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Finite-time blowup of smooth solutions for the relativistic generalized Chaplygin Euler equations

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A B S T R A C T

In this paper, the Cauchy problem of the 3 + 1-dimensional relativistic Euler equations for generalized Chaplygin gas with non-vacuum initial data is considered. It is shown that for large background energy-mass density and small pressure coefficient, the smooth solutions of the relativistic Euler equations for generalized Chaplygin gas with the generalized subluminal condition will blow up on finite time when the initial radial component of the generalized momentum is sufficiently large. Moreover, our blowup condition is independent of the signs of the generalized mass.

1. Introduction, preliminary and statement of main result

The $3+1$-dimensional relativistic Euler equations with fixed space-time coordinates $(t,x) = (t,x_1,x_2,x_3)$ can be written as (reference: equation (1.3) in [20])

$$\begin{cases} \partial_t\left(\dfrac{\rho c^2+p}{c^2-|v|^2}-\dfrac{p}{c^2}\right)+\nabla\cdot\left(\dfrac{\rho c^2+p}{c^2-|v|^2}v\right)=0, \\ \partial_t\left(\dfrac{\rho c^2+p}{c^2-|v|^2}v\right)+\nabla\cdot\left(\dfrac{\rho c^2+p}{c^2-|v|^2}v\otimes v\right)+\nabla p=0, \end{cases} \tag{1}$$

where $\rho \geq 0$, v and p are the mass-energy density, transformed velocity and pressure respectively. $|v|$ is always less than the light speed c so that $c^2-|v|^2$ is always positive.

System (1) plays a prominent role in cosmology: it is often used to model the evolution of the average matter energy content of the universe. In [34], Weinberg gave an explanation of the role that the relativistic

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Euler equations (1) play in the standard model of cosmology. The equations (1) are also widely used in astrophysics and high-energy nuclear physics [22].

The pressure p in $(1)_2$ is determined by the state equation. In this circumstance, the state equation for generalized Chaplygin gas, namely,

$$p = -K\rho^{-\gamma},\ 0 < \gamma \leq 1,\ K > 0 \tag{2}$$

is adopted here.

In terms of physics, since the discovery of dark matter and dark energy as stated in the literature such as [23,29,21] for explaining that the mass of "visible" matter only comprise 4% of the total energy density in the four-dimensional standard cosmology model, the generalized Chaplygin gas model has been introduced in [13] and developed in [35] as a unification of dark matter and dark energy, where the invisible energy component is regarded as a unified dark fluid.

Remark 1. In general theory of relativity, the exact model of system (1) comes from the local conservation of stress–energy for a perfect fluid in the 4-dimensional Minkowski spacetime with metric signature $(-,+,+,+)$. In other words, one has that the covariant derivative with respect to the given metric $g = (g^{ij})$ of the stress-energy tensor T, whose components are given by $T^{ij} = (p + \rho c^2)u^i u^j + pg^{ij}$, is zero, where $i, j = 0, 1, 2, 3$, $g = (g^{ij})$ is the 4 by 4 diagonal matrix with diagonal $(-1, 1, 1, 1)$. When one fixes a space-time coordinates as $(t, x) = (t, x_1, x_2, x_3)$ and write $u = (u^1, u^2, u^3)$ the fluid velocity, then system (1) is derived by letting

$$v = \frac{cu}{\sqrt{1 + |u|^2}}.$$

Subsequently, the negative pressure (2), which is a parameter function in a relativistic system, is assumed as a candidate to explain the phenomenon in the last paragraph resulting from the accelerated expansion of the universe. For more details, readers can refer to [20,30,16].

As far as mathematics is concerned, system (1) returns to the classical Euler equations

$$\begin{cases} \rho_t + \nabla \cdot (\rho v) = 0, \\ (\rho v)_t + \nabla \cdot (\rho v \otimes v) + \nabla p = 0 \end{cases} \tag{3}$$

when c tends to infinity. For the classical Euler equations with state equation (2), the results focuses on low dimensions Riemann problem [1,9,10,26]. To the authors' knowledge, while there is no finite-time blowup result of smooth solutions for the original system without symmetry assumptions, an almost blowup result of the system (3) with pressure (2) was included in [5], where Cheung established a finite blowup result of smooth solutions in a designed non-empty space.

On the other hand, the existing literature of system (1) with state equation (2) also mainly centers on the investigation of the Riemann problem in the 1-dimensional case [25,24,4,3,11,12]. For some results related to the well-posedness and long time behavior, readers can refer to [32,15,33].

Before proceeding, we point out that the result of local-in-time existence of smooth solutions of system (1) with the negative pressure (2) is implied by the corresponding result for system (1) with positive pressure in [18] (Makino and Ukai) by applying the theory of symmetric hyperbolic systems and the theory of Friedrichs-Lax-Kato [14,17]. Readers can refer to the paragraph after remark 1.2 in [32] for more information.

Now, it is clear that with

$$\begin{cases} \widetilde{\rho} := \dfrac{\rho c^2 + p}{c^2 - |v|^2}, \\ \widehat{\rho} := \widetilde{\rho} - \dfrac{p}{c^2} = \dfrac{\rho c^2 + p}{c^2 - |v|^2} - \dfrac{p}{c^2}, \end{cases} \tag{4}$$

(1) is transformed into

$$\begin{cases} \widehat{\rho}_t + \nabla \cdot (\widetilde{\rho} v) = 0, \\ (\widetilde{\rho} v)_t + \nabla \cdot (\widetilde{\rho} v \otimes v) + \nabla p = 0. \end{cases} \tag{5}$$

The Cauchy problem of system (5) (or equivalently system (1)) with the following initial data is considered.

$$\rho(x, 0) = \rho_0(x) > 0, \ v(x, 0) = v_0(x), \tag{6}$$

where the support of $(\rho_0(x) - \bar{\rho}, v_0(x))$ is contained in the open ball centered at origin with radius $R > 0$. Here, $\bar{\rho} > 0$ is the background mass-energy density.

As a special case of Lemma 2.2 in [32], one has the following finite propagation speed property (FPSP).

Lemma 2. *Any smooth solution of system* (1) *with initial data* (6) *and with equation of state* (2) *will satisfy*

$$(\rho - \bar{\rho}, v) \equiv (0, 0)$$

outside

$$B(t) := \{x : |x| \leq R + st\},$$

where

$$s := \sqrt{p'(\bar{\rho})}$$

is the sound speed in the far field.

On the other hand, when the equation of state is given by $p = p(\rho)$ such that i) $p(0) = 0$, ii) $p(\rho) \geq 0$ and iii) $p''(\rho) \geq 0$ for $\rho \in (\rho_*, \rho^*)$, where $0 \leq \rho_* < \rho^* \leq \infty$, the authors in [20] proved a finite-time blowup result for the relativistic Euler equations, namely, system (1), in the case of infinite energy when "subluminal condition" is adopted. More precisely, with the following subluminal condition,

$$0 < p'(\rho) < c^2, \tag{7}$$

where c is the speed of light, the authors in [20] showed, in the spirit of Sideris [27], that finite-time singularity for smooth solutions of system (1) with initial data (6) will be developed for small background energy-density $\bar{\rho}$ if the generalized mass $M(0)$ is positive and the radial component of generalized momentum $F(0)$ is large enough, where

$$M(t) := \int_{\mathbb{R}^N} (\widehat{\rho}(t, x) - \bar{\rho}) dx$$

and

$$F(t) := \int\limits_{\mathbb{R}^3} \widetilde{\rho}(t,x)v(t,x) \cdot x dx.$$

In this paper, with the "generalized subluminal condition", namely,

$$0 < p'(\rho) \leq c^2, \tag{8}$$

the first finite-time blowup result of the three plus one dimensional system (1) for generalized Chaplygin gas without symmetry is established. To be specific, our main result is stated as follows.

Theorem 3. *Consider the Cauchy problem of system (1) with initial data (6) and with the generalized subluminal condition (8). Then, for small pressure coefficient K and large background energy-mass density $\bar{\rho}$, its smooth solutions will blow up on finite time if the initial radial component of the generalized momentum $F(0)$ is sufficiently large. More precisely, if $K \leq 1$ and $\bar{\rho} > \left(\frac{\gamma}{c^2}\right)^{\frac{1}{\gamma+1}}$, where c is the speed of light and γ is the adiabatic index of the pressure (2), then for any given finite $\tau > 0$, the smooth solutions will blow up on or before τ if $F(0) > C(\tau)$, where*

$$F(t) = \int\limits_{\mathbb{R}^3} \widetilde{\rho} v \cdot x dx \tag{9}$$

and

$$C(\tau) := \max \left\{ [2A(\tau)G(\tau)]^{1/2}, \ \left[\int\limits_0^\tau \frac{1}{2A(\mu)} d\mu \right]^{-1} \right\}$$

for some known positive C^1 functions A and G defined in (17) and (18) respectively.

Remark 4. While the generalized momentum $F(t)$ in [20] is adopted in our case, our blowup condition is independent of the signs of the generalized mass $M(0)$. In fact, from (17), one has that

$$M(0) > -\frac{2\pi^2}{5}(R + st)^5.$$

Moreover, Theorem 3 can be extended to a generally weighted generalized momentum $F_1(f,t)$ in Corollary 12 for any f with some mild conditions.

2. Implications of the generalized subluminal condition

In the literature, the subluminal condition (7) was assumed in [20,2,7,8,19] and etc. to establish various results of relativistic systems in fluid dynamics. As $\sqrt{p'(\rho)}$ is the sound speed, (7) means that one requires that the sound speed does not exceed the light speed, which is a common practice in cosmology and astrophysics [6]. Furthermore, it was stated in theorem 2 of [28] that system (1) is strictly hyperbolic if and only if (7) is fulfilled. Under this background, the implications of (7) or more generally (8) in the case that the state equation is given by the generalized Chaplygin gas (2) are presented in this section.

Lemma 5. *In the case that p is negative and given by (2), the generalized subluminal condition (8) implies that ρ and p have lower bounds depending only on c, K and γ. More precisely, one has*

$$\rho \geq \left(\frac{K\gamma}{c^2} \right)^{\frac{1}{\gamma+1}} \tag{10}$$

and

$$p \geq -K^{\frac{1}{\gamma+1}} \left(\frac{c^2}{\gamma} \right)^{\frac{\gamma}{\gamma+1}} . \tag{11}$$

Remark 6. The corresponding results of the above lemma if (7) is assumed are that one has two strict inequalities in (10) and (11).

Proof. From (8), one has

$$0 < p'(\rho) = K\gamma\rho^{-\gamma-1} \leq c^2$$

or

$$\begin{aligned} \rho^{\gamma+1} &\geq K\gamma c^{-2} \\ \rho &\geq (K\gamma)^{\frac{1}{\gamma+1}} c^{-\frac{2}{\gamma+1}} = \left(\frac{K\gamma}{c^2} \right)^{\frac{1}{\gamma+1}} . \end{aligned} \tag{12}$$

Hence,

$$\begin{aligned} \rho^{-\gamma} &\leq (K\gamma)^{-\frac{\gamma}{\gamma+1}} c^{\frac{2\gamma}{\gamma+1}} \\ p = -K\rho^{-\gamma} &\geq -K^{\frac{1}{\gamma+1}} \gamma^{-\frac{\gamma}{\gamma+1}} c^{\frac{2\gamma}{\gamma+1}} = -K^{\frac{1}{\gamma+1}} \left(\frac{c^2}{\gamma} \right)^{\frac{\gamma}{\gamma+1}} . \end{aligned} \tag{13}$$

The proof is completed. □

Next, we show that for small K value, $\widetilde{\rho}$ and $\widehat{\rho}$ are always non-negative.

Lemma 7. *Let $\widehat{\rho}$ and $\widetilde{\rho}$ be smooth solutions of system (5) with $K \leq 1$. Then, the generalized subluminal condition implies $\widehat{\rho} \geq \widetilde{\rho} \geq 0$.*

Proof. From (4), one has

$$\widehat{\rho} = \widetilde{\rho} - \frac{p}{c^2} \geq \widetilde{\rho}$$

as p is negative. Thus, it suffices to show $\widetilde{\rho}$ is non-negative.

Note that

$$\begin{aligned} \widetilde{\rho} \geq 0 &\text{ iff } \rho c^2 + p \geq 0 \\ &\text{ iff } \rho c^2 \geq K\frac{1}{\rho^\gamma} \\ &\text{ iff } \rho \geq \left(\frac{K}{c^2} \right)^{\frac{1}{\gamma+1}} , \end{aligned}$$

where "iff" stands for "if and only if".

From (12), one has

$$\rho \geq c^{-\frac{2}{\gamma+1}} \geq \left(\frac{K}{c^2}\right)^{\frac{1}{\gamma+1}}$$

if $K \leq 1$. The proof is completed. □

3. Proof of finite-time blowup result

The proof of Theorem 3 is divided into 4 intermediate results and a final step.

Lemma 8 *(Intermediate result 1). Consider the smooth solutions $(\widetilde{\rho}, \widehat{\rho}, v)$ of system (5) with initial data (6). Then, one has*

$$M(t) = M(0),$$

where

$$M(t) = \int\limits_{\mathbb{R}^N} (\widehat{\rho} - \bar{\rho}) dx.$$

Proof. Note that $M(t)$ is well-defined by the FPSP.

$$\begin{aligned} M'(t) &= \int\limits_{\mathbb{R}^3} \widehat{\rho}_t dx \\ &= -\int\limits_{\mathbb{R}^3} \nabla \cdot (\widetilde{\rho} v) dx \quad (\text{by } (5)_1) \\ &= 0 \quad (\text{by Divergence Theorem}). \end{aligned}$$

Hence, $M(t) = M(0)$. The proof is completed. □

Then, one has the following lemma which is an unexpected equality in a general setting.

Lemma 9 *(Intermediate result 2). Consider the smooth solutions $(\widetilde{\rho}, \widehat{\rho}, v)$ of system (5) with initial data (6). One has, for any form of $p = p(\rho)$ such that p is a C^1 function of ρ, the derivative of $F(f, t)$ (denoted by $F'(f, t)$) with respect to t is given by the following equality.*

$$F'(f,t) = \int\limits_{B(t)} f \widetilde{\rho} |v|^2 dx + \int\limits_{B(t)} \frac{f'}{r} \widetilde{\rho} (v \cdot x)^2 dx + \int\limits_{B(t)} (p - \bar{p})(3f + f' r) dx, \tag{14}$$

where

$$F(f, t) := \int\limits_{\mathbb{R}^3} f(r) \widetilde{\rho} v \cdot x dx.$$

Here f is any positive C^1 increasing function of $r = |x|$ on $[0, \infty)$.

Proof.

$$
\begin{aligned}
F'(f,t) &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^3} f \widetilde{\rho} v \cdot x dx \\
&= \int_{\mathbb{R}^3} f (\widetilde{\rho} v)_t \cdot x dx \\
&= - \int_{\mathbb{R}^3} f \left(\nabla \cdot (\widetilde{\rho} v \otimes v) + \nabla p \right) \cdot x dx \\
&=: I_1 + I_2,
\end{aligned}
$$

where

$$
I_1 := - \int_{\mathbb{R}^3} f \nabla \cdot (\widetilde{\rho} v \otimes v) \cdot x dx
$$

and

$$
I_2 := - \int_{\mathbb{R}^3} f \nabla p \cdot x dx.
$$

To cope with I_2, one has, for any $p \in C^1$, one has

$$
\begin{aligned}
I_2 &= - \int_{\mathbb{R}^3} \nabla (p - \bar{p}) \cdot f x dx \\
&= \int_{\mathbb{R}^3} (p - \bar{p}) \nabla \cdot (f x) dx \\
&= \int_{\mathbb{R}^3} (p - \bar{p}) (3f + f' r) dx \\
&= \int_{B(t)} (p - \bar{p}) (3f + f' r) dx.
\end{aligned} \tag{15}
$$

To handle I_1, one has

$$
\begin{aligned}
\nabla \cdot (\widetilde{\rho} v \otimes v) \cdot x &= \sum_{i,j=1}^{3} \partial_i (\widetilde{\rho} v_i v_j) x_j \\
&= \sum_{i,j=1}^{3} \left[\partial_i (\widetilde{\rho} v_i v_j) x_j + \widetilde{\rho} v_i v_j \partial_i x_j - \widetilde{\rho} v_i v_j \partial_i x_j \right] \\
&= \sum_{i,j=1}^{3} \left[\partial_i (\widetilde{\rho} v_i v_j x_j) - \widetilde{\rho} v_i v_j \partial_i x_j \right] \\
&= \sum_{j=1}^{3} \left[\nabla \cdot (\widetilde{\rho} v_j x_j) v - \widetilde{\rho} v_j v \cdot \nabla x_j \right] \\
&= \nabla \cdot \left(\sum_{j=1}^{3} \widetilde{\rho} v_j x_j v \right) - \sum_{j=1}^{3} \widetilde{\rho} v_j v \cdot \nabla x_j
\end{aligned}
$$

$$
\begin{aligned}
&= \nabla \cdot \left(\sum_{j=1}^{3} \widetilde{\rho} v_j x_j v \right) - (\widetilde{\rho} v_1 v_1 + \widetilde{\rho} v_2 v_2 + \widetilde{\rho} v_3 v_3) \\
&= \nabla \cdot \left(\sum_{j=1}^{3} \widetilde{\rho} v_j x_j v \right) - \widetilde{\rho} |v|^2 \\
&= \nabla \cdot [\widetilde{\rho}(v \cdot x) v] - \widetilde{\rho} |v|^2
\end{aligned}
$$

where ∂_i is the partial derivative with respect to x_i.

Hence, by Divergence Theorem,

$$
\begin{aligned}
I_1 &= \int\limits_{\mathbb{R}^3} f \widetilde{\rho} |v|^2 dx - \int\limits_{\mathbb{R}^3} f \nabla \cdot [\widetilde{\rho}(v \cdot x) v] \, dx \\
&= \int\limits_{\mathbb{R}^3} f \widetilde{\rho} |v|^2 dx + \int\limits_{\mathbb{R}^3} \nabla f \cdot [\widetilde{\rho}(v \cdot x) v] \, dx \\
&= \int\limits_{\mathbb{R}^3} f \widetilde{\rho} |v|^2 dx + \int\limits_{\mathbb{R}^3} \frac{f'}{r} x \cdot [\widetilde{\rho}(v \cdot x) v] \, dx \\
&= \int\limits_{\mathbb{R}^3} f \widetilde{\rho} |v|^2 dx + \int\limits_{\mathbb{R}^3} \frac{f'}{r} \widetilde{\rho} (v \cdot x)^2 dx \\
&= \int\limits_{B(t)} f \widetilde{\rho} |v|^2 dx + \int\limits_{B(t)} \frac{f'}{r} \widetilde{\rho} (v \cdot x)^2 dx.
\end{aligned}
\tag{16}
$$

Combining (15) and (16), one completes the proof. $\square$

Next, by Cauchy's inequality, the following lemma is achieved.

Lemma 10 *(Intermediate result 3). Consider the smooth solutions $(\widetilde{\rho}, \widehat{\rho}, v)$ of system (5) with initial data (6), with $K \leq 1$ and the generalized subluminal condition. Then, one has*

$$
\int\limits_{B(t)} \widetilde{\rho} |v|^2 dx \geq \frac{F^2(t)}{\int\limits_{B(t)} \widehat{\rho} |x|^2 dx}.
$$

Proof. By Cauchy inequality, one has

$$
\begin{aligned}
F^2(t) &= \left(\int\limits_{B(t)} \widetilde{\rho} v \cdot x dx \right)^2 \\
&\leq \left(\int\limits_{B(t)} \widetilde{\rho} |x|^2 dx \right) \left(\int\limits_{B(t)} \widetilde{\rho} |v|^2 dx \right)
\end{aligned}
$$

$$\leq \left(\int\limits_{B(t)} \widehat{\rho}|x|^2 dx \right) \left(\int\limits_{B(t)} \widetilde{\rho}|v|^2 dx \right)$$

The result follows. □

For the second last step, the following auxiliary functions A and G are analyzed.

Lemma 11 *(Intermediate result 4). Set*

$$A(t) := M(0) + \frac{2\pi^2}{5}(R + st)^5 \tag{17}$$

and

$$G(t) := 4\pi(R + st)^3 \left(K^{\frac{1}{\gamma+1}} \left(\frac{c^2}{\gamma} \right)^{\frac{\gamma}{\gamma+1}} - K\bar{\rho}^{-\gamma} \right). \tag{18}$$

Then, under the conditions of the Theorem 3, both A and G are positive C^1 increasing functions of t for any value of $M(0)$. Moreover,

$$A(t) = \int\limits_{B(t)} \widehat{\rho}|x|^2 dx.$$

Proof. Note that

$$\begin{aligned} 0 &< \int\limits_{B(t)} \widehat{\rho}|x|^2 dx \\ &= M(0) + \bar{\rho} \int\limits_{B(t)} |x|^2 dx \\ &= M(0) + \frac{2\pi^2}{5}(R + st)^5 \\ &= A(t). \end{aligned}$$

Thus, $A(t)$ is positive C^1 increasing for any value of $M(0)$.

On the other hand, note also that

$$\begin{aligned} & K^{\frac{1}{\gamma+1}} \left(\frac{c^2}{\gamma} \right)^{\frac{\gamma}{\gamma+1}} - K\bar{\rho}^{-\gamma} > 0 \\ & \text{iff } \bar{\rho} > \left(\frac{K\gamma}{c^2} \right)^{\frac{1}{\gamma+1}}. \end{aligned}$$

So, by the conditions of Theorem 3, namely,

$$\bar{\rho} > \left(\frac{\gamma}{c^2} \right)^{\frac{1}{\gamma+1}}$$

and $K \leq 1$, G is an increasing positive C^1 function. □

Final step.

Putting $f = 1$ in (14), one has

$$\begin{aligned} F'(t) &= \int\limits_{B(t)} \widetilde{\rho}|v|^2 dx + 3\int\limits_{B(t)} (p-\bar{p})dx \\ &\geq \frac{F^2(t)}{A(t)} + 3\int\limits_{B(t)} (p-\bar{p})dx \\ &\geq \frac{F^2(t)}{A(t)} + 3|B(t)|\left(-K^{\frac{1}{\gamma+1}}\left(\frac{c^2}{\gamma}\right)^{\frac{\gamma}{\gamma+1}} - \bar{p}\right) \\ &= \frac{F^2(t)}{A(t)} + 3|B(t)|\left(-K^{\frac{1}{\gamma+1}}\left(\frac{c^2}{\gamma}\right)^{\frac{\gamma}{\gamma+1}} + K\bar{\rho}^{-\gamma}\right). \end{aligned}$$

By (13), which is given by

$$p \geq -K^{\frac{1}{\gamma+1}}\left(\frac{c^2}{\gamma}\right)^{\frac{\gamma}{\gamma+1}},$$

one has

$$F'(t) \geq \frac{F^2(t)}{A(t)} - G(t).$$

The result follows from Lemma A.1 in the Appendix with $N = 2$.

4. Blowup mechanism

In this section, the blowup mechanism of solutions in Theorem 3 is discussed. We remark here that the corresponding mechanism for Corollary 12 is similar as $f(r)$ is a C^1 function.

First, the singularity type of system (1) in Theorem 3 is different from shock singularity (also known as wave breaking), which means the solution remains bounded but one of its higher derivatives blows up. It is because we will show that under the generalized subliminal condition (8), negative pressure (2) and the condition that the light speed c is the limit speed of v, the blowup phenomenon has to be occurred within the mass-energy density ρ or the velocity of the fluid u. The details are as follows.

From Lemma A.1, one knows that it is the $F(t)$ in (9) that becomes unbounded in finite time. More precisely, one has $F(t) \to +\infty$ as $t \to \tau^-$ if the conditions in Theorem 3 are satisfied (Note that F is increasing from the proof of Lemma A.1). Moreover, as the magnitude $|v|$ of v is bounded by the light speed c, one must have that $\widetilde{\rho}$ becomes unbounded in finite time. From $(4)_1$, one has

$$\widetilde{\rho} = \frac{\rho c^2 + p}{c^2 - |v|^2} = \frac{\rho c^2 - K\dfrac{1}{\rho^\gamma}}{c^2 - |v|^2}.$$

Note that by Lemma 5, ρ has a positive lower bound $\left(\frac{K\gamma}{c^2}\right)^{\frac{1}{\gamma+1}}$. Thus, the term $1/\rho^\gamma$ will not become infinite for all time. It follows that either ρc^2 tends to infinity in finite time or $|v|$ tends to c in finite time. However, as in the setting in [20], v is given by

$$v = \frac{cu}{\sqrt{1+|u|^2}},$$

where $u = (u^1, u^2, u^3)$ is the velocity of the fluid. Hence, $|v|$ tends to c is equivalent to $|u|$ tends to infinity. In conclusion, one must have either infinite mass-energy density ρ or infinite fluid velocity u in finite time for the blowup phenomenon in Theorem 3.

It is worth mentioning that in [31], the authors constructed exact singular solutions for the general $d+1$-dimensional relativistic Chaplygin (corresponds to $\gamma = 1$ in our setting) Euler equations in radial symmetry. The type of singularity therein is described as light-like singularity, from which the blowup mechanism is consistent with the loss of the strict hyperbolicity. Moreover, for $d \geq 2$, the exact singular solution is given by

$$(u^0, u, \rho) = \left(\frac{1}{\sqrt{1-\xi^2}}, \pm \frac{\xi}{\sqrt{1-\xi^2}} \frac{x}{r}, \infty \right),$$

where $\xi = \frac{r}{T-t}$, $r = |x|$, $u = u(t,x)$ is the fluid velocity in coordinates $(t,x) = (t, x^1, x^2, ..., x^d)$ and T is a positive constant. One can see that the solution blows up for all time t and u becomes infinity on the lines $\xi = \pm 1$. In contrast, our result demonstrates that within the framework of strict hyperbolicity (because of the subluminal condition), the smooth solutions of system (1) with $\gamma \in (0,1]$ and without symmetry can develop singularity in finite time.

5. Extension of Theorem 3

In this section, we have the following extension of Theorem 3.

Corollary 12. *Consider the Cauchy problem of system (1) with initial data (6) and with the generalized subluminal condition (8). Then, for small pressure coefficient K and large background energy-mass density $\bar{\rho}$, its smooth solutions will blow up on finite time if the initial radial component of the generalized weighted momentum $F_1(0)$ is sufficiently large. More precisely, if $K \leq 1$ and $\bar{\rho} > \left(\frac{\gamma}{c^2}\right)^{\frac{1}{\gamma+1}}$, where c is the speed of light and γ is the adiabatic index of the pressure (2), then for any given finite $\tau > 0$, the smooth solutions will blow up on or before τ if $F_1(0) > C_1(\tau)$, where*

$$F_1(t) := \int_{\mathbb{R}^3} f(r) \widetilde{\rho} v \cdot x dx$$

for any positive C^1 increasing function f of $r := |x|$ on the non-negative real line and

$$C_1(\tau) := \max \left\{ [2A_1(\tau)G_1(\tau)]^{1/2}, \left[\int_0^\tau \frac{1}{2A_1(\mu)} d\mu \right]^{-1} \right\}$$

for some known positive C^1 functions A_1 and G_1 defined in (19) and (20) respectively.

Proof. First, one has, by Lemma 9,

$$\begin{aligned} {F_1}'(t) &= \int_{B(t)} f \widetilde{\rho} |v|^2 dx + \int_{B(t)} \frac{f'}{r} \widetilde{\rho} (v \cdot x)^2 dx + \int_{B(t)} (p - \bar{p})(3f + f'r) dx \\ &\geq \int_{B(t)} f \widetilde{\rho} |v|^2 dx + \int_{B(t)} (p - \bar{p})(3f + f'r) dx. \end{aligned}$$

As in the proof of Lemma 10, one has

$$\begin{aligned}
F_1^2(t) &= \left(\int_{B(t)} f\widetilde{\rho} v \cdot x dx \right)^2 \\
&\le \left(\int_{B(t)} f\widetilde{\rho}|x|^2 dx \right) \left(\int_{B(t)} f\widetilde{\rho}|v|^2 dx \right) \\
&\le f(R+st) \left(\int_{B(t)} \widehat{\rho}|x|^2 dx \right) \left(\int_{B(t)} f\widetilde{\rho}|v|^2 dx \right).
\end{aligned}$$

Thus,

$$\int_{B(t)} f\widetilde{\rho}|v|^2 dx \ge \frac{F_1^2(t)}{A_1(t)},$$

where

$$A_1(t) := f(R+st)A(t). \tag{19}$$

Here, $A(t)$ is previously defined in (17).

Hence,

$$\begin{aligned}
{F_1}'(t) &\ge \int_{B(t)} f\widetilde{\rho}|v|^2 dx + \int_{B(t)} (p-\bar{p})(3f+f'r)dx \\
&\ge \frac{F_1^2(t)}{A_1(t)} + \int_{B(t)} (p-\bar{p})(3f+f'r)dx \\
&\ge \frac{F_1^2(t)}{A_1(t)} + \left(-K^{\frac{1}{\gamma+1}} \left(\frac{c^2}{\gamma}\right)^{\frac{\gamma}{\gamma+1}} + K\bar{\rho}^{-\gamma} \right) \int_{B(t)} (3f+f'r)dx \\
&= \frac{F_1^2(t)}{A_1(t)} + \left(-K^{\frac{1}{\gamma+1}} \left(\frac{c^2}{\gamma}\right)^{\frac{\gamma}{\gamma+1}} + K\bar{\rho}^{-\gamma} \right) \left[2\pi^2 (R+st)^3 f(R+st) \right].
\end{aligned}$$

Set

$$G_1(t) := -\left(-K^{\frac{1}{\gamma+1}} \left(\frac{c^2}{\gamma}\right)^{\frac{\gamma}{\gamma+1}} + K\bar{\rho}^{-\gamma} \right) \left[2\pi^2 (R+st)^3 f(R+st) \right]. \tag{20}$$

Then, one has

$${F_1}'(t) \ge \frac{F_1^2(t)}{A_1(t)} - G_1(t)$$

and both A_1 and G_1 are positive increasing C^1 functions of t. Now, the result follows from Lemma A.1 in the Appendix with $N = 2$ again. The proof is completed. □

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Appendix A

Lemma A.1. *Consider the following differential inequality in $F(t)$ with $N > 1$ (for integral N, it means $N \geq 2$).*

$$F'(t) \geq \frac{F^N(t)}{A(t)} - G(t). \tag{21}$$

Suppose the C^1 coefficients $A(t)$ and $G(t)$ are positive and increasing. Then, for any given finite $\tau > 0$, $F(t)$ will blow up on or before τ if $F(0) > C(\tau)$, where

$$C(\tau) := \max\left\{ [2A(\tau)G(\tau)]^{1/N}, \left[\int_0^\tau \frac{N-1}{2A(\mu)} d\mu\right]^{-\frac{1}{N-1}} \right\}. \tag{22}$$

Proof. Fix $\tau > 0$, for any $0 \leq t \leq \tau$, (21) becomes

$$\begin{aligned} F'(t) &\geq \frac{F^N(t)}{A(t)} - G(t) \\ &= \frac{F^N(t)}{2A(t)} + \left[\frac{F^N(t)}{2A(t)} - G(t)\right] \\ &\geq \frac{F^N(t)}{2A(t)} + \left[\frac{F^N(t)}{2A(\tau)} - G(\tau)\right] \\ &=: \frac{F^N(t)}{2A(t)} + Q(t). \end{aligned}$$

By (22), $Q(0) > 0$. Hence $Q(t) \geq 0$ on $[0, \tau]$ and

$$F'(t) \geq \frac{F^N(t)}{2A(t)} \tag{23}$$

on $[0, \tau]$. Note that $F(0) > 0$. Hence, $F(t) > 0$ on $[0, \tau]$ by (22) as F is increasing by (23). Therefore,

$$\frac{1}{F^{N-1}(0)} - \frac{1}{F^{N-1}(t)} \geq \int_0^t \frac{N-1}{2A(\mu)} d\mu$$

$$0 < \frac{1}{F^{N-1}(t)} \leq \frac{1}{F^{N-1}(0)} - \int_0^t \frac{N-1}{2A(\mu)} d\mu. \tag{24}$$

By (22), the right hand side of (24) is negative when $t = \tau$. Thus, the solutions blow up on or before time τ. The proof is completed. □

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