding theory by analyzing the consolidation of a soil column under one-dimensional situation. Then, Biot [2] generalized Terzaghi's work to three-dimensional case. Since then, on account of its ubiquity and special properties, quasi-static Biot's model [3] which is degenerated through fully dynamic Biot's model by ignoring the second order temporal derivative term, has been widely used and studied in different fields of scientific research and engineering applications, including materials science [6], reservoir engineering [26], environmental engineering [14] and bio-mechanical engineering applications [4]. There are some related researches to the quasi-static Biot's model. Zenisek [27] built the well-posedness of this Biot's model. Then, Phillips and Wheeler [17,18] addressed some efficient numerical scheme to solve this problem based on both continuous and discontinuous Galerkin methods. However, there is locking phenomenon [19] in this problem under certain circumstances, which presents a pseudo pressure when

☆ This work was supported by the National Natural Science Foundation of China Grant No. 11771348.
* Corresponding author.
E-mail address: hbchen@xjtu.edu.cn (H. Chen).

https://doi.org/10.1016/j.jmaa.2019.123837
0022-247X/

the reservoir coefficient c_o is verge or equal to zero and the permeability tensor is small. For avoiding the locking phenomenon, people utilized the non-conforming and discontinuous Galerkin methods to this model, Rivière [22] used the interior penalty discontinuous Galerkin methods to discrete the Biot's model based on mixed formulations, and Feng et al. [10,11] rearranged the Biot's model equations and novelly proposed a multi-physics scheme, both of which are well overcome the locking phenomenon. For the original fully dynamic Biot's model, Showalter slightly refer to this model problem in [24], but Showalter does not keep work on relevant study. Recently, Lotfian and Sivaselvan [16] proposed a scheme combining mixed finite element method and difference formula and validated the scheme is efficient by some numerical experiments. Particularly, it is meaningful and representative that to study this fully dynamic Biot's model, because some related coupling problems with the fully dynamic Biot's model have been introduced, such as the coupled Navier-Stokes problems/Biot model [7] and fluid-structure interaction problems [5].

In this paper, we apply the interior penalty discontinuous Galerkin method to discrete the fully dynamic Biot's model. Discontinuous Galerkin methods introduced by Reed and Hill [20] in neutron transport equations have become very popular in theoretical study and engineering applications [8,15,18,23] in the past four decades, because it has some particular and remarkable features e.g., arbitrary order accuracy, ability to deal with hanging nodes, local mass conservation, ready parallelization and adaption. To build a well-posed and efficient numerical scheme for the fully dynamic Biot's model, we have to resort the numerical scheme of wave equation, because the wave equation exists a second order time derivative term that is similar with fully dynamic Biot's model problem. For the wave equation, Grote and Schötzau [12] proposed a explicit fully discrete scheme, in which a CFL condition is required. Then, in [13], Han et al. presented an implicit-explicit fully discrete scheme for the wave equation, which does not need the CFL condition. Based on these thoughts we put forward a implicit-explicit fully discrete scheme for the fully dynamic model problem, which is constructed by using interior penalty discontinuous Galerkin method for the spatial approximation and a tailor difference formula to approximate the first and second order time derivative terms. However, it certainly is a big difference from the fully discrete scheme of wave equation, the main difference includes two parts, one is we have to introduce a proper difference formula to discrete the first order temporal derivative terms and the other one is the selection of initial values. The theoretical analysis and numerical results show that the proposed fully discrete scheme is proved to be efficient.

The outline of the rest of this article is arranged as follows: Section 2 introduces and states the fully dynamic Biot's model, and some basic notions and useful results used throughout this article are given. In Section 3, based on original model problem we propose the fully discrete scheme and prove the existence and uniqueness of solutions. Section 4 is contributed to a priori error estimates. In Section 5, we show some numerical tests to examine and support the convergence analysis results. Section 6 presents some conclusions.

2. Preliminaries

In this section, we introduce some preliminary preparations. Firstly, the fully dynamic Biot's model problem is given. Then, we recall some basic notions and list some inequalities.

2.1. Model problem

Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be a open bounded domain with Lipschitz continuous boundary $\partial\Omega$ and $[0, T)$ is an time interval with $T > 0$, the space-time domain is defined by $Q = \Omega \times [0, T)$. There are two model equations with respect to the fully dynamic Biot's problem, one (1) is called the momentum equation and the other one (2) is termed as the mass equation, which are coupled by Biot-Willis constant α. The initial-boundary valued problem of fully dynamic Biot's model we consider is to find displacement $\mathbf{u}(\mathbf{x}, t)$ and pore pressure $p(\mathbf{x}, t)$ such that

Table 1
Summary of physical parameters.

Parameters	Description
λ, μ	Positive Lamé constants
$c_o \geq 0$	Storage coefficient
α	Biot-Willis constant
$\mathbf{K}$	Symmetric permeability tensor

Table 2
Summary of constitutive relations.

$\tilde{\sigma}(\mathbf{u}, p) = \sigma(\mathbf{u}) - \alpha p\mathbf{I}$	Total stress
$\sigma(\mathbf{u}) = \lambda tr(\epsilon(\mathbf{u}))\mathbf{I} + 2\mu\epsilon(\mathbf{u})$	Effective stress
$\epsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$	Strain tensor

$$\frac{\partial^2\mathbf{u}}{\partial t^2} - \nabla \cdot \sigma(\mathbf{u}) + \alpha\nabla p = \mathbf{f} \qquad \text{in} \quad Q, \tag{1}$$

$$\frac{\partial}{\partial t}(c_o p + \alpha\nabla \cdot \mathbf{u}) - \nabla \cdot (\mathbf{K}\nabla p) = l \qquad \text{in} \quad Q, \tag{2}$$

with boundary conditions

$$p = p_D \quad \text{on} \quad \Gamma_p \times [0, T), \tag{3}$$

$$-\mathbf{K}\nabla p \cdot \mathbf{n} = q_N \quad \text{on} \quad \Gamma_f \times [0, T), \tag{4}$$

$$\mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma_o \times [0, T), \tag{5}$$

$$-\tilde{\sigma}\mathbf{n} = \mathbf{t}_N \quad \text{on} \quad \Gamma_t \times [0, T), \tag{6}$$

and initial-valued conditions

$$p|_{t=0} = p^0 \qquad \text{in} \quad \Omega, \tag{7}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}^0 \qquad \text{in} \quad \Omega, \tag{8}$$

$$\mathbf{u}'|_{t=0} = \mathbf{u}'_0 \qquad \text{in} \quad \Omega. \tag{9}$$

In (1)-(9), $\mathbf{f}$, l and $\mathbf{n}$ are body force, source term and unit outward normal vector to $\partial\Omega$, respectively. In addition, some notions corresponding to the model problem are presented in Table 1 and Table 2 in which "tr" stands for trace of the given functions and $\mathbf{I}$ denotes the unit matrix. $\partial\Omega = \Gamma_p \cup \Gamma_f$ and $\partial\Omega = \Gamma_t \cup \Gamma_o$ respectively, represent the boundary subdivision of pore pressure and displacement. The permeability tensor $\mathbf{K}$ is the uniformly elliptic, that is, there exist two positive constants λ_{min} and λ_{max}, for any $\xi \in \mathbb{R}^d$ such that

$$\lambda_{min}\xi^2 \leq \xi^T\mathbf{K}\xi \leq \lambda_{max}\xi^2.$$

2.2. Basic notation and inequalities

Let $\mathcal{T}_h = \{K\}$ be a family of non-overlap and shape-regular subdivision of Ω parameterized by $h > 0$, where h denotes the discrete spatial mesh-size and triangle/tetrahedron K stands for physical computation element, we define $h_K = diam(K)$ and $h = \max_{K\in\mathcal{T}_h} h_K$. Besides, all element faces are denoted by $\mathcal{E}_h$ and $\mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^\partial$. $\mathcal{E}_h^i$ and $\mathcal{E}_h^\partial$ represent the inner and boundary edges, respectively. Similarly, for each element face e of $K \in \mathcal{T}_h$, we define $h_e = diam(e)$. Let e be an interior face shared by two elements K_i and $K_j (i > j)$, $\mathbf{n}$ is unit normal vector from K_i to K_j. For a scalar function p, let $p_i = p|_{\partial K_i}$, $p_j = p|_{\partial K_j}$, and we define

$$\{p\} = \frac{1}{2}(p|_i + p|_j), \quad [p] = (p|_i - p|_j).$$

For a vector function $\mathbf{u}$, let $\mathbf{u}_i = \mathbf{u}|_{\partial K_i}$, $\mathbf{u}_j = \mathbf{u}|_{\partial K_j}$, and we define

$$\{\mathbf{u}\} = \frac{1}{2}(\mathbf{u}|_i + \mathbf{u}|_j), \quad [\mathbf{u}] = (\mathbf{u}|_i - \mathbf{u}|_j).$$

Moreover, if $e \in \partial\Omega$,

$$[p] = \{p\} = p|_e, \qquad [\mathbf{u}] = \{\mathbf{u}\} = \mathbf{u}|_e.$$

More details about the interior penalty discontinuous Galerkin methods one can see references [15,21].

The standard functional spaces $W^{k,\tilde{p}}$ in classical Sobolev spaces theory (see [1,9]) are frequently used in this paper, with norm $\|\cdot\|_{k,\tilde{p}}$ and semi-norm $|\cdot|_{k,\tilde{p}}$. Particularly, for a given region $D \in \mathbb{R}^d (d = 1, 2, 3)$, when $\tilde{p} = 2$, we have $W^{k,2}(D) = H^k(D)$ with norm $\|\cdot\|_{k,D}$ and when $D = \Omega$, we ignore the index D. Furthermore, for $k = 0$ and $\tilde{p} = 2$, we define the norm of $W^{0,2}(\Omega)$ by $\|\cdot\|_{L^2(\Omega)}$, which is the standard L^2-norm, and $(\cdot,\cdot)_D$ denotes the $L^2(D)$ inner product. In addition, we consider spaces

$$L^s(0,T;Y) = \{z : (0,T) \to Y, \int_0^T \|z\|_Y^s dt < \infty\},$$

with

$$\|z\|^2_{L^s(0,T;Y)} = \int_0^T \|z\|_Y^s dt,$$

where $s \geq 1$ and Y is a normed space equipped with the norm $\|\cdot\|_Y$. Similarly, we can define $C^s(0,T;Y)$. In this paper, it is noticeable that the bold fonts stand for the corresponding vector-valued functions or functional spaces.

Some discontinuous/broken Sobolev spaces (see [21]) on the decomposition $\mathcal{T}_h$ shall be recalled. For some nonnegative integer s, define

$$H^s(\mathcal{T}_h) = \{v \in L^2(\Omega) : p|_K \in H^s(K), \quad \forall K \in \mathcal{T}_h\}.$$

Based on the discontinuous Sobolev spaces, the energy norms of relevant spaces are defined by

$$\forall p \in H^s(\mathcal{T}_h), \qquad \|p\|_\epsilon^2 = \sum_{K\in\mathcal{T}_h} \int_K \mathbf{K}\nabla p \cdot \nabla p d\mathbf{x} + h_e^{-1} \sum_{e\in\mathcal{E}_h} \int_e [p][p] ds,$$

$$\forall \mathbf{u} \in \mathbf{H}^s(\mathcal{T}_h), \qquad \|\mathbf{u}\|_\epsilon^2 = \sum_{K\in\mathcal{T}_h} \int_K \nabla\mathbf{u} : \nabla\mathbf{u} d\mathbf{x} + h_e^{-1} \sum_{e\in\mathcal{E}_h} \int_e [\mathbf{u}] \cdot [\mathbf{u}] ds.$$

For some nonnegative integer r, $P_r(K)$ stands for a family of polynomials with polynomials degree no more than r. Then, we introduce the discontinuous finite element spaces $\mathcal{Q}_h$ and $\mathbf{V}_h$ used in this article, define

$$\mathcal{Q}_h = \{q \in H^s(\mathcal{T}_h) : q|_K \in P_r(K), \quad \forall K \in \mathcal{T}_h\},$$

also, define

$$\mathbf{V}_h = [\mathcal{Q}_h]^d = \{\mathbf{u} \in \mathbf{H}^s(\mathcal{T}_h) : \mathbf{u}|_K \in \mathbf{P}_r(K), \quad \forall K \in \mathcal{T}_h\}.$$

Note that, $\mathcal{Q}_h$ is a approximation space of pore pressure and $\mathbf{V}_h$ is a approximation space of displacement. For convenience, $\forall p \in H^s(\mathcal{T}_h)$ we always consider

$$\|p\|^2 = \sum_{K\in\mathcal{T}_h} \int_K p^2 d\mathbf{x}, |p|_1^2 = \sum_{K\in\mathcal{T}_h} \int_K (\nabla p)^2 d\mathbf{x}, \|p\|_1^2 = \|p\|^2 + |p|_1^2,$$

and

$$\forall p, q \in H^s(\mathcal{T}_h), (p, q) = \sum_{K\in\mathcal{T}_h} \int_K pq d\mathbf{x}.$$

Next, some useful and classical inequalities shall be listed (see [21]).

Lemma 2.1. (*Discrete Gronwall inequality*). *Let constants Δt, B, $C > 0$ and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be a sequences of nonnegative real series satisfying*

$$a_n + \Delta t \sum_{i=0}^{n} b_i \le B + C\Delta t \sum_{i=0}^{n} a_i + \Delta t \sum_{i=0}^{n} c_i, \quad \forall n \ge 0.$$

Then, if $C\Delta t < 1$,

$$a_n + \Delta t \sum_{i=0}^{n} b_i \le e^{C(n+1)\Delta t}(B + \Delta t \sum_{i=0}^{n} c_i), \quad \forall n \ge 0.$$

Lemma 2.2. (*Trace inequality*). *There exists a constant c_{tr} independent of mesh-size h, such that*

$$\forall p \in P_r(K), \forall e \subset \partial K, \|p\|_{L^2(e)} \le c_{tr}|e|^{1/2}|K|^{-1/2}\|p\|_{L^2(K)},$$

if $p \in H^1(K)$,

$$\forall e \subset \partial K, \|p\|_{L^2(e)} \le c_{tr}|e|^{1/2}|K|^{-1/2}(\|p\|_{L^2(K)} + h_e\|p\|_{H^1(K)}),$$

where $|\cdot|$ represents the Lebesgue measure of relevant area.

Lemma 2.3. (*Broken Poincaré inequality*). *Let constant c_p be independent of mesh-size h, and if we assume that the Dirichlet boundary $\mathcal{T}_D$ of p is non-empty subset of $\partial\Omega$, then it satisfies*

$$\forall p \in H^1(\mathcal{T}_h), \|p\| \le c_p(\|\nabla p\|^2 + \sum_{e\in\mathcal{E}_h^i\cup\mathcal{T}_D} \frac{1}{|e|^{\frac{1}{d-1}}}\|[p]\|_{L_2(e)}^2)^{1/2}.$$

Lemma 2.4. (*Korn inequality*). *Let constant c_{ko} be independent of mesh-size h, and if we assume that the Dirichlet boundary $\mathcal{T}_D$ of $\mathbf{u}$ is non-empty subset of $\partial\Omega$, we then get*

$$\forall \boldsymbol{u} \in \boldsymbol{H}^1(\mathcal{T}_h), \|\nabla \boldsymbol{u}\| \le c_{ko}(\|\epsilon(\boldsymbol{u})\|^2 + \sum_{e\in\mathcal{E}_h^i\cup\mathcal{T}_D} \frac{1}{|e|^{\frac{1}{d-1}}}\|[\boldsymbol{u}]\|_{L_2(e)}^2)^{1/2}.$$

Indeed, these inequalities defined by scalar-valued functions in this section, can be extended to vector-valued functions.

3. Numerical scheme

In this section, firstly, a fully discrete scheme of the model problem is proposed, we then prove the existence and unique solvability of the numerical scheme.

3.1. Fully discrete scheme

Let Δt represent the discrete time step-size, for some positive integer N, which satisfies the relation $\Delta t = \frac{T}{N}$, and $t_n = n\Delta t$ for $n = 1, 2, \cdots, N$. Before giving the fully discrete scheme, some bilinear operators shall be introduced. For $\forall p, q \in H^1(\mathcal{T}_h)$, the bilinear operator a_p is defined by

$$\begin{aligned} a_p(p,q) = &\sum_{K\in\mathcal{T}_h}\int_K \mathbf{K}\nabla p\cdot\nabla q d\mathbf{x} - \sum_{e\in\mathcal{E}_h}\int_e \{\mathbf{K}\nabla p\}\cdot\mathbf{n}[q]ds \\ &- \sum_{e\in\mathcal{E}_h}\int_e \{\mathbf{K}\nabla q\}\cdot\mathbf{n}[p]ds + \beta_1\sum_{e\in\mathcal{E}_h} h_e^{-1}\int_e [p][q]ds, \end{aligned} \tag{10}$$

where $\beta_1 \geq 1$ is a stability coefficient. For $\forall \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\mathcal{T}_h)$, the bilinear operator $a_{\mathbf{u}}$ is defined by

$$\begin{aligned} a_{\mathbf{u}}(\mathbf{u},\mathbf{v}) = &\sum_{K\in\mathcal{T}_h}\int_K \sigma(\mathbf{u}):\epsilon(\mathbf{v})d\mathbf{x} - \sum_{e\in\mathcal{E}_h}\int_e \{\sigma(\mathbf{u})\}\mathbf{n}\cdot[\mathbf{v}]ds \\ &- \sum_{e\in\mathcal{E}_h}\int_e \{\sigma(\mathbf{v})\}\mathbf{n}\cdot[\mathbf{u}]ds + \beta_2\sum_{e\in\mathcal{E}_h} h_e^{-1}\int_e [\mathbf{u}]\cdot[\mathbf{v}]ds, \end{aligned} \tag{11}$$

where $\beta_2 \geq 1$ is a stability constant. For $\forall q \in H^1(\mathcal{T}_h)$, $\mathbf{u} \in \mathbf{H}^1(\mathcal{T}_h)$, define

$$b(\mathbf{u},q) = \sum_{K\in\mathcal{T}_h}\int_K \nabla\cdot\mathbf{u} q d\mathbf{x} - \sum_{e\in\mathcal{E}_h^i}\int_e [\mathbf{u}]\cdot\mathbf{n}\{q\}ds. \tag{12}$$

For simplicity, we only consider the homogeneous Dirichlet boundary conditions, that is boundary conditions (4) and (6) are not involved in this paper. Then, based on bilinear forms (10), (11) and (12), we present the following numerical fully discrete scheme of the fully dynamic Biot's problem: Find $p_h^{n+1} \in \mathcal{Q}_h$ and $\mathbf{u}_h^{n+1} \in \mathbf{V}_h$ $(n \geq 2)$, for any $q_h \in \mathcal{Q}_h$ and $\mathbf{v}_h \in \mathbf{V}_h$, such that

$$\begin{aligned} &(\frac{\mathbf{u}_h^{n+1} - 2\mathbf{u}_h^n + \mathbf{u}_h^{n-1}}{\Delta t^2}, \mathbf{v}_h) + a_{\mathbf{u}}(\frac{\mathbf{u}_h^{n+1} + \mathbf{u}_h^{n-1}}{2}, \mathbf{v}_h) \\ &\qquad - \alpha b(\mathbf{v}_h, \frac{p_h^{n+1} + p_h^{n-1}}{2}) = (\mathbf{f}(\cdot, t_n), \mathbf{v}_h), \end{aligned} \tag{13}$$

$$\begin{aligned} &c_o(\frac{p_h^{n+1} - p_h^{n-1}}{2\Delta t}, q_h) + \alpha b(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^{n-1}}{2\Delta t}, q_h) \\ &+ a_p(\frac{p_h^{n+1} + p_h^{n-1}}{2}, q_h) = (\frac{l(\cdot, t_{n+1}) + l(\cdot, t_{n-1})}{2}, q_h), \end{aligned} \tag{14}$$

and

$$\mathbf{u}_h^0 \in \mathbf{V}_h, \quad (\mathbf{u}^0 - \mathbf{u}_h^0, \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{15}$$

$$p_h^0 \in \mathcal{Q}_h, \quad (p^0 - p_h^0, q_h) = 0, \quad \forall q_h \in \mathcal{Q}_h, \tag{16}$$

$$\mathbf{u}_{0,h} \in \mathbf{V}_h, \quad (\mathbf{u}_0' - \mathbf{u}_{0,h}, \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{17}$$

$$\mathbf{u}_h^1 = \mathbf{u}_h^0 + \mathbf{u}_{0,h}\Delta t + \mathbf{u}_{0,h}^0 \Delta t^2, \tag{18}$$

where $\mathbf{u}_{0,h}^0 \in \mathbf{V}_h$ and it satisfies

$$(\mathbf{u}_{0,h}^0, \mathbf{v}_h) = (\mathbf{f}(\cdot, t_0), \mathbf{v}_h) - a_{\mathbf{u}}(\mathbf{u}^0, \mathbf{v}_h) + \alpha b(\mathbf{v}_h, p^0), \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \tag{19}$$

$$\begin{aligned} p_h^1 \in \mathcal{Q}_h, c_o(\frac{p_h^1 - p_h^0}{\Delta t}, q_h) + a_p(\frac{p_h^1 + p_h^0}{2}, q_h) \\ = (\frac{l(\cdot, t_0) + l(\cdot, t_1)}{2}, q_h) - \alpha b(\frac{\mathbf{u}_h^1 - \mathbf{u}_h^0}{\Delta t}, q_h), \quad \forall q_h \in \mathcal{Q}_h. \end{aligned} \tag{20}$$

It is noticeable that the numerical scheme (13)-(14) is nothing but a general numerical scheme. (13) is a classical discretization formulation for the second order derivative with respect to temporal variable (see [13]) and (14) is the Crank-Nicolson formulation with two time step-size. Particularly, a Taylor expansion to derive $\mathbf{u}_h^1$ (18) and a standard Crank-Nicolson formulation (20) to solve p_h^1.

Hereafter, we always use constant c to denote a constant varying with different occurrences, which is independent of mesh-size h.

3.2. Existence and uniqueness

Lemma 3.1. (*Coercivity and Boundedness*). *If the penalty parameters β_1 and β_2 are sufficiently large, there exist two positive constants c_{coe1} and c_{coe2}, such that*

$$\forall p_h, q_h \in \mathcal{Q}_h, c_{coe1}\|p_h\|_\epsilon^2 \le a_p(p_h, p_h), \quad a_p(p_h, q_h) \le c\|p_h\|_\epsilon \|q_h\|_\epsilon,$$
$$\forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_h, c_{coe2}\|\mathbf{u}_h\|_\epsilon^2 \le a_{\mathbf{u}}(\mathbf{u}_h, \mathbf{u}_h), \quad a_{\mathbf{u}}(\mathbf{u}_h, \mathbf{v}_h) \le c\|\mathbf{u}_h\|_\epsilon \|\mathbf{v}_h\|_\epsilon.$$

Theorem 3.1. (*Existence and Uniqueness*). *There exist solutions $(\mathbf{u}_h^{n+1}, p_h^{n+1})$ satisfying the fully discrete scheme (13)-(20), and solutions $(\mathbf{u}_h^{n+1}, p_h^{n+1})$ is unique.*

Proof. The existence and uniqueness of the fully discrete scheme and its homogeneous problem is equivalent, because it is a linear, finite dimensional and square system, therefore, we only to prove the corresponding homogeneous problem has unique null solutions. The homogeneous problem of fully discrete scheme is find $p_h^{n+1} \in \mathcal{Q}_h$ and $\mathbf{u}_h^{n+1} \in \mathbf{V}_h$, for any $q_h \in \mathcal{Q}_h$ and $\mathbf{v}_h \in \mathbf{V}_h$, such that

$$(\frac{\mathbf{u}_h^{n+1}}{\Delta t^2}, \mathbf{v}_h) + a_{\mathbf{u}}(\frac{\mathbf{u}_h^{n+1}}{2}, \mathbf{v}_h) - \alpha b(\mathbf{v}_h, \frac{p_h^{n+1}}{2}) = 0, \tag{21}$$

$$c_o(\frac{p_h^{n+1}}{2\Delta t}, q_h) + \alpha b(\frac{\mathbf{u}_h^{n+1}}{2\Delta t}, q_h) + a_p(\frac{p_h^{n+1}}{2}, q_h) = 0. \tag{22}$$

Choosing $\mathbf{v}_h = \mathbf{u}_h^{n+1}$ in (21), we obtain

$$(\frac{\mathbf{u}_h^{n+1}}{\Delta t^2}, \mathbf{u}_h^{n+1}) + a_{\mathbf{u}}(\frac{\mathbf{u}_h^{n+1}}{2}, \mathbf{u}_h^{n+1}) - \alpha b(\mathbf{u}_h^{n+1}, \frac{p_h^{n+1}}{2}) = 0. \tag{23}$$

Taking $q_h = \Delta t p_h^{n+1}$ in (22), we have

$$c_o(\frac{p_h^{n+1}}{2}, p_h^{n+1}) + \alpha b(\frac{\mathbf{u}_h^{n+1}}{2}, p_h^{n+1}) + \Delta t a_p(\frac{p_h^{n+1}}{2}, p_h^{n+1}) = 0. \tag{24}$$

Adding (23) to (24) and Multiplying by 2, we get

$$2\|\frac{\mathbf{u}_h^{n+1}}{\Delta t}\|^2 + a_{\mathbf{u}}(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}) + c_o\|p_h^{n+1}\|^2 + \Delta t a_p(p_h^{n+1}, p_h^{n+1}) = 0,$$

by the coercivity of bilinear forms $a_{\mathbf{u}}$ and a_p and $\Delta t > 0$, $c_o \geq 0$, we have $\|\mathbf{u}_h^{n+1}\|_\epsilon = \mathbf{0}$ and $\|p_h^{n+1}\|_\epsilon = 0$, and complete the proof of existence and uniqueness of solutions. □

4. Priori error estimates

In this section, a priori error estimates based on fully discrete scheme shall be derived. Before giving the error estimates, the projections and estimates for the initial values are presented [21].

Lemma 4.1. (L^2*-projection*). *Let* $\forall p \in H^s(\Omega)(s \geq 1)$, *there exists a unique* $p_h \in \mathcal{Q}_h$, *such that*

$$\forall q_h \in \mathcal{Q}_h, \quad (p - p_h, q_h) = 0,$$

with the approximation property

$$\|p - p_h\| + h\|p - p_h\|_1 \leq ch^\mu \|p\|_{H^s(\Omega)},$$

where $\mu = \min(r+1, s)$.

Lemma 4.2. (*Elliptic projection*). *Let* $\forall \mathbf{u} \in \boldsymbol{H}^s(\Omega)(s \geq 0)$, *there exist a unique* $\mathbf{u}_h \in \mathbf{V}_h$, *such that*

$$\forall \mathbf{v}_h \in \mathbf{V}_h, \quad a_{\mathbf{u}}(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) = 0,$$

with the approximation properties

$$\|\mathbf{u} - \mathbf{u}_h\|_\epsilon \leq ch^\mu \|\mathbf{u}\|_{\boldsymbol{H}^s(\Omega)},$$
$$\|\mathbf{u} - \mathbf{u}_h\| \leq ch^{\mu_1} \|\mathbf{u}\|_{\boldsymbol{H}^s(\Omega)},$$

where $\mu = \min(r+1, s) - 1$ *and* $\mu_1 = \min(r+1, s) - 1$, *specially,* $\mu_1 = \min(r+1, s)$, *if* Ω *is convex.*

Lemma 4.3. *Let function* $z(\cdot, t)$ *be sufficient smoothness, then, the following estimates holds*

$$\|z(\cdot, t_{n+1}) - z(\cdot, t_n)\|^2 \leq \Delta t \int_{t_n}^{t_{n+1}} \|z_t(\cdot, t)\|^2 dt,$$

$$\|\frac{z(\cdot, t_{n+1}) - 2z(\cdot, t_n) + z(\cdot, t_{n-1})}{\Delta t}\|^2 \leq \Delta t \int_{t_{n-1}}^{t_{n+1}} \|z_{tt}(\cdot, t)\|^2 dt,$$

where $z_t(\cdot, t)$ *and* $z_{tt}(\cdot, t)$ *represent the first and second order derivative of* $z(\cdot, t)$ *with respect to temporal variable, respectively.*

Proof. The first estimate is the direct result of the Cauchy-Schwarz inequality

$$\|z(\cdot, t_{n+1}) - z(\cdot, t_n)\|^2 = \|\int_{t_n}^{t_{n+1}} z_t(\cdot, t) dt\|^2 \leq \Delta t \int_{t_n}^{t_{n+1}} \|z_t(\cdot, t)\|^2 dt.$$

Similarly, by the Taylor expansion and Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&\|z(\cdot,t_{n+1})-2z(\cdot,t_n)+z(\cdot,t_{n-1})\|^2 \\
&=\|\int_{t_n}^{t_{n+1}}(t_{n+1}-t)z_{tt}(\cdot,t)dt+\int_{t_{n-1}}^{t_n}(t_n-t)z_{tt}(\cdot,t)dt\|^2 \\
&\le \frac{1}{3}\Delta t^3\int_{t_{n-1}}^{t_{n+1}}\|z_{tt}(\cdot,t)\|^2dt. \quad \square
\end{aligned}
$$

Lemma 4.4. *Let $z\in H^{r+1}(\Omega)\cap H^2(\mathcal{T}_h)$ and its L^2-projection $z_h\in\mathcal{Q}_h$, we then have*

$$
\forall \tilde{z}_h\in\mathcal{Q}_h,\quad a_p(z-z_h,\tilde{z}_h)\le\tilde{c}\|\tilde{z}_h\|_\epsilon^2+\bar{c}h^{2r},
$$

where $\tilde{c}$ is a small constant and $\bar{c}$ is a bounded constant, both of which are independent of mesh-size h.

Proof. This inequality is a general conclusion after one use the approximation property of L^2-projection, and Cauchy-Schwarz, Young and trace inequalities to it, the specific proof one can see [21]. $\square$

Lemma 4.5. *Let initial values satisfy*

$$
\boldsymbol{u}\in C^2([0,t_1];\boldsymbol{H}^{r+1}(\Omega)),\mathbf{u}_{ttt}\in L^2(0,t_1;\boldsymbol{H}^1(\Omega)),p_{ttt}\in L^2(0,t_1;L^2(\Omega)),
$$

we have the following error estimates

$$
\begin{aligned}
\|\boldsymbol{u}^0-\mathbf{u}_h^0\|+\|p^0-p_h^0\|+\|\mathbf{u}_0'-\mathbf{u}_{0,h}\|&\le ch^{r+1},\\
\|\mathbf{u}^0-\mathbf{u}_h^0\|_\epsilon+\|p^0-p_h^0\|_\epsilon+\|\mathbf{u}_0'-\mathbf{u}_{0,h}\|_\epsilon&\le ch^r,\\
\|\mathbf{u}(\cdot,t_1)-\mathbf{u}_h^1\|+\|\frac{\mathbf{u}(\cdot,t_1)-\mathbf{u}^0}{\Delta t}-\frac{(\mathbf{u}_h^1-\mathbf{u}_h^0)}{\Delta t}\|&\le c(h^{r+1}+\Delta t^2),\\
\|\mathbf{u}(\cdot,t_1)-\mathbf{u}_h^1\|_\epsilon+\|\frac{\mathbf{u}(\cdot,t_1)-\mathbf{u}^0}{\Delta t}-\frac{(\mathbf{u}_h^1-\mathbf{u}_h^0)}{\Delta t}\|_\epsilon&\le c(h^r+\Delta t^2),\\
c_o\|p(\cdot,t_1)-p_h^1\|+\|p(\cdot,t_1)-p_h^1\|_\epsilon&\le c(h^r+\Delta t^2).
\end{aligned}
$$

Proof. In light of the approximation property of L^2-projection, we obtain

$$
\begin{aligned}
\|\mathbf{u}^0-\mathbf{u}_h^0\|+\|p^0-p_h^0\|+\|\mathbf{u}_0'-\mathbf{u}_{0,h}\|&\le ch^{r+1},\\
\|\mathbf{u}^0-\mathbf{u}_h^0\|_1+\|p^0-p_h^0\|_1+\|\mathbf{u}_0'-\mathbf{u}_{0,h}\|_1&\le ch^r.
\end{aligned}
$$

By trace inequality, we have

$$
\|\mathbf{u}^0-\mathbf{u}_h^0\|_\epsilon+\|p^0-p_h^0\|_\epsilon+\|\mathbf{u}_0'-\mathbf{u}_{0,h}\|_\epsilon\le ch^r.
$$

Note that, by the original model equation (1), it satisfies

$$
(\mathbf{u}_{tt}(\cdot,t_0),\mathbf{v}_h)=(\mathbf{f}(\cdot,t_0),\mathbf{v}_h)-a_{\mathbf{u}}(\mathbf{u}^0,\mathbf{v}_h)+\alpha b(\mathbf{v}_h,p^0),\forall\mathbf{v}_h\in\mathbf{V}_h, \tag{25}
$$

subtracting (19) from (25) yields

$$(\mathbf{u}_{tt}(\cdot, t_0) - \mathbf{u}_{0,h}^0, \mathbf{v}_h) = 0, \forall \mathbf{v}_h \in \mathbf{V}_h,$$

thus, $\mathbf{u}_{0,h}^0$ is the L^2-projection of $\mathbf{u}_{tt}(\cdot, t_0)$, by the approximation property of L^2-projection and trace inequality, we obtain

$$\|\mathbf{u}_{tt}(\cdot, t_0) - \mathbf{u}_{0,h}^0\| + h\|\mathbf{u}_{tt}(\cdot, t_0) - \mathbf{u}_{0,h}^0\|_1 + h\|\mathbf{u}_{tt}(\cdot, t_0) - \mathbf{u}_{0,h}^0\|_\epsilon \le ch^{r+1}.$$

According to the Taylor expansion, there holds

$$\mathbf{u}(\cdot, t_1) = \mathbf{u}^0 + \Delta t \mathbf{u}_0' + \Delta t^2 \mathbf{u}_{tt}(\cdot, t_0) + \frac{1}{2}\int_{t_0}^{t_1} (t_1 - t)^2 \mathbf{u}_{ttt}(\cdot, t) dt. \tag{26}$$

Subtracting (18) from (26), it satisfies

$$\begin{aligned} \mathbf{u}(\cdot, t_1) - \mathbf{u}_h^1 =& (\mathbf{u}^0 - \mathbf{u}_h^0) + \Delta t(\mathbf{u}_0' - \mathbf{u}_{0,h}) \\ & + \Delta t^2(\mathbf{u}_{tt}(\cdot, t_0) - \mathbf{u}_{0,h}^0) + \frac{1}{2}\int_{t_0}^{t_1} (t_1 - t)^2 \mathbf{u}_{ttt}(\cdot, t) dt, \end{aligned} \tag{27}$$

immediately, the triangle inequality implies

$$\begin{aligned} \|\mathbf{u}(\cdot, t_1) - \mathbf{u}_h^1\| \le& \|\mathbf{u}^0 - \mathbf{u}_h^0\| + \Delta t\|\mathbf{u}_0' - \mathbf{u}_{0,h}\| \\ & + \Delta t^2\|\mathbf{u}_{tt}(\cdot, t_0) - \mathbf{u}_{0,h}^0\| + \frac{1}{2}\|\int_{t_0}^{t_1} (t_1 - t)^2 \mathbf{u}_{ttt}(\cdot, t) dt\|, \\ \|\mathbf{u}(\cdot, t_1) - \mathbf{u}_h^1\|_1 \le& \|\mathbf{u}^0 - \mathbf{u}_h^0\|_1 + \Delta t\|\mathbf{u}_0' - \mathbf{u}_{0,h}\|_1 \\ & + \Delta t^2\|\mathbf{u}_{tt}(\cdot, t_0) - \mathbf{u}_{0,h}^0\|_1 + \frac{1}{2}\|\int_{t_0}^{t_1} (t_1 - t)^2 \mathbf{u}_{ttt}(\cdot, t) dt\|_1, \end{aligned}$$

by applying the corresponding approximation property, we obtain

$$\begin{aligned} \|\mathbf{u}(\cdot, t_1) - \mathbf{u}_h^1\| &\le c(h^{r+1} + \Delta t^2), \\ \|\mathbf{u}(\cdot, t_1) - \mathbf{u}_h^1\|_1 &\le c(h^r + \Delta t^2), \end{aligned}$$

also, by trace inequality, we have

$$\|\mathbf{u}(\cdot, t_1) - \mathbf{u}_h^1\|_\epsilon \le c(h^r + \Delta t^2).$$

Similarly, from (27), it holds

$$\begin{aligned} \|\frac{\mathbf{u}(t_1) - \mathbf{u}(t_0)}{\Delta t} - \frac{(\mathbf{u}_h^1 - \mathbf{u}_h^0)}{\Delta t}\| &\le c(h^{r+1} + \Delta t^2), \\ \|\frac{\mathbf{u}(t_1) - \mathbf{u}(t_0)}{\Delta t} - \frac{(\mathbf{u}_h^1 - \mathbf{u}_h^0)}{\Delta t}\|_\epsilon &\le c(h^r + \Delta t^2). \end{aligned}$$

Based on the standard Crank-Nicolson scheme, there holds

$$\forall q_h \in \mathcal{Q}_h, \quad c_o(\frac{p(\cdot,t_1)-p^0}{\Delta t}, q_h) + a_p(\frac{p(\cdot,t_1)+p^0}{2}, q_h) \tag{28}$$
$$= (\frac{l(\cdot,t_1)+l(\cdot,t_0)}{2}, q_h) - \alpha b(\frac{\mathbf{u}(\cdot,t_1)-\mathbf{u}^0}{\Delta t}, q_h) - (F_p(0,t)+F_{\mathbf{u}}(0,t), q_h),$$

where

$$F_p(0,t) = -\frac{1}{4\Delta t}\int_{t_0}^{t_1}(t_1-s)^2 p_{ttt}(\cdot,s)ds + \frac{\Delta t}{4}\int_{t_0}^{t^1} p_{ttt}(\cdot,s)ds$$
$$-\frac{1}{4\Delta t}\int_{t_0}^{t_1}(t_0-s)^2 p_{ttt}(\cdot,s)ds,$$

$$F_{\mathbf{u}}(0,t) = -\frac{1}{4\Delta t}\int_{t_0}^{t_1}(t_{n+1}-s)^2 \nabla\cdot\mathbf{u}_{ttt}(\cdot,s)ds + \frac{\Delta t}{4}\int_{t_{n-1}}^{t^1} \nabla\cdot\mathbf{u}_{ttt}(\cdot,s)ds$$
$$-\frac{1}{4\Delta t}\int_{t_0}^{t_1}(t_0-s)^2 \nabla\cdot\mathbf{u}_{ttt}(\cdot,s)ds.$$

By Cauchy-Schwarz inequality we have

$$\|F_p(0,t)\| \le c\Delta t^2, \quad \|F_{\mathbf{u}}(0,t)\| \le c\Delta t^2.$$

Subtracting (20) from (28), the error equation holds

$$c_o(\frac{p(\cdot,t_1)-p_h^1-(p^0-p_h^0)}{\Delta t}, q_h) + a_p(\frac{p(\cdot,t_1)-p_h^1+p^0-p_h^0}{2}, q_h)$$
$$= -\alpha b(\frac{\mathbf{u}(\cdot,t_1)-\mathbf{u}_h^1-(\mathbf{u}^0-\mathbf{u}_h^0)}{\Delta t}, q_h) - (F_p(0,t)+F_{\mathbf{u}}(0,t), q_h),$$

multiplying by Δt and rearranging it, we get

$$c_o(p(\cdot,t_1)-p_h^1, q_h) + \Delta t a_p(\frac{p(\cdot,t_1)-p_h^1}{2}, q_h)$$
$$= -\alpha b(\mathbf{u}(\cdot,t_1)-\mathbf{u}_h^1-(\mathbf{u}^0-\mathbf{u}_h^0), q_h) - \Delta t(F_p(0,t)+F_{\mathbf{u}}(0,t), q_h)$$
$$+ c_o(p^0-p_h^0, q_h) - \Delta t a_p(\frac{p^0-p_h^0}{2}, q_h).$$

Then, in light of general error estimates of parabolic problem (see [21]) and previous results, immediately, we can obtain

$$c_o\|p(\cdot,t_1)-p_h^1\| + \|p(\cdot,t_1)-p_h^1\|_\epsilon \le c(h^r+\Delta t^2). \quad \square$$

Theorem 4.1. *(A priori error estimates). Let $(\mathbf{u}_h^{n+1}, p_h^{n+1})$ be the solutions of fully discrete scheme (13)-(14) and $(\mathbf{u}(\mathbf{x},t), p(\mathbf{x},t))$ be the solutions of fully dynamic Biot's model (1)-(9). In addition, we assume*

$$\boldsymbol{u} \in C^2(0,T;\boldsymbol{H}^{r+1}(\Omega)), \boldsymbol{u}_{tt} \in L^2(0,T;\boldsymbol{H}^1(\Omega)), \boldsymbol{u}_{ttt} \in L^2(0,T;\boldsymbol{H}^2(\Omega)),$$
$$\boldsymbol{u}_{tttt} \in L^2(0,T;\boldsymbol{L}^2(\Omega)), p_{tt} \in L^2(0,T;H^1(\Omega)), p_{ttt} \in L^2(0,T;L^2(\Omega)).$$

Then, for $\forall 2 \leq m \leq N$*, we have*

$$\|\mathbf{u}(\cdot,t_m) - \mathbf{u}_h^m\|_\epsilon^2 \leq c(h^{2r} + \Delta t^4),$$
$$\sum_{n=1}^{m-1} \Delta t \|p(\cdot,t_n) - \frac{p_h^{n+1}+p_h^{n-1}}{2}\|_\epsilon^2 \leq c(h^{2r} + \Delta t^4).$$

Proof. To simply the notions, for a given function $z(\cdot,t)$, define

$$\partial_t z^n = \frac{z^{n+1}-z^{n-1}}{2\Delta t} = \frac{\partial_t z^{n+\frac{1}{2}} + \partial_t z^{n-\frac{1}{2}}}{2}, \qquad \partial_t z^{n+\frac{1}{2}} = \frac{z^{n+1}-z^n}{\Delta t},$$
$$\partial_{tt} z^n = \frac{z^{n+1}-2z^n+z^{n-1}}{\Delta t^2} = \frac{\partial_t z^{n+\frac{1}{2}} - \partial_t z^{n-\frac{1}{2}}}{\Delta t}, \qquad z^{n:\frac{1}{2}} = \frac{z^{n+1}+z^{n-1}}{2},$$

where z^n stands for $z(\cdot,t_n)$. Based on these notions, at t_n moment, the momentum equation (1) satisfies

$$\begin{aligned}(\partial_{tt}\mathbf{u}^n, \mathbf{v}_h) + a_{\mathbf{u}}(\mathbf{u}^{n:\frac{1}{2}}, \mathbf{v}_h) - \alpha b(\mathbf{v}_h, p^{n:\frac{1}{2}}) - (\partial_{tt}\mathbf{u}^n - \mathbf{u}_{tt}^n, \mathbf{v}_h)\\ = (\mathbf{f}^n, \mathbf{v}_h) + (F(\mathbf{u},n,t) + F(p,n,t), \mathbf{v}_h),\end{aligned} \tag{29}$$

where

$$F(\mathbf{u},n,t) = \Delta\mathbf{u}^n - \Delta\mathbf{u}^{n:\frac{1}{2}},$$
$$F(p,n,t) = \nabla p^{n:\frac{1}{2}} - \nabla p^n.$$

Then, by the idea of Taylor expansion and the process of construction of Crank-Nicolson scheme, the mass equation (2) satisfies

$$\begin{aligned}c_o(\partial_t p^n, q_h) + \alpha b(\partial_t \mathbf{u}^n, q_h) + a_p(p^{n:\frac{1}{2}}, q_h)\\ = (l^{n:\frac{1}{2}}, q_h) - (F_p(n,t) + F_{\mathbf{u}}(n,t), q_h),\end{aligned} \tag{30}$$

where

$$\begin{aligned}F_p(n,t) = &-\frac{1}{8\Delta t}\int_{t_{n-1}}^{t_{n+1}} (t_{n+1}-s)^2 p_{ttt}(\cdot,s)ds + \frac{\Delta t}{2}\int_{t_{n-1}}^{t^{n+1}} p_{ttt}(\cdot,s)ds\\ &-\frac{1}{8\Delta t}\int_{t_{n-1}}^{t_{n+1}} (t_{n-1}-s)^2 p_{ttt}(\cdot,s)ds,\\ F_{\mathbf{u}}(n,t) = &-\frac{1}{8\Delta t}\int_{t_{n-1}}^{t_{n+1}} (t_{n+1}-s)^2 \nabla\cdot\mathbf{u}_{ttt}(s)ds + \frac{\Delta t}{2}\int_{t_{n-1}}^{t^{n+1}} \nabla\cdot\mathbf{u}_{ttt}(\cdot,s)ds\\ &-\frac{1}{8\Delta t}\int_{t_{n-1}}^{t_{n+1}} (t_{n-1}-s)^2 \nabla\cdot\mathbf{u}_{ttt}(\cdot,s)ds.\end{aligned}$$

Note that, similar to Lemma 4.3, the following estimates holds

$$\|F(\mathbf{u},n,t)\|^2 \le \Delta t^3 \int\limits_{t_{n-1}}^{t_{n+1}} \|\Delta \mathbf{u}_{tt}(\cdot,t)\|^2 dt, \tag{31}$$

$$\|F(p,n,t)\|^2 \le \Delta t^3 \int\limits_{t_{n-1}}^{t_{n+1}} \|\nabla p_{tt}(\cdot,t)\|^2 dt,$$

$$\|F_p(n,t)\|^2 \le \Delta t^3 \int\limits_{t_{n-1}}^{t_{n+1}} \|p_{ttt}(\cdot,t)\|^2 dt,$$

$$\|F_{\mathbf{u}}(n,t)\|^2 \le \Delta t^3 \int\limits_{t_{n-1}}^{t_{n+1}} \|\nabla \mathbf{u}_{ttt}(\cdot,t)\|^2 dt.$$

Now, subtracting (13)-(14) from (29)-(30), respectively, we can obtain the error equations. For $\forall \mathbf{v}_h \in \mathbf{V}_h$ and $\forall q_h \in \mathcal{Q}_h$, such that

$$\begin{aligned}
&(\partial_{tt}\mathbf{u}^n - \partial_{tt}\mathbf{u}_h^n, \mathbf{v}_h) + a_{\mathbf{u}}(\mathbf{u}^{n:\frac{1}{2}} - \mathbf{u}_h^{n:\frac{1}{2}}, \mathbf{v}_h) - \alpha b(\mathbf{v}_h, p^{n:\frac{1}{2}} - p_h^{n:\frac{1}{2}}) \\
&= (F(\mathbf{u},n,t) + F(p,n,t), \mathbf{v}_h) + (\partial_{tt}\mathbf{u}^n - \mathbf{u}_{tt}^n, \mathbf{v}_h), \\
&c_o(\partial_t p^n - \partial_t p_h^n, q_h) + \alpha b(\partial_t \mathbf{u}^n - \partial_t \mathbf{u}_h^n, q_h) + a_p(p^{n:\frac{1}{2}} - p_h^{n:\frac{1}{2}}, q_h) \\
&= -(F_p(n,t) + F_{\mathbf{u}}(n,t), q_h).
\end{aligned}$$

For simplicity, we introduce the following notions

$$\begin{aligned}
\chi_p^n &= \Pi_p p^n - p_h^n, & \xi_p^n &= \Pi_p p^n - p^n, \\
\chi_{\mathbf{u}}^n &= \mathbf{\Pi}_{\mathbf{u}} \mathbf{u}^n - \mathbf{u}_h^n, & \xi_{\mathbf{u}}^n &= \mathbf{\Pi}_{\mathbf{u}} \mathbf{u}^n - \mathbf{u}^n,
\end{aligned} \tag{32}$$

where L^2-projection Π_p and elliptic projection $\mathbf{\Pi}_{\mathbf{u}}$ are given in Lemma 4.1 and Lemma 4.2. Then, applying (32) to error equations, we have

$$\begin{aligned}
&(\partial_{tt}\chi_{\mathbf{u}}^n - \partial_{tt}\xi_{\mathbf{u}}^n, \mathbf{v}_h) + a_{\mathbf{u}}(\chi_{\mathbf{u}}^{n:\frac{1}{2}} - \xi_{\mathbf{u}}^{n:\frac{1}{2}}, \mathbf{v}_h) - \alpha b(\mathbf{v}_h, \chi_p^{n:\frac{1}{2}} - \xi_p^{n:\frac{1}{2}}) \\
&= (F(\mathbf{u},n,t) + F(p,n,t), \mathbf{v}_h) + (\partial_{tt}\mathbf{u}^n - \mathbf{u}_{tt}^n, \mathbf{v}_h),
\end{aligned} \tag{33}$$

$$\begin{aligned}
&c_o(\partial_t \chi_p^n - \partial_t \xi_p^n, q_h) + \alpha b(\partial_t \chi_{\mathbf{u}}^n - \partial_t \xi_{\mathbf{u}}^n, q_h) + a_p(\chi_p^{n:\frac{1}{2}} - \xi_p^{n:\frac{1}{2}}, q_h) \\
&= -(F_p(n,t) + F_{\mathbf{u}}(n,t), q_h).
\end{aligned} \tag{34}$$

By the definition of L^2-projection Π_p and elliptic projection $\mathbf{\Pi}_{\mathbf{u}}$, (33) and (34) can be rewritten as

$$\begin{aligned}
&(\partial_{tt}\chi_{\mathbf{u}}^n, \mathbf{v}_h) - \alpha b(\mathbf{v}_h, \chi_p^{n:\frac{1}{2}}) + a_{\mathbf{u}}(\chi_{\mathbf{u}}^{n:\frac{1}{2}}, \mathbf{v}_h) - (\partial_{tt}\xi_{\mathbf{u}}^n, \mathbf{v}_h) + \alpha b(\mathbf{v}_h, \xi_p^{n:\frac{1}{2}}) \\
&= (F(\mathbf{u},n,t) + F(p,n,t), \mathbf{v}_h) + (\partial_{tt}\mathbf{u}^n - \mathbf{u}_{tt}^n, \mathbf{v}_h),
\end{aligned} \tag{35}$$

$$\begin{aligned}
&c_o(\partial_t \chi_p^n, q_h) + \alpha b(\partial_t \chi_{\mathbf{u}}^n, q_h) + a_p(\chi_p^{n:\frac{1}{2}}, q_h) - a_p(\xi_p^{n:\frac{1}{2}}, q_h) - \alpha b(\partial_t \xi_{\mathbf{u}}^n, q_h) \\
&= -(F_p(n,t) + F_{\mathbf{u}}(n,t), q_h).
\end{aligned} \tag{36}$$

Taking $\mathbf{v}_h = \partial_t \chi_{\mathbf{u}}^n$ in (35), it satisfies

$$
\begin{aligned}
(\partial_{tt}\chi_{\mathbf{u}}^n, \partial_t\chi_{\mathbf{u}}^n) - \alpha b(\partial_t\chi_{\mathbf{u}}^n, \chi_p^{n:\frac{1}{2}}) + a_{\mathbf{u}}(\chi_{\mathbf{u}}^{n:\frac{1}{2}}, \partial_t\chi_{\mathbf{u}}^n) \qquad\qquad (37)\\
= (F(\mathbf{u}, n, t) + F(p, n, t), \partial_t\chi_{\mathbf{u}}^n) + (\partial_{tt}\xi_{\mathbf{u}}^n, \partial_t\chi_{\mathbf{u}}^n)\\
- \alpha b(\partial_t\chi_{\mathbf{u}}^n, \xi_p^{n:\frac{1}{2}}) + (\partial_{tt}\mathbf{u}^n - \mathbf{u}_{tt}^n, \partial_t\chi_{\mathbf{u}}^n).
\end{aligned}
$$

Choosing $q_h = \chi_p^{n:\frac{1}{2}}$ in (36), it produces

$$
\begin{aligned}
&c_o(\partial_t\chi_p^n, \chi_p^{n:\frac{1}{2}}) + \alpha b(\partial_t\chi_{\mathbf{u}}^n, \chi_p^{n:\frac{1}{2}}) + a_p(\chi_p^{n:\frac{1}{2}}, \chi_p^{n:\frac{1}{2}}) \qquad (38)\\
&= -(F_p(n, t) + F_{\mathbf{u}}(n, t), \chi_p^{n:\frac{1}{2}}) + a_p(\xi_p^{n:\frac{1}{2}}, \chi_p^{n:\frac{1}{2}}) + \alpha b(\partial_t\xi_{\mathbf{u}}^n, \chi_p^{n:\frac{1}{2}}).
\end{aligned}
$$

Adding (37) to (38) yields

$$
\begin{aligned}
&(\partial_{tt}\chi_{\mathbf{u}}^n, \partial_t\chi_{\mathbf{u}}^n) + a_{\mathbf{u}}(\chi_{\mathbf{u}}^{n:\frac{1}{2}}, \partial_t\chi_{\mathbf{u}}^n) + c_o(\partial_t\chi_p^n, \chi_p^{n:\frac{1}{2}}) + a_p(\chi_p^{n:\frac{1}{2}}, \chi_p^{n:\frac{1}{2}}) \qquad (39)\\
&= (F(\mathbf{u}, n, t) + F(p, n, t), \partial_t\chi_{\mathbf{u}}^n) + a_p(\xi_p^{n:\frac{1}{2}}, \chi_p^{n:\frac{1}{2}})\\
&\quad + (\partial_{tt}\xi_{\mathbf{u}}^n, \partial_t\chi_{\mathbf{u}}^n) - \alpha b(\partial_t\chi_{\mathbf{u}}^n, \xi_p^{n:\frac{1}{2}}) + \alpha b(\partial_t\xi_{\mathbf{u}}^n, \chi_p^{n:\frac{1}{2}})\\
&\quad - (F_p(n, t) + F_{\mathbf{u}}(n, t), \chi_p^{n:\frac{1}{2}}) + (\partial_{tt}\mathbf{u}^n - \mathbf{u}_{tt}^n, \partial_t\chi_{\mathbf{u}}^n).
\end{aligned}
$$

To this end, we need to estimate the left and right-hand side terms of (39), respectively. For the left-hand side terms, from the symmetry of $a_{\mathbf{u}}$, we have

$$
\begin{aligned}
&(\partial_{tt}\chi_{\mathbf{u}}^n, \partial_t\chi_{\mathbf{u}}^n) + a_{\mathbf{u}}(\chi_{\mathbf{u}}^{n:\frac{1}{2}}, \partial_t\chi_{\mathbf{u}}^n) + c_o(\partial_t\chi_p^n, \chi_p^{n:\frac{1}{2}}) + a_p(\chi_p^{n:\frac{1}{2}}, \chi_p^{n:\frac{1}{2}}) \qquad (40)\\
&= \frac{1}{2\Delta t}(\|\partial_t\chi_{\mathbf{u}}^{n+\frac{1}{2}}\|^2 - \|\partial_t\chi_{\mathbf{u}}^{n-\frac{1}{2}}\|^2) + \frac{c_o}{4\Delta t}(\|\chi_p^{n+1}\|^2 - \|\chi_p^{n-1}\|^2)\\
&\quad + \frac{1}{4\Delta t}a_{\mathbf{u}}(\chi_{\mathbf{u}}^{n+1}, \chi_{\mathbf{u}}^{n+1}) - \frac{1}{4\Delta t}a_{\mathbf{u}}(\chi_{\mathbf{u}}^{n-1}, \chi_{\mathbf{u}}^{n-1}) + a_p(\chi_p^{n:\frac{1}{2}}, \chi_p^{n:\frac{1}{2}}).
\end{aligned}
$$

Then, we only to estimate the right-hand side terms, term by term. For convenience, we define by I_1, I_2, $\cdot\cdot\cdot$, I_7 the right-hand side terms of (39), respectively. I_1 can be bounded by using Cauchy-Schwarz, Young and broken Poincaré inequalities and (31),

$$
\begin{aligned}
I_1 &= -(F_p(n, t) + F_{\mathbf{u}}(n, t), \chi_p^{n:\frac{1}{2}}) \le (\|F_p(n, t)\| + \|F_{\mathbf{u}}(n, t)\|)\|\chi_p^{n:\frac{1}{2}}\|\\
&\le c(\|F_p(n, t)\|^2 + \|F_{\mathbf{u}}(n, t)\|^2) + \frac{c_{coe1}}{16}\|\chi_p^{n:\frac{1}{2}}\|_\epsilon^2\\
&\le c\Delta t^3 \int_{t_{n-1}}^{t_{n+1}} (\|p_{ttt}(\cdot, t)\|^2 + \|\nabla\mathbf{u}_{ttt}(\cdot, t)\|^2)dt + \frac{c_{coe1}}{16}\|\chi_p^{n:\frac{1}{2}}\|_\epsilon^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_2 &= (F_1(\mathbf{u}, n, t) + F_1(p, n, t), \partial_t\chi_{\mathbf{u}}^n)\\
&\le (\|F(\mathbf{u}, n, t)\| + \|F(p, n, t)\|)\|\partial_t\chi_{\mathbf{u}}^n\|\\
&\le \|F(\mathbf{u}, n, t)\|^2 + \|F(p, n, t)\|^2 + \|\partial_t\chi_{\mathbf{u}}^n\|^2.
\end{aligned}
$$

Note that, $2\partial_t\chi_{\mathbf{u}}^n = \partial_t\chi_{\mathbf{u}}^{n+\frac{1}{2}} + \partial_t\chi_{\mathbf{u}}^{n-\frac{1}{2}}$, thus

$$I_2 \le \Delta t^3 \int\limits_{t_{n-1}}^{t_{n+1}} (\|\Delta \mathbf{u}_{tt}(\cdot,t)\|^2 + \|\nabla p_{tt}(\cdot,t)\|^2)dt + \|\partial_t \chi_{\mathbf{u}}^{n+\frac{1}{2}}\|^2 + \|\partial_t \chi_{\mathbf{u}}^{n-\frac{1}{2}}\|^2.$$

I_3 can be bounded by applying the Cauchy-Schwarz inequality, Young inequality, Lemma 4.3 and approximation property of elliptic projection,

$$I_3 = (\partial_{tt}\xi_{\mathbf{u}}^n, \partial_t \chi_{\mathbf{u}}^n) \le \|\partial_{tt}\xi_{\mathbf{u}}^n\|\|\partial_t\chi_{\mathbf{u}}^n\| \le \|\partial_{tt}\xi_{\mathbf{u}}^n\|^2 + \|\partial_t\chi_{\mathbf{u}}^n\|^2,$$

in light of $\mathbf{u} \in C^2(0,T;\mathbf{H}^{r+1}(\Omega))$ (see [13]), we get

$$I_3 \le \frac{ch^{2(r+1)}}{\Delta t} \int\limits_{t_{n-1}}^{t_{n+1}} \|\mathbf{u}_{tt}(\cdot,t)\|_{r+1}^2 dt + \|\partial_t \chi_{\mathbf{u}}^{n+\frac{1}{2}}\|^2 + \|\partial_t \chi_{\mathbf{u}}^{n-\frac{1}{2}}\|^2.$$

For the estimate of I_4, by Cauchy-Schwarz, trace, Young and broken Poincaré inequalities and the approximation properties of elliptic projection, we have

$$\begin{aligned}
I_4 &= \alpha b(\partial_t \xi_{\mathbf{u}}^n, \chi_p^{n:\frac{1}{2}}) = \alpha(\nabla\cdot\partial_t\xi_{\mathbf{u}}^n, \chi_p^{n:\frac{1}{2}}) - \alpha\sum_{e\in\mathcal{E}_h^i}([\partial_t\xi_{\mathbf{u}}^n]\cdot\mathbf{n}, \{\chi_p^{n:\frac{1}{2}}\})_e \\
&\le \alpha\|\nabla\cdot\partial_t\xi_{\mathbf{u}}^n\|\|\chi_p^{n:\frac{1}{2}}\| + \alpha\sum_{e\in\mathcal{E}_h^i}\|[\partial_t\xi_{\mathbf{u}}^n]\|_{L^2(e)}\|\{\chi_p^{n:\frac{1}{2}}\}\|_{L^2(e)} \\
&\le \alpha\|\nabla\cdot\partial_t\xi_{\mathbf{u}}^n\|\|\chi_p^{n:\frac{1}{2}}\| + c(h^{-1}\|\partial_t\xi_{\mathbf{u}}^n\| + \|\partial_t\xi_{\mathbf{u}}^n\|_1)\|\chi_p^{n:\frac{1}{2}}\| \\
&\le c(\|\nabla\cdot\partial_t\xi_{\mathbf{u}}^n\|^2 + h^{-2}\|\partial_t\xi_{\mathbf{u}}^n\|^2 + \|\partial_t\xi_{\mathbf{u}}^n\|_1^2) + \frac{c_{coe1}}{16}\|\chi_p^{n:\frac{1}{2}}\|_\epsilon^2 \\
&\le ch^{2r} + \frac{c_{coe1}}{16}\|\chi_p^{n:\frac{1}{2}}\|_\epsilon^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_5 &= -\alpha b(\partial_t\chi_{\mathbf{u}}^n, \xi_p^{n:\frac{1}{2}}) \\
&= \alpha\sum_{e\in\mathcal{E}_h^i}([\partial_t\chi_{\mathbf{u}}^n]\cdot\mathbf{n}, \{\xi_p^{n:\frac{1}{2}}\})_e - \alpha(\nabla\cdot\partial_t\chi_{\mathbf{u}}^n, \xi_p^{n:\frac{1}{2}}) \\
&= \alpha\sum_{e\in\mathcal{E}_h^i}([\partial_t\chi_{\mathbf{u}}^n]\cdot\mathbf{n}, \{\xi_p^{n:\frac{1}{2}}\})_e \\
&\le \alpha\sum_{e\in\mathcal{E}_h^i}\|[\partial_t\chi_{\mathbf{u}}^n]\cdot\mathbf{n}\|_{L^2(e)}\|\{\xi_p^{n:\frac{1}{2}}\}\|_{L^2(e)} \\
&\le c(h^{-2}\|\xi_p^{n:\frac{1}{2}}\|^2 + \|\xi_p^{n:\frac{1}{2}}\|_1^2) + \|\partial_t\chi_{\mathbf{u}}^n\|^2 \\
&\le ch^{2r} + \|\partial_t\chi_{\mathbf{u}}^{n+\frac{1}{2}}\|^2 + \|\partial_t\chi_{\mathbf{u}}^{n-\frac{1}{2}}\|^2.
\end{aligned}$$

I_6 can be bounded by utilizing Lemma 4.4,

$$I_6 = a_p(\xi_p^{n:\frac{1}{2}}, \chi_p^{n:\frac{1}{2}}) \le ch^{2r} + \frac{c_{coe1}}{16}\|\chi_p^{n:\frac{1}{2}}\|_\epsilon^2.$$

Then, we only to consider I_7, by Cauchy-Schwarz and Young inequalities, we get

$$
\begin{aligned}
I_7 = (\partial_{tt}\mathbf{u}^n - \mathbf{u}_{tt}^n, \partial_t \chi_{\mathbf{u}}^n) &\le \|\partial_{tt}\mathbf{u}^n - \mathbf{u}_{tt}^n\| \|\partial_t \chi_{\mathbf{u}}^n\| \\
&\le \|\partial_{tt}\mathbf{u}^n - \mathbf{u}_{tt}^n\|^2 + \|\partial_t \chi_{\mathbf{u}}^{n+\frac{1}{2}}\|^2 + \|\partial_t \chi_{\mathbf{u}}^{n-\frac{1}{2}}\|^2,
\end{aligned}
$$

by the identities (see [13])

$$
\partial_{tt}\mathbf{u}^n = \frac{1}{\Delta t^2} \int_{t_{n-1}}^{t_{n+1}} (\Delta t - |t - t_n|)\mathbf{u}_{tt}^2(\cdot, t)dt,
$$

$$
\partial_{tt}\mathbf{u}^n - \mathbf{u}_{tt}^n = \frac{1}{6\Delta t^2} \int_{t_{n-1}}^{t_{n+1}} (\Delta t - |t - t_n|)^3 \mathbf{u}_{tttt}^2(\cdot, t)dt,
$$

and by Cauchy-Schwarz inequality, we have

$$
\|\partial_{tt}\mathbf{u}^n - \mathbf{u}_{tt}^n\|^2 \le c\Delta t^3 \int_{t_{n-1}}^{t_{n+1}} \|\mathbf{u}_{tttt}(\cdot, t)\|^2 dt,
$$

therefore

$$
I_7 \le c\Delta t^3 \int_{t_{n-1}}^{t_{n+1}} \|\mathbf{u}_{tttt}(\cdot, t)\|^2 dt + \|\partial_t \chi_{\mathbf{u}}^{n+\frac{1}{2}}\|^2 + \|\partial_t \chi_{\mathbf{u}}^{n-\frac{1}{2}}\|^2.
$$

Based on the estimates of left and right-hand side terms, there holds

$$
\begin{aligned}
&\frac{1}{2\Delta t}(\|\partial_t \chi_{\mathbf{u}}^{n+\frac{1}{2}}\|^2 - \|\partial_t \chi_{\mathbf{u}}^{n-\frac{1}{2}}\|^2) + \frac{1}{4\Delta t} a_{\mathbf{u}}(\chi_{\mathbf{u}}^{n+1}, \chi_{\mathbf{u}}^{n+1}) \\
&-\frac{1}{4\Delta t} a_{\mathbf{u}}(\chi_{\mathbf{u}}^{n-1}, \chi_{\mathbf{u}}^{n-1}) + \frac{c_o}{4\Delta t}(\|\chi_p^{n+1}\|^2 - \|\chi_p^{n-1}\|^2) + a_p(\chi_p^{n:\frac{1}{2}}, \chi_p^{n:\frac{1}{2}}) \\
\le& c\Delta t^3 \int_{t_{n-1}}^{t_{n+1}} (\|\nabla p_{tt}(\cdot, t)\|^2 + \|p_{ttt}(\cdot, t)\|^2)dt + \frac{c_{coe1}}{4}\|\chi_p^{n:\frac{1}{2}}\|_\epsilon^2 \\
&+ c\Delta t^3 \int_{t_{n-1}}^{t_{n+1}} (\|\Delta \mathbf{u}_{tt}(\cdot, t)\|^2 + \|\mathbf{u}_{tttt}(\cdot, t)\|^2 + \|\nabla \mathbf{u}_{ttt}(\cdot, t)\|^2)dt \\
&+ \frac{ch^{2(r+1)}}{\Delta t} \int_{t_{n-1}}^{t_{n+1}} \|\mathbf{u}_{tt}(\cdot, t)\|_{r+1}^2 dt + ch^{2r} + \|\partial_t \chi_{\mathbf{u}}^{n+\frac{1}{2}}\|^2 + \|\partial_t \chi_{\mathbf{u}}^{n-\frac{1}{2}}\|^2.
\end{aligned} \tag{41}
$$

Multiplying (41) by $4\Delta t$, and in light of the coercivity, we have

$$
\begin{aligned}
&2(\|\partial_t \chi_{\mathbf{u}}^{n+\frac{1}{2}}\|^2 - \|\partial_t \chi_{\mathbf{u}}^{n-\frac{1}{2}}\|^2) + a_{\mathbf{u}}(\chi_{\mathbf{u}}^{n+1}, \chi_{\mathbf{u}}^{n+1}) \\
&+c_o(\|\chi_p^{n+1}\|^2 - \|\chi_p^{n-1}\|^2) + 3c_{coe1}\Delta t\|\chi_p^{n:\frac{1}{2}}\|_\epsilon^2 - a_{\mathbf{u}}(\chi_{\mathbf{u}}^{n-1}, \chi_{\mathbf{u}}^{n-1}) \\
\le& c\Delta t^4 \int_{t_{n-1}}^{t_{n+1}} (\|\nabla p_{tt}(\cdot, t)\|^2 + \|p_{ttt}(\cdot, t)\|^2)dt + c\Delta t h^{2r}
\end{aligned} \tag{42}
$$

$$+ c\Delta t^4 \int_{t_{n-1}}^{t_{n+1}} (\|\Delta \mathbf{u}_{tt}(\cdot,t)\|^2 + \|\nabla \mathbf{u}_{ttt}(\cdot,t)\|^2 + \|\mathbf{u}_{tttt}(\cdot,t)\|^2)dt$$

$$+ ch^{2(r+1)} \int_{t_{n-1}}^{t_{n+1}} \|\mathbf{u}_{tt}(\cdot,t)\|_{r+1}^2 dt + c\Delta t(\|\partial_t \chi_{\mathbf{u}}^{n+\frac{1}{2}}\|^2 + \|\partial_t \chi_{\mathbf{u}}^{n-\frac{1}{2}}\|^2).$$

Summation over (41) from $n = 1$ to $m(m \le N-1)$, we get

$$2\|\partial_t \chi_{\mathbf{u}}^{m+\frac{1}{2}}\|^2 + a_{\mathbf{u}}(\chi_{\mathbf{u}}^{m+1}, \chi_{\mathbf{u}}^{m+1}) + c_o\|\chi_p^{m+1}\|^2 + c_{coe1}\Delta t \sum_{n=1}^{m} \|\chi_p^{n:\frac{1}{2}}\|_\epsilon^2 \tag{43}$$

$$\le c(\Delta t^4 + h^{2r}) + c\Delta t \sum_{n=1}^{m} (\|\partial_t \chi_{\mathbf{u}}^{n+\frac{1}{2}}\|^2 + \|\partial_t \chi_{\mathbf{u}}^{n-\frac{1}{2}}\|^2) + 2\|\partial_t \chi_{\mathbf{u}}^{\frac{1}{2}}\|^2$$
$$+ a_{\mathbf{u}}(\chi_{\mathbf{u}}^0, \chi_{\mathbf{u}}^0) + a_{\mathbf{u}}(\chi_{\mathbf{u}}^1, \chi_{\mathbf{u}}^1) + c_o(\|\chi_p^0\|^2 + \|\chi_p^1\|^2). \tag{44}$$

From the boundedness and Lemma 4.5, we have

$$\begin{aligned} &a_{\mathbf{u}}(\chi_{\mathbf{u}}^0, \chi_{\mathbf{u}}^0) + a_{\mathbf{u}}(\chi_{\mathbf{u}}^1, \chi_{\mathbf{u}}^1) + c_o(\|\chi_p^0\|^2 + \|\chi_p^1\|^2) \\ &\le c(\|\chi_{\mathbf{u}}^0\|_\epsilon^2 + \|\chi_{\mathbf{u}}^1\|_\epsilon^2) + c_o(\|\chi_p^0\|^2 + \|\chi_p^1\|^2) \\ &\le c(\Delta t^4 + h^{2r}). \end{aligned}$$

Note that, according to the definition of $\partial_t \chi_{\mathbf{u}}^{\frac{1}{2}}$, it satisfies

$$\begin{aligned} \|\partial_t \chi_{\mathbf{u}}^{\frac{1}{2}}\|^2 &= \|\frac{\mathbf{u}_h^1 - \mathbf{u}_h^0}{\Delta t} - \frac{\mathbf{\Pi_u u}^1 - \mathbf{\Pi_u u}^0}{\Delta t}\| \\ &\le \|\frac{\mathbf{u}^1 - \mathbf{u}^0 - (\mathbf{u}_h^1 - \mathbf{u}_h^0)}{\Delta t}\|^2 + \|\frac{\mathbf{u}^1 - \mathbf{u}^0 - \mathbf{\Pi_u}(\mathbf{u}^1 - \mathbf{u}^0)}{\Delta t}\|^2. \end{aligned}$$

Lemma 4.5 gives

$$\|\frac{\mathbf{u}^1 - \mathbf{u}^0 - (\mathbf{u}_h^1 - \mathbf{u}_h^0)}{\Delta t}\| \le c\Delta t^2 + ch^{r+1},$$

and by Cauchy-Schwarz inequality and approximation property of elliptic projection, we get

$$\begin{aligned} &\|\frac{\mathbf{u}^1 - \mathbf{u}^0 - \mathbf{\Pi_u}(\mathbf{u}^1 - \mathbf{u}^0)}{\Delta t}\| \\ &= \|(\mathbf{I} - \mathbf{\Pi_u})\frac{\mathbf{u}^1 - \mathbf{u}^0}{\Delta t}\| = \|\frac{(\mathbf{I} - \mathbf{\Pi_u})}{\Delta t} \int_{t_0}^{t_1} \mathbf{u}_t dt\| \\ &= \|\frac{1}{\Delta t} \int_{t_0}^{t_1} (\mathbf{I} - \mathbf{\Pi_u})\mathbf{u}_t dt\| \le \|(\mathbf{I} - \mathbf{\Pi_u})\mathbf{u}_t\| \le ch^{2(r+1)}, \end{aligned}$$

where $\mathbf{I}$ is identity operator. Therefore, it satisfies

$$\|\partial_t \chi_{\mathbf{u}}^{\frac{1}{2}}\|^2 \le c(\Delta t^4 + h^{2(r+1)}).$$

Then, we obtain

$$
\begin{aligned}
2\|\partial_t \chi_{\mathbf{u}}^{m+\frac{1}{2}}\|^2 + a_{\mathbf{u}}(\chi_{\mathbf{u}}^{m+1}, \chi_{\mathbf{u}}^{m+1}) + c_o\|\chi_p^{m+1}\|^2 + c_{coe1}\Delta t \sum_{n=1}^{m} \|\chi_p^{n:\frac{1}{2}}\|_\epsilon^2 \\
\leq c\Delta t^4 + ch^{2r} + c\Delta t \sum_{n=1}^{m} (\|\partial_t \chi_{\mathbf{u}}^{n+\frac{1}{2}}\|^2 + \|\partial_t \chi_{\mathbf{u}}^{n-\frac{1}{2}}\|^2).
\end{aligned}
$$

By the coercivity of $a_{\mathbf{u}}$ and a_p, we get

$$
\begin{aligned}
&\|\partial_t \chi_{\mathbf{u}}^{m+\frac{1}{2}}\|^2 + c_{coe2}(\|\chi_{\mathbf{u}}^{m+1}\|_\epsilon^2 + \|\chi_{\mathbf{u}}^{m}\|_\epsilon^2) \\
&+ c_o(\|\chi_p^{m+1}\|^2 + \|\chi_p^{m}\|^2) + c_{coe1}\Delta t \sum_{n=1}^{m} \|\chi_p^{n:\frac{1}{2}}\|_\epsilon^2 \\
&\leq c\Delta t^4 + ch^{2r} + c\Delta t \sum_{n=0}^{m} \|\partial_t \chi_{\mathbf{u}}^{n+\frac{1}{2}}\|^2.
\end{aligned}
$$

In light of the Gronwall inequality, there holds

$$
\begin{aligned}
&c_o\|\chi_p^{m+1}\|^2 + \|\partial_t \chi_{\mathbf{u}}^{m+\frac{1}{2}}\|^2 \\
&\qquad + \|\chi_{\mathbf{u}}^{m+1}\|_\epsilon^2 + \Delta t \sum_{n=1}^{m} \|\chi_p^{n:\frac{1}{2}}\|_\epsilon^2 \leq c\Delta t^4 + ch^{2r}.
\end{aligned}
$$

Using triangle inequality, we have

$$
\begin{aligned}
&\|\mathbf{u}^{m+1} - \mathbf{u}_h^{m+1}\|_\epsilon^2 + \sum_{n=1}^{m} \Delta t\|p^{n:\frac{1}{2}} - p_h^{n:\frac{1}{2}}\|_\epsilon^2 \\
&\quad = \|\chi_{\mathbf{u}}^{m+1} - \xi_{\mathbf{u}}^{m+1}\|_\epsilon^2 + \sum_{n=1}^{m} \Delta t\|\chi_p^{n:\frac{1}{2}} - \xi_p^{n:\frac{1}{2}}\|_\epsilon^2 \\
&\quad \leq \|\chi_{\mathbf{u}}^{m+1}\|_\epsilon^2 + \|\xi_{\mathbf{u}}^{m+1}\|_\epsilon^2 + \sum_{n=1}^{m} \Delta t(\|\chi_p^{n:\frac{1}{2}}\|_\epsilon^2 + \|\xi_p^{n:\frac{1}{2}}\|_\epsilon^2) \\
&\quad \leq ch^{2r} + c\Delta t^4.
\end{aligned}
$$

Particularly, we use $p_h^{n:\frac{1}{2}}$ to approximate p^n, thus

$$
\begin{aligned}
\sum_{n=1}^{m} \Delta t\|p^n - p_h^{n:\frac{1}{2}}\|_\epsilon^2 &= \sum_{n=1}^{m} \Delta t\|p^n - p^{n:\frac{1}{2}} + p^{n:\frac{1}{2}} - p_h^{n:\frac{1}{2}}\|_\epsilon^2 \\
&\leq \sum_{n=1}^{m} \Delta t\|p^n - p^{n:\frac{1}{2}}\|_\epsilon^2 + \sum_{n=1}^{m} \Delta t\|p^{n:\frac{1}{2}} - p_h^{n:\frac{1}{2}}\|_\epsilon^2 \\
&\leq ch^{2r} + c\Delta t^4.
\end{aligned}
$$

Based on the above analysis, we finish the proof of the priori error estimates by replace $m+1$ with m. □

Note that, the estimate for pore pressure p is reasonable, because we use $p_h^{n:\frac{1}{2}}$ to approximate $p(\cdot, t_n)$ not p_h^n in fully discrete scheme (13)-(14).

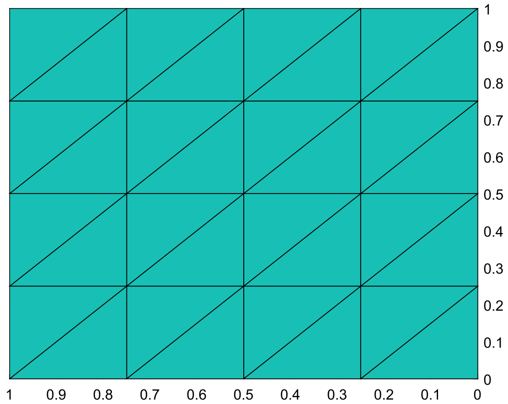

Fig. 1. Uniform mesh-grid.

5. Numerical experiments

In this section, we list some numerical tests to examine the convergence analysis, both spatial and temporal convergence rates are validated by the prescribed examples. Some suitable parameters are chosen to verify the theoretical results in numerical examples. For simplicity, we consider the computation domain with $\Omega = [0,1] \times [0,1]$ and set parameters $c_o = 1$, $\alpha = 1$, $\lambda = 1$, $\mu = 1$ and $\mathbf{K} = \mathbf{I}$ (unit matrix). Particularly, the rest of parameters may vary with different numerical tests. For the finite element spaces, $r = 1$ and $r = 2$, are chosen in the following numerical experiments. Let $\mathbf{u} = (u_1, u_2)$, then we consider the problem has exact solutions and the exact solutions of original problem (1)-(9) are defined by

$$\begin{aligned} p =& sin(t)sin(\pi x)sin(\pi y), \\ u_1 =& sin(t)sin(\pi x)sin(\pi y), \\ u_2 =& sin(t)sin(\pi x)sin(\pi y), \end{aligned}$$

where source terms $\mathbf{f}$ and l can be determined by model equations (1) and (2), respectively. Moreover, the initial-valued conditions p^0, $\mathbf{u}^0$ and $\mathbf{u}_0'$ are obtained by taking values at $t = 0$, respectively, and we can get the Dirichlet boundary conditions by using the corresponding exact solutions to confined to the boundary $\partial\Omega$. Particularly, we consider the uniform mesh-grid Fig. 1 in the following numerical tests. For simplicity, we introduce the norms which is discontinuous in time

$$|||\phi|||^2_{L^2(0,T;X)} = \Delta t \sum_{n=1}^{N} \|\phi\|^2_X.$$

5.1. Spatial convergence order

In this subsection, we verify the spatial convergence orders of fully discrete scheme by choosing a small time step-size $\Delta t = 0.00001$ and terminal time $T = 0.00101$. In addition, we take $\beta_1 = \beta_2 = 10$ when $r = 1$ and $\beta_1 = \beta_2 = 20$ when $r = 2$.

From Table 3 and Table 4, we can observe that the spatial convergence rate is the first order when $r = 1$ and square order when $r = 2$, both of which are all optimal and accord with theoretical convergence analysis.

Table 3
Spatial convergence rate with $r = 1$.

h	$\|\|\|p - p_h\|\|\|_{L^2(0,T;\|\cdot\|_\epsilon)}$	Rate	$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|_{L^2(0,T;\|\cdot\|_\epsilon)}$	Rate
$\frac{1}{2}$	1.6620e–04	–	3.1028e–05	–
$\frac{1}{4}$	1.1156e–04	–	1.6478e–05	–
$\frac{1}{8}$	4.6555e–05	1.26	8.3670e–06	0.97
$\frac{1}{16}$	1.3345e–05	1.80	4.1998e–06	0.99
$\frac{1}{32}$	3.6874e–06	1.85	2.1024e–06	0.99

Table 4
Spatial convergence rate with $r = 2$.

h	$\|\|\|p - p_h\|\|\|_{L^2(0,T;\|\cdot\|_\epsilon)}$	Rate	$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|_{L^2(0,T;\|\cdot\|_\epsilon)}$	Rate
$\frac{1}{2}$	5.4091e–05	–	1.2706e–05	–
$\frac{1}{4}$	1.0267e–05	2.39	3.3623e–06	1.91
$\frac{1}{8}$	1.5403e–06	2.77	8.5233e–07	1.98
$\frac{1}{16}$	2.2482e–07	2.74	2.1352e–07	1.99
$\frac{1}{32}$	3.9845e–08	2.50	5.3114e–08	2.00

Table 5
Temporal convergence rate with $r = 1$.

Δt	h^2	$\|\|\|p - p_h\|\|\|_{L^2(0,T;\|\cdot\|_\epsilon)}$	Rate	$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|_{L^2(0,T;\|\cdot\|_\epsilon)}$	Rate
$\frac{1}{10}$	$\frac{1}{4}$	2.9958e–01	–	5.3730e–01	–
$\frac{1}{20}$	$\frac{1}{16}$	8.7184e–02	1.78	1.3047e–01	2.04
$\frac{1}{40}$	$\frac{1}{64}$	2.2936e–02	1.92	3.1672e–02	2.04

Table 6
Temporal convergence rate with $r = 2$.

Δt	h	$\|\|\|p - p_h\|\|\|_{L^2(0,T;\|\cdot\|_\epsilon)}$	Rate	$\|\|\|\mathbf{u} - \mathbf{u}_h\|\|\|_{L^2(0,T;\|\cdot\|_\epsilon)}$	Rate
$\frac{1}{10}$	$\frac{1}{2}$	1.9184e–01	–	2.997e–01	–
$\frac{1}{20}$	$\frac{1}{4}$	5.5064e–02	1.80	8.3238e–02	1.84
$\frac{1}{40}$	$\frac{1}{8}$	1.4571e–02	1.91	2.1388e–02	1.96
$\frac{1}{80}$	$\frac{1}{16}$	3.7264e–03	1.96	5.3858e–03	1.98
$\frac{1}{160}$	$\frac{1}{32}$	9.4061e–04	1.98	1.3496e–03	1.99

5.2. Temporal convergence order

In this subsection we report the temporal convergence rate of proposed fully discrete scheme with a fixed $T = 1$. For efficiently validate the square convergence order of temporal variable, we shall make a proper match between step-size Δt and mesh-size h. Particularly, we choose $5\Delta t = h$ and $\beta_1 = \beta_2 = 10$, when $r = 1$ and set $5\Delta t = h$ and $\beta_1 = \beta_2 = 20$, when $r = 2$.

From Table 5 and Table 6, we can make a conclusion that the fully discrete scheme has the square temporal convergence rate according to the relation between Δt and h, and these numerical results well support the theoretical analysis.

6. Conclusions

In this paper, we propose a fully discrete scheme for the dynamic Biot's model by using discontinuous Galerkin method and proper difference formulations. The proposed scheme is proved to be efficient in both theoretical analysis and numerical examples by taking ingenious initial values for both pore pressure and displacement.

References

[1] R.A. Adams, Sobolev Spaces, Academic Press, New York, 2003.
[2] M.A. Biot, Theorey of elasticity and consolidation for a porous anisotropic solid, J. Appl. Phys. 26 (2) (1955) 82–185.
[3] M.A. Biot, General solutions of the equations of elasticity and consolidation for a porous material, J. Appl. Phys. 223 (1) (1956) 91.
[4] H. Bryne, L. Preziosi, Modeling solid tumour growth using the theory of mixtures, Math. Med. Biol. 20 (4) (2003) 341–366.
[5] E. Burman, M.A. Fernández, Stabilization of explicit coupling in fluid-structure interaction involving fluid incompressibility, Comput. Methods Appl. Math. 198 (5–8) (2009) 766–784.
[6] P.C. Carman, Permeability of saturated sands, soils and clays, J. Agric. Sci. 29 (02) (1939) 262.
[7] A. Cesmelioglu, Analysis of the coupled Navier-Stokes/Biot problem, J. Math. Anal. Appl. 456 (2017) 970–991.
[8] B. Cockburn, G. Kanschat, D. Schötzau, C. Schwab, Local discontinuous Galerkin methods for the Stokes systems, SIAM J. Numer. Anal. 40 (1) (2003) 319–343.
[9] L.C. Evans, Partial Differential Equations, American Mathematical Society, 2010.
[10] X.B. Feng, Z.H. Ge, Y.K. Li, Analysis of a multiphysics finite element method for a poroelasticity model, IMA J. Numer. Anal. 38 (2018) 330–359.
[11] Z.H. Ge, M.X. Ma, Multirate iterative scheme based on multiphysics discontinuous Galerkin method for a poroelasticity model, Appl. Numer. Math. 128 (2018) 125–138.
[12] M. Grote, D. Schötzau, Optimal error estimates for the fully discrete interior penalty DG method for the wave equation, J. Sci. Comput. 40 (1–3) (2019) 257–272.
[13] W.M. Han, L.M. He, F. Wang, Optimal order error estimates for discontinuous Galerkin methods for the wave equation, J. Sci. Comput. 78 (2019) 121–144.
[14] J.A. Hudson, O. Stephansson, J. Andersson, et al., Coupled t-h-m issues relating to radioactive waste repository design and performance, Int. J. Rock Mech. Min. Sci. 38 (1) (2001) 143–161.
[15] F.F. Jing, W.M. Han, W.J. Yan, F. Wang, Discontinuous Galerkin methods for a stationary Navier-Stokes problem with a nonlinear slip boundary condition of friction type, J. Sci. Comput. 76 (2018) 888–912.
[16] Z. Lotfian, M.V. Sivaselvan, Mixed finite element formulation for dynamics of porous media, Int. J. Numer. Math. Eng. 115 (2018) 141–171.
[17] P.J. Phillips, M.F. Wheeler, A coupling of mixed and continuous Galerkin finite element methods for poroelasticity: the continuous in time case, Comput. Geosci. 11 (2) (2007) 131–144.
[18] P.J. Phillips, M.F. Wheeler, A coupling of mixed and discontinuous Galerkin finite element methods for poroelasticity, Comput. Geosci. 12 (4) (2008) 417–435.
[19] P.J. Phillips, M.F. Wheeler, Overcoming the problem of locking in linear elasticity and poroelasticity: a heuristic approach, Comput. Geosci. 13 (1) (2009) 5–12.
[20] W.H. Reed, T.R. Hill, Triangular mesh methods for neutron transport equations, Los Alamous Scientific Laboratory report LA-UR-73-479, Los Alamous, NM, 1973.
[21] B. Rivière, Discontinuous Galerkin Methods for Solving Elliptic and Parabolic Equations: Theory and Implementation, SIAM, Philadelphia, 2008.
[22] B. Rivière, J. Tan, T. Thompson, Error analysis of primal discontinuous Galerkin methods for a mixed formulation of the Biot equations, Comput. Math. Appl. 73 (4) (2017) 666–683.
[23] B. Rivière, M.F. Wheeler, A posteriori error estimates for a discontinuous Galerkin method applied to elliptic problems, Comput. Math. Appl. 46 (1) (2003) 141–163.
[24] R.E. Schwalter, Diffusion in poro-elastic media, J. Math. Anal. Appl. 251 (1) (2000) 310–340.
[25] K. Terzaghi, Erdbaumechanik auf Bodenphysikalischer Grundlage, Duticke F., Vienna, 1925.
[26] Y. Wang, M.B. Dusseault, A coupled conductive-convective thermo-poroelastic solution and implications for wellbore stability, J. Pet. Sci. Eng. 38 (3) (2003) 187–198.
[27] A. Zenisek, The existence and uniqueness theorem in Biot's consolidation theory, Apl. Mat. 29 (1984) 194–210.