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Complete bifurcation diagram and global phase portraits of Liénard differential equations of degree four

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ABSTRACT

Li and Llibre in [*J. Differential Equations* **252** (2012) 3142–3162] proved that a Liénard system of degree four: $\frac{dx}{dt} = y - (ax + bx^2 + cx^3 + x^4)$, $\frac{dy}{dt} = -x$ has at most one limit cycle. Moreover, the limit cycle is stable and hyperbolic if it exists. Based on their works, the aim of this paper is to give the complete bifurcation diagram and global phase portraits in the Poincaré disc of this system further. First we analyze the equilibria at both finity and infinity. Then, a necessary and sufficient condition for existence of separatrix loop is founded by the rotation property. Moreover, a necessary and sufficient condition of the existence of limit cycles is also obtained. Finally, we show that the complete bifurcation diagram includes one Hopf bifurcation surface and one bifurcation surface of separatrix loop.

1. Introduction and main results

In nonlinear applied science and engineering, the Liénard system $\ddot{x} + f(x)\dot{x} + g(x) = 0$ is very common. Therefore, mathematicians and physicists et al. are very interested in the Liénard system, see [23,26,33] and the references therein. Lins, de Melo and Pugh in [22] planned to study the number of limit cycles of

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -x, \end{cases} \tag{1}$$

where $F(x) := \int_0^x f(s)ds$ is a polynomial. Lins, de Melo and Pugh have proved completely that system (1) has at most one limit cycle when $\deg F := m = 3$ and no limit cycles for the cases $m \leq 2$. Moreover, they conjectured that the upper bound of the number of limit cycle of system (1) is $[\frac{k-1}{2}]$ when $m \geq 4$. This problem has also attracted Smale, see [30]. Until 2012, Li and Llibre in [21] proved the uniqueness of limit

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cycle for the case $m = 4$. However, the conjecture is incorrect for the cases $m \geq 6$ since Dumortier et al. in [16] provided one additional limit cycle so that it is not true when $m \geq 7$, and Maesschalck and Dumortier in [25] providing two additional limit cycles when $m \geq 6$, De Maesschalck and Huzak in [9] proved further that the maximum number of limit cycles of system (1) is at least $m-2$ when $m \geq 6$. As far as we know, we do not know whether the conjecture for the case $m = 5$ is correct. Summarizing the aforementioned results, Llibre and Zhang in [24] gave the following improved open problem:

What is the maximum number of limit cycles for the Liénard differential systems (1) *when* $\deg F = 5$*?*

Besides, many mathematicians were also interested in

$$\begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -g(x), \end{cases} \tag{2}$$

where $g(x) \neq x$ and $F(x)$ are polynomials. For simplicity, let $(\deg F, \deg g) := (m, n)$. By Dulac criterion, it is clear that system (2) has no limit cycle as $m = 1$. For the simplest case $(m, n) = (2, 2)$, note that system (2) is a famous Bogdnov-Takens system, see [7,19]. In this case, Coppel in [6] proved the uniqueness of limit cycle. Based on the result of the uniqueness for limit cycle, Perko in [27] gave the complete bifurcation diagram and all global phase portraits of system (2) and Gasull et al. in [18] solved the Perko's two conjectures about the properties of the homoclinic bifurcation curve. For the case $(m, n) = (2, 3)$, Dumortier and Rousseau investigated the complete bifurcation diagram and all global phase portraits of system (2) except they conjectured the number of limit cycle surrounding three equilibria is at most one. Later, Dumortier himself and Li in [10] solved this conjecture. For the case $(m, n) = (3, 2)$, Dumortier and Li in [11] also investigated the complete bifurcation diagram and all global phase portraits of system (2). However, for the cases $m + n \geq 6$, the dynamics are more complex than the cases $m + n \leq 5$. For example, the maximum number of limit cycles are larger than one for the cases $m+n \geq 6$. Up to now, the maximum number of limit cycles for any case $m+n \geq 6$ is still unknown. We can still find many articles about the case $(m, n) = (3, 3)$, see [1–5,8,12–15,17,20,31].

By the aforementioned introduction, we know that the bifurcation diagrams and global phase portraits of the cases $m + n \leq 5$ have been solved completely except the case $(m, n) = (4, 1)$. The purpose of this paper is to give the bifurcation diagram and all global phase portraits of the case $(m, n) = (4, 1)$. The Liénard system of the case $(4, 1)$ is

$$\begin{cases} \frac{dx}{dt} = y - (a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4), \\ \frac{dy}{dt} = -x, \end{cases} \tag{3}$$

where $(a_1, a_2, a_3) \in \mathbb{R}^3$ and $a_4 \neq 0$. To reduce the number of parameters, with the following transformation:

$$x \to \frac{1}{\sqrt[3]{a_4}} x, \quad y \to \frac{1}{\sqrt[3]{a_4}} y,$$

we change (3) to

$$\begin{cases} \frac{dx}{dt} = y - (ax + bx^2 + cx^3 + x^4), \\ \frac{dy}{dt} = -x, \end{cases} \tag{4}$$

where

$$(a, b, c) := (a_1, \frac{a_2}{\sqrt[3]{a_4}}, \frac{a_3}{\sqrt[3]{a_4^{\,2}}}) \in \mathbb{R}^3.$$

With $(x, t, c) \to (-x, -t, -c)$, system (4) is invariant. Therefore, we only need to discuss the case $c \geq 0$.

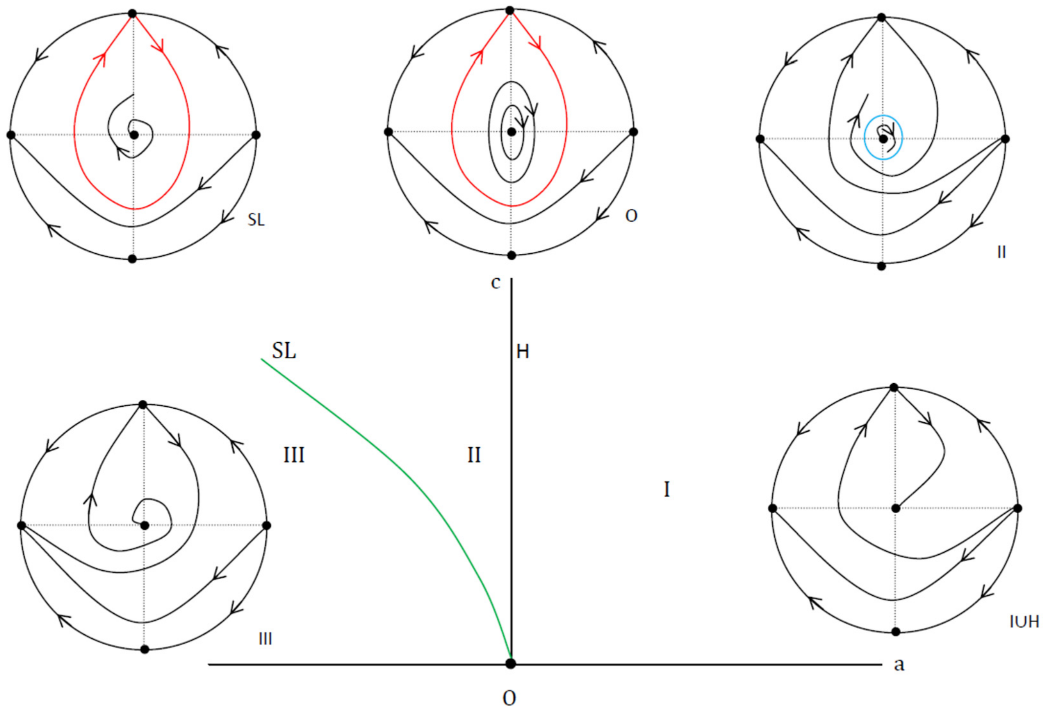


Fig. 1. The cross-section $b = b^*$ of the bifurcation diagram and global phase portraits of system (4) for any fixed b^*.

The main results of this paper are the following theorem.

Theorem 1.1. *The global bifurcation diagram of system* (4) *consists of the origin and two bifurcation surfaces*

(a) *Bifurcation of separatrix loop surface* $SL = \{(a, b, c) \in \mathbb{R}^2 \times [0, +\infty)|\ c = \varphi(a, b), a \leq 0\}$,
(b) *Hopf bifurcation surface* $H = \{(a, b, c) \in \mathbb{R}^2 \times [0, +\infty)|\ a = 0, c > 0\}$,

where φ *is continuous in* a, b *and decreasing in* a. *The cross-section* $b = b^*$ *of the bifurcation diagram and all global phase portraits of* (4) *are shown in* Fig. 1, *where*

$$O = \{(a, b, c) \in \mathbb{R}^2 \times [0, +\infty)|\ a = c = 0\},$$
$$I = \{(a, b, c) \in \mathbb{R}^2 \times [0, +\infty)|\ a > 0,\ c \geq 0\},$$
$$II = \{(a, b, c) \in \mathbb{R}^2 \times [0, +\infty)|\ a < 0,\ c > \varphi(a)\},$$
$$III = \{(a, b, c) \in \mathbb{R}^2 \times [0, +\infty)|\ a < 0,\ 0 \leq c < \varphi(a)\}.$$

Remark. By Figure 7 of [22] and Figure 2 of [24], we can obtain the global phase portrait in the region O directly.

In the rest of this paper, we study the global bifurcation diagram and the global phase portraits of system (4). In section 2, the equilibria at both finity and infinity are analyzed, Hopf bifurcation at the origin point is investigated. We give a necessary and sufficient condition for existence of separatrix loop and also a necessary and sufficient condition for existence of limit cycle in section 3. In section 4, we classify the global phase portraits and obtain the complete global bifurcation diagram. Moreover, to show the existence of the bifurcation of separatrix loop surface numerically, we give two numerical phase portraits in II and III.

2. Local dynamics of system (4)

In this section we analyze the local dynamics of system (4). Firstly, we discuss the finite equilibria.

Lemma 2.1. *System (4) has a unique finite equilibrium $O:(0,0)$. O is a stable (resp. an unstable) node for $a \geq 2$ (resp. $a \leq -2$); a stable (resp. an unstable) focus when $0 < a < 2$(resp. $-2 < a < 0$); a center for $a = c = 0$; a stable weak focus of order one for $a = 0, c > 0$.*

Proof. It is easy to know that there is only one equilibrium O of system (4). Clearly, the Jacobian matrix at O is

$$J = \begin{pmatrix} -a & 1 \\ -1 & 0 \end{pmatrix}.$$

Obviously, $\mathrm{det}J = 1$, $\mathrm{tr}J = -a$. Then, we have $\mathrm{tr}^2 J - 4\mathrm{det}J = a^2 - 4$. Therefore, when either $a \geq 2$ or $a \leq -2$, O is a node. O is stable (resp. unstable) when $a \geq 2$ (resp. $a \leq -2$). Moreover, O is a focus and stable (resp. unstable) when $0 < a < 2$ (resp.$-2 < a < 0$). When $a = 0$, i.e., $\mathrm{tr}J = 0$, we only know that O is a center or focus. In other words, we need to judge O further when $a = 0$. Considering $a = 0$, we rewrite system (4) as follows

$$\begin{cases} \dot{x} = y + p(x), \\ \dot{y} = -x + q(x), \end{cases}$$

where $p(x) = -bx^2 - cx^3 - x^4$ and $q(x) = 0$. By [19, p. 152], we can obtain the first focal value at $(x, y) = (0, 0)$ of system (4)

$$\begin{aligned} g_3 &= \frac{1}{16}\{(p_{xxx} + p_{xyy} + q_{xxy} + q_{yyy}) - [p_{xy}(p_{xx} + p_{yy}) - q_{xy}(q_{xx} + q_{yy}) - p_{xx}q_{xx} + p_{yy}q_{yy}]\} \\ &= \frac{-3c}{8} < 0 \end{aligned}$$

for $c > 0$. Thus, O is a stable weak focus of order one when $c > 0$.

Considering the other case $a = c = 0$, O is a center by the symmetric property of vector field $(y - bx^2 - x^4, -x)$ to the y-axis. $\square$

By Lemma 2.1, it follows that O is a stable weak focus of order one for $a = 0, c > 0$ and a source for $a < 0$. Therefore, when a crosses the value $a = 0$ into the second quadrant, O loses the stability. In other words, a Hopf bifurcation occurs at O. Then, in the following lemma we will prove it.

Lemma 2.2. *Assume that $c > 0$. System (4) undergoes a Hopf bifurcation at $(0, 0)$ and generates a unique, stable, limit cycle at a small neighborhood of $(0, 0)$ as a decreases from zero.*

Proof. When $a = \varepsilon$, the eigenvalues of J at O are $\lambda_{1,2} = -\frac{\varepsilon}{2} \pm \sqrt{\frac{\varepsilon^2}{4} - 1}$, where $|\varepsilon| > 0$ is small. It is easy to see that $\lambda_{1,2}$ are a simple pair of pure imaginary roots when $\varepsilon = 0$. Thus, it is easy to obtain

$$\frac{d}{d\varepsilon}\left(\mathrm{Re}\,\lambda_{1,2}(\varepsilon)\right)\big|_{\varepsilon=0} = \frac{d(-\varepsilon/2)}{d\varepsilon} = -\frac{1}{2} \neq 0$$

and

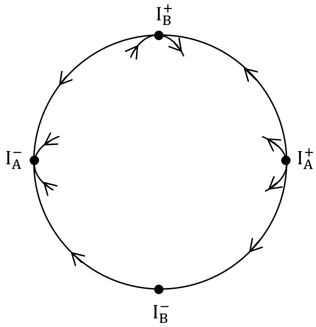


Fig. 2. Equilibria at infinity of system (4) on the Poincaré disc.

$$\kappa(\varepsilon) := \varepsilon g_3 \frac{d}{d\varepsilon} \left(\operatorname{Re} \lambda_{1,2}(\varepsilon)\right)|_{\varepsilon=0} = \frac{3c\varepsilon}{16}.$$

By Theorem 2.4 of [7, Chapter 3], $\kappa(\varepsilon) < 0$ implies that system (4) undergoes a Hopf bifurcation at $(0, 0)$ when $\varepsilon < 0$ and a stable limit cycle bifurcates from O. $\square$

In order to study the dynamics of large $|x| + |y|$ of system (4), we study the qualitative properties of equilibria at infinity.

Using a Poincaré transformation $x = 1/z, y = u/z$, we change system (4) to

$$\begin{cases} \frac{du}{dt_1} = u + cuz + buz^2 - z^3 + auz^3 - u^2z^3, \\ \frac{dz}{dt_1} = z + cz^2 + bz^3 + az^4 - uz^4, \end{cases} \tag{5}$$

where $dt = z^3 dt_1$. Obviously, system (5) has a unique equilibrium $A : (0, 0)$ on the u-axis. Clearly, A is an unstable star node. A on the (u, z)-plane corresponds to two nodes I_A^+ and I_A^- at infinity on the x-axis on the (x, y)-plane, where I_A^+ is an unstable node and I_A^- is a stable node. With the other Poincaré transformation $x = v/z, y = 1/z$, we can write system (4) in the following form

$$\begin{cases} \frac{dv}{dt_1} = z^3 - v^4 - avz^3 - cv^3z - bv^2z^2 + v^2z^3, \\ \frac{dz}{dt_1} = vz^4, \end{cases} \tag{6}$$

where $dt = z^3 dt_1$. One needs to investigate the unique equilibrium $B : (0, 0)$ on the (v, z)-plane. It is obvious that B is degenerate. B on (v, z)-plane corresponds to I_B^+ and I_B^- at infinity of the positive and negative y-axis on (x, y)-plane.

Definition 2.1. An equilibrium is called a saddle when it has exactly two stable manifolds and two unstable manifolds. An equilibrium is called a cusp when it has exactly one stable manifold and one unstable manifold.

By [22] and [24], we can obtain the following lemma directly. However, a proof of this lemma is sketchy in [22], and Llibre and Zhang in [24] did not prove it. Therefore, we intend to give a different complete proof of this lemma.

Lemma 2.3. *I_B^+ is a degenerate saddle and I_B^- is a cusp. Moreover, the equilibria at infinity on the Poincaré disc are shown in* Fig. 2.

Proof. By the polar coordinates $v = r\cos\theta$ and $z = r\sin\theta$, system (6) can be written in polar form

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{H_1(\theta) + O(r)}{G_1(\theta) + O(r)}$$

where $G_1(\theta) = -\sin^4\theta$ and $H_1(\theta) = \cos\theta\sin^3\theta$. It is easy to obtain that $G_1(\theta) = 0$ has two real roots 0 and π, and $H_1(0) = H_1(\pi) = 0$. By [33], we need $H(\theta) \neq 0$ in any exceptional direction so that we can use the normal sector method. Therefore, we intend to blow up this degenerate equilibrium. With the Briot-Bouquet transformation $v \to v, z \to z_1 v$, we rewrite system (6) as

$$\begin{cases} \frac{dv}{dt_2} = -v^2 - cz_1v^2 - bz_1^2v^2 + z_1^3v - az_1^3v^2 + z_1^3v^3, \\ \frac{dz_1}{dt_2} = z_1v + cz_1^2v - z_1^4 + bz_1^3v + az_1^4v, \end{cases} \tag{7}$$

where $v^2dt_1 = dt_2$. We need only to analyze equilibrium $(0, 0)$ of system (7). Obviously, it is degenerate. With polar coordinates, by (7) we obtain

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{H_2(\theta) + O(r)}{G_2(\theta) + O(r)}$$

where $G_2(\theta) = 2\sin\theta\cos^2\theta$ and $H_2(\theta) = \cos\theta\left(\sin^2\theta - \cos^2\theta\right)$. It is easy to compute that the roots of $G_1(\theta) = 0$ are $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$. Since

$$G_2'(0)H_2(0) = G_2'(\pi)H_2(\pi) = -2 < 0,$$

system (7) has a unique orbit connecting $(0, 0)$ along $\theta = 0$ as $t \to +\infty$ and a unique orbit connecting $(0, 0)$ along $\theta = \pi$ as $t \to -\infty$. Because $H_2(\frac{\pi}{2}) = H_2(\frac{3\pi}{2}) = 0$, we need to use Briot-Bouquet transformation further. With $v \to v_1z_1, z_1 \to z_1$, we change system (7) to

$$\begin{cases} \frac{dv_1}{dt_3} = -2v_1^2 - 2cv_1^2z_1 - 2bv_1^2z_1^2 - 2av_1^2z_1^3 + 2v_1z_1^2 + v_1^3z_1^4, \\ \frac{dz_1}{dt_3} = v_1z_1 + cv_1z_1^2 - z_1^3 + bv_1z_1^3 + av_1z_1^4, \end{cases} \tag{8}$$

where $z_1dt_2 = dt_3$. Here, we need only to study $(0, 0)$ of system (8). Clearly, the origin of system (8) is degenerate. By $v_1 = r\cos\theta, z_1 = r\sin\theta$, from (8) we obtain

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{H_3(\theta) + O(r)}{G_3(\theta) + O(r)}$$

where $G_3(\theta) = 3\sin\theta\cos^2\theta$ and $H_3(\theta) = \cos\theta\left(\sin^2\theta - 2\cos^2\theta\right)$. Clearly, $G_3(\theta) = 0$ has exactly four roots $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and $G_3'(0)H_3(0) = G_3'(\pi)H_3(\pi) = -6 < 0$. Then, system (8) has a unique orbit connecting $(0, 0)$ along $\theta = 0$ as $t \to +\infty$ and a unique orbit connecting $(0, 0)$ along $\theta = \pi$ as $t \to -\infty$. However, $H_3(\frac{\pi}{2}) = H_3(\frac{3\pi}{2}) = 0$. Thus, we need to use Briot-Bouquet transformation further in the directions $\theta = \pi/2, 3\pi/2$. Repeating the aforementioned process, with $v_1 \to v_2z_1, z_1 \to z_1$, (8) is changed into

$$\begin{cases} \frac{dv_2}{dt_4} = -3v_2^2 + 3v_2z_1 - 3cv_2^2z_1 - 3bv_2^2z_1^2 - 3av_2^2z_1^3 + v_2^3z_1^5, \\ \frac{dz_1}{dt_4} = v_2z_1 - z_1^2 + cv_2z_1^2 + bv_2z_1^3 + av_2z_1^4, \end{cases} \tag{9}$$

where $z_1dt_3 = dt_4$. Here, we need only to study the degenerate equilibrium $(0, 0)$ of system (9). By $v_2 = r\cos\theta, z_1 = r\sin\theta$, from (9) we obtain

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{H_4(\theta) + O(r)}{G_4(\theta) + O(r)}$$

where

$$G_4(\theta) = 4\cos^2\theta\sin\theta - 4\cos\theta\sin^2\theta \text{ and } H_4(\theta) = -3\cos^3\theta + 3\cos^2\theta\sin\theta + \cos\theta\sin^2\theta - \sin^3\theta.$$

Then, $G_4(\theta) = 0$ has exactly six roots $0, \frac{\pi}{4}, \frac{\pi}{2}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}$. Moreover,

$$G_4'(0)H_4(0) = G_4'(\pi)H_4(\pi) = -12 < 0 \text{ and } G_4'(\frac{\pi}{2})H_4(\frac{\pi}{2}) = G_4'(\frac{3\pi}{2})H_4(\frac{3\pi}{2}) = -4 < 0.$$

Therefore, system (9) has a unique orbit connecting $(0,0)$ along $\theta = 0$ as $t \to +\infty$, a unique orbit connecting $(0,0)$ along $\theta = \pi/2$ as $t \to +\infty$, a unique orbit connecting $(0,0)$ along $\theta = \pi$ as $t \to -\infty$ and a unique orbit connecting $(0,0)$ along $\theta = 3\pi/2$ as $t \to -\infty$. But $H_3(\frac{\pi}{4}) = H_3(\frac{5\pi}{4}) = 0$. Thus, we need to blow up the degenerate equilibrium. By transformation $v_2 \to v_2, z_1 \to z_2 v_2$, (9) can be rewritten as

$$\begin{cases} \frac{dv_2}{dt_5} = -3v_2 + 3v_2 z_2 - 3cv_2^2 z_2 - 3bv_2^3 z_2^2 - 3av_2^4 z_2^3 + v_2^7 z_2^5, \\ \frac{dz_2}{dt_5} = 4z_2 - 4z_2^2 + 4cvz_2^2 + 4bv_2^2 z_2^3 + 4av_2^3 z_2^4 - v_2^6 z_2^6, \end{cases} \tag{10}$$

where $v_2 dt_4 = dt_5$. To study the exceptional directions $\pi/4$ and $5\pi/4$ of system (9), we need only to study the degenerate equilibrium $(0,1)$ of system (10). For simplicity, we move the point $(0,1)$ to the origin. In other words, with transformation $v_2 \to v_2, z_2 \to z_3 + 1$, system (10) becomes

$$\begin{cases} \frac{dv_2}{dt_5} = 3v_2 z_3 - 3cv_2^2 (z_3+1) - 3bv_2^3 (z_3+1)^2 - 3av_2^4 (z_3+1)^3 + v_2^7 (z_3+1)^5, \\ \frac{dz_3}{dt_5} = 4cv_2 - 4z_3 - 4z_3^2 + 8cv_2 z_3 + 4cv_2 z_3^2 + 4bv_2^2 (z_3+1)^3 + 4av_2^3 (z_3+1)^4 - v_2^6 (z_3+1)^6. \end{cases} \tag{11}$$

With transformation $v_2 \to v_2, z_3 \to -z_4 + cv_2, t_5 \to -t_5/4$, system (11) becomes

$$\begin{cases} \dfrac{dv_2}{dt_5} = -\dfrac{3}{4} v_2 (-z_4 + cv_2) + \dfrac{3}{4} cv_2^2 (-z_3 + cv_2 + 1) + \dfrac{3}{4} bv_2^3 (-z_4 + cv_2 + 1)^2 \\ \qquad + \dfrac{3}{4} av_2^4 (-z_4 + cv_2 + 1)^3 - \dfrac{1}{4} v_2^7 (-z_4 + cv_2 + 1)^5, \\ \dfrac{dz_4}{dt_5} = z_4 - z_4^2 + \dfrac{3}{4} cz_4 v_2 + c^2 v_2^2 + cv_2 z_4^2 - \dfrac{11}{4} c^2 v_2^2 z_4 + \dfrac{7}{4} c^3 v_2^3 \\ \qquad + bv_2^2 (-z_4 + cv_2 + 1)^3 + av_2^3 (-z_4 + cv_2 + 1)^4 - \dfrac{1}{4} v_2^6 (-z_4 + cv_2 + 1)^6 \\ \qquad + \dfrac{3}{4} bcv_2^3 (-z_4 + cv_2 + 1)^2 + \dfrac{3}{4} acv_2^4 (-z_4 + cv_2 + 1)^3 - \dfrac{c}{4} v_2^7 (-z_4 + cv_2 + 1)^5. \end{cases} \tag{12}$$

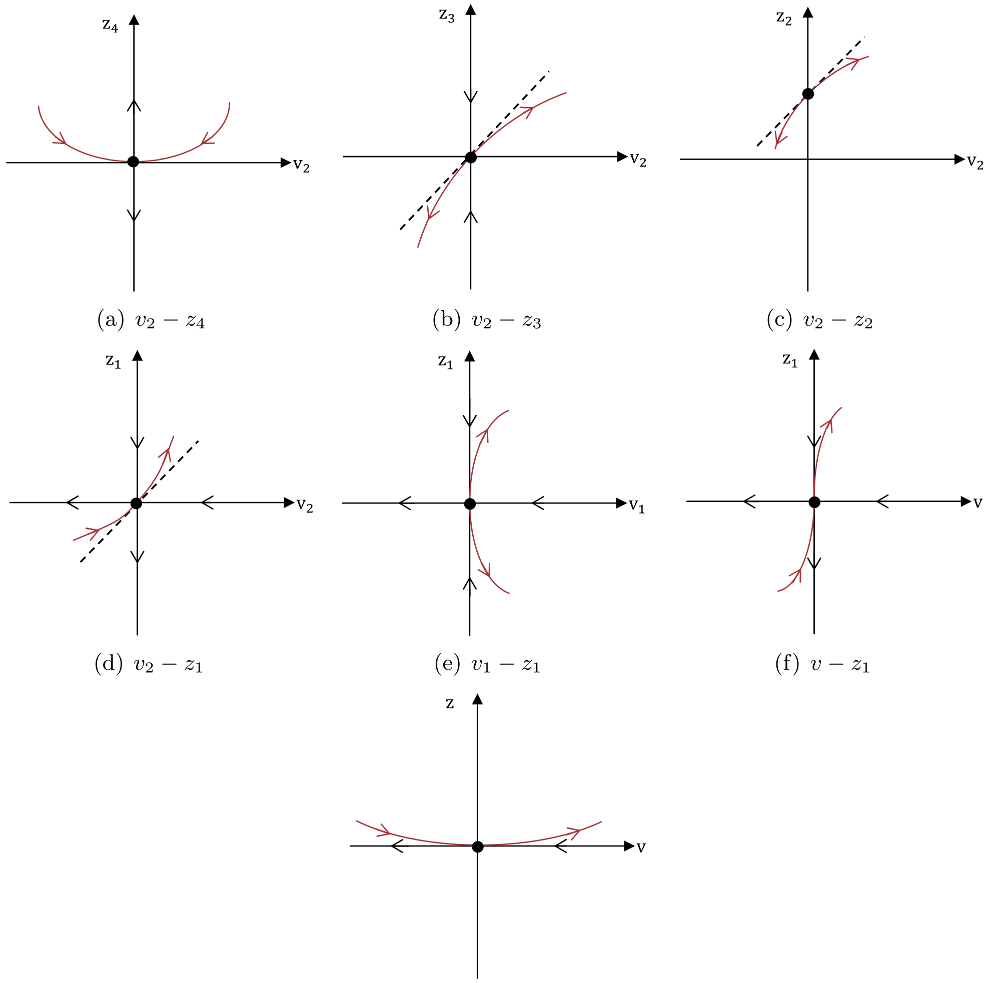
By the implicit function theorem and $dz_4/dt_5 = 0$, we can compute

$$z_4 = \Upsilon(v_2) = a_2 v_2^2 + a_3 v_2^3 + a_4 v_2^4 + a_5 v_2^5 + a_6 v_2^6 + o(v_2^6), \tag{13}$$

where $a_2 = -c^2 - b, a_3 = -c^3 - 3\,bc - a, a_4 = -c^4 - 6\,bc^2 - 4\,ac - 2\,b^2, a_5 = -c^5 - 10\,bc^3 - 10\,ac^2 - 10\,b^2 c - 5\,ab, a_6 = -c^6 - 15\,bc^4 - 20\,ac^3 - 30\,b^2 c^2 - 30\,abc - 5\,b^3 - 3\,a^2 + 1/4$. Therefore, by (13), we have

$$\frac{dv_2}{dt_5} = -\frac{1}{16} v_2^7 + O\left(|v_2|^7\right).$$

By Theorem 7.1 of [33, Chapter 2], we can conclude that $(0,0)$ of system (12) is a saddle, as shown in Fig. 3(a). Therefore, the origin of system (11) is also a saddle, as shown in Fig. 3(b). In other words, $(0,1)$ of system (10) is still a saddle, as shown in Fig. 3(c). Now, the number of orbits connecting the origin along $\theta = \pi/4, 5\pi/4$ of system (9) is clear. Then, the origin of system (9) is shown in Fig. 3(d). So, the origin of



(a) $v_2 - z_4$ (b) $v_2 - z_3$ (c) $v_2 - z_2$

(d) $v_2 - z_1$ (e) $v_1 - z_1$ (f) $v - z_1$

(g) $v - z$

Fig. 3. Orbits changing under the Briot-Bouquet transformations.

system (8) is shown in Fig. 3(e), the origin of system (7) is shown in Fig. 3(f) and the origin of system (6) is shown in Fig. 3(g). Thus, I_B^+ is a saddle and I_B^- is a cusp by Definition 2.1. Combining the qualitative properties of $I_A^\pm$ and $I_B^\pm$, these equilibria at infinity on the Poincaré disc are shown in Fig. 2. □

3. Nonlocal dynamics

As said in Section 1, system (4) has at most one limit cycle by [21]. However, the exact number of limit cycle is still unclear. In this section, we will give the necessary and sufficient condition of the existence of limit cycles of system (4).

Lemma 3.1. *A necessary condition of the existence of limit cycles of system* (4) *is* $ac < 0$.

Proof. First, we claim that system (4) has no limit cycle for $a = c = 0$. Assume that system (4) exhibits a limit cycle Γ for $a = c = 0$. In other words, Γ is a isolated closed orbit. Then, all orbits except O are closed by the symmetry of $(y - F(x), -x)$ for $a = c = 0$. However, a limit set of a orbit in a small neighborhood of Γ is Γ since Γ is a limit cycle. This is a contradiction.

Second, we claim that system (4) has no limit cycle for $ac \geq 0$ and $a^2 + c^2 > 0$. Since

$$F(x) - F(-x) = 2ax + 2cx^3 > 0$$

for $x > 0$, the assertion can be proven by [22] and [24, Proposition 4]. Thus, this proof is finished. □

When $ac < 0$, we give further the following proposition.

Proposition 3.1. *When $ac < 0$, there is a continuous function $c = \varphi(a, b)$ which is strictly decreasing on a, for which the following statements hold:*

(a) *System (4) has a separatrix loop if and only if $c = \varphi(a, b)$.*
(b) *System (4) has a unique limit cycle if and only if $c > \varphi(a, b)$.*
(c) *System (4) has no limit cycles for $c < \varphi(a, b)$.*

Proof. First, we will prove that vector fields of system (4) are rotated about a and c. It is clear that

$$\begin{vmatrix} y - (a_1x + bx^2 + cx^3 + x^4), & -x \\ y - (a_2x + bx^2 + cx^3 + x^4), & -x \end{vmatrix} = (a_1 - a_2)x^2 > 0, \tag{14}$$

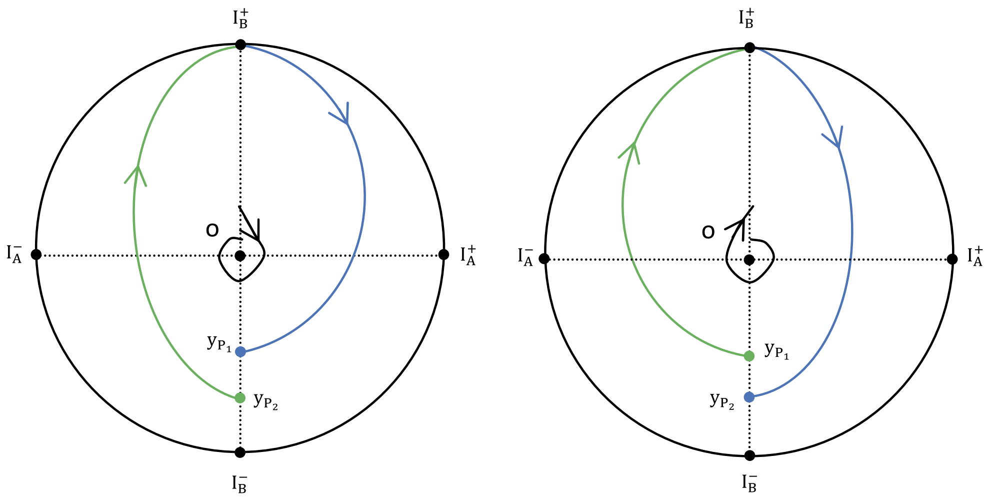
where $x > 0$ and $a_1 > a_2$ are arbitrary. Thus, by [32, Section 3] and [33, Section 3 of Chapter 4], the vector field $(y - (ax + bx^2 + cx^3 + x^4), -x)$ is a generalized vector field with respect to parameter a. It is also obvious that

$$\begin{vmatrix} y - (ax + bx^2 + c_1x^3 + x^4), & -x \\ y - (ax + bx^2 + c_2x^3 + x^4), & -x \end{vmatrix} = (c_1 - c_2)x^4 > 0, \tag{15}$$

where $x > 0$ and $c_1 > c_2$ are arbitrary. Then, the vector field $(y - (ax + bx^2 + cx^3 + x^4), -x)$ is rotated about c.

Second, we claim that the unstable and stable manifolds of I_B^+ either intersect the negative y-axis or connect with O. On the one hand, by Lemma 2.3, the unstable manifolds of I_B^+ cannot connect with I_A^+ and I_B^- since I_A^+ is unstable and I_B^- is a cusp. On the other hand, the unstable manifolds of I_B^+ cannot intersect anticlockwise the positive y-axis since $\dot{x} > 0$ in the positive y-axis. The assertion is proven.

Let $a - c + k = 0$, where $k > 0$ is a any constant. By Lemma 3.1, system (4) has no limit cycle for $ac \geq 0$. Assume that the first intersection points of the unstable and stable manifolds of I_B^+ and the negative y-axis are respectively P_1 and P_2, where one of P_1 and P_2 allows to be O. On the one hand, we consider $a = 0$ and $c = k$. Since O is a stable focus by Lemma 2.1, we claim that $y_{P_1} > y_{P_2}$, as shown in Fig. 4(a). Otherwise, when $y_{P_1} < y_{P_2}$, system (4) has an unstable limit cycle by a Poincaré-Bendixson theorem; when $y_{P_1} = y_{P_2}$, i.e., system (4) has a separatrix loop, system (4) has an unstable limit cycle for small $a < 0$ by the rotated parameter a and homoclinic bifurcation. This is a contradiction. Thus, the assertion is proven. On the other hand, we can prove similarly that $y_{P_1} < y_{P_2}$ for $c = 0$ and $a = -k$, as shown in Fig. 4(b). It is easy to check that the vector field $(y - (ax + bx^2 + (a + k)x^3 + x^4), -x)$ is rotated about a. By [28,29], the unstable and stable manifolds of I_B^+ vary monotonically as a vary monotonically. Further, y_{P_1} decreases and y_{P_2} increases as a increases. By the mean value theorem, there is a unique value $a_0 \in (-k, 0)$ for fixed b and k such that $y_{P_1} = y_{P_2}$, i.e., system (4) has a separatrix loop. Note that k is arbitrary and the vector fields of system (4) is rotated about a and c. Therefore, there is a continuous decreasing function $c = \varphi(a, b)$ such that system (4) has a separatrix loop. Thus, the statement **(a)** is proven.



(a) for $a = 0$ and $c = k$ (b) for $a = -k$ and $c = 0$

Fig. 4. The first intersection points of the unstable and stable manifolds of I_B^+ and the negative y-axis.

Consider $c > \varphi(a, b)$, i.e., $y_{P_1} > y_{P_2}$. With $y_{P_1} > y_{P_2}$ and the instability of O, system (4) has at least one stable limit cycle by the Poincaré-Bendixson theorem. By [21], system (4) has at most one limit cycle. Moreover, the limit cycle is stable and simple if it exists. Thus, the statement **(b)** is proven.

Finally, consider $c < \varphi(a, b)$, i.e., $y_{P_1} < y_{P_2}$. Assume that system (4) exhibits one stable limit cycle. By $y_{P_1} < y_{P_2}$ and the Poincaré-Bendixson theorem, system (4) has at least one unstable limit cycle surrounding the stable limit cycle. This is a contradiction. Thus, the statement **(c)** is proven. Consequently, this proof is finished. □

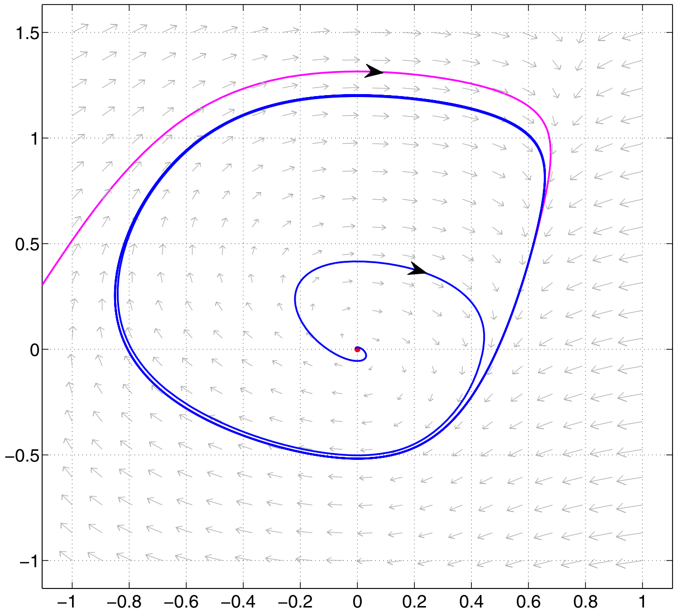
4. Proof of Theorem 1.1 and numerical examples

Proof of Theorem 1.1. By the aforementioned lemmas and propositions, system (4) occurs a Hopf bifurcation H and a bifurcation of separatrix loop SL. Then, the upper half ac-plane is divided by H and SL into three regions I, II, III when b is fixed. In other words, the complete bifurcation diagram is given, as shown in Fig. 1.

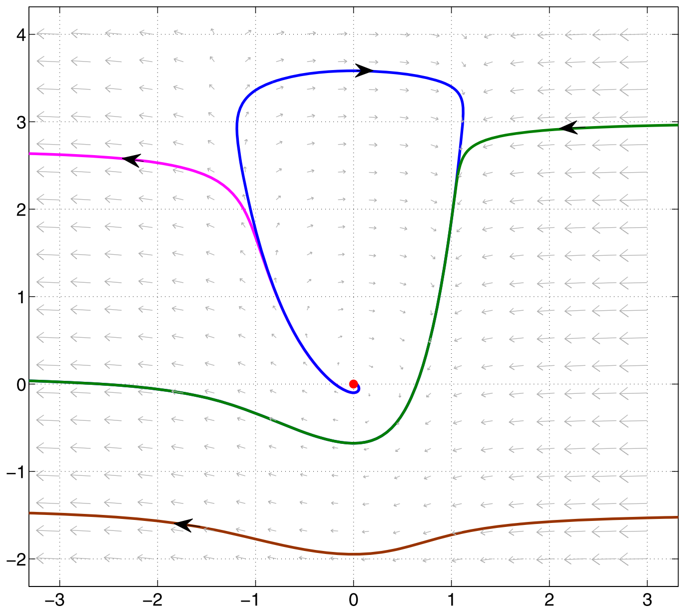
It is note that the equilibria at infinity of system (4) are fixed for all $(a, b, c) \in \mathbb{R} \times \mathbb{R} \times [0, \infty)$ by Lemma 2.3. In $I \cup H$, the origin of system (4) is a sink by Lemma 2.1 and system (4) has no limit cycle by Lemma 3.1. Therefore, the global phase portrait in the region $I \cup H$ can be obtained. In II, the origin of system (4) is a source by Lemma 2.1 and system (4) has a unique limit cycle by Proposition 3.1. Therefore, the global phase portrait in the region II can be obtained. In III, the origin of system (4) is a source by Lemma 2.1 and system (4) has no limit cycle by Proposition 3.1. Therefore, the global phase portrait in the region III can be obtained. In O, the global phase portrait has been given in [22] and [24]. In SL, the origin of system (4) is a source by Lemma 2.1 and system (4) has a separatrix loop and no limit cycle by Proposition 3.1. Therefore, the global phase portrait in the region SL can be obtained. Thus, we have completed this proof. □

Then, to demonstrate the analytical results of the existence of the bifurcation of separatrix loop surface, we give the following two numerical examples.

Example 1. When $(a, b, c) = (-1, 1, 3)$, the numerical phase portrait is shown in Fig. 5(a). We can find that system (4) has a unique limit cycle by numerical simulations, implying $(-1, 1, 3) \in II$.



(a) for $(a, b, c) = (-1, 1, 3)$



(b) for $(a, b, c) = (-1, 1, 1)$

Fig. 5. The numerical phase portraits.

Example 2. When $(a, b, c) = (-1, 1, 1)$, the numerical phase portrait is shown in Fig. 5(b). We can find that system (4) has no limit cycle by numerical simulations, implying $(-1, 1, 1) \in III$.

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